CONJUGACY CLASS NUMBERS AND π-SUBGROUPS

GUNTER MALLE, GABRIEL NAVARRO, AND GEOFFREY R. ROBINSON

Abstract. The number $l(G)$ of p-regular classes of a finite group G is a key invariant in modular representation theory. Several outstanding conjectures propose that this number can be calculated or bounded in terms of certain invariants of some subgroups of G. Our main question here is if $l(G)$ can be bounded by the number of conjugacy classes of some subgroup of G of order not divisible by p. This would have consequences for the Malle–Robinson $l(B)$-conjecture. Furthermore, we investigate a π-version of this, for sets of primes π.

As part of our investigations, we study finite groups that have more conjugacy classes than any of their proper subgroups. These groups naturally appear in questions on bounding from above the number of conjugacy classes of a group, and were considered by G. R. Robinson and J. G. Thompson in the context of the $k(GV)$-problem. We classify the almost Abelian groups G with $F^*(G)$ quasi-simple. Our results should be of use in several related questions.

1. Introduction

We investigate relations between the number of conjugacy classes of a finite group and those of its subgroups. More specifically, let π be a set of primes. We write $k(G)$ for the number of conjugacy classes of a finite group G, and $k_\pi(G)$ for its number of conjugacy classes of π-elements, that is, elements whose order is only divisible by primes in π. We say that G is *π-bounded*, if there is a π-subgroup $H \leq G$ such that $k_\pi(H) = k(H) \geq k_\pi(G)$. In this case, H is called a *π-witness* for G.

Clearly, any group is $\pi(G)$-bounded, where $\pi(G)$ denotes the set of prime divisors of $|G|$. Also, G is $\{p\}$-bounded for every prime p, with p-witness a Sylow p-subgroup of G. More generally, if G has a Hall π-subgroup H and every cyclic π-subgroup of G is contained in a G-conjugate of H, then G is π-bounded with π-witness H. We see that, in particular, π-separable groups (and therefore all solvable groups) are π-bounded for all π.

In our first main theorem, we show that the same result also holds in important families of groups at the opposite end of the spectrum.

Theorem 1. *The following groups are π-bounded for all π:*

(1) *the symmetric and alternating groups;*
(2) *the sporadic simple groups and their automorphism groups; and*

Date: March 10, 2021.

2010 *Mathematics Subject Classification.* 20D20, 20C20, 20D60.

Key words and phrases. number of conjugacy classes, π-subgroups, almost Abelian finite groups, $l(B)$-conjecture.

The first author gratefully acknowledges financial support by SFB TRR 195. The research of the second author is supported by Ministerio de Ciencia e Innovación PID2019-103854GB-I00 and FEDER funds.

(3) *the full covering groups of the simple groups of Lie type.*

A particularly interesting case, in modular representation theory, is when $\pi = \pi(G)\setminus\{p\}$ for some prime $p \in \pi(G)$. In this case we ask for a p'-subgroup H such that $k(H) \geq k_{p'}(G) = l(G)$, with $l(G)$ the number of irreducible p-Brauer characters of G. This question is of interest in trying to prove the Malle–Robinson $l(B)$-conjecture [10]. Here we show:

Theorem 2. *Let G be a finite quasi-simple group. Then G is p'-bounded for every prime p.*

At the time of writing, we do not know if all finite groups are p'-bounded for every p. One of the reasons why we are interested in this question is the following implication for the Malle–Robinson conjecture [10] on the number of characters in a p-block:

Theorem 3. *Let G be a p-constrained finite group that is p'-bounded. Then the Malle–Robinson $l(B)$-conjecture holds for G.*

Here, G is p-constrained if its generalised Fitting subgroup $F^*(G)$ is a p-group.

Notice that if a finite group G happens to have a subgroup $H < G$ such that $k(H) \geq k(G)$, then G is π-bounded for all π containing $\pi(H)$. This leads us to study finite groups G for which $k(H) < k(G)$ for all proper subgroups $H < G$, baptised *almost Abelian* by John Thompson (as clearly all Abelian groups have this property). In our fourth main result, we obtain a classification of almost simple almost Abelian groups. As it turns out there are no (non-Abelian) simple almost Abelian groups, yet there are (a few) almost simple and very few quasi-simple almost Abelian groups.

Theorem 4. *Let G be a finite group with $F^*(G)$ quasi-simple. Then G is almost Abelian if and only if*

(1) *G is almost simple and isomorphic to one of* $\mathrm{PGL}_2(q)$ *with q odd,* $\mathrm{PGL}_3(q)$ *with* $q \equiv 1 \pmod 3$*,* $\mathrm{L}_3(3).2$*,* $\mathrm{L}_3(4).3$*,* $\mathrm{L}_3(4).6$*,* $\mathrm{L}_3(4).D_{12}$*,* $\mathfrak{S}_6$*,* $\mathfrak{A}_6.2^2$*, or* $M_{12}.2$*; or*
(2) *G is quasi-simple and isomorphic to one of* $3.\mathfrak{A}_6$*,* $6.\mathfrak{A}_6$ *or* $2.M_{12}$*; or*
(3) *G is neither almost simple nor quasi-simple and isomorphic to one of* $3.\mathrm{L}_3(4).6$*,* $4_1.\mathrm{L}_3(4).2_3$ *or* $2.M_{12}.2$*.*

As is well-known, there is a great deal of research on the problem of bounding above the number of conjugacy classes of finite groups, many times in terms of the size (not the number of conjugacy classes) of certain specific subgroups, or simply in terms of specific ad-hoc functions. (See for instance [7], [8], [12], etc.) Of course, some of this work, but not all, is directly motivated by Brauer's $k(B)$-conjecture. As a consequence of our present work, we point out that if one wishes to establish a certain upper bound for $k(G)$, which is known to hold for all proper subgroups of G and which can be reduced to groups with $F^*(G)$ quasi-simple, only the groups in Theorem 4 now have to be checked.

The paper is structured as follows. In Section 2 we investigate almost Abelian nearly simple groups and prove Theorem 4. In Section 3 we study π-bounded almost simple groups and prove Theorem 1. Finally, in Section 4 we specialise to the question of p'-boundedness and prove Theorems 2 and 3.

Acknowledgement: We thank Thomas Breuer for furnishing character tables of extensions of certain covering groups of $\mathfrak{A}_n$ and constructions for extensions of $\mathrm{L}_3(4)$, as well as for his comments on an earlier version.

2. Almost Abelian nearly simple groups

In this section we investigate finite groups G such that $k(H) < k(G)$ for all proper subgroups $H < G$. Recall from the introduction that we call such a group *almost Abelian*, a term coined by John Thompson (private communication, 1993). If G is not almost Abelian, then any subgroup $H < G$ with $k(H) \geq k(G)$ is called a *witness to G not being almost Abelian.*

Some related questions of interest are: which finite groups do not have a solvable subgroup $H \leq G$ with $k(H) \geq k(G)$, or, which groups do not possess a nilpotent subgroup $H \leq G$ with $k(H) \geq k(G)$?

Example 2.1. Observe that all extra-special p-groups, for p a prime, are almost Abelian. The smallest non-nilpotent almost Abelian group is $\mathrm{SL}_2(3)$ of order 24. The smallest solvable but not nilpotent almost Abelian group of order not divisible by 6 is the group $[320, 1012]$ in the Small Groups Library of `GAP` [14], with chief factors $2^2.2^4.5$.

Almost Abelian groups do not behave well with respect to quotients by central subgroups. For example $\mathrm{SL}_2(3)$ is almost Abelian, but $\mathfrak{A}_4$ is not. Conversely, $D_{16}/Z(D_{16}) = D_8$ is almost Abelian, but D_{16} is not.

Let us observe the following for possible future use.

Proposition 2.2. (a) *A direct product of almost Abelian groups is almost Abelian.*

(b) *If $G = ZH$, where $[Z, H] = 1$ and Z is abelian, then G is almost Abelian if and only if H is almost Abelian.*

Proof. (a) Let A, B be almost Abelian finite groups, and let H be a proper subgroup of $A \times B$. Then $k(H) \leq k(H \cap A)k(H/(H \cap A))$ and

$$H/(H \cap A) \cong HA/A \cong B \cap HA.$$

If $H \cap A < A$, then $k(H \cap A) < k(A)$ since A is almost Abelian. We also have $k(B \cap HA) \leq k(B)$ since B is almost Abelian, so we obtain $k(H) < k(A)k(B)$.

If $H \cap A = A$, that is to say, if $A \leq H$, we have $H = A \times (H \cap B)$ by Dedekind's modular law. But H is a proper subgroup of G, so that $H \cap B < B$ and then the previous argument applies with the roles of A, B interchanged.

In any case, then, we have $k(H) < k(A \times B)$, so that $A \times B$ is almost Abelian, as H was an arbitrary proper subgroup of $A \times B$.

(b) Recall that in this case $k(G) = k(H)|Z : Z \cap H|$. Assume that G is almost Abelian. If $X \leq H$, then $k(X)|Z : Z \cap H| \leq k(X)|Z : Z \cap X| = k(XZ) \leq k(G) = k(H)|Z : Z \cap H|$, with equality if and only if $Z \cap H \leq X$ and $XZ = G$, that is if $X = H$. Hence, if $X < H$, then $k(H) < k(X)$. The other implication is shown similarly. □

2.1. Almost Abelian decorated sporadic groups.

We now start our classification of nearly simple almost Abelian groups. This relies on the classification of finite simple groups. For certain small groups, like $\mathfrak{A}_6$ and $\mathrm{L}_3(4)$ we also use extensive explicit computations with character tables as well as inside permutation representations of various extensions in the `GAP` system [14], partly provided to us by Thomas Breuer. Throughout, G will be a finite group such that $F^*(G)$ is quasi-simple, that is, $F^*(G)$ is perfect, $F^*(G)/Z(F^*(G))$ is non-Abelian simple, and furthermore, $G/Z(F^*(G))$ is almost simple.

We fix the following notation. We set $Z := Z(F^*(G))$, $\bar{G} := G/Z$, and $S := F^*(G)/Z = \bar{G}^{(\infty)}$ the non-Abelian composition factor of G.

The rationale behind studying nearly simple almost Abelian groups is the following. When working with linear groups X, we often end up via Clifford theory in the following situation: X is an almost Abelian subgroup of $Y = \mathrm{GL}(V)$ for some finite vector space over a finite field, X has a unique component E which is an absolutely irreducible subgroup of $\mathrm{GL}(V)$. We know that X is a subgroup of $N := N_Y(E)$, so we seek to understand almost Abelian subgroups of N. Let Z be the group of non-zero scalars in Y. By Proposition 2.2(b), we have that XZ is almost Abelian. Since E is absolutely irreducible, we know that XZ is maximal subject to inducing the same group of outer automorphisms on E as X does. Also, for any subgroup W of Z, the group XW is almost Abelian by the same argument. In other words, the subgroup X (containing E) of N is almost Abelian if and only if XZ is, so it is a question of which subgroups of $N_Y(E)/EC_Y(E)$ have almost Abelian full preimages in N.

Note that our results fall short of providing a complete classification of nearly simple almost abelian groups. This would seem to require a much more extensive case-by-case study.

We start with the sporadic groups.

Proposition 2.3. *Let G be such that $F^*(G)$ is quasi-simple with S a sporadic simple group or ${}^2F_4(2)'$. Then G is almost Abelian if and only if $G = M_{12}.2$, $G = 2.M_{12}$ or $2.M_{12}.2$. In all other cases a witness can be chosen to be solvable, and even nilpotent if $G \neq J_2.2$.*

Proof. From the GAP tables [14] and the list of p-ranks in [5, Tab. 5.6.1] it follows that most quasi-simple sporadic groups G have an elementary Abelian section H for some prime p with $k(H) = |H| > k(G)$. Now if $H = X/Y$, say, then $k(X) \geq k(H) = |H| > k(G)$, whence in this case G cannot be almost Abelian. The exceptions are the groups in the following list, with a witness as indicated

G	$3.M_{22}$	$4.M_{22}$	$6.M_{22}$	$12.M_{22}$	$2.J_2$	$2.HS$
$k(G)$	34	39	65	109	38	42
H	C_6^2	C_{44}	C_{66}	C_{132}	$C_{10} \times C_5$	$C_{10} \times C_5$

and $G = M_{12}$ and $G = 2.M_{12}$. The group M_{12}, with $k(M_{12}) = 15$, has a subgroup C_4^2, while for $G = 2.M_{12}$ with $k(G) = 26$ the maximal $k(H) = 23$ for proper subgroups $H < G$ is attained for example for H a Sylow 2-subgroup. So this group is almost Abelian.

