

ON A MINIMAL COUNTEREXAMPLE TO BRAUER'S $k(B)$ -CONJECTURE

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ABSTRACT. We study Brauer's long-standing $k(B)$ -conjecture on the number of characters in p -blocks for finite quasi-simple groups and show that their blocks do not occur as a minimal counterexample for $p \geq 5$ nor in the case of abelian defect. For $p = 3$ we obtain that the principal 3-blocks do not provide minimal counterexamples. We also determine the precise number of irreducible characters in unipotent blocks of classical groups for odd primes.

1. INTRODUCTION

In his 1954 address at the International Congress in Amsterdam Richard Brauer posed a list of fundamental problems in representation theory of finite groups [2], many of which are still open. One among them is his conjecture on the number $k(B)$ of irreducible complex characters in a p -block B of a finite group: this number $k(B)$ should be at most equal to the order $|D|$ of a defect group D of B . Brauer and Feit [3] already showed that $k(B) \leq p^{2d-2}$ if $|D| = p^d$, but to the present day, no general upper bound linear in $|D|$ is known for $k(B)$. For p -solvable groups the $k(B)$ -conjecture was shown by Nagao to reduce to the coprime $k(GV)$ -problem, which was finally settled in 2004, see [14]. In fact, using this result Robinson [25, Thm. 1] showed that for p -solvable groups we even have $k(B) < |D|$ whenever D is non-abelian.

This motivates to consider the following strong form of Brauer's $k(B)$ -conjecture:

*Let B be a block of a finite group with defect group D .
Then $k(B) \leq |D|$, with strict inequality unless D is abelian.*

In this paper we investigate a possible minimal counterexample to this conjecture, which by the results cited above must necessarily be non-solvable. This focuses attention on the non-abelian simple groups and their covering groups. Our main result is:

Theorem 1. *Let $p \geq 5$ be a prime and B a p -block of a finite quasi-simple group G . Then B is not a minimal counterexample to the strong form of Brauer's $k(B)$ -conjecture.*

Here, (G, B) is called a minimal counterexample if the conjecture holds for all p -blocks B_1 of groups G_1 with $|G_1/Z(G_1)|$ strictly smaller than $|G/Z(G)|$ having defect groups isomorphic to those of B . We also show that blocks with abelian defect cannot lead to minimal counterexamples (in view of Theorem 1, this concerns the primes $p = 2, 3$):

Date: September 3, 2018.

1991 Mathematics Subject Classification. 20C15, 20C33.

Key words and phrases. Number of simple modules, Brauer's $k(B)$ -conjecture, blocks of simple groups. The author gratefully acknowledges financial support by SFB TRR 195.

Theorem 2. *Let p be a prime and B a p -block of a finite quasi-simple group G with abelian defect groups. Then B is not a minimal counterexample to Brauer's $k(B)$ -conjecture.*

We also obtain strong restrictions for the primes $p = 2, 3$:

Theorem 3. *Assume that a p -block B of a finite quasi-simple group G is a minimal counterexample to Brauer's $k(B)$ -conjecture. Then $p \leq 3$, G is of Lie type in characteristic not p , B is an isolated block of G , the defect groups of B are non-abelian, and either $p = 2$ or B is not unipotent. In particular, the principal 3-block is not a minimal counterexample.*

The proofs will be given in the subsequent sections, using the classification of finite simple groups in conjunction with Lusztig's theory of characters of finite reductive groups. While our methods fall short of verifying the $k(B)$ -conjecture for all blocks of quasi-simple groups, since they partly rely on Bonnafé–Rouquier type reduction arguments to rule out a minimal counterexample, in the most interesting case of unipotent blocks with $p > 2$ and non-abelian defect our arguments actually show that these do satisfy the $k(B)$ -conjecture.

Remark 1.1. Let us comment on the cases left open by our results. For $p = 2, 3$ the isolated blocks of all quasi-simple groups of Lie type remain to be considered. For $p = 2$, already the case of unipotent blocks of $\mathrm{SL}_n(q)$ seems hard.

Remark 1.2. If B is a block with abelian defect group D , then according to (the proven direction of) Brauer's height zero conjecture all characters in $\mathrm{Irr}(B)$ are of height zero. By the Alperin–McKay conjecture they should be in bijection with the height zero characters of the Brauer corresponding block b of $N_G(D)$ (which also has defect group D), whence $k(B) = k(b)$. Thus, assuming the validity of the Alperin–McKay conjecture, a block B with non-normal abelian defect group cannot be a minimal counterexample to the $k(B)$ -conjecture. The interesting situation hence rather seems to be the one of non-abelian defect groups; see also the recent result of Sambale recalled in Theorem 2.1.

No reduction of the general conjecture to the case of (quasi-)simple groups has been found so far; see Navarro's article [22] for some thoughts and ideas in that direction.

The paper is built up as follows: In Section 2 we settle the case of quasi-simple groups not of Lie type, mostly by collecting results from the literature. In Section 3 we prove our main theorem for groups of Lie type in their defining characteristic. The by far most complicated case, groups of Lie type in cross characteristic, is then considered, after some general reductions in Section 4, in Section 5 where we deal with classical groups at odd primes, and in Section 6 where we concentrate on groups of exceptional type. On the way we also extend the results of Olsson [23] to derive explicit formulas for the number of characters in unipotent blocks, which may be of independent interest (see Propositions 5.4 and 5.5). The proof of Theorems 1, 2 and 3 is given at the end of Section 6.2.

Acknowledgement: I thank Gabriel Navarro for helpful correspondence on the topic of the paper and Jay Taylor for his remarks on an earlier version. I also thank Jonathan Gruber for pointing out an inaccuracy.

2. GROUPS NOT OF LIE TYPE

In this section we consider the quasi-simple groups G such that $G/Z(G)$ is not of Lie type. Here, and later on, we will make use of the following result [26, Thm. A]:

Theorem 2.1 (Sambale). *The $k(B)$ -conjecture holds for all blocks B with abelian defect groups of rank at most 3.*

Brauer had previously shown his conjecture for abelian defect groups of rank at most 2. Moreover, we have the following reduction, relying on a result of Navarro:

Theorem 2.2 (Navarro). *The $k(B)$ -conjecture (in strong form) holds for all p -blocks of a quasi-simple group G if it holds for the faithful p -blocks of p' -covering groups of $G/Z(G)$.*

Proof. Let B be a p -block of G . Set $G_1 = G/O_p(G)$ and let B_1 be the unique p -block of G_1 contained in B . Then by [22, Thm. C], B satisfies the $k(B)$ -conjecture if B_1 does. So it suffices to consider p' -coverings of $G/Z(G)$. For G such a covering group let K be the intersection of the kernels of the characters in B and B_1 the p -block of $G_1 = G/K$ contained in B . Then we have $k(B) = k(B_1)$ and the defect groups are isomorphic as K is a p' -subgroup, so again the validity of the $k(B)$ -conjecture for B_1 implies it for B . \square

Proposition 2.3. *Let G be quasi-simple such that $S = G/Z(G)$ is one of:*

- (1) *a sporadic simple group;*
- (2) *an alternating group \mathfrak{A}_6 or \mathfrak{A}_7 ;*
- (3) *an exceptional covering group of a simple group of Lie type; or*
- (4) *the Tits group ${}^2F_4(2)'$.*

Then the $k(B)$ -conjecture in the strong form holds for all p -blocks of G for all primes p .

Proof. The character tables of all the groups G in the statement are known and available in GAP [27]; since the block-subdivision of $\text{Irr}(G)$ and the defect of a block can be computed from a knowledge of the character table, the claim can easily be verified automatically. In fact, by Theorem 2.1 the $k(B)$ -conjecture holds for blocks with abelian defect of rank at most 3, so only very few cases remain to be considered. \square

Theorem 2.4 (Olsson). *Let G be a covering group of the alternating group \mathfrak{A}_n , $n \geq 5$. Then the $k(B)$ -conjecture in the strong form holds for all p -blocks of G for all primes p .*

Proof. By Proposition 2.3 we may assume that $n \neq 6, 7$. The case of $G = \mathfrak{A}_n$, as well as of its double cover $\tilde{\mathfrak{A}}_n$ when $p \neq 2$ is due to Olsson [24]. The validity for the faithful 2-blocks of the double cover now follows from Theorem 2.2. \square

3. GROUPS OF LIE TYPE IN DEFINING CHARACTERISTIC

In this section we show that the p -blocks of quasi-simple groups of Lie type in characteristic p satisfy the $k(B)$ -conjecture.

Proposition 3.1. *Let G be a covering group of $\text{PSL}_2(q)$, with $q = p^f \geq 4$. Then the $k(B)$ -conjecture (in the strong form) holds for all p -blocks of G .*

Proof. By Proposition 2.3 we may assume that G is not an exceptional covering group of $S := \text{PSL}_2(q)$. Note that Sylow p -subgroups of G are (elementary) abelian. First assume that $p = 2$. Then $G = S$, $|\text{Irr}(G)| = q + 1$, and all but the Steinberg character of G lie in the principal block B of G , with defect groups the Sylow p -subgroups of S of order q . So we have equality $k(B) = |D|$.

