ON BLOCKS WITH ONE MODULAR CHARACTER

GUNTER MALLE, GABRIEL NAVARRO, AND BRITTA SPÄTH

ABSTRACT. Suppose that B is a Brauer p-block of a finite group G with a unique modular character φ. We prove that φ is liftable to an ordinary character of G (which moreover is p-rational for odd p). This confirms the basic set conjecture for these blocks.

1. INTRODUCTION

Let G be a finite group and let p be a prime. While the Brauer p-blocks of G consisting of one ordinary irreducible character were understood by Richard Brauer himself a long time ago, the blocks containing just one modular character are still a mystery. The study of the Broué–Puig nilpotent blocks [3] (a canonical example of blocks with one modular character and, in some sense, the easiest) is already quite complicated. But not every block with one modular character is nilpotent (except possibly in quasi-simple groups.)

From the point of view of algebras, blocks with one modular character are the most natural. If F is an algebraically closed field of characteristic p, and B is a p-block of G (that is, an indecomposable two-sided ideal of the group algebra FG), then the algebra B has a unique irreducible F-representation if and only if $B/\mathbf{J}(B)$ is a matrix algebra, where $\mathbf{J}(B)$ is the Jacobson radical of B. We see that from several perspectives, the blocks with one modular character constitute a fundamental object that deserves study.

If B has a unique irreducible modular character, say φ, then the decomposition numbers of the algebra B are simply the numbers $\chi(1)/\varphi(1)$, where χ runs over the irreducible complex characters in B. No general interpretation of these numbers is known. (If B is nilpotent, for instance, these are the irreducible character degrees of any defect group of B, see [3, Thm. 1.2(2)].) Our main result is that one of these numbers should always be 1. In other words, φ should always lift to an ordinary (complex) character of G.

Main Theorem. *Let G be a finite group and let p be a prime. Assume that B is a p-block of G and $\mathrm{IBr}(B) = \{\varphi\}$. Then there exists $\chi \in \mathrm{Irr}(B)$ such that $\chi(1) = \varphi(1)$. If $p \neq 2$, χ can be chosen to be p-rational.*

Our Main Theorem was known for nilpotent blocks (by the work of Broué–Puig [3]). It is also known for p-solvable groups by the celebrated Fong–Swan Theorem (while the p-rationality part follows from a classical theorem of M. Isaacs [13]). Our proof of the

---

Date: December 8, 2016.

1991 *Mathematics Subject Classification.* 20C20, 20C05, 20C33.

Key words and phrases. blocks with one irreducible module, nilpotent blocks.

The first and third author gratefully acknowledge financial support by ERC Advanced Grant 291512. The second author is partially supported by the Spanish Ministerio de Educación y Ciencia Proyectos MTM2013-40464-P and Prometeo II/Generalitat Valenciana. We thank the referee for some questions that led to clarifications in some proofs.

general case uses the Classification of Finite Simple Groups, and the main obstacle is to understand the p-blocks B of quasi-simple groups G such that $\mathrm{IBr}(B)$ is contained in a single $\mathrm{Aut}(G)$-orbit. We believe that their classification will turn out to be fundamental when further studying blocks with one modular character.

This work started from an observation by R. Kessar and M. Linckelmann. Recall that the basic set conjecture asserts that for every block B of a finite group G, there exists a subset $\mathcal{B}$ of $\mathrm{Irr}(G)$ such that the set $\mathcal{B}^0$ of their restrictions to p-regular elements is a $\mathbb{Z}$-basis of the ring $\mathbb{Z}[\mathrm{IBr}(B)]$ of generalized Brauer characters of B. (See for instance [9].) In the case where $\mathrm{IBr}(B) = \{\varphi\}$, as pointed out by Kessar and Linckelmann, the basic set conjecture is equivalent to proving that φ is liftable, which is our Main Theorem.