For the case when $G \neq F^*(G)$ is almost simple, again the table of p-ranks shows that we only need to consider

G	$M_{12}.2$	$M_{22}.2$	$J_2.2$	$HS.2$	$J_3.2$	${}^2F_4(2)$
$k(G)$	21	21	27	39	30	29
H	–	C_2^5	[96]	2_+^{1+6}	C_{34}	C_2^5

for all of which a solvable subgroup H as listed does the job (with [96] denoting an unspecified group of order 96; observe that the only witnesses in this group, of orders 96 and 384, are not nilpotent), except for $G = M_{12}.2$. Here the maximal $k(H) = 20$ for $H < G$ is attained, for example, for an Abelian subgroup $H = C_{10} \times C_2$.

Finally, assume that G is neither perfect nor $Z(F^*(G)) = 1$, so G is a non-trivial covering group of $M_{12}.2$, $M_{22}.2$, $J_2.2$, $J_3.2$, $McL.2$, $Suz.2$, $ON.2$, $Fi_{22}.2$ or Fi_{24}. For all of these but the first, the preimage in G of a suitable Sylow p-subgroup of $G/Z(F^*(G))$ has more conjugacy classes than G has. On the other hand, $G = 2.M_{12}.2$ has $k(G) = 34$ while any proper subgroup has at most 32 conjugacy classes. □

2.2. **Almost Abelian decorated alternating groups.** To deal with symmetric and alternating groups, we use the following upper bound

$$p(n) < \frac{\pi}{\sqrt{6(n-1)}} \exp\left(\pi\sqrt{2(n-1)/3}\right) \qquad \text{for } n > 1$$

for the number $p(n)$ of partitions of n (see [15, Thm. 15.7]).

Proposition 2.4. *Let G be such that $F^*(G)$ is a covering group of an alternating group $\mathfrak{A}_n$, $n \geq 5$. Then G is almost Abelian if and only if $G = \mathfrak{S}_5 \cong \mathrm{PGL}_2(5)$, $\mathfrak{S}_6$, $\mathfrak{A}_6.2_2 \cong \mathrm{PGL}_2(9)$, $\mathfrak{A}_6.2^2$, $3.\mathfrak{A}_6$ or $6.\mathfrak{A}_6$. In all other cases apart from $3.\mathfrak{A}_6.2_1$ and $3.\mathfrak{A}_6.2_2$ there is a nilpotent witness.*

Proof. We discuss the various possibilities for G, starting with the almost simple ones. The symmetric group $\mathfrak{S}_n$ has an elementary Abelian subgroup $H = 3^k$ with $k = \lfloor n/3 \rfloor$. The above estimate for $p(n)$, respectively the explicit value of $p(n)$ for small n shows that H is a witness when $n \geq 12$ and $n \neq 13, 14, 17$. For the remaining values $n \geq 7$ there exists a nilpotent witness H according to the following table:

n	7	8	9	10	11	13	14	17
$k(\mathfrak{S}_n)$	15	22	30	42	56	101	135	297
H	D_8C_3	D_8^2	D_8C_6	$D_8^2C_2$	$D_8^2C_3$	D_8^3	$C_3^3C_5$	$C_3^4C_5$

For $\mathfrak{S}_5 \cong \mathrm{PGL}_2(5)$ and $\mathfrak{S}_6$ there are no proper subgroups with sufficiently many (i.e., 7 respectively 11) conjugacy classes.

For $G = \mathfrak{A}_n$ the elementary Abelian 3-subgroup H of order 3^k, with $k = \lfloor n/3 \rfloor$, has $k(H) \geq k(G)$ whenever $n \notin \{5, 8, 11\}$, using that $k(\mathfrak{A}_n) < k(\mathfrak{S}_n)$ for $n \geq 5$ (see e.g. [3, Cor. 2.7]) and explicit values for small n. For the remaining values we can choose:

n	5	8	11
$k(\mathfrak{A}_n)$	5	14	31
H	C_5	C_{15}	$C_3^2C_5$

Finally, for $\mathfrak{A}_6.2_3$ we may take $H = C_3^2$, while $k(\mathfrak{A}_6.2^2) = 13$ is larger than the class number of any proper subgroup, and $\mathfrak{A}_6.2_2 \cong \mathrm{PGL}_2(9)$ is also almost Abelian by inspection.

In all of the cases above for which we found a witness H of odd order, the 2-coverings $2.\mathfrak{S}_n$ and $2.\mathfrak{A}_n$ can also not be almost Abelian, with witness the full preimage $\hat{H}$ of H, since, $k(\hat{H}) = 2k(H) \geq 2k(S_n) \geq k(2.\mathfrak{S}_n)$. Comparing with the precise value of $k(2.\mathfrak{S}_n)$, only $2.\mathfrak{S}_n$ with $n \in \{5, 7, 8, 10\}$ needs to be discussed, where we find witnesses as follows:

n	5	7	8	10
$k(2.\mathfrak{S}_n)$	12	23	31	57
H	C_6C_2	C_{24}	$C_{12}C_3$	$C_{30}C_2$

It remains to discuss the case when $S = \mathfrak{A}_6$ and when $F^*(G)$ is an exceptional cover of $\mathfrak{A}_7$. In the latter case, we find witnesses as follows:

n	$3.\mathfrak{A}_7$	$3.\mathfrak{A}_7.2$	$6.\mathfrak{A}_7$	$6.\mathfrak{A}_7.2$
$k(G)$	23	22	40	35
H	C_6^2	C_6^2	C_{42}	C_{42}

For various extensions of $\mathfrak{A}_6$ witnesses are given by:

n	$\mathfrak{A}_6.2_3$	$2.\mathfrak{A}_6.2_2$	$3.\mathfrak{A}_6.2_1$	$3.\mathfrak{A}_6.2_2$	$3.\mathfrak{A}_6.2_3$	$3.\mathfrak{A}_6.2^2$	$6.\mathfrak{A}_6.2_1$	$6.\mathfrak{A}_6.2_2$
$k(G)$	8	20	16	16	22	20	26	29
H	C_3^2	C_{20}	$3.\mathfrak{A}_6$	$3.\mathfrak{A}_6$	C_{24}	C_{24}	C_{30}	C_{30}

Note that for $3.\mathfrak{A}_6.2_1$ and $3.\mathfrak{A}_6.2_2$ there do not exist solvable witnesses. Finally, the covering groups of $\mathfrak{A}_6$ have $k(3.\mathfrak{A}_6) = 17$, $k(6.\mathfrak{A}_6) = 31$, while the (smallest) proper subgroups with maximal class number are cyclic of order 15, 30 respectively. □

2.3. **Lie type groups: some small cases.** For most groups of Lie type, we will exhibit a unipotent witness. But there are a number of small groups, for which such a witness does not exist. These are discussed in this section.

Proposition 2.5. *Let G be such that $F^*(G)$ is a covering group of* $\mathrm{L}_2(q)$ $(9 \neq q \geq 4)$ *or* ${}^2B_2(q^2)$ $(q^2 \geq 8)$. *Then G is almost Abelian if and only if $G \cong \mathrm{PGL}_2(q)$ with q odd. In the other cases, there exists an Abelian unipotent witness for $\bar{G}$ unless $\bar{G}$ is one of* $\mathrm{PGL}_2(q)$ *with q even,* $\mathrm{L}_2(8).3$, $\mathrm{L}_2(16).4$ *or* ${}^2B_2(8).3$.

Proof. Let first $S = \mathrm{L}_2(q)$ with $q \geq 5$. We start with the almost simple case. The outer automorphism group of S is generated by a field automorphism γ of order f, where $q = p^f$, and, if q is odd, a diagonal automorphism δ of order 2. Thus we need to discuss extensions of S by γ^d for $d|f$, by $\delta\gamma^d$ for $d|f$ with f/d even, and by $\langle \delta, \gamma^d \rangle$ for $d|f$. Let's consider these in turn.

If $q \geq 5$ is odd then $k(\mathrm{L}_2(q)) = (q+5)/2 \leq q$, so here a Sylow subgroup in the defining characteristic, of order q, provides a witness. For even $q \geq 8$ we have $k(\mathrm{L}_2(q)) = q+1$, which is attained by a cyclic subgroup of that order. For odd $q \geq 5$ we have $k(\mathrm{PGL}_2(q)) = q+2$, but all proper subgroups have fewer conjugacy classes, so these groups are almost Abelian. If q is the square of an odd prime power, there is a further extension $G = S.\langle \delta\gamma^{f/2} \rangle$ of S of degree 2, which has $k(G) = (q + 6\sqrt{q} + 5)/4 < q$.

Now assume that G is a proper extension of one of the groups $G_0 = G_0(q)$ considered before by a field automorphism, say $|G : G_0| = e > 1$ with $e|f$, and write $q = r^e$. The conjugacy classes of G are parametrised as follows: for any class with representative in $G_0(r^d)$, $d|e$, we obtain at most $e/d \cdot k(G_0(r^d))/d$ classes. Thus

$$k(G) \leq \sum_{d|e} e \cdot k(G_0(r^d))/d^2 = k(G_0(q))/e + e \sum_{d|e, d\neq e} k(G_0(r^d))/d^2.$$

This can be seen to be less than or equal to q unless $q \in \{4, 8, 9, 16, 25\}$, and even less than $q/2$ when $4|e$. Now $\mathrm{L}_2(4).2 \cong \mathrm{PGL}_2(5)$ is in our exceptions, and $q = 9$ was excluded, while $G = \mathrm{L}_2(8).3$ has an extra-special subgroup $H = 3^{1+2}$ with $k(H) = k(G)$, $k(\mathrm{L}_2(16).2) = 16$, $G = \mathrm{L}_2(16).4$ with $k(G) = 17$ has a subgroup C_{17}, and the extensions of $\mathrm{L}_2(25)$ different from $\mathrm{PGL}_2(25)$ have less than 25 classes. Finally, for $G = S.\langle \delta\gamma^d \rangle$ with $d|f$, $f/(2d)$ even, we have $k(G) \leq 2k(S.\langle\gamma^{2d}\rangle) < q$ by the above. This completes the discussion of almost simple extensions of $\mathrm{L}_2(q)$.

By our previous considerations, $\bar{G}$ has a unipotent witness H, in which case its full preimage in G will give a witness in G, unless $\bar{G}$ is one of $\mathrm{PGL}_2(q)$, $\mathrm{L}_2(8).3$ or $\mathrm{L}_2(16).4$. The latter two groups do not have proper covering groups, nor does $\mathrm{PGL}_2(q)$ for q even. If q is odd, then $k(2.\mathrm{L}_2(q).2) = 2(q+1)$, which is also the order of the Abelian preimage of a maximal torus of $\mathrm{PGL}_2(q)$ in $2.\mathrm{L}_2(q).2$. Note that we excluded $\mathrm{L}_2(4) \cong \mathrm{L}_2(5)$ and $\mathrm{L}_2(9) \cong \mathfrak{A}_6$, so there are no exceptional covering groups.

Now assume that $S = {}^2B_2(q^2)$, $q^2 \geq 8$, is a Suzuki group. First observe that S has a unipotent witness H, of order $2q^2$, as well: let $P \leq S$ be a Sylow 2-subgroup. Then $Z(P)$ contains (and in fact is equal to) an elementary Abelian subgroup of order q^2 (viz., $\Omega_1(Z(P))$). We may take $H > Z(P)$ with $|H/Z(P)| = 2$, then H is Abelian as $H/Z(P)$ is cyclic. Since $k(S) = q^2 + 3$, H is a witness. Here, the outer automorphism group consists solely of field automorphisms, which are of odd order. By the argument employed for $\mathrm{L}_2(q)$ above, we find that $k(G) \leq q^2$ for all such extensions $G \neq S$ unless $q^2 = 8$. Here $k({}^2B_2(8).3) = 17$, but there a (unipotent) Sylow 2-subgroup U has $k(U) = 22$.

The only Suzuki group with a non-trivial coverings is $S = {}^2B_2(8)$, with Schur multiplier 2^2. Now $G = 2.{}^2B_2(8)$ with $k(G) = 19$ has a unipotent subgroup of order 32 with 20 conjugacy classes, while for $G = 2^2.{}^2B_2(8)$ with $k(G) = 35$ and $G = 2^2.{}^2B_2(8).3$ with $k(G) = 25$ we may take the full preimage in G of a subgroup of order 13 of S as witness. □

Proposition 2.6. *Let G be such that $F^*(G)$ is a covering group of* $\mathrm{L}_3(q)$ $(q \neq 2, 4)$ *or* $\mathrm{U}_3(q)$ $(q \neq 2)$. *Then G is almost Abelian if and only if* $G \cong \mathrm{PGL}_3(q)$ *with* $q \equiv 1 \pmod 3$ *or* $G = \mathrm{L}_3(3).2$.

In the other cases, there exists a unipotent witness for $\bar{G}$ unless $\bar{G}$ is one of $\mathrm{PGL}_3(q)$ *with* $q \not\equiv 1 \pmod 3$ *or* $\mathrm{PGU}_3(q)$.