Now assume that $p > 2$. Here we may assume that $G = \text{SL}_2(q)$, since $|Z(\text{SL}_2(q))| = 2$ is prime to p . Now G has three p -blocks, one of defect zero, and the other two lying above

the two central characters of G . The latter have full defect and each contain $(q+3)/2$ irreducible characters, while Sylow p -subgroups of G have order q . \square

Theorem 3.2. *Let G be quasi-simple of Lie type in characteristic p . Then the $k(B)$ -conjecture in the strong form holds for all p -blocks of G .*

Proof. Again by Proposition 2.3 we may assume that G is not an exceptional covering group of $S := G/Z(G)$. Thus, $|Z(G)|$ is prime to p and hence we may assume that G is the universal p' -covering group of S , whence it can be obtained as the group of fixed points under a Steinberg map F of a simple simply connected linear algebraic group \mathbf{G} over an algebraic closure of \mathbb{F}_p . Let q be the positive real number such that some power F^d of F acts as q^d on the character group of an F -stable maximal torus of \mathbf{G} . The order of a Sylow p -subgroup of G is then given by q^N , where N is the number of positive roots in the root system of \mathbf{G} (see e.g. [20, Prop. 24.3]). By a theorem of Humphreys [16] any p -block of G either has full defect, or defect zero. We now employ results on upper bounds for the number of conjugacy classes $k(G)$. By Fulman and Guralnick [12, Thm. 1.1] we have $k(G) \leq 27.2q^r$, where r is the rank of \mathbf{G} . Comparing with the size q^N of a Sylow p -subgroup we see that our claim holds whenever $q^{N-r} > 27.2$, so certainly if $N-r \geq 5$. Since G is not of type A_1 by Proposition 3.1, this only leaves the groups of types A_2 , A_3 , 2A_2 , 2A_3 , B_2 , G_2 , 2B_2 and 2G_2 . For the latter three series of exceptional groups, [12, Table 1] gives the precise value of $k(G)$, from which our desired inequality readily follows. For $G = \mathrm{SL}_3(q)$ or $\mathrm{SU}_3(q)$ we have $k(G) \leq q^2 + q + 8$ (see [12, p. 3032]), while Sylow p -subgroups have order q^3 , so only the case $q = 2$ remains. But $k(\mathrm{SL}_3(2)) = 6$, while $\mathrm{SU}_3(2)$ is solvable. For types A_3 and 2A_3 we have $N-r = 3$, thus we are done when $q^3 > 27.2$, that is, for $q > 3$. But $k(\mathrm{SL}_4(2)) = 14 < 2^6$, $k(\mathrm{SU}_4(2)) = 20 < 2^6$, $k(\mathrm{SL}_4(3)) = 51 < 3^6$, $k(\mathrm{SU}_4(3)) = 71 < 3^6$.

Finally, for G of type B_2 we have $N-r = 2$, so we only need to check $q \leq 5$, which is readily done with GAP. \square

4. GROUPS OF LIE TYPE IN NON-DEFINING CHARACTERISTIC

In this section we start our investigation of groups of Lie type for non-defining primes. We consider the following setup. Let \mathbf{G} be a simple algebraic group of simply connected type over an algebraic closure of \mathbb{F}_p , with a Steinberg map $F : \mathbf{G} \rightarrow \mathbf{G}$. We write $G := \mathbf{G}^F$ for the group of fixed points, a finite group of Lie type, which is quasi-simple except in finitely many cases. Throughout, ℓ is a prime dividing $|G|$ but different from p .

Lemma 4.1. *The $k(B)$ -conjecture holds in its strong form for all blocks of the Suzuki and Ree groups.*

Proof. For these groups, Sylow ℓ -subgroups for $\ell \neq p$ are abelian of rank at most 3, hence the conjecture holds by Theorems 3.2 and 2.1, unless $\ell = 3$ and $G = {}^2F_4(2^{2f+1})$. The assertion in the latter case was shown in [18, Kor.]. \square

From now on we may hence assume that G is not a Suzuki or Ree group and so that F is a Frobenius map with respect to some \mathbb{F}_q -structure, with q a power of p . For a prime ℓ different from p we denote by $d_\ell(q)$ the order of q modulo ℓ , if ℓ is odd, respectively the order of q modulo 4 if $\ell = 2$.

We recall some results on the block distribution of characters of finite reductive groups. This is closely related to the partition of $\text{Irr}(G)$ into Lusztig series $\mathcal{E}(G, s)$, where s runs over semisimple elements of the dual group $G^* := \mathbf{G}^{*F}$ modulo conjugation. Let B be an ℓ -block of G . Then there exists a semisimple ℓ' -element $s \in G^*$ such that B is contained in

$$\mathcal{E}_\ell(G, s) := \coprod_t \mathcal{E}(G, st), \quad (*)$$

where t runs over a system of representatives of conjugacy classes of semisimple ℓ -elements in $C_G(s)$ (see [7, Thm. 9.12]). Recall that a semisimple element $s \in \mathbf{G}^{*F}$ is called *isolated* if the connected component $C_{\mathbf{G}^*}^\circ(s)$ of the centraliser $C_{\mathbf{G}^*}(s)$ is not contained in any proper F -stable Levi subgroup $\mathbf{L}^* < \mathbf{G}^*$.

We now argue that most ℓ -blocks of G are Morita equivalent to blocks of smaller groups and so cannot give rise to a minimal counterexample.

Lemma 4.2. *Let $s \in G^*$ be a non-isolated semisimple ℓ' -element. Then no ℓ -block of G in $\mathcal{E}_\ell(G, s)$ is a minimal counterexample to the strong form of the $k(B)$ -conjecture.*

Proof. By assumption there is some proper F -stable Levi subgroup $\mathbf{L}^* < \mathbf{G}^*$ such that $C_{\mathbf{G}^*}^\circ(s) \leq \mathbf{L}^*$. Then by a deep result of Bonnafé, Dat and Rouquier [1, Thm. 7.7], any ℓ -block B in $\mathcal{E}_\ell(G, s)$ is Morita equivalent to a suitable ℓ -block b of \mathbf{N}^F , where $\mathbf{N} \geq \mathbf{L}$ is dual to $C_{\mathbf{G}^*}(s)^F \mathbf{L}$. Moreover this is induced by Jordan decomposition, hence compatible with central characters. Then $k(B) = k(b)$ and the defect groups of B and b have the same order (and are in fact isomorphic), whence the $k(B)$ -conjecture holds for B by the minimality assumption. \square

We are thus left with the situation that B lies in a series $\mathcal{E}_\ell(G, s)$ with $s \in G^*$ an isolated semisimple element. In this case we can utilise a result of Enguehard:

Proposition 4.3. *Let B be an isolated, non-unipotent ℓ -block of a quasi-simple group of Lie type H for a prime $\ell \geq 3$ that is good for H . Then B is not a minimal counterexample to the strong form of the $k(B)$ conjecture.*

Proof. By Proposition 2.3 we may assume that H is not an exceptional covering group. Thus, $H = G/Z$, where $G = \mathbf{G}^F$ is as above and $Z \leq Z(G)$. By Lemma 4.2 we may assume that \mathbf{G} is not of type A as the only isolated element in type A is the identity, which corresponds to the unipotent blocks. But then ℓ good implies that ℓ does not divide $|Z(G)|$, so we may consider B as an ℓ -block of G . By the main result of Enguehard [11, Thm. 1.6] there is a group G_1 , with $|G_1/Z(G_1)|$ strictly smaller than $|G/Z(G)|$ (since B is not unipotent) with an ℓ -block B_1 having the same invariants (number of irreducible characters and defect group) as B . Moreover, there is a bijection $\text{Irr}(B) \rightarrow \text{Irr}(B_1)$ preserving central characters. In particular, B and any block of H dominated by B is not a minimal counterexample. \square

We note one further reduction which will be used for isolated 5-blocks of $E_8(q)$:

Lemma 4.4. *Let $s \in G^*$ be a non-central semisimple ℓ' -element with connected centraliser $\mathbf{C}^* = C_{\mathbf{G}^*}(s)$ such that $\mathcal{E}_\ell(G, s)$ is a single ℓ -block B , and $\mathcal{E}_\ell(C, 1)$ also is a single ℓ -block b , where \mathbf{C} is dual to \mathbf{C}^* . Then B is not a minimal counterexample to the strong form of the $k(B)$ conjecture.*

Proof. We have that $\text{Irr}(B) = \coprod_t \mathcal{E}(G, st)$ and $\text{Irr}(b) = \coprod_t \mathcal{E}(C, t)$ where both disjoint unions run over $C^* := C_{G^*}(s)$ -conjugacy classes of ℓ -elements t in C^* . Application of Jordan decomposition in G as well as in C puts both $\mathcal{E}(G, st)$ and $\mathcal{E}(C, t)$ into bijection with the same Lusztig series $\mathcal{E}(C_t, 1)$ of $C_t := C_{G^*}(st) = C_{C^*}(t)$ for any such t . Thus, if $\chi \in \mathcal{E}(G, s)$ and $\chi' \in \mathcal{E}(C, t)$ correspond to the same character in $\mathcal{E}(C_t, 1)$ we have $\chi(1)/\chi'(1) = |G^* : C_{G^*}(s)|_{p'}$. So comparing character degrees of height zero characters in the two Lusztig series we see that B and b have defect groups of the same order. Thus the inequality holds for B if (and only if) it holds for b . \square

For unipotent blocks, that is, ℓ -blocks in $\mathcal{E}_\ell(G, 1)$, we have the following partial reduction:

Lemma 4.5. *Let ℓ be a good prime for \mathbf{G} which does not divide $|Z(G)|$, and let B be a unipotent ℓ -block of G/Z with abelian defect groups, where $Z \leq Z(G)$. Then B is not a minimal counterexample to the strong form of the $k(B)$ -conjecture.*

Proof. By our assumption on ℓ we may assume that B is a unipotent block of G . Then under the stated conditions, by [5, Thm. 3.1(2)] the block B is isotypic with a block of the normaliser of a defect group, and hence can't be a counterexample by minimality. \square

To deal with unipotent blocks with non-abelian defect groups we will need some information on conjugacy classes of ℓ -elements and their Lusztig series. The following was essentially shown by Cabanes and Enguehard:

Proposition 4.6. *Let $\sigma : \mathbf{G} \rightarrow \mathbf{H}$ be an isogeny of connected reductive groups in characteristic p equivariant with respect to Frobenius endomorphisms F, F' of \mathbf{G}, \mathbf{H} respectively, with dual isogeny $\sigma^* : \mathbf{H}^* \rightarrow \mathbf{G}^*$. Set $r := |Z(\mathbf{G})/Z^\circ(\mathbf{G})|$. Let ℓ be a prime different from p not dividing r and $s \in H^*$ of order prime to $r\ell$. Then σ^* induces a bijection between the conjugacy classes of ℓ -elements in $C_{H^*}(s)$ and $C_{G^*}(\sigma^*(s))$, and for any such ℓ -element t we have a bijection $\mathcal{E}(H, st) \rightarrow \mathcal{E}(G, \sigma^*(st))$.*

Proof. First consider the case that $\sigma : \mathbf{G} \rightarrow \mathbf{G}_{\text{ad}}$ is the adjoint quotient map. Then the claim follows from [7, Prop. 17.4]. Application of the same result to the adjoint quotient $\sigma' : \mathbf{H} \rightarrow \mathbf{H}_{\text{ad}} = \mathbf{G}_{\text{ad}}$, and then composing the obtained bijections gives the statement. \square

5. CLASSICAL GROUPS IN NON-DEFINING CHARACTERISTIC

In this section we consider unipotent ℓ -blocks of classical groups for odd primes ℓ different from their defining characteristic.