This liftability of φ in our main result is, once again, a shadow of some deeper, structural conjecture that R. Kessar and M. Linckelmann have proposed: every block with one modular irreducible character is Morita equivalent over the ring of p-adics to a block of a p-solvable group. (This is consistent with their results in [17] and [18], and it would imply the liftability statement in our Main Theorem, using the Fong–Swan Theorem and the fact that blocks which are Morita equivalent over the ring of p-adics have the same decomposition matrix, see [1, Sect. 2.2, Ex. 3].) This deeper conjecture, however, seems out of reach with the present methods.

Finally, for p odd, we show that in the situation of the Main Theorem one can also choose the lift χ to be p-rational. Nevertheless we can't answer the question of G. R. Robinson whether this p-rational lift is unique, although we believe that this is the case. None of this seems to be implied by the general Kessar–Linckelmann conjecture.

2. A result on simple groups

In this section and the following we start our investigation of blocks of finite quasi-simple groups all of whose irreducible Brauer characters lie in a single orbit under the automorphism group. In Section 5 it will turn out that exactly those blocks are relevant for our inductive proof. One can see the importance of those blocks already in the following situation: assume that in the situation of the Main Theorem there exists a quasi-simple normal subgroup X of G. Then the blocks of X covered by a block of G with one Brauer character have the above mentioned property.

Let p be a prime. Our notation for blocks and characters of finite groups follows [24]. Hence we have fixed a maximal ideal M of the ring of algebraic integers containing p with respect to which Brauer characters of finite groups are constructed.

Theorem 2.1. *Suppose that G is a quasi-simple group, p is a prime with $p \nmid |\mathbf{Z}(G)|$, and B is a p-block of G. Let $A = \mathrm{Aut}(G)$. If $\mathrm{IBr}(B)$ consists of A-conjugates of φ, then there exists some $\chi \in \mathrm{Irr}(G)$ such that*

(1) $\chi^0 = \varphi$;
(2) *χ is p-rational if $p \neq 2$;*
(3) *the stabilisers $I_A(\chi) = I_A(\varphi) =: I$ coincide; and*
(4) *for $P \in \mathrm{Syl}_p(I)$, the character χ extends to some $\tilde\chi \in \mathrm{Irr}(G \rtimes P)$ with $p \nmid |\mathbf{C}_{G\rtimes P}(G) : \ker(\tilde\chi|_{\mathbf{C}_{G\rtimes P}(G)})|$.*

The proof will be given in this section and the next, based on the classification of finite simple groups. Assertions (3) and (4) will be crucial in our inductive approach to arbitrary groups. Note that assertion (2) implies the statement in (4) in case of odd p by [14, Thm. (6.30)].

2.1. **General Observations.** We start by collecting some general observations, relating to our Theorem 2.1, as well as about nilpotent blocks and their characters.

Lemma 2.2. *Suppose that $E \lhd G$. Let P be a p-subgroup of G such that $P \cap E = Q \in \mathrm{Syl}_p(E)$, and assume that $\mathbf{C}_P(E) = 1$. Let $\theta \in \mathrm{Irr}(E)$ be P-invariant. Then θ extends to EP if and only if θ has an extension $\tilde{\theta}$ to $H := E \rtimes P$ such that $p \nmid |\mathbf{C}_H(E) : \ker(\tilde{\theta}|_{\mathbf{C}_H(E)})|$.*

Proof. Since E is normal, P acts on E as automorphisms by conjugation. Let $H = E \rtimes P$ be the corresponding semidirect product, and view E and P as subgroups of H. Now, the map $f : H \to EP$ given by $f(e, x) = ex$ is a surjective group homomorphism with kernel $\Delta Q = \{(x, x^{-1}) \mid x \in Q\}$, a normal p-subgroup of H isomorphic to Q, which intersects trivially with both E and P. In particular, $\Delta Q \subseteq \mathbf{C}_H(E)$. By orders, $(\Delta Q)P$ is a Sylow p-subgroup of H and $(\Delta Q)P \cap \mathbf{C}_H(E) = (\Delta Q)\mathbf{C}_P(E) = \Delta Q$, so we see that ΔQ is a Sylow p-subgroup of $\mathbf{C}_H(E)$. Using that f induces an isomorphism $H/\Delta Q \to EP$, we easily see that θ extends to EP if and only if θ has an extension $\tilde{\theta}$ to H that contains ΔQ in its kernel. Since $\Delta Q \in \mathrm{Syl}_p(\mathbf{C}_H(E))$, the rest of the claim easily follows. □