Proof. We first discuss the almost simple case, that is $F^*(G) = S$. The outer automorphism groups of the groups in question are generated by a diagonal automorphism δ, a graph automorphism σ and a field automorphism γ. Here the first two have order at most 3, respectively 2, while the latter has order f when $q = p^f$ for $\mathrm{L}_3(q)$ and order $2f$ for $\mathrm{U}_3(q)$ and then its fth power is σ. Again we first consider extensions not involving field automorphisms.

For inner-diagonal extensions G the class numbers are given by

G	$\mathrm{L}_3(q)$ $q \equiv 1\,(3)$	$\mathrm{PGL}_3(q)$ $q \equiv 1\,(3)$	$\mathrm{PGL}_3(q)$ $q \not\equiv 1\,(3)$	$\mathrm{U}_3(q)$ $q \equiv 2\,(3)$	$\mathrm{PGU}_3(q)$ $q \equiv 2\,(3)$	$\mathrm{PGU}_3(q)$ $q \not\equiv 2\,(3)$
$k(G)$	$\frac{1}{3}(q^2+q+10)$	q^2+q+2	q^2+q	$\frac{1}{3}(q^2+q+12)$	q^2+q+4	q^2+q+2
$\lvert H\rvert$	q^2	$-$	q^2+q+1	q^2	$(q+1)^2$	$(q+1)^2$

and the size of a suitable (Abelian) witness H is indicated in the last row, except for $\mathrm{PGL}_3(q)$ with $q \equiv 1 \pmod 3$ which has no proper subgroup with that many classes.

The class numbers of the extensions G of inner-diagonal groups by the graph automorphism of order 2 are given by

G	$\mathrm{L}_3(q).2$ $q\equiv 1\,(3)$	$\mathrm{PGL}_3(q).2$ $q\equiv 1\,(3)$	$\mathrm{PGL}_3(q).2$ $q\not\equiv 1\,(3)$	$\mathrm{U}_3(q).2$ $q\equiv 2\,(3)$	$\mathrm{PGU}_3(q).2$ $q\equiv 2\,(3)$	$\mathrm{PGU}_3(q).2$ $q\not\equiv 2\,(3)$
$k(G)$	$\frac{1}{6}(q^2{+}10q{+}37)$	$\frac{1}{2}(q^2{+}4q{+}11)$	$\frac{1}{2}(q^2{+}4q{+}9)$	$\frac{1}{6}(q^2{+}10q{+}39)$	$\frac{1}{2}(q^2{+}4q{+}13)$	$\frac{1}{2}(q^2{+}4q{+}11)$

when q is odd, and that same numbers minus $3/2$ when q is even. These quantities are larger than q^2 only when $q \le 3$, or when H is one of $\mathrm{PGU}_3(4)$, $\mathrm{PGL}_3(5)$ or $\mathrm{PGU}_3(5)$.(Note that $\mathrm{L}_3(4)$ was excluded.) Of these, only $\mathrm{L}_3(3).2$ is almost Abelian, while $\mathrm{PGU}_3(4).2$ has a unipotent subgroup U of order 32 with $k(U) = k(G) = 20$, and $G = \mathrm{PGL}_3(5).2$, $\mathrm{PGU}_3(5).2$ have unipotent subgroups U of order 5^3 with $k(U) = 29 \ge k(G)$. This completes the discussion of extensions only involving diagonal and graph automorphisms.

When $q = r^2$ is a square there is an extension of $\mathrm{L}_3(q)$ and of $\mathrm{PGL}_3(q)$ by a graph-field automorphism of order 2, with class numbers

G	$\mathrm{L}_3(q).2$ $q\equiv 1\,(3)$	$\mathrm{PGL}_3(q).2$ $q\equiv 1\,(3)$	$\mathrm{PGL}_3(q).2$ $q\not\equiv 1\,(3)$
$k(G)$	$\frac{1}{6}(q^2{+}4q{+}3r{+}46)$	$\frac{1}{2}(q^2{+}4q{+}3r{+}14)$	$\frac{1}{2}(q^2{+}4q{+}3r{+}6)$

hence all smaller than q^2 when $q > 4$, and even smaller than $q^2/3$ when q is a fourth or sixth power. Also, if $q = r^3$ is a third power there is an extension of $\mathrm{L}_3(q)$, $q \equiv 1 \pmod 3$, and of $\mathrm{U}_3(q)$, $q \equiv 2 \pmod 3$, of degree 3, by the product $\delta\gamma^{f/3}$ of a diagonal and a field automorphism, with $k(\mathrm{L}_3(q).3) = (q^2 + q + 24(r^2 + r) - 14)/9$, $k(\mathrm{U}_3(q).3) = (q^2+q+24(r^2+r)+36)/9$ (this can be seen since $\gamma^{f/3}$ fixes exactly three more irreducible characters than $\delta\gamma^{f/3}$, see [9, §3]). Both class numbers are less than $q^2/2$, hence this also covers the extensions of those groups by the graph automorphism of order 2.

Now assume that G is a proper extension of one of the groups $G_0 = G_0(q)$ considered before by a field automorphism. Then the counting argument given in the proof of Proposition 2.5 shows that $k(G) < q^2$ unless $S = \mathrm{L}_3(4)$, which was excluded.

It remains to discuss the case when G contains automorphisms of the form $\delta\gamma^d$ or $\sigma\gamma^d$ but not γ^d, and f/d is divisible by 9, 4 respectively. In these cases G has a normal subgroup G_1 of index 3 containing the non-trivial field automorphism $\gamma^{d/3}$, respectively $\gamma^{d/2}$, and as we argued before, $k(G_1) \le q^2/3$, so $k(G) \le q^2$. This concludes the discussion of the almost simple case.

Now consider the general case. By the first part, there is a unipotent witness for $\bar{G}$, whose full preimage will furnish a witness in G, unless $\bar{G}$ is one of $\mathrm{PGL}_3(q)$, $\mathrm{PGU}_3(q)$ or $\mathrm{L}_3(3).2$. Now none of $\mathrm{PGL}_3(q)$ for $q \not\equiv 1 \pmod 3$, $\mathrm{PGU}_3(q)$ for $q \not\equiv 2 \pmod 3$ and $\mathrm{L}_3(3).2$ does have proper covering groups. Furthermore, for $q \equiv 1 \pmod 3$ we have $k(3.\mathrm{L}_3(q).3) = 3q(q + 1)$, since exactly nine of the $q^2 + q + 8$ irreducible characters of $\mathrm{SL}_3(q)$ (corresponding to semisimple elements of $\mathrm{PGL}_3(q)$ with disconnected centraliser) are not invariant under the diagonal automorphism. The full preimage of a maximal torus of order q^2+q+1 of $\mathrm{PGL}_3(q)$ is Abelian and thus provides a witness. Similarly, for $2 < q \equiv 2 \pmod 3$ we have $k(3.\mathrm{U}_3(q).3) = 3(q^2 + q + 2)$, and the Abelian preimage of a maximal torus of order $(q + 1)^2$ has more classes. Note that all groups $\mathrm{L}_3(q)$ and $\mathrm{U}_3(q)$ with an exceptional Schur multiplier were excluded in our statement. □

We now discuss two further groups with large Schur multiplier.

Proposition 2.7. *Let G be such that $F^*(G)$ is a covering group of $S = \mathrm{L}_3(4)$. Then G is almost Abelian if and only if it is one of $S.3$, $S.6$, $S.D_{12}$, $4_1.S.2_3$, $3.S.6$.*

Proof. The group $S = \mathrm{L}_3(4)$ has Schur multiplier 3×4^2 and $\mathrm{Out}(S) \cong D_{12}$, the dihedral group of order 12. Class numbers for many of the extensions are contained in [14], and for all other extensions, we get at least upper bounds from these by trivial considerations. The group $3.S.3$ has a cyclic subgroup of order 63, and its preimage in G is a witness in all cases. Thus we may assume that either $|Z|$ or $|G : F^*(G)|$ is prime to 3. According to [14] the class numbers of certain quasi-simple groups $Z.S$, (an upper bound for) the maximal class number of any group G with $F^*(G) = Z.S$ and the class number of a suitable 2-subgroup H of $Z.S$ are given below:

Z	1	2	2^2	4_1	4_2	2×4	4^2
$k(Z.S)$	10	18	34	30	32	60	112
$\max k$	28	39	49	48	50	69	180
$k(H)$	19	32	64	40	64	80	88

Thus we obtain witnesses in all groups with $Z = 2^2, 4_2, 4 \times 2$. The group $4^2.S$ has a subgroup $C_4^2C_7$ which provides a witness. For $F^*(G) = 4_1.S$ the only extensions with class number bigger than 40 are of the form $4_1.S.2_3$ and $4_1.S.2^2$. While the latter, of which there are two isomorphism types with 42 resp. 48 classes, contain a unipotent subgroup with 44 resp. 49 classes, the group $4_1.S.2_3$ is almost Abelian. Its 3-fold cover has only 71 classes and thus is not almost Abelian. The only extensions with $F^*(G) = 2.S$ with more than 32 classes are the two types of groups $2.S.2^2$, with 33 resp. 39 classes, and both have a unipotent subgroup H with $k(H) = 40$. Finally, for $F^*(G) = S$, we need to discuss $S.2^2$ with 22 classes and with a unipotent subgroup having 26 classes, and the groups with $|G : F^*(G)|$ divisible by 3. These contain an Abelian subgroup of order 21, which is enough for $S.3.2_2$ and $S.3.2_3$, while the other three possibilities are almost Abelian and occur in the conclusion. Again by direct computation, among their 3-covers only $3.S.6$ is almost Abelian. □

Proposition 2.8. *Let G be such that $F^*(G)$ is a covering group of $\mathrm{U}_4(3)$. Then G is not almost Abelian.*

Proof. The group $S = \mathrm{U}_4(3)$ has Schur multiplier 4×3^2 and $\mathrm{Out}(S) \cong D_8$, the dihedral group of order 8. We have $k(S) = 20$, $k(3_1.S) = 52$, $k(3_2.S) = 46$, and $k(3^2.S) = 136$. Also, $3^2.S$ has an elementary Abelian subgroup U of order 3^6, and (thus) $3_1.S$ and $3_2.S$ have such a subgroup U of order 3^5 and S has such a subgroup U of order 3^4. It follows that the full preimage of U in G is a witness at least if $|G : F^*(G)| \leq 4$. It thus only remains to discuss groups G with $G/F^*(G) \cong D_8 = \mathrm{Out}(S)$. But then the Sylow 3-subgroup of Z must either be trivial, or the full 3^2 of the Schur multiplier, and $k(S.D_8) = 61$ and $k(3^2.S.D_8) = 110 < 3^6$. □

2.4. **Lie type groups: the generic case.**

Theorem 2.9. *Let G be almost simple with socle $S = F^*(G)$ of Lie type. Then G has a unipotent subgroup U with $k(U) \geq k(G)$ unless S is one of $\mathrm{L}_2(q)$, $\mathrm{L}_3(q)$, $\mathrm{U}_3(q)$, $\mathrm{U}_4(2)$ or $\mathrm{S}_4(2)' \cong \mathfrak{A}_6$.*

Proof. Let G be as in the statement. Let n denote the Lie rank of $S = F^*(G)$, q the underlying field size and p the characteristic of S. According to [3, Cor. 1.2] we have $k(G) \le 100\, q^n$, which is itself bounded above by q^{n+7}. On the other hand Table 3.3.1 in [5] shows that S has a large elementary Abelian p-subgroup U. A quick comparison gives that for S of type $E_6, {}^2E_6, E_7$ or E_8 the size of U is larger than q^{n+7}.

To deal with the remaining types we consider better bounds for $k(G)$. For $S = \mathrm{L}_{n+1}(q)$, $n \ge 3$, the outer automorphism group is generated by diagonal, graph and field automorphisms. For G an extension by a diagonal automorphism we have $k(G) \le q^n + 5q^{n-1}$ by [3, Cor. 3.7(2)], so in general we obtain $k(G) \le 2\log_p(q)(q^n + 5q^{n-1})$. By [5, Tab. 3.3.1] there is an elementary Abelian p-subgroup E of G of size q^k where $k = \lfloor (n+1)^2/4 \rfloor$. Thus we are done unless either $n = 3$, $q \le 5$, or $(n, q) = (4, 2)$. For $S = \mathrm{L}_5(2)$ we have $|\mathrm{Out}(S)| = 2$ and $2k(S) = 54 < 2^6$, for $S = \mathrm{L}_4(5)$ we have $|\mathrm{Out}(S)| = 8$ and $8k(S) = 392 < 5^4$. For $S = \mathrm{L}_4(4)$ we have $|\mathrm{Out}(S)| = 4$, $k(S) = 84$ and $k(S.2_1) = 63$, so $k(G) \le 168 < 4^4$. For $S = \mathrm{L}_4(3)$ we have $k(G) \le 56 < 3^4$, and for $S = \mathrm{L}_4(2) \cong \mathfrak{A}_8$ with $k(\mathfrak{A}_8) = 14$ we can take $U = C_2^4$, for $\mathrm{Aut}(S) = \mathfrak{S}_8$ there is a subgroup U of order 2^6 with $k(U) = 25 > k(\mathfrak{S}_8) = 22$.