5.1. Linear and unitary groups. We start by recalling the parametrisation of conjugacy classes of ℓ -elements in $\text{GL}_n(q)$. Let ℓ be an odd prime not dividing q . We write \mathcal{F}_ℓ for the set of irreducible monic polynomials over \mathbb{F}_q whose roots have ℓ -power order in $\overline{\mathbb{F}_q}^\times$. Let $d = d_\ell(q)$ be the order of q modulo ℓ , and let ℓ^a be the precise power of ℓ dividing $q^d - 1$. For $f \in \mathcal{F}_\ell$ we set $c_f = \max\{0, i - a\}$ if the roots of f have order ℓ^i . Write $n = wd + r$ with $0 \leq r < d$. With this notation the conjugacy classes of ℓ -element in $\text{GL}_n(q)$ are parametrised by ℓ -weight vectors of w , that is, by functions

$$m : \mathcal{F}_\ell \rightarrow \mathbb{Z}_{\geq 0}, \quad f \mapsto m_f, \quad \text{with} \quad \sum_{f \in \mathcal{F}_\ell} m_f \ell^{c_f} = w.$$

The corresponding ℓ -elements have characteristic polynomial $(X - 1)^{dm_1+r} \prod_{f \neq X-1} f^{m_f}$ in $\mathrm{GL}_n(q)$, where we have set $m_1 := m_{X-1}$. The centraliser of such an element is then a direct product $\mathrm{GL}_{dm_1+r}(q) \prod_{f \neq X-1} \mathrm{GL}_{m_f}(q^{d\ell^{e_f}})$.

To treat the special linear and unitary groups we need to determine the number $k(B)$ for their principal ℓ -block B ; to do this, we use of the result of Olsson [23] who verified the $k(B)$ -conjecture for all ℓ -blocks of $\mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q)$, where $2 < \ell \neq p$.

For integers $s, t \geq 1$ let $k(s, t)$ denote the number of s -tuples of partitions of t , and for $\ell, a, w \geq 1$ let

$$k(\ell, a, w) := \sum_{\mathbf{w}} k(\ell^a, w_0) \prod_{i \geq 1} k(\ell^a - \ell^{a-1}, w_i)$$

where the sum runs over all ℓ -compositions of w , that is, all tuples $\mathbf{w} = (w_0, w_1, \dots)$ of non-negative integers satisfying $\sum_{i \geq 0} w_i \ell^i = w$. As customary, we write $\mathrm{GL}_n(-q) := \mathrm{GU}_n(q)$ and $\mathrm{SL}_n(-q) := \mathrm{SU}_n(q)$.

Theorem 5.1. *Let $\mathrm{SL}_n(\epsilon q) \leq G \leq \mathrm{GL}_n(\epsilon q)$ with $\epsilon \in \{\pm 1\}$, and $\ell > 2$ be a prime dividing $q - \epsilon$. Set $\ell^a := (q - \epsilon)_\ell$, $\ell^m := |Z(\mathrm{SL}_n(\epsilon q))|_\ell = \gcd(n, q - \epsilon)_\ell$, $\ell^g := |\mathrm{GL}_n(\epsilon q) : G|_\ell$ and $u := \min\{m, g\}$. Let B denote the principal ℓ -block of G . Then*

$$k(B) = \left(k(\ell, a, n) + \sum_{i=1}^u \ell^{2i-2} (\ell^2 - 1) k(\ell, a, n/\ell^i) \right) / \ell^g.$$

Proof. Let first $\epsilon = 1$, so $\mathrm{SL}_n(q) \leq G \leq \mathrm{GL}_n(q) =: \tilde{G}$. Let \tilde{B} be the principal ℓ -block of \tilde{G} ; thus $\mathrm{Irr}(\tilde{B})$ is the union of the Lusztig series $\mathcal{E}(\tilde{G}, t)$ where t runs over ℓ -elements of $\tilde{G}^* = \mathrm{GL}_n(q)$. Then the characters in the principal ℓ -block B of G are precisely the constituents of the restrictions to G of the $\chi \in \mathrm{Irr}(\tilde{B})$. Now by Lusztig's Jordan decomposition an irreducible character χ lying in the Lusztig series $\mathcal{E}(\tilde{G}, t)$ of a semisimple element $t \in \tilde{G}^*$ restricts irreducibly to $\mathrm{SL}_n(q)$ unless the image \bar{t} of t in $\mathrm{PGL}_n(q)$ has disconnected centraliser in PGL_n . Now assume that $t \in \mathrm{GL}_n(q)$ is an ℓ -element. If the centraliser of \bar{t} in PGL_n is disconnected, then $\gcd(n, q - 1)$ and $o(t)$ are not coprime (see [20, Prop. 14.20]), that is, ℓ divides n . In this case, $\chi \in \mathcal{E}(\tilde{G}, t)$ splits into d_χ distinct irreducible characters of $\mathrm{SL}_n(q)$, with d_χ dividing $A_t := |C_{\mathrm{PGL}_n}(\bar{t})^F : C_{\mathrm{PGL}_n}^o(\bar{t})^F|$ where F is the standard Frobenius endomorphism on PGL_n . By Clifford's theorem this implies that χ splits into exactly $\ell^u = \gcd\{|\tilde{G} : G|, d_\chi\}$ distinct constituents upon restriction to G .

Thus in order to calculate $k(B)$ we need to determine the centraliser of \bar{t} for ℓ -elements $t \in \tilde{G}$. Let $i \geq 1$. Then $A_t \geq \ell^i$ if and only if t and ζt have the same eigenvalues for an ℓ^i th root of unity $\zeta \in \mathbb{F}_q^\times$. That is, the characteristic polynomial $f \in \mathbb{F}_q[X]$ of t satisfies $f(X) = f(\zeta X)$. Now multiplication by ζ makes orbits of length ℓ^i on $\overline{\mathbb{F}_q}^\times$, and such an orbit is a subset of the roots of a polynomial $f \in \mathcal{F}_\ell$ if and only if $\deg(f)$ is divisible by ℓ^i . Thus, the ℓ -elements t with $\ell^i | A_t$ are parametrised by maps $m : \mathcal{F}_\ell \rightarrow \mathbb{Z}_{\geq 0}$ as above that are constant on ζ -orbits. Arguing as in [23, proof of Prop. 6] we see that the number of irreducible characters of \tilde{G} in the union of the corresponding Lusztig series is $k(\ell, a, n/\ell^i)$. The claim now follows by induction over the ℓ -part ℓ^g of the index of G in \tilde{G} . \square

Lemma 5.2. *Let $\ell \geq 2$ and $w \geq 1$. The number $p_\ell(w)$ of ℓ -compositions of w (partitions into parts of ℓ -power order) satisfies $p_\ell(w) \leq \ell^{\binom{u+1}{2}}$ where $u = \lfloor \log_\ell w \rfloor$.*

Proof. The claim is easily verified for $w < \ell^2$. Else, arranging the sought for partitions according to their number of parts of length 1 we indeed get, arguing by induction

$$p_\ell(w) \leq \sum_{j=0}^{\lfloor w/\ell \rfloor} p_\ell(\lfloor w/\ell \rfloor - j) \leq \frac{w}{\ell} p_\ell(\lfloor w/\ell \rfloor) \leq \prod_{i=1}^u \frac{w}{\ell^i} = \frac{w^u}{\ell^{\binom{u+1}{2}}} \leq \ell^{\binom{u+1}{2}}. \quad \square$$

Theorem 5.3. *Let $H = G/Z$ with $G \in \{\mathrm{SL}_n(q), \mathrm{SU}_n(q)\}$ and $Z \leq Z(G)$. Let $\ell > 2$ be a prime. Then the unipotent ℓ -blocks of H are not counterexamples to Brauer's $k(B)$ -conjecture.*

Proof. We embed $G := \mathrm{SL}_n(q) \trianglelefteq \tilde{G} := \mathrm{GL}_n(q)$. Let B be a unipotent ℓ -block of G with defect group D . Then there is a unipotent ℓ -block \tilde{B} of \tilde{G} covering B , with defect group $\tilde{D} \geq D$, and by [23, Thm.] we have $k(\tilde{B}) \leq |\tilde{D}|$. As \tilde{G}/G is cyclic, restriction of characters from \tilde{G} to G is multiplicity-free. Moreover, as seen in the previous proof, an irreducible character $\chi \in \mathrm{Irr}(\tilde{B})$ restricts irreducibly to G unless ℓ divides $r := \mathrm{gcd}(n, q-1)$.