Before considering the blocks of specific simple groups we prove that Theorem 2.1 holds whenever B is nilpotent or is covered by a nilpotent block (in a specific situation).

Proposition 2.3. *Let G be a finite group, $N \lhd G$ and B be a G-invariant nilpotent p-block of N. Then for the unique $\varphi \in \mathrm{IBr}(B)$ there exists some G-invariant $\psi \in \mathrm{Irr}(B)$ with $\psi^0 = \varphi$ such that ψ extends to Q, where Q/N is a Sylow p-subgroup of G/N. Moreover if $p \neq 2$, ψ can be chosen to be p-rational.*

Proof. By the characterisation of characters in nilpotent p-blocks when $p > 2$ there is a unique character $\psi \in \mathrm{Irr}(B)$ that is p-rational, see [3, Thm. 1.2]. This character has the claimed properties according to [15, Thm. (6.30)].

When $p = 2$ we use the following considerations: Let D be a defect group of B and B' the Brauer correspondent of B in $\mathbf{N}_N(D)$. Let $\psi' \in \mathrm{Irr}(B')$ be the unique character with $D \subseteq \ker(\psi')$. Hence ψ' is $H := \mathbf{N}_G(D)$-invariant. Note that ψ' extends to $\mathbf{N}_Q(D)$ since ψ' corresponds to a defect zero character of $\mathbf{N}_N(D)/D$ that extends to $\mathbf{N}_Q(D)/D$ according to [24, Ex. (3.10)]. According to the proof of [24, Thm. (8.28)] there exists a central extension $\hat{H}$ of H by a p'-group U with $\mathbf{N}_N(D) \lhd \hat{H}$ such that ψ' extends to some $\tilde{\psi}' \in \mathrm{Irr}(\hat{H})$. Using the cocycle defining $\hat{H}$ as central extension of $\mathbf{N}_G(D)$ by U we define $\hat{G}$ as a central extension of G such that $N \lhd \hat{G}$ and $\hat{H} \subseteq \hat{G}$. Moreover let $\mathcal{O}$ be the complete discrete valuation ring of a p-modular system (that is big enough), $e \in \mathcal{O}N$ and $e' \in \mathcal{O}\mathbf{N}_N(D)$ be the central-primitive idempotents of B and B'. According to [21, 1.20.3] the algebras $\mathcal{O}\hat{G}e$ and $\mathcal{O}\mathbf{N}_{\hat{G}}(D)e'$ are Morita equivalent.

Since the irreducible module affording ψ' has multiplicity one in $\mathcal{O}\mathbf{N}_{\hat{G}}(D)e'$ and dimension $\psi'(1)^2$ there exists a character $\tilde{\psi} \in \mathrm{Irr}(\hat{G})$ afforded by a submodule of $\mathcal{O}\hat{G}e$ that has multiplicity one in $\mathcal{O}\hat{G}e$. Hence $\psi := \tilde{\psi}_N$ is irreducible, G-invariant and $\psi^0 = \varphi$. It remains to prove that ψ extends to Q, but by the construction $\widehat{G}$ is a central extension of G by a p'-group, hence Q is (isomorphic to) a subgroup of $\hat{G}$ and ψ extends to Q. □