For $S = \mathrm{U}_{n+1}(q)$, $n \ge 3$, we have $k(G) \le q^n + 8q^{n-1}$ for any extension G of S only containing diagonal automorphisms by [3, Cor. 3.11(2)], so $k(G) \le 2\log_p(q)(q^n + 8q^{n-1})$ in general. Now S has a unipotent subgroup $U = q^{1+2(n-1)}$ with $k(U) \ge q^{2(n-1)}$, so the claim holds unless either $n = 3$ and $q \le 8$, or $(n, q) = (4, 2)$. The group $S = \mathrm{U}_5(2)$ has $k(G) \le 47 < 2^6$, the group $S = \mathrm{U}_4(8)$ has $|\mathrm{Out}(S)| = 6$ and $6k(S) = 3612 < 8^4$, the group $S = \mathrm{U}_4(7)$ has $|\mathrm{Out}(S)| = 8$ and $8k(S) = 928 < 7^4$, the group $S = \mathrm{U}_4(5)$ has $|\mathrm{Out}(S)| = 4$ and $4k(S) = 388 < 5^4$, the group $S = \mathrm{U}_4(4)$ has $k(S) = 94$, $k(\mathrm{Aut}(S)) = 73$, so $k(G) \le 4^4$. The group $S = \mathrm{U}_4(3)$ was discussed in Proposition 2.8.

For $S = \mathrm{S}_{2n}(q)$, $n \ge 2$, and G only involving diagonal automorphisms, by [3, Thms. 3.12 and 3.13] we have $k(G) \le q^n + 62\, q^{n-1}$ when q is odd, and $k(G) \le q^n + 29\, q^{n-1}$ when q is even. Furthermore, $|\mathrm{Out}(\mathrm{S}_{2n}(q))| = \gcd(2, q-1)\log_p(q)$ for $n > 2$ or q odd, and $|\mathrm{Out}(\mathrm{S}_4(2^f))| = 2f$. There is an elementary Abelian unipotent subgroup of S of order $q^{n(n+1)/2}$. So we are done unless $n = 2$ and $q \le 13$, or $(n, q) = (3, 2)$. For $S = \mathrm{Sp}_6(2)$ we have $|\mathrm{Out}(S)| = 1$ and $k(S) = 30 < 2^6$. The extension $\mathrm{PCSp}_4(q)$ of S by a diagonal automorphism has $q^2 + 3q + 7 < q^3$ classes for odd q, and $q^2 + 2q + 3$ classes when q is even, which gives our claim for $q > 4$. Now $S = \mathrm{Sp}_4(4)$ has $k(S) = 27$, $k(S.2) = 30$, thus $k(G) \le 4^3$. Also, $S = \mathrm{S}_4(3)$ has $k(G) \le 25 < 3^3$. Finally, $\mathrm{S}_4(2) \cong \mathfrak{S}_6$ is almost Abelian.

For $S = \mathrm{O}_{2n+1}(q)$ of type B_n we may assume $n \ge 3$ and q is odd, as otherwise it is isomorphic to a symplectic group. Here $k(G) \le 7.1\, q^n$ by [3, Thm. 3.17(1)] for any G inducing diagonal automorphisms, and thus $k(G) \le \log_p(q)(7.1\, q^n)$, while there is an elementary Abelian p-subgroup of order q^{2n-1}, which gives a witness in all cases.

For $S = \mathrm{O}_{2n}^+(q)$ an even-dimensional orthogonal group of rank $n \ge 4$, we have $k(S) \le q^n/2 + 9\, q^{n-1}$ when $q > 3$ by [3, Tab. 2], respectively $k(G) \le \min\{27.2\, q^n, q^n + 68\, q^{n-1}\}$ when $q \le 3$ by [3, Thm. 1.1], when G only involves diagonal automorphisms. Now $|\mathrm{Out}(S)| \le 2\gcd(q-1, 2)^2 \log_p(q)$ for $n > 4$, and 3 times that number for $n = 4$. On the other hand, S has an elementary Abelian p-subgroup of order $q^{\binom{n}{2}}$ by [5, Tab. 3.3.1]. This proves our claim for $\mathrm{O}_{2n}^+(q)$ unless $n = 4$, $q \le 5$. For the remaining groups, for $S = \mathrm{O}_8^+(5)$ we have $|\mathrm{Out}(S)| = 24$ and $24k(S) = 24 \cdot 360 < 5^6$, for $S = \mathrm{O}_8^+(4)$ we have $|\mathrm{Out}(S)| = 12$ and $k(S) = 405$, $k(S.2) = 375$, so $k(G) < 4^6$ for all relevant G, for $S = \mathrm{O}_8^+(3)$ we have

$k(\mathrm{Aut}(S)) = 171$, $k(S.3) = 94$, $k(S.2) \le 144$, and thus $k(G) \le 432 < 3^6$. Finally, for $S = \mathrm{O}_8^+(2)$ we have $k(S) = 53$, $k(S.2) = 67$, $k(S.3) = 55$ and $k(S.\mathfrak{S}_3) = 68$, while $S.2$ contains a subgroup $\mathrm{S}_6(2) \times 2$ and hence an elementary Abelian subgroup C_2^7.

For $S = \mathrm{O}_{2n}^-(q)$ we have $|\mathrm{Out}(S)| = 2\gcd(q^n+1,4)\log_p(q)$, and there is an elementary Abelian p-subgroup U of order $q^{2+\binom{n-1}{2}}$, respectively of order q^6 when $n = 4$. Thus we are done unless $n = 4, 5$, $q \le 3$. As $k(\mathrm{O}_{10}^-(2)) = 115$ and $k(\mathrm{O}_{10}^-(3)) = 226$, these two groups do satisfy the conclusion. Further, $S = \mathrm{O}_8^-(3)$ has $|\mathrm{Out}(S)| = 4$ and $4k(S) = 448 < 3^6$, and $S = \mathrm{O}_8^-(2)$ has $k(S) = 39$, $k(\mathrm{Aut}(S)) = 60$, both less than 2^6.

For the remaining exceptional type groups we use the explicit formulas for $k(S)$ given in [3, Tab. 1]. For ${}^2G_2(q^2)$, $q^2 = 3^{2f+1}$ $(f \ge 1)$, we have $k(G) \le (2f+1)(q^2+8)$ while there is an elementary Abelian p-subgroup U of order $|U| = q^4$, for ${}^2F_4(q^2)$ $(q^2 = 2^{2f+1} \ge 8)$ we have $k(G) \le 2(2f+1)q^4$ while $|U| = q^{10}$, for $G_2(q)$ $(q \ge 3)$ we have $k(G) \le \log_p(q)(q^2+2q+9)$ while $|U| = q^3$, for ${}^3D_4(q)$ we have $k(G) \le \log_p(q)(q^4+q^3+q^2+q+6)$ while $|U| = q^5$ (and $k({}^3D_4(2)) = 35$, $k({}^3D_4(2).3) = 49$, while there is a subgroup $U = C_4^2 \times C_2^2$), by [5, Table 3.3.1]. For $S = F_4(q)$ we have $k(S) \le 2\log_p(q)(7.6\,q^4)$ and there is an elementary Abelian $U < G$ with $|U| = q^9$. This completes the proof. □

We can thus complete the proof of Theorem 4(1):

Corollary 2.10. *Let G be almost simple with $F^*(G)$ of Lie type. Then G is almost Abelian if and only if $G = \mathrm{PGL}_2(q)$ with q odd, or $G = \mathrm{PGL}_3(q)$ with $q \equiv 1 \pmod 3$, or G is one of $\mathrm{L}_3(3).2$, $\mathrm{L}_3(4).6$, $\mathrm{L}_3(4).D_{12}$, $\mathrm{L}_2(9).2_1 \cong \mathfrak{S}_6$ or $\mathrm{Aut}(\mathrm{L}_2(9)) \cong \mathrm{Aut}(\mathfrak{A}_6)$.*

Proof. By Theorem 2.9 we only need to discuss the groups $\mathrm{L}_2(q)$, $\mathrm{L}_3(q)$, $\mathrm{U}_3(q)$, ${}^2B_2(q^2)$, and $\mathrm{U}_4(2)$. For $S = \mathrm{U}_4(2) \cong \mathrm{S}_4(3)$ we already saw a 3-group witness, and the other groups were treated in Propositions 2.5–2.7. □

For more general extensions of groups of Lie type we have the following immediate consequence of Theorem 2.9:

Corollary 2.11. *Let G be such that $F^*(G)$ is quasi-simple of Lie type in characteristic p with $|Z(F^*(G))|$ prime to p and $F^*(G)/Z(F^*(G))$ none of $\mathrm{L}_2(q)$, $\mathrm{L}_3(q)$, or $\mathrm{U}_3(q)$. Then G is not almost Abelian.*

Proof. For all groups in the statement except $\mathrm{U}_4(2) \cong \mathrm{S}_4(3)$, $G/Z(F^*(G))$ has a unipotent witness U, and as by assumption $|Z(F^*(G))|$ is prime to the characteristic p, the full preimage of U is a witness in G. For $\mathrm{U}_4(2)$ the claim is checked directly. □

But more is true, namely for most quasi-simple groups we can find a unipotent witness:

Theorem 2.12. *Let G be quasi-simple of Lie type. Then there is a unipotent subgroup $U \le G$ with $k(U) \ge k(G)$ unless $S = G/Z(G)$ is one of $\mathrm{L}_2(q)$, $\mathrm{L}_3(q)$, $\mathrm{U}_3(q)$, $\mathrm{U}_4(2)$, $\mathrm{S}_4(3)$, or $G = 2.\mathrm{L}_4(2)$.*

Proof. First assume that $|Z(G)|$ is prime to the underlying characteristic p of G. Then, we may assume that G is the universal p'-covering group of $S = G/Z(G)$. The Suzuki and Ree groups do not have proper p'-covering groups, so they are handled by Theorem 2.9. So, $G = \mathbf{G}^F$ for a simple algebraic group $\mathbf{G}$ of simply connected type with a Frobenius endomorphism F. Recall that we exclude the groups $S = \mathrm{L}_2(q)$, $\mathrm{L}_3(q)$ and $\mathrm{U}_3(q)$. The tables 1, 2 and 3 in [3] give upper bounds for the class numbers $k(G)$. For groups of exceptional type, these coincide with the bounds for adjoint type groups used in the proof of Theorem 2.9, so the claim in that case follows. For $G = \mathrm{SL}_n(q)$ the bounds on $k(G)$ are better than for $\mathrm{PGL}_n(q)$, so we only need to look at the three groups $\mathrm{SL}_5(2)$, $\mathrm{SL}_4(5)$ and $\mathrm{SL}_4(3)$ violating the bound in the proof of Theorem 2.9. In fact, the first is also of adjoint type, and for the other two, $k(G)$ is small enough. For $G = \mathrm{SU}_n(q)$, the bound for $q = 2$ is worse than for $\mathrm{PGU}_n(q)$, but still small enough except for $G = \mathrm{SU}_6(2)$. There, $k(G) = 132 < 2^8$ as required. Also, the exceptional cases $\mathrm{SU}_4(7)$, $\mathrm{SU}_4(5)$, $\mathrm{SU}_4(3)$ can be inspected individually to verify the claim.

For the symplectic and orthogonal groups, we may assume $p \neq 2$, as otherwise G is as in Theorem 2.9. Among symplectic groups, only $\mathrm{Sp}_4(q)$ with $q \leq 13$ are left by the proof of Theorem 2.9. Here $k(G) = q^2 + 5q + 10$, which is less than q^3 unless $q = 3$. For $\mathrm{Sp}_4(3)$ there is no unipotent witness. For $\mathrm{Spin}_{2n+1}(q)$, $n \geq 3$ and q odd, the bound from [3] is sufficient except for $G = \mathrm{Spin}_7(3)$, but there $k(G) = 88 < 3^5$. Finally, for the even dimensional spin groups, only $\mathrm{Spin}_8^+(3)$, $\mathrm{Spin}_8^+(5)$, $\mathrm{Spin}_8^-(3)$ and $\mathrm{Spin}_{10}^-(3)$ need to be considered. For all four of them, the actual class number is smaller than the size of our Abelian unipotent subgroup.