First assume that ℓ does not divide r . Then all $\chi \in \mathrm{Irr}(\tilde{B})$ restrict irreducibly to G so $k(B) = k(\tilde{B})/\ell^a$. As $|D| = |\tilde{D}|/\ell^a$ the conjecture follows for the block B of $\mathrm{SL}_n(q)$. Furthermore, $|Z(G)| = r$ is not divisible by ℓ in this case, so all characters in $\mathrm{Irr}(B)$ have $Z(G)$ in their kernel, and the claim also follows for $H = G/Z$ for any $Z \leq Z(G)$.

So now assume that $\ell \mid \mathrm{gcd}(n, q-1)$, and write ℓ^a for the precise power of ℓ dividing $q-1$, and ℓ^m for the ℓ -part of $|Z(\mathrm{SL}_n(q))|$. Then all unipotent characters of G lie in the principal ℓ -block (see e.g. [6, Thm.]), so the defect group D of B is a Sylow ℓ -subgroup of G . From Theorem 5.1 we now get, arguing as in the proof of [23, (I)]

$$k(B) \leq \left(p_\ell(n)\ell^{an} + \sum_{i=1}^m p_\ell(n/\ell^i)\ell^{an/\ell^i+2i} \right) / \ell^a \leq p_\ell(n)(\ell^{an} + \sum_{i=1}^m \ell^{an/\ell^i+2i}) / \ell^a.$$

First assume that $n \geq 2\ell$. Then $an/\ell^i + 2i \leq an - 2i$ for $i \geq 1$ and so by Lemma 5.2

$$k(B) \leq p_\ell(n)\ell^{an-a+1/(\ln(\ell)(\ell^2-1))} \leq \ell^{an-a+1/(\ln(\ell)(\ell^2-1))+\binom{u+1}{2}}$$

with $u = \lfloor \log_\ell(n) \rfloor$. As we have $|D| = \ell^{an-a+(n!)_\ell}$ and $|\tilde{D}| = \ell^{an-a+(n!)_{\ell-m}}$ where $m \leq u$, the claim follows except when $n = 2\ell$ or $\ell = 3, n \leq 12$. In the first case we use that $p_\ell(2\ell) = 3$, and the latter cases can be checked individually.

Finally assume that $n = \ell$. Then by [23, p. 46] we have $k(\tilde{B}) \leq \ell^{a\ell} + \ell^a$ and by Theorem 5.1, $k(B) \leq \ell^{a\ell-a} + \ell^2 < \ell^{a\ell-a+1} = |\tilde{D}|/\ell^a = |D|$. Furthermore, a character parametrised by a semisimple ℓ -element $t \in \mathrm{GL}_n(q)$ is trivial on the center if and only if t lies in the derived subgroup $\mathrm{SL}_n(q)$. From this it can then be checked that the principal ℓ -block \bar{B} of $\mathrm{PSL}_\ell(q)$ has $k(\bar{B}) = (k(B) + \ell - 1)/\ell \leq |D|/\ell = |\bar{D}|$, with $\bar{D} = D/Z(\mathrm{SL}_\ell(q))$ a Sylow ℓ -subgroup of $\mathrm{PSL}_\ell(q)$, so our claim follows for $H = G/Z(G)$ as well.

The arguments for $G = \mathrm{SU}_n(q)$ are entirely analogous, again relying on the explicit formula for $k(\tilde{B})$ in [23, Prop. 6] and for $k(B)$ in Theorem 5.1. \square

5.2. Classical groups. We now turn to the quasi-simple groups of symplectic and orthogonal type. Here, in the spirit of Olsson's result for linear and unitary groups, we first derive a formula for the number of characters in unipotent blocks which may be of independent interest.

Let $G_n(q)$ be one of $\mathrm{Sp}_{2n}(q)$ or $\mathrm{SO}_{2n+1}(q)$ and $\ell \neq 2$ an odd prime not dividing q . We write $d = d_\ell(q)$ for the order of q modulo ℓ and let $d' := d/\mathrm{gcd}(d, 2)$. The unipotent ℓ -blocks of $G_n(q)$ are parametrised by d -cuspidal pairs in $G_n(q)$, that is, by pairs (L, λ) where $L = G_{n-wd'}(q) \times T_d^w$ is a d -split Levi subgroup of $G_n(q)$ (with a torus $T_d \cong \mathrm{GL}_1(q^d)$ if d is odd, respectively $T_d \cong \mathrm{GU}_1(q^{d'})$ if $d = 2d'$ is even), and λ is a d -cuspidal unipotent character of L , and hence of $G_{n-wd'}(q)$ (see [6, Thm.]). We then write $b(L, \lambda)$ for this block, and call w its *weight*. The unipotent characters in the block $b(L, \lambda)$ are then the members of the d -Harish-Chandra series above (L, λ) , so by [4, Thm. 3.2] they are in bijection with the irreducible characters of the relative Weyl group of this d -cuspidal pair, which in this case is the imprimitive complex reflection group $G(2d', 1, w)$. In particular their number is given by $k(2d', w)$.

The following result bears a strong resemblance to [23, Prop. 6]; and again it expresses $k(B)$ only in terms of the weight w of B , of $d = d_\ell(q)$ and the ℓ -part of $q^d - 1$:

Proposition 5.4. *Let $G \in \{\mathrm{Sp}_{2n}(q), \mathrm{SO}_{2n+1}(q)\}$, let $\ell \neq 2$ and B be a unipotent ℓ -block of G of weight w . Let $d = d_\ell(q)$, $d' := d/\mathrm{gcd}(d, 2)$ and write ℓ^a for the precise power of ℓ dividing $q^d - 1$. Then*

$$k(B) = \sum_{\mathbf{w}} k(2d' + (\ell^a - 1)/2d', w_0) \prod_{i \geq 1} k((\ell^a - \ell^{a-1})/2d', w_i),$$

where the sum runs over all sequences $\mathbf{w} = (w_0, w_1, \dots)$ of non-negative integers satisfying

$$\sum_{i \geq 0} w_i \ell^i = w.$$

Proof. First assume that $G = \mathrm{SO}_{2n+1}(q)$. We count the characters in B lying in Lusztig series $\mathcal{E}(G, t)$, with $t \in G^* = \mathrm{Sp}_{2n}(q)$ an ℓ -element. The conjugacy classes of semisimple elements in $\mathrm{Sp}_{2n}(q)$ are uniquely determined by the characteristic polynomials of their elements in the natural $2n$ -dimensional matrix representation. Furthermore, there is a semisimple element with characteristic polynomial a given monic polynomial $f \in \mathbb{F}_q[X]$ of degree $2n$ with non-zero constant coefficient if and only if any root of f in $\overline{\mathbb{F}}_q$ has the same multiplicity as its inverse. That is to say, f must be invariant under the transformation

$$f \mapsto f^* := X^{\deg(f)} f(X^{-1})/f(0).$$

Let \mathcal{F} denote the set of monic irreducible polynomials $f \in \mathbb{F}_q[X]$ whose roots in $\overline{\mathbb{F}}_q$ are of ℓ -power order. We choose a system of representatives $\overline{\mathcal{F}}$ of $*$ -orbits in \mathcal{F} . Write \mathcal{F}^i for the subset of polynomials in \mathcal{F} whose roots have order ℓ^i , $i \geq 0$. Then $f \in \mathcal{F}$ has degree

$$c_f := \deg(f) = \begin{cases} 1 & \text{if } f \in \mathcal{F}^0 = \{X - 1\}, \\ d & \text{if } f \in \mathcal{F}^i \text{ with } 1 \leq i \leq a, \\ d\ell^i & \text{if } f \in \mathcal{F}^{a+i} \text{ with } i \geq 1. \end{cases}$$

Now first assume that d is odd. Then no $1 \neq \alpha \in \overline{\mathbb{F}}_q^\times$ of ℓ -power order is Galois conjugate to its inverse, as otherwise $\alpha^{q^c} = \alpha^{-1}$, that is, $\alpha^{q^{c+1}} = 1$ for some $c \geq 0$, which is absurd. Hence, all orbits of $*$ on \mathcal{F}^i , $i \geq 1$, have length 2. Thus the classes of ℓ -elements t in G^* are in bijection with maps $m : \overline{\mathcal{F}} \rightarrow \mathbb{Z}_{\geq 0}$, $f \mapsto m_f$, with $\sum_{f \in \overline{\mathcal{F}}} m_f \deg(f) = n$, such that t has characteristic polynomial $(X - 1)^{2m_1} \prod_{X-1 \neq f \in \overline{\mathcal{F}}} (f f^*)^{m_f}$, where for notational

convenience we write $m_1 := m_{X-1}$. The centraliser $C = C_{G^*}(t)$ of $t \in G^*$ corresponding to the map m is then isomorphic to $\mathrm{Sp}_{2m_1}(q) \times \prod_{X-1 \neq f \in \mathcal{F}} \mathrm{GL}_{m_f}(q^{c_f})$.

According to [6, Thm.] a character of $\mathcal{E}(G, t)$ lies in the block B parametrised by (L, λ) if its Jordan corresponding unipotent character of C lies in the unipotent block B_C corresponding to the same pair (L, λ) . For this to happen we must have in particular that L is isomorphic to a d -split Levi subgroup of C^* , hence $m_1 = n - (w - u)d$ for some $u \geq 0$. The unipotent characters of C in B_C are then the outer tensor products of unipotent characters in the block corresponding to (L, λ) in $\mathrm{Sp}_{2m_1}(q)$ times arbitrary unipotent characters in the other factors $\mathrm{GL}_{m_f}(q^{c_f})$, so their number is given by $k(2d, u) \prod_{f \neq X-1} \pi(m_f)$ with $\pi(m_f) = k(1, m_f)$ the number of partitions of m_f . Now clearly the number of elements in \mathcal{F}^{a+i} is $(\ell^a - \ell^{a-1})/(2d)$. Then the combinatorial argument in [23, p. 45], with $2d$ replacing e' , applies to show the stated formula for $k(B)$.