Proposition 2.4. *Let A be a finite group with normal subgroups $G \lhd \tilde{G}$. Let p be an odd prime, B a p-block of G covered by a nilpotent block of $\tilde{G}$. Suppose that every ordinary character of G extends to its inertia group in $\tilde{G}$ and that $\tilde{G}/G$ is abelian with $p \nmid |\tilde{G} : G\mathbf{C}_{\tilde{G}}(G)|$. Let $\varphi \in \mathrm{IBr}(B)$. Then there exists some p-rational A_φ-invariant $\psi \in \mathrm{Irr}(B)$ with $\psi^0 = \varphi$.*

Proof. Let $\tilde{B}$ be a nilpotent block of $\tilde{G}$ covering B. Let $\widetilde{\varphi} \in \mathrm{IBr}(\widetilde{B})$ and $\widetilde{\psi} \in \mathrm{Irr}(\widetilde{B})$ be p-rational with $\widetilde{\psi}^0 = \widetilde{\varphi}$ from Proposition 2.3. Now any $\varphi \in \mathrm{IBr}(B)$ is a constituent of $\widetilde{\varphi}_G$.

Note that φ extends to its inertia group in $\tilde{G}$: let $\psi \in \mathrm{Irr}(B)$ such that ψ is a constituent of $\widetilde{\psi}_G$ and φ is a constituent of ψ^0. Let $\tilde{G}_1$ be the maximal subgroup of $\tilde{G}$ such that $\tilde{G}_1$ has p-index in $\tilde{G}$ and $\tilde{G}_1/G$ is a p'-group. Then $\widetilde{\varphi}_{\tilde{G}_1}$ and hence $\widetilde{\psi}_{\tilde{G}_1}$ is irreducible. Since ψ extends to its inertia group in $\tilde{G}_1$ the character $\widetilde{\psi}_{\tilde{G}_1}$ has multiplicity 1 in $\psi^{\tilde{G}_1}$. Hence $\widetilde{\varphi}_{\tilde{G}_1}$ has multiplicity 1 in $(\psi^{\tilde{G}_1})^0$ and φ extends to its inertia group in $\tilde{G}$.

Since all ordinary characters of G extend to their inertia group in $\tilde{G}$ and $\tilde{G}/G$ is abelian the number of constituents of $\widetilde{\psi}_G$ coincides with the number of linear characters μ of $\tilde{G}/G$ with $\widetilde{\psi}\mu = \widetilde{\psi}$. Analogously the number of constituents of $\widetilde{\varphi}_G$ coincides with the number of linear Brauer characters μ of $\tilde{G}/G$ with $\widetilde{\varphi}\mu = \widetilde{\varphi}$ because $p \nmid |\tilde{G} : G\mathbf{C}_{\tilde{G}}(G)|$. Every constituent of $\widetilde{\varphi}_G$ lifts to a unique constituent of $\widetilde{\psi}_G$ since both characters have the same number of constituents. Let $\psi \in \mathrm{Irr}(B)$ be the constituent of $\widetilde{\psi}_G$ with $\psi^0 = \varphi$.

For every $a \in A_\varphi$ we see that $\widetilde{\psi}^a$ is a p-rational character of the block $\tilde{B}^a$. Note that $\tilde{B}^a$ is nilpotent as well and contains a p-rational character $\mu\widetilde{\psi}$ for some linear p-rational character μ of $\tilde{G}/G$. Since in a nilpotent block there exists a unique p-rational character, we see $\widetilde{\psi}^a = \mu\widetilde{\psi}$. Then ψ^a is a constituent of $\widetilde{\psi}_G = (\mu\widetilde{\psi})_G$. This implies $\psi^a = \psi$.

Note that since $\widetilde{\psi}$ is p-rational, ψ is p-rational according to the definition. $\square$

To finish this paragraph let us point out nilpotent blocks are not the only ones containing just one modular character:

Remark 2.5. A typical way of constructing blocks with one modular character which are not nilpotent is the following. Let H be a p'-group of central type (that is, H is a group possessing $\lambda \in \mathrm{Irr}(\mathbf{Z}(H))$ such that $\lambda^H = e\theta$ for some $\theta \in \mathrm{Irr}(H)$ and $e \geq 1$). Suppose that $H/\mathbf{Z}(H) \neq 1$ acts faithfully on a p-group V. Then the semidirect product VH has a unique (non-nilpotent) block covering λ with a unique modular character.