Finally, assume that $Z(G)$ has order divisible by p, which means that G is an exceptional covering group of S (see [11, Tab. 24.3]). We deal with these in turn. For G not a covering group of one of the excluded groups, that is, G one of $2^2.\,\mathrm{U}_6(2)$, $2.\,\mathrm{S}_6(2)$, $3.\,\mathrm{O}_7(3)$, $3.G_2(3)$, $2.G_2(4)$, $2.F_4(2)$ or $2^2.{}^2E_6(2)$ one checks that $k(G)$ is always less than $|U|$ for the unipotent witness U in $G/Z(G)$ from Theorem 2.9. The group $S = \mathrm{O}_8^+(2)$ has a 2-subgroup U of order 2^8 with $k(U) = 130$, larger than $k(2.S) = 83$. Moreover, $2^2.S$ contains a subgroup $2 \times 2.\,\mathrm{Sp}_6(2)$, which in turn has a unipotent subgroup with 152 classes while $k(2^2.S) = 143$. Note that ${}^2B_2(8)$ and $\mathrm{U}_4(3)$ were treated in Propositions 2.5 and 2.8. □

Finally, we complete the proof of Theorem 4:

Proof of Theorem 4. By Propositions 2.3–2.7 all that is left to do is to investigate exceptional covering groups of almost simple, non-simple groups of Lie type. We may discard those with S an alternating group, or with S isomorphic to a group of Lie type in different characteristic or for which $\mathrm{Out}(S) = 1$.

For $S = \mathrm{U}_6(2)$ we have $k(S.2) = 65$, $k(S.3) = 114$ and $k(S.\mathfrak{S}_3) = 99$, while a suitable maximal torus of S, $3.S$, $S.3$ and $3.S.3$ has order 3^4, 3^5, 3^5, and 3^6 respectively. The full preimage of this in G will thus provide a witness. The group $S = \mathrm{O}_7(3)$ has an Abelian subgroup of order 2^6, so its preimage in $3.S.2$ and $6.S.2$ provides a witness as $k(3.S.2) = 132$, $k(6.S.2) = 186$. For $S = \mathrm{O}_8^+(2)$ the class number $k(2^2.S) = 143$ is larger than $k(2^2.S.A)$ for any $1 \neq A \leq \mathrm{Out}(S) \cong \mathfrak{S}_3$, while the class number $k(2.S.2) = 112$ is smaller than that of a 2-subgroup H of S with $k(H) = 130$.

Further, $k(3.G_2(3).2) = 39 < k(3.G_2(3)) = 45$, $k(2.G_2(4).2) = 65 < k(P) = 118$ with $P \in \mathrm{Syl}_2(G_2(4))$, $k(2.F_4(2).2) = 151 < k(U) = 2^9$, with U an elementary Abelian unipotent subgroup of $F_4(2)$, and for $S = {}^2E_6(2)$, $k(S.A) \leq 266$ for $A \leq \mathrm{Out}(S) \cong \mathfrak{S}_3$, while S has an Abelian unipotent subgroup of order $2^{13} > 266|Z|$ since $|Z| \leq 12$. This deals with all remaining possibilities for S. □

3. π-BOUNDED FINITE GROUPS

Let π be a set of primes. Recall that we say that a finite group G *is π-bounded* if there exists a π-subgroup $H \leq G$ with $k_\pi(G) \leq k_\pi(H) = k(H)$. Here, $k_\pi(G)$ denotes the number of π-conjugacy classes of G. In this case, H is called a *(π-)witness* for G. Clearly, if there is a π-witness for G at all, then there is an almost Abelian such witness.

Of course solvable groups are π-bounded for any π by the fact that all their maximal π-subgroups are conjugate (these are then Hall π-subgroups). Also, the extreme cases $|\pi| = 1$ (by the Sylow theorems) and $\pi = \pi(G)$ are obviously fine. By a theorem of Wielandt [16] any group with a nilpotent Hall π-subgroup is π-bounded.

3.1. General observations.

Let us start out with a few easy but nonetheless useful observations:

Lemma 3.1. *Let G be a finite group and let N be a normal π'-subgroup of G. Then G is π-bounded if and only if G/N is π-bounded.*

Proof. Using the Schur–Zassenhaus theorem, we have $k_\pi(G) = k_\pi(G/N)$. Thus if H is a π-witness for G, then HN/N is a π-witness for G/N. Conversely, if J/N is a π-witness for G/N and H is a complement of N in J, then H is a π-witness for G. □

Lemma 3.2. *Let $N \trianglelefteq G$ be finite groups with G/N a π'-group. If N is π-bounded then so is G.*

Proof. Let $H \leq N$ be a π-subgroup with $k(H) \geq k_\pi(N)$. As all π-elements of G are contained in N, we have $k_\pi(G) \leq k_\pi(N)$, whence $k(H) \geq k_\pi(G)$ as well. □

We next record some restrictions on potential π-witnesses for future use. (We shall not use these in the sequel.) If we have $k_\pi(G) \leq k(H)$ for some π-subgroup H of G, we may suppose that $k(H)$ is maximised over π-subgroups of G, and that $|H|$ is minimised over π-subgroups with the maximal possible number of conjugacy classes. Thus we may suppose that H is almost Abelian, and with $k(H)$ maximised over π-subgroups of G.

Note also that we may restrict attention to the case that $k_\pi(G) > k_\pi(H)$ whenever $H < G$ (for in any other case, a π-witness for a suitable proper subgroup H would be a witness for G too). Likewise, the same applies if we are seeking an Abelian witness, (or nilpotent, solvable, etc). We call such a group G *almost π-Abelian.*

In fact the statement that every finite group is π-bounded is equivalent to the statement that every almost π-Abelian group is an almost Abelian π-group. For if G is an almost π- Abelian group which is not a π-group, then G is certainly not π-bounded, for G itself is then not a π-witness, but no proper subgroup of G has as many classes of π-elements as G does, so there is no π-witness for G at all. If G is an almost π-Abelian π-group, then G is almost Abelian, since G has more classes of π-elements that any of its proper subgroups, so G has more conjugacy classes than any of its proper subgroups, as all relevant subgroups are π-groups.

On the other hand, if G is a finite group of minimal order subject to not being π-bounded, then G is certainly not a π-group, but as we remarked above, G must be almost π-Abelian, otherwise a proper subgroup H of G with $k_\pi(H) \geq k_\pi(G)$ is π-bounded by the minimality of G, and then a π-witness for H is a π-witness for G, contrary to G not being π-bounded.

While it seems impractical to classify almost π-Abelian groups G for general π, or even the almost simple such G, the following properties of almost π-Abelian groups seem noteworthy:

Lemma 3.3. *Let G be an almost π-Abelian finite group for some set of primes π. Then:*

(a) *$O_{\pi'}(G)$ is contained in the Frattini subgroup of G.*
(b) *$k(G) \leq k_\pi(G)k_{\pi'}(G)$, and furthermore, equality holds if and only if G is an almost Abelian π-group. In particular, if equality holds, then G is π-bounded.*

Proof. Assume that $N = O_{\pi'}(G)$ is not contained in the Frattini subgroup of G. Then there is a maximal subgroup M with $G = NM$. Then $G/N \cong M/O_{\pi'}(M)$, so M has the same number of π-classes as G does, contrary to G being almost π-Abelian. Thus we have (a).

For (b), we have $k(G) = \sum_x k_\pi(C_G(x))$, where x runs through a set of representatives for the conjugacy classes of π'-elements of G. For each such x, we have (by hypothesis) that $k_\pi(C_G(x)) \leq k_\pi(G)$, and the inequality is strict unless $x \in Z(G)$. Hence the first claimed inequality follows. Furthermore, it is clear that the inequality is strict unless $C_G(x) = G$ for each π'-element $x \in G$.

Hence if equality holds, then every π'-element of G is central, so $G \cong O_{\pi'}(G) \times O_\pi(G)$ by Schur–Zassenhaus. Then (a) forces $O_{\pi'}(G) = 1$. On the other hand, an almost Abelian π-group is clearly almost π-Abelian, so (b) follows. □

Proposition 3.4. *Let G be a finite group, and let G^* be a central extension of G by a finite Abelian subgroup Z. Assume that H is a π-witness for G, and that either H is cyclic, or the order of H is prime to (the π-part of) the order of the Schur multiplier of G. Then $O_\pi(H^*)$ is a π-witness for G^*, where H^* is the full preimage of H in G^*.*

Proof. If Z is π'-group, then we have $k_\pi(G^*) = k_\pi(G)$, so it is no loss of generality to suppose that Z is a π-group (for otherwise, we may just extend in two stages, first by $O_{\pi'}(Z)$ (which does not change the number of π-classes) and then by $O_\pi(Z)$).

Each π-element x of G has $|Z|$ preimages in G^*, all of which are π-elements, and for every π-element y^* of G^* the image y^*Z is a π-element of G. Hence we have $k_\pi(G^*) \leq |Z|k_\pi(G)$, and equality can only hold if whenever y^* is a π-element of G^*, no two elements of the coset y^*Z are conjugate in G^*.

If H is cyclic, then its full preimage H^* in G^* is Abelian of order $|Z||H|$, and then

$$k(H^*) = |Z|k(H) \geq |Z|k_\pi(G) \geq k_\pi(G^*).$$

If H has order coprime to (the π-part of) the order of the Schur multiplier of G, then the full preimage of H in G^* has order $|Z||H|$ and has $|Z|k(H)$ conjugacy classes, so we still obtain $k(O_\pi(H^*)) \geq k_\pi(G^*)$. □

In the next result, which is due to the third author and was used in [13, Prop. 3] without proof, a related weaker variant of π-boundedness is proven:

Lemma 3.5. *Let G be a finite group and π be a set of prime divisors of $|G|$, say $\pi = \{p_i \mid 1 \leq i \leq n\}$. Then there are p_i-subgroups Q_i of G such that*

$$k_\pi(G) \leq \prod_{i=1}^{n} k(Q_i).$$

In particular, $k_\pi(G) \leq |G|_\pi$.

Proof. For each i, choose a p_i-subgroup Q_i of G with $k(Q_i)$ maximal among p_i-subgroups. We proceed by induction on $|\pi|$, the result being clear when $|\pi| = 1$.

Suppose then that $|\pi| > 1$ and let $\pi_1 = \pi \backslash \{p_1\}$. Then

$$k_\pi(G) = \sum_x k_{\pi_1}(C_G(x)),$$

where x runs over a set of representatives for the conjugacy classes of p_1-elements of G. By induction, for each such x, we have

$$k_{\pi_1}(C_G(x)) \leq \prod_{i=2}^{n} k(Q_i)$$

(for we may assume the relevant inequality in $C_G(x)$ by induction, but with Q_i replaced by some p_i-subgroup R_i of $C_G(x)$ with its number of conjugacy classes maximised. However, our choice of Q_i certainly yields $k(R_i) \leq k(Q_i)$). It is clear that the number of conjugacy classes of p_1-elements of G is at most $k(Q_1)$, by the case $|\pi| = 1$. Hence we do have

$$k_\pi(G) \leq \prod_{i=1}^{n} k(Q_i),$$

as claimed. □

Remark 3.6. (1) Note that the subgroups Q_i may be chosen to be almost Abelian (if any Q_i is not almost Abelian, then it has a proper subgroup R_i with $k(R_i) \geq k(Q_i)$, and then we can replace Q_i by R_i).

(2) Notice that $k_\pi(G) = |G|_\pi$ if and only if G has an Abelian Hall π-subgroup H and a normal π-complement K:

If H and K exist, then the equality clearly holds. Conversely, if equality holds, then the argument of the proof of Lemma 3.5 shows that we must have $k_p(G) = |P|$ for each prime $p \in \pi$ and each Sylow p-subgroup P of G. In particular, this requires that any two elements of P conjugate in G are already conjugate in P, in which case G has a normal p-complement by Burnside's transfer theorem. Since $p \in \pi$ was arbitrary, we see that G has a normal π-complement, L say. Then

$$[G : L] = k_\pi(G) = k_\pi(G/L) = k(G/L)$$

and G/L is Abelian. By the Schur–Zassenhaus theorem, we know that G has a Hall π-subgroup M, and $M \cong G/L$ is Abelian.

(3) Let us recall a very useful fact proved by Fulman and Guralnick in [3, Lemma 2.3]: If G is a finite group and $N \trianglelefteq G$ then $k_\pi(G) \leq k_\pi(N)\, k_\pi(G/N)$.

3.2. **Symmetric and alternating groups are π-bounded.** We now investigate π-boundedness of nearly simple groups. For symmetric and alternating groups, we have a conceptual approach for large primes. We say that a finite tuple $T = (b_1, b_2, \ldots, b_t)$ of integers is *separable* if for each $i \geq 2$, there is a prime q_i which divides b_i, such that q_i does not divide $\prod_{j<i} b_j$. Note that T is certainly separable if its elements are pairwise coprime.