Next assume that $d = 2d'$ is even. Then all ℓ -elements in $\overline{\mathbb{F}}_q^\times$ are Galois conjugate to their inverses, and so $f = f^*$ for all $f \in \mathcal{F}$. Thus the classes of ℓ -elements $t \in G^*$ are in bijection with maps $m : \mathcal{F} \rightarrow \mathbb{Z}_{\geq 0}$, $f \mapsto m_f$, with $\sum_{f \in \mathcal{F}} m_f \deg(f) = 2n$, such that t has characteristic polynomial $(X - 1)^{m_1} \prod_{X-1 \neq f \in \mathcal{F}} f^{m_f}$, where again we write $m_1 := m_{X-1}$. The centraliser $C = C_{G^*}(t)$ of $t \in G^*$ corresponding to the map m is then isomorphic to $\mathrm{Sp}_{m_1}(q) \times \prod_{X-1 \neq f \in \mathcal{F}} \mathrm{GU}_{m_f}(q^{c_f/2})$. Again by [6, Thm.] a character of $\mathcal{E}(G, t)$ lies in the block B parametrised by (L, λ) if its Jordan corresponding unipotent character of C lies in the unipotent block B_C corresponding to (L, λ) . In that case, L is isomorphic to a d -split Levi subgroup of C , hence $m_1 = 2n - 2(w - u)d'$ for some $u \geq 0$. As before the unipotent characters of C in B_C are the outer tensor products of unipotent characters in the block corresponding to (L, λ) in $\mathrm{Sp}_{m_1}(q)$ times arbitrary unipotent characters in the other factors $\mathrm{GU}_{m_f}(q^{c_f/2})$, so their number is given by $k(d, u) \prod_{f \neq X-1} \pi(m_f/2)$. The number of elements in \mathcal{F}^{a+i} is $(\ell^a - \ell^{a-1})/d$. Again we conclude as in [23, p. 45], using that $d = 2d'$.

To treat $G = \mathrm{Sp}_{2n}(q)$ we need to consider ℓ -elements in the dual group $\mathrm{SO}_{2n+1}(q)$. But according to [13, Prop. 4.2] there is a bijection between conjugacy classes of ℓ -elements in G and G^* sending centralisers to their duals. Since centralisers of odd order elements in both G, G^* are always connected, they have the same numbers of unipotent characters. So the count for $\mathrm{Sp}_{2n}(q)$ is exactly the same as for $\mathrm{SO}_{2n+1}(q)$ and we are done. \square

5.3. Even-dimensional orthogonal groups. We next consider the even dimensional orthogonal groups. Let $G_n^\epsilon(q) = \mathrm{SO}_{2n}^\epsilon(q)$, with $\epsilon \in \{\pm\}$, $n \geq 4$. (Here, as customary, we write SO_{2n} for the connected component of the identity in the general orthogonal group GO_{2n} .) We recall some facts on blocks of $G_n^\epsilon(q)$ from [6, Thm.]. Let ℓ be an odd prime and $d = d_\ell(q)$. The unipotent ℓ -blocks of $G_n^\epsilon(q)$ are again parametrised by d -cuspidal pairs (L, λ) , where $L = G_{n-wd}^\delta(q) \times T_d^w$, with either $T_d = \mathrm{GL}_1(q^d)$ for odd $d = d'$, or $T_d = \mathrm{GU}_1(q^{d'})$ for $d = 2d'$ even, and $\delta = \epsilon$ if d is odd or w is even, and $\delta = -\epsilon$ else, and λ is a d -cuspidal unipotent character of L . We write $B = b(L, \lambda)$ for the corresponding block; and call w the *weight* of B . A defect group of $b(L, \lambda)$ is then obtained as a Sylow ℓ -subgroup of $C_G([\mathbf{L}, \mathbf{L}])$, which in our case is $\mathrm{GO}_{2wd}^{\epsilon\delta}(q)$. Observe that by the parity condition on the sign $\epsilon\delta$, a Sylow ℓ -subgroup of $\mathrm{GO}_{2wd}^{\epsilon\delta}(q)$ is already a Sylow ℓ -subgroup of $\mathrm{SO}_{2wd+1}(q)$. The number of unipotent characters of $G_n^\epsilon(q)$ in the

ℓ -block B then equals the number of irreducible characters of the relative Weyl group of (L, λ) , which is $G(2d', 1, w)$ unless λ is parametrised by a degenerate symbol, in which case it is the normal subgroup $G(2d', 2, w)$ (see [4, p. 51]).

We first derive a closed formula for the number of characters in blocks of the (disconnected) general orthogonal groups. Let \tilde{B} be a block of $\mathrm{GO}_{2n}^\epsilon(q)$ lying above the unipotent block $B = b(L, \lambda)$ of $\mathrm{SO}_{2n}^\epsilon(q)$. Then either \tilde{B} lies above a unique unipotent block of $\mathrm{SO}_{2n}^\epsilon(q)$, in which case the tensor product of \tilde{B} with the non-trivial linear character of $\mathrm{GO}_{2n}^\epsilon(q)$ is another block above B , or else the cuspidal pair (L, λ) is such that λ is labelled by a degenerate symbol, in which case \tilde{B} lies above the two blocks parametrised by the two unipotent characters labelled by this degenerate symbol. In either case, the unipotent characters in \tilde{B} are in bijection with the irreducible characters of $G(2d', 1, w)$, which is the relative Weyl group in $\mathrm{GO}_{2n}^\epsilon(q)$ of the cuspidal pair (L, λ) .

Proposition 5.5. *Let $\ell \neq 2$ and B be an ℓ -block of $\mathrm{GO}_{2n}^\epsilon(q)$, $n \geq 4$, lying above a unipotent ℓ -block of $\mathrm{SO}_{2n}^\epsilon(q)$ of weight w . Let $d = d_\ell(q)$, $d' := d/\mathrm{gcd}(d, 2)$ and write ℓ^a for the precise power of ℓ dividing $q^d - 1$. Then*

$$k(B) = \sum_{\mathbf{w}} k(2d' + (\ell^a - 1)/2d', w_0) \prod_{i \geq 1} k((\ell^a - \ell^{a-1})/2d', w_i),$$

where the sum runs over all sequences $\mathbf{w} = (w_0, w_1, \dots)$ of non-negative integers satisfying

$$\sum_{i \geq 0} w_i \ell^i = w.$$

Proof. Let $\mathbf{G} = \mathrm{SO}_{2n} \leq \hat{\mathbf{G}} := \mathrm{GO}_{2n} \leq \tilde{\mathbf{G}} := \mathrm{SO}_{2n+1}$ embedded in the natural way and write $G = \mathbf{G}^F$, $\hat{G} = \hat{\mathbf{G}}^F$ and $\tilde{G} = \tilde{\mathbf{G}}^F$. Let B be a unipotent ℓ -block lying above the block $b(L, \lambda)$ of G of weight w , for a d -cuspidal unipotent character of some d -split Levi subgroup $\mathbf{L} \leq \mathbf{G}$. To investigate $\mathcal{E}_\ell(G, 1) \cap \mathrm{Irr}(B)$, let $t \in G^*$ be an ℓ -element. Let $(m_f)_{f \in \mathcal{F}}$ describe the characteristic polynomial of t in the natural matrix representation of $G^* \cong G$. Then the centraliser of t in G^* has the same form as in $\tilde{G} = \mathrm{SO}_{2n+1}(q)$, except that the factor $G_{n-wd'}(q)$ is replaced by a group $\mathrm{GO}_{2(n-wd')}^\delta(q)$. Now again by [6, Thm.] such an ℓ -element t contributes to B if and only if $C_{G^*}(t)$ contains \mathbf{L}^* , so if

$$C_{\hat{G}^*}(t) \cong \mathrm{GO}_{2m_1}^\delta(q) \times \prod_{X-1 \neq f \in \mathcal{F}} \mathrm{GL}_{m_f}(q^{c_f})$$

with $m_1 = n - (w - u)d'$ for some $u \geq 0$.

Now first assume that d is odd. Then by the remarks preceding this proposition the number of unipotent characters of the first factor of this centraliser in the ℓ -block above λ is given by $|\mathrm{Irr}(G(2d, 1, u))| = k(2d, u)$, just as in the case of $\mathrm{SO}_{2n+1}(q)$, and so the number of unipotent characters above λ in this centraliser is given by $k(2d, u) \prod_{f \neq X-1} \pi(m_f)$, exactly as in the previous proof. Now note that any two ℓ -elements with this shape of centraliser lie inside a subgroup $\mathrm{GO}_{2m_1}^\delta(q) \times \mathrm{GL}_{m_0}(q)$, with $m_0 = n - m_1$, and thus are conjugate already inside this group. So the classes of ℓ -elements with Lusztig series contributing to B are parametrised exactly as in \tilde{G}^* . Thus, we see that $k(B)$ is given by the same expression as the one we obtained in Proposition 5.4 for $\mathrm{SO}_{2n+1}(q)$. The case of even d is entirely similar. \square

To descend to the special orthogonal groups, we need an auxiliary result on characters of certain imprimitive complex reflection groups.

Lemma 5.6. *Let $d, n \geq 1$. Then $|\text{Irr}(G(2d, 2, n))| \leq |\text{Irr}(G(2d, 1, n))| = k(2d, n)$.*

Proof. The irreducible characters of the complex reflection group $G(2d, 1, n)$, which is the wreath product $C_{2d} \wr \mathfrak{S}_n$, are naturally indexed by $2d$ -tuples of partitions of n . This shows that $|\text{Irr}(G(2d, 1, n))| = k(2d, n)$. An irreducible character of $G(2d, 1, n)$ parametrised by such a $2d$ -tuple $(\lambda_1, \dots, \lambda_{2d})$ stays irreducible upon restriction to the normal subgroup $G(2d, 2, n)$ of index 2 unless $\lambda_i = \lambda_{i+d}$ for $i = 1, \dots, d$. In this case, $(\lambda_1, \dots, \lambda_d)$ is a d -tuple of partitions of $n/2$, and in particular n must be even, which we assume from now on. In this case our preceding discussion gives $|\text{Irr}(G(2d, 2, n))| = (k(2d, n) - k(d, n/2))/2 + 2k(d, n/2)$. So our claim is proven when we can show that $k(2d, n) \geq 3k(d, n/2)$.