Let $T = (b_1, b_2, \ldots, b_t)$ be a fixed separable tuple of integers and let n be a fixed positive integer. Each partition λ of n with all its parts drawn from T corresponds to a unique ordered t-tuple of non-negative integers $(a_1, \ldots, a_t)$ with $\sum_{i=1}^t a_i b_i = n$, and each such t-tuple does yield a unique partition of n with parts drawn from T (with a_i parts of size b_i for $1 \le i \le t$). In the spirit of [4] we define the *Gödel number* $g_T(\lambda)$ of λ with respect to T to be the positive integer $\prod_{i=1}^t b_i^{a_i}$. Note that when λ is a partition of n with all parts drawn from T, then $\mathfrak{S}_n$ has an Abelian subgroup of order $g_T(\lambda)$, which is isomorphic to

$$C_{b_1}^{a_1} \times \ldots \times C_{b_t}^{a_t}.$$

Theorem 3.7. *For a separable tuple of integers $T \neq \emptyset$, the number of partitions of n with all their parts drawn from T is less than or equal to the maximal value M of $g_T(\lambda)$ as λ runs through partitions of n with all parts drawn from T, and equality can only hold when $T = (1)$ or $2 \in T$. Furthermore, $\mathfrak{S}_n$ has an Abelian subgroup of order M.*

Proof. Since T is separable, it follows (using uniqueness of prime factorization) that if λ and μ are different partitions of n with all their parts drawn from T, then $g_T(\lambda) \neq g_T(\mu)$. For the t-tuple $(a_1, \ldots, a_t)$ associated to the partition λ of n may be recovered from the Gödel number $g_T(\lambda)$, since for $i \ge 2$, $a_i \nu_{q_i}(b_i) = \nu_{q_i}(g_T(\lambda)) - \sum_{j>i} a_j \nu_{q_i}(b_j)$, and then $a_1 = (n - \sum_{i=2}^t a_i b_i)/b_1$.

Thus, the number of such partitions is the number of associated Gödel numbers. All possible Gödel numbers are less than or equal to M. Furthermore, 2 cannot occur as the Gödel number of a relevant partition when $2 \notin T$, so the number of such partitions is strictly less than M in that case unless $T = (1)$. □

Corollary 3.8. *Let $n \ge 9$ be an integer and let π be a set of primes each greater than $\sqrt{n}$. Then $\mathfrak{S}_n$ has an Abelian π-subgroup of order greater than $2k_\pi(\mathfrak{S}_n)$.*

Proof. Let $\pi = \{p_1 < \ldots < p_t\}$, where $p_1 > \sqrt{n}$. Note that $\sigma \in \mathfrak{S}_n$ is a π-element if and only if the lengths of its disjoint cycles are all drawn from $T := (1, p_1, \ldots, p_t)$, and T is clearly separable. Since $n > 8$, we have $2, 3 \notin \pi$. Theorem 3.7 allows us to conclude that the number of conjugacy classes of π-elements of $\mathfrak{S}_n$ is less than the maximal order M of an Abelian π-subgroup of $\mathfrak{S}_n$. Furthermore, all relevant Gödel numbers $g_T(\lambda)$ are odd integers as $2 \notin \pi$. As also $3 \notin \pi$, we may exclude at least $\frac{M+1}{2}$ integers less than M as candidates for Gödel numbers, and the claim follows. □

More precisely, under the hypotheses of Corollary 3.8, the number of conjugacy classes of π-elements of $\mathfrak{S}_n$ is bounded above by the number of π-numbers less than or equal to M, where M is the maximum order of an Abelian π-subgroup of $\mathfrak{S}_n$.

Remark 3.9. It is not difficult to prove that under the hypotheses of the corollary, we have $M \le p_1^{n/p_1}$, but it need not be the case that $M \le p_1^{\lfloor n/p_1 \rfloor}$. For example, if $n = 19$ and $\pi = \{5, 7\}$, then we have

$$M = 5 \cdot 7^2 > 5^{\lfloor \frac{19}{5} \rfloor} = 5^3.$$

Similarly, it need not be the case under the hypotheses of the corollary that the "obvious" Abelian π-subgroup of order $p_1^{\lfloor n/p_1 \rfloor}$ has order greater than $2k_\pi(\mathfrak{S}_n)$. For example, when $n = 9$ and $\pi = \{5, 7\}$, we see that $\mathfrak{S}_9$ has three conjugacy classes of π-elements containing elements of respective orders $1, 5$ and 7. In this case, we have $M = 7 > 2k_\pi(\mathfrak{S}_9)$, but we do not have

$$2k_\pi(\mathfrak{S}_9) < p_1^{\lfloor \frac{n}{p_1} \rfloor} = 5.$$

We now deal with the small primes.

Proposition 3.10. *The symmetric group $\mathfrak{S}_n$, $n \geq 5$, is π-bounded for all π containing a prime $p \leq \sqrt{n}$, with a nilpotent witness when $\pi \subsetneq \pi(\mathfrak{S}_n)$.*

Proof. Let $\pi \subseteq \pi(\mathfrak{S}_n)$ be a set of primes. Let p be the smallest prime in π. Then $\mathfrak{S}_n$ has an elementary Abelian subgroup C_p^k with $k = \lfloor n/p \rfloor$. On the other hand,

$$k_\pi(\mathfrak{S}_n) \leq k(\mathfrak{S}_n) \leq \exp(\pi\sqrt{2n/3}) \leq \exp(2.6\sqrt{n})$$

by the upper bound for $p(n)$ in Section 2.2. Thus, to conclude, we need that $p^{(n-p)/p} \geq \exp(2.6\sqrt{n})$, that is, $((n-p)/p)\ln p \geq 2.6\sqrt{n}$, that is, $\sqrt{n}/p \geq 2.6/\ln p + 1/\sqrt{n}$. This is satisfied whenever $p \geq 17$ or $n \geq 195$. For $n \leq 194$, using the precise value of $p(n)$ one sees that the claim holds whenever $n \geq 30$. For the remaining values of n and p, we use the exact number of π_p-partitions of n, where $\pi_p := \{r \in \pi(\mathfrak{S}_n) \mid r \geq p\}$ to see the assertion unless $p = 2$ and $n \leq 17$. For the latter cases, a subgroup $D_8^k C_2^l$ with $\lfloor n/2 \rfloor = 2k + l$, $l \in \{0, 1\}$ has sufficiently many classes unless $n \in \{5, 7, 9, 11\}$ and π contains 3. We can then choose the following nilpotent witness H:

n	5	7	9	11
$\min \pi'$	5	5	7	7
$k_\pi(\mathfrak{S}_n) \leq$	6	14	28	55
H	C_3C_2	D_8C_3	$D_8C_3C_2$	$D_8^2C_3$

□

As already observed in Proposition 2.4, the groups $\mathfrak{S}_5$ and $\mathfrak{S}_6$ are almost Abelian and hence there can be no nilpotent witness for π the set of all primes.

Corollary 3.11. *The alternating group $\mathfrak{A}_n$, for $n \geq 5$, is π-bounded for any π.*

Proof. Let $p \in \pi$ be minimal. Note that if $p > 2$ then any π-subgroup of $\mathfrak{S}_n$ is contained in $\mathfrak{A}_n$, so the assertion follows from Corollary 3.8 if $p > \sqrt{n}$. If $2 < p \leq \sqrt{n}$, since $k(\mathfrak{A}_n) < k(\mathfrak{S}_n)$ for $n \geq 5$ (see e.g. [3, Cor. 2.7]), the argument in the proof of Proposition 3.10 shows the claim for $n \geq 30$. For smaller values of n, we get the result by using the precise number of π_p-partitions.

Finally assume that $p = 2$. For $n \geq 18$ and $n = 16$ an elementary Abelian 2-subgroup does the job. For $10 \leq n \leq 17$, $n \neq 13$, we can take a subgroup $D_8^k C_2^l$ with $\lfloor (n-2)/2 \rfloor = 2k + l$, $l \in \{0, 1\}$, while for $n = 13$ a Sylow 2-subgroup has $56 = k(\mathfrak{A}_{13}) + 1$ classes, and for $n = 8$ a Sylow 2-subgroup has $16 = k(\mathfrak{A}_n) + 2$ classes. For the remaining values of n there is no suitable 2-subgroup. But for $n \in \{6, 7, 9\}$, if $3 \in \pi$ then we may choose an elementary Abelian 3-subgroup, while if $3 \notin \pi$ then $k_\pi(\mathfrak{A}_n)$ is small enough for a 2-subgroup to work. Finally, for $n = 5$, we take $H = C_5$ if $5 \in \pi$ and $H = C_3$ if $\pi \subseteq \{2, 3\}$. □

Proposition 3.12. *Let G be a covering group of an alternating group $\mathfrak{A}_n$, $n \geq 5$. Then G is π-bounded for all π.*

Proof. By Corollary 3.11 we only need to consider proper covering groups. We will deal with the exceptional covers of $\mathfrak{A}_6$ and $\mathfrak{A}_7$ later, so for the moment we have $|Z(G)| = 2$ and thus may assume $2 \in \pi$ by Lemma 3.1. Then

$$k_\pi(2.\mathfrak{A}_n) \le k(2.\mathfrak{A}_n) \le 2k(\mathfrak{A}_n) \le 2k(\mathfrak{S}_n) = 2p(n),$$

where the last inequality comes from [3, Cor. 2.7]. Now $\mathfrak{A}_n$ has an elementary Abelian subgroup of order 2^k, where $k = \lfloor (n-2)/2 \rfloor$ and then its full preimage in $2.\mathfrak{A}_n$ has at least 2^k conjugacy classes. Comparison with the asymptotic formula for $p(n)$ from Section 2.2 shows that 2^k is at least equal to our bound on $k_\pi(2.\mathfrak{A}_n)$ for all $n \ge 50$, whence the claim in these cases.

Note that our estimate for $k(2.\mathfrak{A}_n)$ is very crude. The precise number of irreducible characters (and hence of conjugacy classes) of $2.\mathfrak{A}_n$ is given by the number of even partitions of n, plus the number of partitions all of which parts are odd and distinct, plus the number of partitions of n into distinct parts, plus the number of partitions into m distinct parts such that $n \equiv m \pmod 2$. Using this precise value, we obtain the claim for $n \ge 22$.

Assume $3 \in \pi$. Then we use that $k(2.\mathfrak{A}_n)$ is less than the order of the full preimage of an elementary Abelian 3-subgroup of $\mathfrak{A}_n$ in $2.\mathfrak{A}_n$ unless $(n,p) = (5,5)$ or $(8,7)$. If $n = 5$ then a subgroup C_6 does the job, for $n = 8$ we can take C_{30}. For $3 \notin \pi$ the maximal elementary Abelian 2-subgroups are sufficient whenever $n \ge 14$, or $n = 10, 12$. For the remaining values of n we take H as indicated below:

n	5	6	7	8	9	11	13
$k_\pi(2.\mathfrak{A}_n) \le$	7	9	11	12	11	23	37
H	C_{10}	C_{10}	C_{14}	C_{14}	C_{14}	C_5^2	C_{70}

Finally, for the exceptional covering groups of $\mathfrak{A}_6$ we take $H = C_5$ when $5 \in \pi$, and $H = 3^2 : 4$ when $5 \notin \pi$ (respectively their full preimages), and for the exceptional covering groups of $\mathfrak{A}_7$ we take $H = C_7$ when $7 \in \pi$, the full preimage of C_6 when $2, 3 \in \pi$, of C_5 when $5 \in \pi$, and an Abelian subgroup C_4 or C_3^2 otherwise. □

3.3. **Sporadic groups are π-bounded.**

Proposition 3.13. *Let G be almost simple with $F^*(G)$ a sporadic simple group or the Tits simple group. Then G is π-bounded for all π, with a nilpotent witness if $\pi \subsetneq \pi(G)$.*

Proof. For p a prime let us denote $\pi_p := \{r \in \pi(G) \mid r \ge p\}$. Observe that if $k_{\pi_p}(G) \le p$, then G is π-bounded for all $\{p\} \subseteq \pi \subseteq \pi_p$, with witness $H \cong C_p$. Using the explicitly known character tables of the groups in question, we determine those primes p such that $k_{\pi_p}(G) > p$. For these cases, we compare $k_{\pi_p}(G)$ to the order of a maximal elementary Abelian p-subgroup H of G given in [5, Tab. 5.6.1]. This leaves 27 pairs (G,p), mostly with $p = 2$. We now discuss these.

For M_{11} we need to consider $p = 2$. If $3 \in \pi$ then $H = C_3^2$ does the job, if $11 \in \pi$ then we take $H = C_{11}$, and else a Sylow 2-subgroup can be chosen. For M_{12} again $p = 2$ is open, and we take $H = C_4^2$. For $M_{12}.2$ at $p = 2$ we take $C_{10} \times C_2$ when $5 \in \pi$, and a Sylow 2-subgroup H (with $k(H) = 17$) else. For J_1 we take C_{19} if $19 \in \pi$, C_{15} if $p = 3$ and $5 \in \pi$, C_7 if $p = 3$ and $7 \in \pi$, C_{11} if $p = 3$ and $11 \in \pi$; C_{15} if $\{2,3,5\} \subseteq \pi$, C_{11} if $p = 2$ and $\{3,5\} \not\subseteq \pi$, and C_2^3 else.

For M_{22} with $p = 5$ we take C_{11} if $11 \in \pi$ and C_5 else; for J_2 with $p = 2$ take C_5^2 when $5 \in \pi$, and C_2^4 else, for $J_2.2$ with $p = 2$ take C_5^2 if $5 \in \pi$, and a Sylow 2-subgroup H (with $k(H) = 22$) else; for M_{23} with $p = 3, 5$ take C_{11} if $11 \in \pi$, take C_3^2 if $p = 3$ and $11 \notin \pi$, take C_{23} when $p = 5$ and $23 \in \pi$, and C_5 otherwise. For HS or $HS.2$ with $p = 2$ take a Sylow 2-subgroup. For J_3 with $p = 5$ take C_{17} if $17 \in \pi$, and C_5 else; when $p = 2$ take C_3^3 when $3 \in \pi$ and C_2^4 else; for $J_3.2$ with $p = 2$ take C_{34} if $17 \in \pi$ and a suitable subgroup H of order 128 with $k(H) = 26$ else.

For M_{24} with $p = 3, 5$ take C_{23} if $23 \in \pi$, C_{15} if $5 \in \pi$, and C_3^2 else. For McL with $p = 2$ take C_3^4 when $3 \in \pi$, and C_2^4 else; for $McL.2$ with $p = 2$ take again C_3^4 when $3 \in \pi$, C_5^2 when $5 \in \pi$, and C_2^4 otherwise. For He and $He.2$ with $p = 3$ take C_7^2 if $7 \in \pi$ and C_3^2 otherwise. For $Suz.2$ with $p = 2$ take C_3^5 when $3 \in \pi$ and C_2^6 else. For ON and $ON.2$ with $p = 5$ take C_{19} or C_{31} or C_5^2, for ON and $ON.2$ with $p = 2$ take $H = C_4^3$. For Co_3 with $p = 2$ we may take C_4^3, for $HN.2$ with $p = 2$ take C_3^4 if $3 \in \pi$ and C_2^6 else. For Ly with $p = 2$ take C_3^5 when $3 \in \pi$, C_5^3 when $5 \in \pi$, C_{31} when $31 \in \pi$, C_{67} when $67 \in \pi$ and C_2^4 else. For Ly with $p = 7$ or $p = 11$ we take C_r, $r \in \{31, 67\}$ if $r \in \pi$, and C_p else. For Th with $p = 2$ take $H = 2^{1+8}$, for Fi_{23} with $p = 7$ either take C_{11} or C_7.

For J_4 with $p \geq 3$ we may take C_r for the maximal $r \in \pi$ provided that $r \geq 23$, we take C_{15} when $p = 3$ and $5 \in \pi$, and C_3^2 else, we take C_{35} when $p = 5$ and $7 \in \pi$, and C_5 else, and we take C_7 when $p = 7$. □

3.4. **Groups of Lie type.** For groups of Lie type, we start out with the following trivial consequence of Theorems 2.9 and 2.12 in conjunction with Lemma 3.1:

Corollary 3.14. *Let S be simple of Lie type in characteristic p, not one of $\mathrm{L}_2(q)$, $\mathrm{L}_3(q)$, $\mathrm{U}_3(q)$, $\mathrm{U}_4(2)$ or $\mathrm{S}_4(2)'$. If G is either almost simple with $F^*(G) = S$ or quasi-simple with $G/Z(G) = S$ and $(|Z(G)|, p) = 1$, then G is π-bounded for any π containing p.*

The following easy generalisation of a well-known result deals with $p \notin \pi$ for groups of simply connected type.

Theorem 3.15. *Let $\mathbf{G}$ be a connected reductive algebraic group such that $[\mathbf{G}, \mathbf{G}]$ is simply connected. Let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius map and $\mathbf{T} \leq \mathbf{G}$ be an F-stable maximal torus. Let π be a set of primes not containing the defining prime of $\mathbf{G}$. Then*

$$k_\pi(\mathbf{G}^F) = \frac{1}{|W|} \sum_{w \in W} |\mathbf{T}^{wF}|_\pi$$

where $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ is the Weyl group of $\mathbf{G}$. In particular, $(\mathbf{T}^{wF})_\pi$ is a π-witness for $\mathbf{G}^F$ for any $w \in W$ with $|\mathbf{T}^{wF}|_\pi$ maximal.

Proof. Our assumption on π means we are only interested in semisimple conjugacy classes of $G := \mathbf{G}^F$. By [2, Prop. 3.7.3] there is a natural bijection between semisimple conjugacy classes of G and F-stable W-orbits in $\mathbf{T}$ preserving element orders, so $k_\pi(G) = |(\mathbf{T}/W)_\pi^F|$. Now we have

$$|(\mathbf{T}/W)^F| = \frac{1}{|W|} \sum_{w \in W} |\mathbf{T}^{wF}|$$

[2, Prop. 3.7.4], and again the underlying bijection descends to π-elements. The stated formula follows, as well as the fact that the maximal $|\mathbf{T}^{wF}|_\pi$ gives a π-witness. □

Proposition 3.16. *Let G be an exceptional covering group of a simple group of Lie type. Then G is π-bounded for any π.*

Proof. In the proof of Theorem 2.12 we found witnesses H for most of these covering groups not being almost Abelian, and these also provide π-witnesses whenever H is a π-group. It thus only remains to consider the following: Coverings of $\mathrm{U}_4(3)$, $\mathrm{U}_6(2)$, $\mathrm{S}_6(2)$, $\mathrm{O}_7(3)$, $\mathrm{O}_8^+(2)$, $G_2(3)$, $G_2(4)$, $F_4(2)$ or ${}^2E_6(2)$ with π not containing the defining characteristic, and coverings of $\mathrm{L}_3(4)$ and ${}^2B_2(8)$.

Now $S = \mathrm{Sp}_6(2), \mathrm{O}_8^+(2), G_2(3), G_2(4), F_4(2)$ are of simply connected type, and $|Z(G)|$ is a p-power, so we are done with Theorem 3.15 when $p \notin \pi$.

For $S = \mathrm{U}_4(3)$ if $3 \notin \pi$ then $k_\pi(G) \leq 8$, and we may take $H = C_2^4$ when $2 \in \pi$, respectively C_p for $p \in \pi$ when $2, 3 \notin \pi$.

The group $S = \mathrm{U}_6(2)$ has an elementary Abelian subgroup C_3^4, which is large enough for all covering groups G of S with $3 \in \pi$. The maximal $k_\pi(G)$ for $2, 3 \notin \pi$ is 5, less or equal to any $p \in \pi$.

The group $S = \mathrm{O}_7(3)$ has an elementary Abelian subgroup C_2^5, which is large enough for all covering groups G of S with $2 \in \pi$. If $2, 3 \notin \pi$ then $k_\pi(G) \leq 5$.

The group $S = {}^2E_6(2)$ has an elementary Abelian subgroup C_3^5, which works when $3 \in \pi$, a subgroup C_5^2 for $2, 3 \notin \pi$, $5 \in \pi$, C_7^2 for $2, 3, 5 \notin \pi$, and for $2, 3, 5, 7 \notin \pi$, $k_\pi(G) \leq 8$.

The various 2-covers G of $\mathrm{L}_3(4)$ different from $4^2.\mathrm{L}_3(4)$ possess a 2-subgroup U with $k(U) \geq k(G)$, and thus the preimage of U in any $2'$-covering shows the claim when $2 \in \pi$. For $G = 4^2.\mathrm{L}_3(4)$ with $k(G) = 112$ a subgroup $C_4^2 C_7$ is a witness when $7 \in \pi$, else a 2-subgroup U with $k(U) = 88$ does the job. When $2 \notin \pi$ then $k_\pi(G) \leq 6$, and in fact $k_\pi(G) = 2$ if $5, 7 \notin \pi$, so we are done again.

Finally, the group $S = {}^2B_2(8)$ has a subgroup C_{13} which works when $13 \in \pi$. For $13 \notin \pi$ a Sylow 2-subgroup of G does the job, and when $2, 13 \notin \pi$ then $k_\pi(G) \leq 5$. □

We thus obtain Theorem 1(3):

Corollary 3.17. *Let G be the full covering group of a simple group of Lie type in characteristic p. Then G is π-bounded for any π.*

Proof. By Proposition 3.16 we may assume that G is not an exceptional covering and hence $|Z(G)|$ is prime to p. Then, if $p \notin \pi$ we argue as follows: there is a simple algebraic group $\mathbf{G}$ of simply connected type such that $G = \mathbf{G}^F$ and the claim follows by Theorem 3.15.

For $p \in \pi$ this is essentially Corollary 3.14; that is, only the exceptions mentioned in that result have to be discussed. It follows from the proof of Propositions 2.5 and 2.6 that $\mathrm{SL}_2(q)$ is π-bounded when $2 \in \pi$, and that $\mathrm{SL}_3(q), \mathrm{SU}_3(q)$ are π-bounded when $3 \in \pi$. For $G = \mathrm{SL}_2(q)$ with $2 \notin \pi$ we have $k_\pi(G) \leq q$, so a Sylow p-subgroup provides a witness. Similarly, for $G = \mathrm{SL}_3(q)$ or $\mathrm{SU}_3(q)$ with $3 \notin \pi$ we have $k_\pi(G) < q^2$ and so again a Sylow p-subgroup can be chosen as witness. For $G = \mathrm{SU}_4(2)$ if $3 \in \pi$ there is an Abelian witness of order 27, when $3 \notin \pi$ then $k_\pi(G) \leq 6$ and an elementary Abelian subgroup C_2^3 does the job. □

We do not see how to treat the case of almost simple groups of Lie type when $p \notin \pi$, but we offer some partial results for certain sets π:

Corollary 3.18. *Let* $\mathbf{G}$ *be connected reductive with a Frobenius map* F *and let* π *be a set of primes not containing the defining prime of* $\mathbf{G}$ *and such that the centralisers of all* π*-elements of* $\mathbf{G}^F$ *are connected. Then* $G = \mathbf{G}^F$ *is* π*-bounded.*

Proof. The proof of Theorem 3.15 extends to this case, since the only place at which the assumption on $[\mathbf{G}, \mathbf{G}]$ was used is to ensure that centralisers of semisimple (π-) elements are connected, which here we guarantee by our assumption on π. □

Corollary 3.19. *Let* G *be simple of Lie type, or the group of* F*-fixed points of a simple algebraic group* $\mathbf{G}$ *of adjoint type in characteristic* p*. Assume either* $p \in \pi$*, or one of:*

(1) *G has type ${}^3D_4$, F_4 or E_8;*
(2) *$2 \notin \pi$, and G has type B_n, C_n $(n \geq 2)$, D_n, ${}^2D_n$ $(n \geq 4)$ or E_7; or*
(3) *$3 \notin \pi$ and G has type E_6 or ${}^2E_6$.*

Then G *is* π*-bounded.*

Proof. If $p \in \pi$ we may invoke Corollary 3.14. So assume $p \notin \pi$. If G is of adjoint type, then the claim follows from Corollary 3.18 as centralisers of semisimple elements of order prime to 2 in case (2), respectively prime to 3 in case (3), are connected. For the simple groups we may take a π-witness for the simply connected covering $\tilde{G}$ whose image in G is then also a π-witness for G by Lemma 3.1 as $Z(\tilde{G})$ is a π'-group. □

4. p'-bounded groups

We now specialise to the case when π contains all but one prime divisor of $|G|$. If G is π-bounded for $\pi(G) \setminus \pi = \{p\}$ we also say that G *is* p'*-bounded.* Recall that G is p'-bounded if there is a p'-subgroup $H \leq G$ with $k(H) \geq l(G)$, where $l(G)$ denotes the number of p'-classes of G, that is, the number of irreducible p-Brauer characters of G. One motivation for our study is the application to the $l(B)$-conjecture, see Corollary 4.6.

Remark 4.1. We are not aware of any finite group that is not p'-bounded for all primes p. Recall that such a group necessarily has to be non-solvable.

4.1. Quasi-simple groups are p'-bounded.

We first investigate p'-boundedness for the finite quasi-simple groups G and in particular prove Theorem 2.

Proposition 4.2. *Let* G *be a covering group of a sporadic simple group or of the Tits simple group. Then* G *is* p'*-bounded for all primes* p*.*

Proof. By Proposition 3.13 we only need to consider proper coverings, with $Z(G) \neq 1$. From the list of ranks of covering groups of sporadic groups in [5, Tab. 5.6.1] it follows that there are at least two primes r such that there exist elementary Abelian r-subgroups H with $k(H) \geq \max_{p \neq r}\{k_{p'}(G)\}$, for $G/Z(G)$ one of Co_1, Fi_{24}', B. Furthermore, there is an elementary Abelian r-subgroup of sufficient rank for at least one prime r for $G/Z(G)$ one of $McL, Suz, Ru, ON, Fi_{22}$, while this argument does not apply for any prime for $G/Z(G) \in \{M_{12}, M_{22}, J_2, J_3, HS\}$.

For the groups $G/Z(G) \in \{McL, Suz, Ru, ON, Fi_{22}\}$ we have that $k_{p'}(G)$ for the one missing prime p is still bounded above by the size of some elementary Abelian p'-subgroup.