Now according to [23, Lemma 1(ii)] we have

$$k(2d, n) = \sum_{1 \leq t \leq n} k(d, t)k(d, n-t) \geq k(d, n/2)^2,$$

so we are done if $k(d, n/2) \geq 3$. Using [23, Prop. 5], for example, one sees that this holds unless $(d, n/2) \in \{(1, 1), (1, 2), (2, 1)\}$. For these three cases the claim can be checked directly. \square

Corollary 5.7. *Let $G = \text{SO}_{2n}^\pm(q)$ with $n \geq 4$, let $\ell \neq 2$ and B be a unipotent ℓ -block of G of weight w . Let $d = d_\ell(q)$, $d' := d/\text{gcd}(d, 2)$ and write ℓ^a for the precise power of ℓ dividing $q^d - 1$. Then*

$$k(B) \leq \sum_{\mathbf{w}} k(2d' + (\ell^a - 1)/2d', w_0) \prod_{i \geq 1} k((\ell^a - \ell^{a-1})/2d', w_i),$$

where the sum runs over all sequences $\mathbf{w} = (w_0, w_1, \dots)$ of non-negative integers satisfying

$$\sum_{i \geq 0} w_i \ell^i = w.$$

As the proof shows, the inequality will often be strict, but this form will suffice for our purpose.

Proof. Let $\mathbf{G} = \text{SO}_{2n} \leq \hat{\mathbf{G}} := \text{GO}_{2n}$ embedded in the natural way. Let B be a unipotent ℓ -block of $G = \mathbf{G}^F$ and let \hat{B} be an ℓ -block of $\hat{G} = \hat{\mathbf{G}}^F$ lying above it. First assume that \hat{B} covers two blocks of G . Then clearly $k(\hat{B}) = k(B)$ and we are done by Proposition 5.5. So B is the only block covered by \hat{B} . In the proof of Proposition 5.5 we determined the contribution of the various Lusztig series $\mathcal{E}(G, t)$, $t \in G^*$ an ℓ -element, to $\text{Irr}(\hat{B})$, in terms of the unipotent characters of the centraliser $C_{G^*}(t)$. Now the number of unipotent characters in a given ℓ -block of a factor $\text{SO}_{2m_1}^\pm(q)$, with $m_1 = n - (w - u)d'$ for some $u \geq 0$, is given by $|\text{Irr}(G(2d, 2, u))|$, while the number of characters lying above it in $\text{GO}_{2m_1}^\pm(q)$ equals $|\text{Irr}(G(2d, 1, u))|$. According to Lemma 5.6 this second number is always at least as big as the former, so any Lusztig series $\mathcal{E}(G, t)$ contributes at most as many characters to B as the characters above it contribute to \hat{B} , whence $k(B) \leq k(\hat{B})$. Thus our claim follows with Proposition 5.5. \square

Theorem 5.8. *Let H be quasi-simple of classical Lie type in characteristic p and assume that $\ell \neq 2, p$. Then the unipotent ℓ -blocks of H are not minimal counterexamples to the strong form of Brauer's $k(B)$ -conjecture.*

Proof. By Proposition 2.3 we do not have to consider exceptional covering groups. As the order of the non-exceptional part of the Schur multiplier of $H/Z(H)$ is a power of 2, and all unipotent blocks have $Z(H)$ in their kernel, we in fact only need to consider the case when H is simple. Moreover by Lemma 4.5 we can restrict attention to unipotent blocks with non-abelian defect groups.

Let us first consider $G = \mathrm{SO}_{2n+1}(q)$ with $n \geq 2$. Let $d = d_\ell(q)$ and let B be a unipotent ℓ -block of G parametrised by the d -cuspidal pair (L, λ) , with L of semisimple rank $n - wd'$ where $d' = d/\mathrm{gcd}(d, 2)$. By [6, Thm. 4.4(ii)] the defect groups of B are isomorphic to Sylow ℓ -subgroups of $C_G([\mathbf{L}, \mathbf{L}])$. Now $[\mathbf{L}, \mathbf{L}] = \mathrm{SO}_{2(n-wd')+1}$ has centraliser $\mathrm{GO}_{2wd'}^\pm(q)$ in G , where the “+” sign occurs if and only if d is odd. A Sylow ℓ -subgroup of $\mathrm{GO}_{2wd'}^\pm(q)$ is isomorphic to the wreath product $C_{\ell^a} \wr P$, with ℓ^a the precise power of ℓ dividing $q^d - 1$ and P a Sylow ℓ -subgroup of the complex reflection group $G(2d', 1, w)$. On the other hand, the number $k(B)$ was computed in Proposition 5.4. Application of the precise same estimates as in [23, p. 46] now shows that $k(B) < |D|$ whenever D is non-abelian (which we may assume by Lemma 4.5). Since the simple group $H = [G, G]$ has index at most 2 in G , and restriction of characters in ℓ -series of G is irreducible by Lusztig's parametrisation (see [7, Prop. 15.6]), we obtain our claim for the unipotent blocks of H .

We can argue as in the previous case to obtain the desired inequality for unipotent blocks of $G = \mathrm{Sp}_{2n}(q)$, just replacing the centraliser of $[\mathbf{L}, \mathbf{L}] = \mathrm{Sp}_{2(n-wd')}$ in G by $\mathrm{Sp}_{2wd'}(q)$, whose Sylow ℓ -subgroups have the same order. Then the claim also holds for the unipotent blocks of the simple factor group $\mathrm{S}_{2n}(q)$.

Finally, consider $G = \mathrm{SO}_{2n}^\pm(q)$, $n \geq 4$. Let B be a unipotent ℓ -block of G parametrised by the d -cuspidal pair (L, λ) . As observed above, we may assume that B has non-abelian defect, that is, $w \geq \ell$. It is now easy to check that the bound for $k(B)$ given in Corollary 5.7 is less than $|D|$ in all cases. Now the derived subgroup $[G, G]$ has index at most 2 in G , and characters corresponding to ℓ -elements restrict irreducibly, and the simple group H is obtained as $[G, G]Z(G)/Z(G)$, with $|Z(G)| \leq 2$, which completes the proof. \square

6. EXCEPTIONAL GROUPS IN NON-DEFINING CHARACTERISTIC

We now deal with the unipotent blocks of exceptional groups of Lie type. The strong form of the $k(B)$ -conjecture for ${}^3D_4(q)$ was shown in [8], and for $G_2(q)$ the statement can be read off from [15]. So in view of Lemma 4.1 we only need to concern ourselves with the five series of exceptional groups of rank at least 4. Continuing our previous notation let \mathbf{G} be simple of simply connected exceptional type and of rank at least 4 with a Frobenius endomorphism F , and $G = \mathbf{G}^F$. In this section we also allow $\ell = 2$.

6.1. Unipotent blocks.

Proposition 6.1. *The principal ℓ -block of a quasi-simple exceptional group of Lie type G/Z , where $Z \leq Z(G)$, is not a counterexample to the $k(B)$ -conjecture in strong form.*

Proof. We first consider the assertion for the principal ℓ -block B_0 of G . Then clearly it will follow for the principal ℓ -block of G/Z for all $Z \leq Z(G)$ such that ℓ does not divide

$|Z|$. Note that unless ℓ divides the order of the Weyl group of G , the Sylow ℓ -subgroups of G are abelian and ℓ is good for \mathbf{G} (see [20, Thm. 25.14]). In the latter case, in particular ℓ does not divide the order of the center of G . So B_0 is isotypic to a block of the normaliser of a Sylow ℓ -subgroup of G by Lemma 4.5, hence of an ℓ -solvable group, for which the $k(B)$ -conjecture has been shown to hold.

So we may assume that ℓ divides the order of the Weyl group of \mathbf{G} . Let \mathbf{G}^* be dual to \mathbf{G} and set $G^* = \mathbf{G}^{*F}$. According to [19, Thm. 5.14] the normaliser of a Sylow ℓ -subgroup P of G^* embeds into the normaliser $N := N_{G^*}(\mathbf{T})$ of an F -stable maximal torus \mathbf{T} of \mathbf{G}^* containing a Sylow d -torus of \mathbf{G}^* , where $d = d_\ell(q)$. Now the principal ℓ -block B_0 of G satisfies

$$\mathrm{Irr}(B_0) \subseteq \mathcal{E}_\ell(G, 1) = \bigcup_t \mathcal{E}(G, t)$$

(see (*) in Section 4), where t runs over a system of representatives of the conjugacy classes of ℓ -elements in G^* . Furthermore, by Lusztig's Jordan decomposition for any such t , $\mathcal{E}(G, t)$ is in bijection with $\mathcal{E}(C_{G^*}(t), 1)$, the unipotent characters of $C_{G^*}(t)$. Since every conjugacy class of ℓ -elements of G^* has a representative in our chosen Sylow ℓ -subgroup P , it hence suffices to show that

$$|P| > \sum_{t \in P/\sim} |\mathcal{E}(G, t)| = \sum_{t \in P} \frac{1}{|t^N \cap P|} |\mathcal{E}(C_{G^*}(t), 1)|,$$

where the first sum runs over a system of representatives of elements in P modulo N -conjugation, and the second over all elements of P . In particular the right hand side has exactly $|P|$ summands. Hence we are done if we show that the average value of the summands is smaller than 1.

For this we split the sum into two parts, depending on whether $t \in P \cap \mathbf{T}$ or not. The centraliser of a semisimple element $t \in \mathbf{T}$ is a subsystem subgroup of \mathbf{G}^* , with Weyl group W_t a reflection subgroup of the relative Weyl group $W = N_G(\mathbf{T})/\mathbf{T}^F$ of \mathbf{T} (see [20, Thm. 14.2]). Thus $|t^N \cap P| \geq |W : W_t|$ and we are done whenever $|W : W_t| > |\mathcal{E}(C_{G^*}(t), 1)|$. The unipotent characters have been classified by Lusztig; in particular he has shown that for a connected reductive group their number is multiplicative over the simple components. For \mathbf{G} of type A_n , $|\mathcal{E}(\mathbf{G}^F, 1)|$ is the number of partitions of $n+1$, for the other simple groups of low rank these numbers are given in Table 1 (see e.g. Chevie [21]).

TABLE 1. Numbers of unipotent characters for simple groups

G	B_2, C_2	B_3, C_3	B_4, C_4	D_4	2D_4	F_4	$D_5, {}^2D_5$	
$ \mathcal{E}(G, 1) $	6	12	25	14	10	37	20	
G	D_6	2D_6	$E_6, {}^2E_6$	$D_7, {}^2D_7$	E_7	D_8	2D_8	E_8
$ \mathcal{E}(G, 1) $	42	36	30	65	76	120	110	166

For $t \in P \setminus \mathbf{T}$ let $\bar{M} = N_N(P)\mathbf{T}^F/\mathbf{T}^F$ and \bar{t} the image in \bar{M} of t . Then t has at least $|\mathbf{T} \cap P|/|C_{\mathbf{T} \cap P}(t)| \cdot |\bar{M}|/|C_{\bar{M}}(\bar{t})|$ conjugates in P , and again we are done if this number

exceeds $|\mathcal{E}(C_{G^*}(t), 1)|$. Note that \bar{t} is a non-trivial ℓ -element of W . The value of this bound on class lengths can easily be computed inside W .

Let first $G = F_4(q)$. Then only $\ell \leq 3$ needs to be considered. Here \mathbf{T} is either maximally split with $|\mathbf{T}^F| = (q-1)^4$ if $d = 1$, or Ennola dual to that with $|\mathbf{T}^F| = (q+1)^4$ if $d = 2$. For $\ell = 2$ we compute the orbits of W on the set of elements t of order $o(t)$ dividing 4 in \mathbf{T} using Chevie [21]; the results are as follows:

$C_{G^*}(t)$	$o(t)$	$ \mathcal{E}(G, t) $	$ P \cap t^N $
$F_4(q)$	1	37	1
$B_4(q)$	2	25	3
$C_3(q).A_1(q)$	2	24	12
$B_3(q).(q-1)$	4	12	24
$C_3(q).(q-1)$	4	12	24
$A_3(q).\tilde{A}_1(q)$	4	10	24
$B_2(q).A_1(q).(q-1)$	4	12	72
$\tilde{A}_2(q).A_1(q).(q-1)$	4	6	96

Visibly, the average value of $|\mathcal{E}(G, t)|/|P \cap t^N|$ is strictly less than 1 on the union of these classes. Now the involutions and the elements of order 4 with centraliser $A_3(q).\tilde{A}_1(q)$ are the only isolated 2-elements in \mathbf{G} , so any other 2-element in $P \cap \mathbf{T}$ has centraliser of semisimple rank at most 3, so at most 12 unipotent characters but at least 24 conjugates in P , whence the average value over all 2-elements in $P \cap \mathbf{T}$ is indeed strictly less than 1.

For $\ell = 3$ the classes of elements in $P \cap \mathbf{T}$ of order dividing 3 are as follows,

$C_{G^*}(t)$	$o(t)$	$ \mathcal{E}(G, t) $	$ P \cap t^N $
$F_4(q)$	1	37	1
$B_3(q).(q-1)$	3	12	24
$C_3(q).(q-1)$	3	12	24
$A_2(q).\tilde{A}_2(q)$	3	9	32

which gives an average value less than 1. Furthermore, $W(F_4)$ has only three classes of non-trivial elements of 3-power order, with orbit length at least 18 in $PW(F_4)$, larger than the number of unipotent characters of any proper centraliser.

Now let $G = E_6(q)$. Here we need to consider $\ell = 2, 3$ and moreover $\ell = 5$ when $5|(q-1)$. Again, the relevant classes of elements of order 4, 3 and 5 can be computed with Chevie, and the claim follows for the group G . If $Z \neq 1$ then we have $3|(q-1)$, $|Z| = 3$ and we may assume that $\ell = 3$. The classes of elements of G^* of order dividing 3 contained in $P \cap \mathbf{T}$ are

$C_{G^*}(t)$	$ \mathcal{E}(G, t) $	$ P \cap t^N $
$E_6(q)$	30	1
$A_5(q).(q-1)$	11	72
$A_2(q)^3.3$	17	80
$D_4(q).(q-1)^2.3$	26	30
${}^3D_4(q).(q^2+q+1).3$	24	60
$D_5(q).(q-1)$ (twice)	20	27
$A_4(q).A_1(q).(q-1)$ (twice)	14	216

and again it follows that the average value under consideration is less than 1. The situation for the twisted groups ${}^2E_6(q)$ is entirely similar except that $\ell = 5$ now has to be considered when $5|(q+1)$.

For $G = E_7(q)$ we need to consider $\ell = 2, 3$, and $\ell = 5, 7$ when either divides $q^2 - 1$. The only centralisers for which the number of unipotent characters is larger than the index of the corresponding Weyl group in $W(E_7)$ are those of types E_7 , $E_6.2$ and $D_6 + A_1$, which correspond to quasi-isolated involutions and those only occur once each. Adding the contributions by the other centralisers of elements of order 4, 3, 5 or 7 respectively, we again get an average value below 1. For G/Z with $|Z| = 2$, we need to consider $\ell = 2$. Here the claim follows by an analogous computation.

Finally, for $G = E_8(q)$ we need to consider $\ell = 2, 3, 5$, and $\ell = 7$ when $7|(q^2 - 1)$. Here, only $t = 1$ and the class of involutions with centraliser of type $E_7 + A_1$ give a too large contribution, but this is again offset by the collection of all elements of order 4. \square

We now turn to general unipotent blocks. At bad primes these were determined by Enguehard [10].

Theorem 6.2. *Let B be a unipotent ℓ -block of a quasi-simple exceptional group of Lie type G/Z , where $Z \leq Z(G)$. Then B is not a minimal counterexample to the $k(B)$ -conjecture in strong form.*

Proof. By Proposition 6.1 we may assume that B is not the principal block of G/Z , and by Lemma 4.5 we only need to consider the case of bad primes ℓ . Furthermore, by Theorem 2.1 we may assume that defect groups have rank at least 4 when they are abelian. For type F_4 , by [10, p. 349] the only non-principal unipotent block of positive defect occurs for $\ell = 3$, but it has abelian defect groups of rank 2.

For $G = E_6(q)$ and $\ell \leq 5$ again by [10, p. 351] the only relevant unipotent block is the one above the cuspidal unipotent character of $D_4(q)$ for $\ell = 3$ and $q \equiv 1 \pmod{3}$, with defect group D of order $3(q-1)_3^2$. For this non-principal 3-block B we use the description of $\text{Irr}(B)$ given in [10, Thm. B]: we have $\text{Irr}(B) \subseteq \mathcal{E}_\ell(G, 1)$, and for any ℓ -element $t \in G^*$, $\text{Irr}(B) \cap \mathcal{E}(G, t)$ is in bijection with the irreducible characters in a corresponding unipotent block of $C_{G^*}(t)$, belonging to the ‘‘same’’ 1-cuspidal pair (apart from certain exceptions as described in [10, Prop. 17]). In our situation, B is associated to the 1-cuspidal pair $(D_4(q), \lambda)$, where λ denotes the unipotent cuspidal character of $D_4(q)$, and thus $\mathcal{E}(G, t)$ with t a non-trivial 3-element contains a character from B only if t has centraliser containing a Levi subgroup of type D_4 . Thus the possibilities for $C_{G^*}(t)$ are $D_4(q).(q-1)^2$, $D_5(q).(q-1)$ and ${}^3D_4(q).(q^2+q+1)$, in which case we have $|\text{Irr}(B) \cap \mathcal{E}(G, t)| = 1, 2, 1$ respectively (in the last case corresponding to the cuspidal unipotent character ${}^3D_4[-1]$ of ${}^3D_4(q)$, see loc. cit.). As $|\text{Irr}(B) \cap \mathcal{E}(G, 1)| = 3$, and any t as before has at least 6 conjugates in P , the claim ensues.

The arguments for the non-principal unipotent blocks of ${}^2E_6(q)$, described in [10, p. 354], are entirely similar except that here $d = d_3(q) = 2$.

For $G = E_7(q)$ by the table in [10, p. 354], the only relevant blocks are for $\ell = 2$ the blocks lying above the d -cuspidal unipotent characters of Levi subgroups of type E_6 , and for $\ell = 3$ a block lying above the d -cuspidal unipotent character of a Levi subgroup of type D_4 . In the case $\ell = 2$ the defect groups are dihedral (see [10, p. 357]) in which case our claim is known to hold, see [26, Cor. 8.2]. For the non-principal unipotent 3-block B

above D_4 the defect groups have order $|D| = 3(q - \epsilon)^3$ where $q \equiv \epsilon \pmod{3}$ (see loc. cit.), and again we can compute the number of characters in $\text{Irr}(B)$ using [10, Thm. B]. The centralisers of 3-elements $1 \neq t \in G^*$ containing a subgroup of type D_4 are of types

$$D_4, {}^3D_4, D_5, D_4 + A_1, D_6, D_5 + A_1, \text{ and } E_6$$

when $d = 1$, or their Ennola duals for $d = 2$, with $|\text{Irr}(B) \cap \mathcal{E}(G, t)| = 1, 1, 2, 2, 5, 4, 3$ respectively, but all of these have at least 28 conjugates in P ; this also makes up for the 10 characters contributed by $\text{Irr}(B) \cap \mathcal{E}(G, 1)$.