Thus we are left with the covering groups of the five groups

$$M_{12}, M_{22}, J_2, J_3, HS.$$

For $2.M_{12}$ we take $H = C_{22}$ for $p = 3, 5$, and a Sylow 2-subgroup for $p = 11$. For the covering groups of M_{22} we take the full preimage of C_{11} for $p \neq 11$, and C_2^4 respectively C_2^5, respectively a Sylow 2-subgroup and its 3-coverings for $p = 11$. For $2.J_2$ we take $H = C_5^2 \times C_2$ for $p = 3, 7$, and $H = C_2^3 \times C_3$ for $p = 5$, for $3.J_3$ we take C_{57} for $p \neq 19$, and C_3^4 for $p = 19$, and for $2.HS$ we take $C_5^2 \times C_2$ for $p \neq 5$, and C_4^3 for $p = 5$. □

Let $\mathbf{G}$ be a simple algebraic group of simply connected type and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism, with finite group of fixed points $G = \mathbf{G}^F$. All finite simple groups of Lie type can be obtained as $G/Z(G)$, except for ${}^2F_4(2)'$ which was discussed in Proposition 4.2.

Proposition 4.3. *Let G be a quasi-simple group of Lie type and p the defining characteristic of G. Then G is p'-bounded.*

Proof. By Lemma 3.1 we may assume that $O_p(G) = 1$ and hence in particular that G is not an exceptional covering group of $G/Z(G)$ (see [11, Tab. 24.3]). First assume that $G = \mathbf{G}^F$ is obtained as above. The irreducible p-modular representations of G are known to be parametrised by q-restricted weights, where q is the absolute value of all eigenvalues of F on the character group of an F-stable maximal torus of the underlying algebraic group $\mathbf{G}$ (see [1, Thm. 3.2]). There are precisely q^r such, where r is the rank of $\mathbf{G}$, so $l(G) = q^r$.

On the other hand, we will exhibit an Abelian p'-subgroup T of G (in fact, a maximal torus) of order $|T| > q^r$, which will complete the proof. For $G = \mathrm{SL}_{r+1}(q)$ we take as T the subgroup generated by a Singer cycle, of order $(q^{r+1} - 1)/(q - 1)$, for $\mathrm{SU}_{r+1}(q)$, $\mathrm{Sp}_{2r}(q)$ and $\mathrm{Spin}_{2r+1}(q)$ with $r \geq 2$, and for $\mathrm{Spin}_{2r}^-(q)$ with $r \geq 4$ a Sylow 2-torus of order $(q + 1)^r$, and for $\mathrm{Spin}_{2r}^+(q)$ with $r \geq 4$ a torus of order $(q^2 + 1)(q^{r-2} + 1)$. For the groups of exceptional type, it is easy to see that there is such a torus T, see Table 1.

Table 1. Large tori in exceptional groups

G	$\|T\|$	G	$\|T\|$
${}^2B_2(q^2)$	$q^2 + \sqrt{2}q + 1$	$F_4(q)$	$(q + 1)^4$
${}^2G_2(q^2)$	$q^2 + \sqrt{3}q + 1$	$E_6(q)$	$(q^2 + q + 1)^3$
$G_2(q)$	$(q + 1)^2$	${}^2E_6(q)$	$(q + 1)^6$
${}^3D_4(q)$	$(q^2 + q + 1)^2$	$E_7(q)$	$(q + 1)^7$
${}^2F_4(q^2)$	$(q^2 + \sqrt{2}q + 1)^2$	$E_8(q)$	$(q + 1)^8$

Now assume that $Z \leq Z(G)$ is a central (p'-)subgroup. Then $|Z| \leq q + 1$ and thus $l(G/Z) = (q^r - 1)/|Z| + 1 \leq (q^r + q)/|Z|$ (see [1, §4.2]), while the image of T in G/Z has order $|T|/|Z|$. It is easily checked that $(q^r + q)/|Z| \leq |T|/|Z|$, showing that T/Z is a witness for G/Z. □

We can now prove Theorem 2:

Proof of Theorem 2. We consider the cases for $S = G/Z(G)$ according to the classification of finite simple groups. If S is an alternating group, then G is p'-bounded for any p by Proposition 3.12. If S is sporadic, the claim is in Proposition 4.2. Now assume that S is of Lie type. If p is the defining characteristic, the assertion was shown in Proposition 4.3

while if p is not the defining characteristic of S, it is in Corollary 3.14, Proposition 3.16 and Corollary 3.17. □

4.2. **p'-bounded almost quasi-simple groups.** We extend the results of the previous section to some classes of almost quasi-simple groups.

Proposition 4.4. *Let G be such that $F^*(G)$ is a covering group of a sporadic simple group or of the Tits simple group. Then G is p'-bounded for all p.*

Proof. By Proposition 4.2 we may assume that $F^*(G) \neq G$. Then $G/F^*(G)$ is of order at most 2, hence the claim follows with Lemma 3.2 when $p = 2$. Now assume that $p \geq 3$. A large elementary Abelian 2-subgroup, or a Sylow 2-subgroup prove our claim for $F^*(G)/Z(F^*(G)) \in \{M_{22}, HS, He, Fi_{22}, F'_{24}, {}^2F_4(2)'\}$. For $M_{12}.2$ we take $C_2^2 \times C_5$ for $p = 3, 11$, and a Sylow 2-subgroup for $p = 5$, for $2.M_{12}.2$ we take a subgroup $C_2^3C_4$. For J_2 we take a Sylow 5- or a Sylow 2-subgroup, for J_3 we take a Sylow 19- or Sylow 3-subgroup, for McL we take a Sylow 5- or a Sylow 3-subgroup, for Suz we take a Sylow 2-subgroup or an elementary Abelian group of order 3^5, for ON a Sylow 19- or Sylow 7-subgroup, and for HN an elementary Abelian subgroup of order 2^6 or 3^4. □

Proposition 4.5. *Let G be such that $F^*(G)$ is a covering group of the alternating group $\mathfrak{A}_n$, $n \geq 5$. Then G is p'-bounded for all p.*

Proof. By Proposition 3.12 we may assume that $F^*(G) \neq G$. Then $G/F^*(G)$ is a 2-group, hence the claim follows with Lemma 3.2 when $p = 2$. Now assume that $p \geq 3$. By Corollary 3.8 and Proposition 3.10 we may assume $Z(G) \neq 1$.

Then $\mathfrak{S}_n$ contains an elementary Abelian 2-subgroup H of order 2^k where $k = \lfloor n/2 \rfloor$. Using the same estimate as in the proof of Proposition 3.12 for $n \geq 26$, and the explicit value of $k(\mathfrak{S}_n)$ one sees that H solves our problem for $n \geq 20$ when $G = 2.\mathfrak{S}_n$. Looking at elementary Abelian 3-subgroups of $\mathfrak{S}_n$ only the following cases are left for $2.\mathfrak{S}_n$, where we can choose H as indicated:

$n =$	5	7	8	9	11	14
$l(2.\mathfrak{S}_n) \leq$	$10, 8$	$21, 13$	25	22	$72, 34$	166
H	C_6C_2, C_8	C_{24}, C_{14}	D_8^2	D_8^2	$D_8^2.C_3, D_8^2.C_2$	$D_8^3C_2$

Finally, the exceptional covering group $G = 3.\mathfrak{S}_7$ has $l(G) \leq 19$, for which we can take $H = C_6^2$, and $G = 6.\mathfrak{S}_7$ has $l(G) \leq 29$, and we can take a cyclic subgroup of order 30 or 42. For $S := F^*(G)/Z(F^*(G)) = \mathfrak{A}_6$, we choose H according to the following list:

G	$2.S.2_2$	$3.\mathfrak{S}_6$	$3.S.2_2$	$3.S.2_3$	$6.\mathfrak{S}_6$	$6.S.2_2$	$3.S.2^2$
$l(G)$	$18, 12$	13	10	19	20	17	17
H	C_{20}, C_{16}	$C_3 \times C_2^3$	C_{12}	C_{24}	C_{24}	C_{24}	C_{24}

□

4.3. **The number of characters in a p-block.** Finally, we discuss the relation of p'-boundedness to the $l(B)$-conjecture put forward by the first and third author [10]. Recall that this stipulates that for any p-block B of a finite group G, with defect group D, we have $l(B) \leq p^{s(D)}$, where $s(D)$ denotes the sectional rank of D.

Proof of Theorem 3. Assume that G is p-constrained, that is, $F^*(G) = O_p(G)$ is a p-group. Then G has a unique p-block, the principal block $B_0(G)$. It is sufficient to see that $l(B_0(G)) = l(G) \leq |V|$, where $P = O_p(G)$ and $V = P/\Phi(P)$. We claim that $\bar{G} = G/P$ acts faithfully on V. Suppose that $x \in G$ acts trivially on V. Then so does $x_{p'}$, and by coprime action, $x_{p'}$ acts trivially on P. So x is a p-element. We conclude that $C_G(V)$ is a p-group, so it is contained in P, proving our claim. We are thus reduced to the situation that $G = V\bar{G}$ with $V = O_p(G)$ elementary Abelian and $\bar{G}$ acting faithfully on it.

Note that $l(G) = l(\bar{G})$ as all simple modules in characteristic p have $O_p(G)$ in their kernel. By assumption there is a p'-subgroup $H \leq \bar{G}$ with $k(H) \geq l(\bar{G})$. Then we get by the (proved) $k(GV)$-conjecture that

$$l(G) = l(\bar{G}) \leq k(H) < k(VH) \leq |V|,$$

as desired. □

Our Main Theorem 2 on quasi-simple groups thus implies:

Corollary 4.6. *Let VG be an extension of an elementary Abelian p-group V by a quasi-simple group G acting faithfully on V. Then $l(VG) < p^{s(D)}$, where D is a Sylow p-subgroup of VG, and $s(D)$ denotes the sectional rank of D.*

Guralnick and Tiep [6] showed that all but finitely many quasi-simple groups G that are not of Lie type in characteristic p do satisfy $k(VG) < |V|/2$ for any irreducible kG-module V, by a detailed analysis of the possible modules V.

Our investigations on p'-boundedness lead us to formulate the following:

Conjecture 4.7. *Let p be any prime. Then every finite group G is p'-bounded, that is,*

$$l(G) \leq \max\{k(H) \mid H \leq G \text{ is a } p'\text{-subgroup}\}.$$

References

[1] O. Brunat, F. Lübeck, On defining characteristic representations of finite reductive groups. *J. Algebra* **395** (2013), 121–141.

[2] R. Carter, *Finite Groups of Lie type: Conjugacy Classes and Complex Characters.* Wiley, Chichester, 1985.

[3] J. Fulman, R. Guralnick, Bounds on the number and sizes of conjugacy classes in finite Chevalley groups with applications to derangements. *Trans. Amer. Math. Soc.* **364** (2012), 3023–3070.

[4] K. Gödel, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme. I. *Monatshefte f. Math.* **38** (1931), 173–198.

[5] D. Gorenstein, R. Lyons, R. Solomon, *The Classification of the Finite Simple Groups.* American Mathematical Society, 40, Number 3. Providence, RI, 1998.

[6] R. M. Guralnick, P. H. Tiep, The non-coprime $k(GV)$ problem. *J. Algebra* **293** (2005), 185–242.

[7] L. G. Kovács, G. R. Robinson, On the number of conjugacy classes of a finite group. *J. Algebra* **160** (1993), 441–460.

[8] M. W. Liebeck, L. Pyber, Upper bounds for the number of conjugacy classes of a finite group. *J. Algebra* **198** (1997), 538–562.

[9] G. Malle, Extensions of unipotent characters and the inductive McKay condition. *J. Algebra* **320** (2008), 2963–2980.

[10] G. Malle, G. R. Robinson, On the number of simple modules in a block of a finite group. *J. Algebra* **475** (2017), 423–438.

[11] G. Malle, D. Testerman, *Linear Algebraic Groups and Finite Groups of Lie Type.* Cambridge Studies in Advanced Mathematics, 133. Cambridge University Press, Cambridge, 2011.

[12] A. Maróti, Bounding the number of conjugacy classes of a permutation group. *J. Group Theory* **8** (2005), 273–289.
[13] A. Maróti, H. N. Nguyen, On the number of conjugacy classes of π-elements in finite groups. *Arch. Math. (Basel)* **102** (2014), 101–108.
[14] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*; 2004, http://www.gap-system.org.
[15] J. H. van Lint, R. M. Wilson, *A Course in Combinatorics.* Cambridge University Press, Cambridge, 1992.
[16] H. Wielandt, Zum Satz von Sylow. *Math. Z.* **60** (1954), 407–408.

FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany.
Email address: malle@mathematik.uni-kl.de

Departament de Matemàtiques, Universitat de València, 46100 Burjassot, València, Spain.
Email address: gabriel@uv.es

Institute of Mathematics, University of Aberdeen, Aberdeen AB24 3UE, Scotland, United Kingdom
Email address: g.r.robinson@abdn.ac.uk