For $G = E_8(q)$ we need to consider $\ell = 2, 3, 5$. The relevant unipotent blocks for $\ell = 2$ are those above the d -cuspidal unipotent characters of $E_6(q)$ (or ${}^2E_6(q)$), and for $\ell = 3, 5$ those above the d -cuspidal unipotent character of $D_4(q)$. The non-principal unipotent block for $\ell = 2$ has defect groups of order $4(q - \epsilon)_2^2$ (see [10, p. 364]) with $q \equiv \epsilon \pmod{4}$, and only the Lusztig series $\mathcal{E}(G, t)$ with t a 2-element with centraliser of type $({}^2)E_6, E_7$ or $({}^2)E_6 + A_1$ will contribute, with $|\text{Irr}(B) \cap \mathcal{E}(G, t)| = 1, 2, 2$ respectively, in addition to the six unipotent characters. Again, the inequality is easily seen to hold. The computations for $\ell = 3, 5$ are similar to those done previously. This completes the discussion of all unipotent blocks of exceptional type groups. \square

6.2. Isolated 5-blocks in $E_8(q)$. The only simple groups of Lie type for which 5 is a bad prime are those of type E_8 . In view of Proposition 4.3 and Theorem 6.2, in order to complete the proof of Theorem 1 for $p = 5$ it therefore remains to treat the isolated 5-blocks of $E_8(q)$. These have been classified in [17, Prop. 6.10 and 6.11], but without determining the precise character distribution.

Proposition 6.3. *Let B be an isolated 5-block of $E_8(q)$. Then B is not a minimal counterexample to the $k(B)$ -conjecture in the strong form.*

Proof. By Theorem 6.2 we may assume that B is not unipotent. Thus B lies in $\mathcal{E}_5(G, s)$ for an isolated semisimple 5'-elements $1 \neq s$ of G^* . According to Lemma 4.4 we may assume that $\mathcal{E}_5(G, s)$ is not a single 5-block. Now by [17, Tab. 7 and 8] the only non-unipotent isolated 5-blocks of $E_8(q)$ for which the defect groups are not abelian of rank at most 3 and for which $\mathcal{E}_5(G, s)$ is not a single 5-block are those collected in Table 2, where we have kept the numbering from loc. cit. The two unnumbered lines correspond to 5-blocks that had inadvertently been omitted. All cases occur when $q \equiv 1 \pmod{5}$, that is, $d = d_5(q) = 1$, except for the one in the last line which occurs when $q \equiv 2, 3 \pmod{5}$. The case of $q \equiv 4 \pmod{5}$ is entirely analogous to the one where $q \equiv 1 \pmod{5}$. In the last column we provide the size of a defect group D as described in [17, Thm. 1.2(b)], where 5^a denotes the precise power of 5 dividing $q^d - 1$.

First, consider cases 1 and 2. Here $\mathcal{E}_5(G, s)$ contains two 5-blocks, and the precise subdivision of characters among these blocks is not known. In case 2, all characters in the corresponding block B_2 must have degree divisible by $|G : D|_5 = 5^2(q - 1)_5^4 = 5^{4a+2}$. But this does not hold for characters in the principal series of G . The only other Harish-Chandra series occurring in a Lusztig series $\mathcal{E}(G, st)$, with $t \in C_{G^*}(s)$ as 5-element, are the ones above a cuspidal unipotent character of a split Levi subgroup of type D_4 . In particular, $\mathcal{E}(G, st)$ only contributes to $\text{Irr}(B_2)$ if $C_{G^*}(st)$ contains $D_4(q)$, that is, if t centralises a subgroup $D_4(q)$ in $C_{G^*}(s) = D_8(q)$. The centraliser of $D_4(q)$ inside $D_8(q)$ is a subgroup $D_4(q)$, as can be seen from the extended Dynkin diagram, so our claim for

TABLE 2. Some isolated 5-blocks in $E_8(q)$

No.	$C_{G^*}(s)^F$	\mathbf{L}	λ	$ D $
1	$D_8(q)$	\emptyset	1	5^{8a+1}
2		D_4	$D_4[1]$	5^{4a}
3	$E_7(q)A_1(q)$	\emptyset	1	5^{8a+1}
4		D_4	$D_4[1]$	5^{4a}
7	$D_5(q)A_3(q)$	\emptyset	1	5^{8a+1}
8		D_4	$D_4[1]$	5^{4a}
11	${}^2A_7(q)A_1(q)$	A_1^3	1	5^{5a}
14	${}^2A_8(q)$	A_1^4	1	5^{4a}
16	$E_6(q)A_2(q)$	\emptyset	1	5^{8a+1}
17		D_4	$D_4[1]$	5^{4a}
19	${}^2E_6(q).{}^2A_2(q)$	A_1^3	1	5^{5a}
20		D_4	$({}^2A_2, \phi_{21})$	5^{4a}
	${}^2A_5(q).{}^2A_2(q)A_1(q)$	A_1^3	1	5^{5a}
		D_4	$({}^2A_2, \phi_{21})$	5^{4a}
25	$D_8(q)$	Φ_4^4	1	5^{4a}

B_2 follows again with Corollary 5.7. The exact same reasoning applies in cases 8 and 17 (note that by [17, Tab. 7] the characters in the Harish-Chandra series above a cuspidal character of $E_6(q)$ do not lie in the 5-block corresponding to case 17). To deal with the case 1 we claim that the total number of characters in $\mathcal{E}_5(G, s)$ is less than $|D|$; for this note that for any possible centraliser $C_{G^*}(st)$ the number of non-principal series characters in $\mathcal{E}(G, st)$ is at most one sixth of the total number of characters. So $|\mathcal{E}_5(G, s)|$ is at most $6/5$ of the number obtained from the principal series characters, which we determined in Corollary 5.7. Evaluation of that formula shows our claim.

Cases 3,7,11,14,16,19 and 25 are settled completely analogously.

For the unnumbered lines, by Proposition 4.6 it suffices to count the characters in a simply connected covering $H = \mathrm{SU}_6(q) \times \mathrm{SU}_3(q) \times \mathrm{SL}_2(q)$ of the centraliser $C_{G^*}(s)$. By Olsson's formula for $\mathrm{GU}_n(q)$ we have that

$$|\mathcal{E}_5(\mathrm{GU}_6(q) \times \mathrm{GU}_3(q) \times \mathrm{GU}_2(q), 1)| \leq 5^{4a} \cdot 5^{2a} \cdot 5^{2a} = 5^{8a}.$$

Now by Lusztig's result [7, Thm. 15.11] all characters of $\mathrm{GU}_6(q) \times \mathrm{GU}_3(q) \times \mathrm{GU}_2(q)$ corresponding to 5-elements restrict irreducibly to H , and since H has index divisible by 5^{3a} it follows that

$$|\mathcal{E}_5(\mathrm{SU}_6(q) \times \mathrm{SU}_3(q) \times \mathrm{SL}_2(q), 1)| \leq 5^{8a}/5^{3a} = 5^{5a}.$$

So the first block with this centraliser (which has abelian defect groups) satisfies the $k(B)$ -conjecture. The characters in the other block must have degree divisible by $|G : D|_5 = 5^{4a+2}$, so they must lie above the cuspidal unipotent character of the factor $\mathrm{SU}_3(q)$. Here, the count for the middle factor is just 5^a , and again the desired inequality follows. A similar argument can be employed to deal with the remaining two cases 4 and 20. \square

6.3. Isolated 3-blocks with abelian defect.

Proposition 6.4. *Let B be an isolated 3-block of a quasi-simple group of Lie type with abelian defect groups. Then B is not a minimal counterexample to the $k(B)$ -conjecture.*

Proof. By Lemma 4.5 and Theorems 5.3 and 6.2 we may assume that B is not unipotent. Thus B lies in $\mathcal{E}_3(G, s)$ for an isolated semisimple $3'$ -element $1 \neq s$ of G^* . Moreover, G is not of classical type by Proposition 4.3. According to Lemma 4.4 we may assume that $\mathcal{E}_3(G, s)$ is not a single 3-block. Now by [17, Tab. 2, 3, 4 and 6] the only non-unipotent isolated 3-block B with abelian defect groups of rank at least 4 for which $\mathcal{E}_3(G, s)$ is not a single 3-block occurs in $E_8(q)$ with $C_{G^*}(s)$ of type ${}^2A_4(q)^2$. Now note that the other two blocks in $\mathcal{E}_3(G, s)$ have smaller defect groups. But then according to the main result of [17] the characters of $\mathcal{E}_3(G, s)$ in B are exactly those of height zero, hence those lying in the principal Harish-Chandra series. For these we showed the validity of the required inequality in Theorem 5.3. \square

We conclude by proving Theorems 1, 2 and 3. Let (G, B) be a minimal counterexample to Brauer's $k(B)$ -conjecture in the strong form with G quasi-simple. Then G must be of Lie type by Proposition 2.3. Theorem 3.2 shows that p is not the defining prime. The block B is not unipotent for $p \neq 2$ by Theorems 5.3, 5.8 and 6.2. It is shown in Lemma 4.2 that B must be isolated. The isolated blocks for good primes $p \geq 3$ are not minimal counterexamples by Enguehard's result in Proposition 4.3, and the isolated 5-blocks of $E_8(q)$ are neither by Proposition 6.3. This achieves the proof of Theorem 1.

Now consider Theorem 2 on blocks with abelian defect groups. In the case $p = 3$ the claim follows with Proposition 6.4. Now assume that B is a quasi-isolated 2-block with abelian defect. Then by [9, Lemma 5.2] we have that either defect groups of B have rank at most 2, or G is of type A and B is quasi-isolated but not isolated. In this case we may conclude by Lemma 4.2. Finally, the principal block of a group of Lie type is unipotent, so the last assertion of Theorem 3 also follows.

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