2.2. **Blocks of sporadic groups.** We now start our investigation of p-blocks of finite quasi-simple groups all of whose irreducible Brauer characters lie in a single orbit under the automorphism group. For groups not of Lie type in cross characteristic we obtain a full classification.

Proposition 2.6. *Let G be quasi-simple such that $G/\mathbf{Z}(G)$ is a sporadic simple group or the Tits group ${}^2F_4(2)'$. Let p be a prime and B a p-block of G, not of central defect, such that $\mathrm{IBr}(B)$ is a single orbit under $\mathrm{Aut}(G)_B$. Then B has defect 1 and the degrees of its irreducible Brauer characters are as given in Table 1.*

Proof. From the known character tables of covering groups of sporadic simple groups it is easy to determine the block distribution and the number of modular Brauer characters in

Table 1. Blocks in sporadic groups

G	p	$\mathrm{IBr}(B)$	G	p	$\mathrm{IBr}(B)$
J_1	2	76	Co_3	2	129536
$2.J_2$	3	$126, 126$	Fi_{22}	2	2555904
M_{23}	3	231	Ly	3	18395586
$2.HS$	3	$924, 924$	Fi_{23}	2	73531392
McL	2	3520	$2.Co_1$	3	59153976
$3.McL$ $(2\times)$	2	6336	J_4	3	786127419

each p-block. If B is a block of defect zero, or if $l(B) := |\mathrm{IBr}(B)|$ is larger than $|\mathrm{Out}(G)_B|$, the block can be discarded. Also, for many of the smaller groups, the Brauer character tables are available in GAP [30]. This leaves only the cases in Table 1, and a 5-block B of Fi_{24}' of defect 1 with $l(B) = 2$. But the block B' of $Fi_{24}'.2$ covering B has $l(B') = 4$, so this does not give an example. $\square$

Inspection of the known character tables for the groups in Table 1 shows:

Corollary 2.7. *In the situation of Proposition 2.6 every $\varphi \in \mathrm{IBr}(B)$ is liftable to an $\mathrm{Aut}(G)_\varphi$-invariant character in $\mathrm{Irr}(G)$, and either φ has a unique lift, which then is p-rational, or $p = 3$, there are exactly three distinct lifts two of which are algebraically conjugate, or $p = 2$ and φ has exactly two lifts.*

2.3. **Blocks of alternating groups.** We next consider covering groups of alternating groups.

Theorem 2.8. *Let G be a covering group of an alternating group $\mathfrak{A}_n$, $n \geq 5$. Let p be a prime and B a p-block of G of weight w, not of central defect, such that $\mathrm{IBr}(B)$ is a single orbit under $\mathrm{Aut}(G)_B$. Then B has defect one and one of the following occurs:*
(1) $G = \mathfrak{A}_n$, $p = 3$, $w = 1$, $l(B) = 1$ and B is self-conjugate;
(2) $G = 2.\mathfrak{A}_n$, $p = 3$, $w = 1$, $l(B) \leq 2$;
(3) $G = 2.\mathfrak{A}_6$, $p = 5$, $w = 1$, $l(B) = 2$, $\{\varphi(1) \mid \varphi \in \mathrm{IBr}(B)\} = \{4\}$; or
(4) $G = 6.\mathfrak{A}_6$, $p = 5$, $l(B) = 2$, $\{\varphi(1) \mid \varphi \in \mathrm{IBr}(B)\} = \{6\}$ (two blocks).

Proof. For $n \leq 7$ we consult the known Brauer character tables. Thus we may assume that $n \geq 8$, the full covering group of $\mathfrak{A}_n$ has center of order 2, and $|\mathrm{Out}(\mathfrak{A}_n)| = 2$. We use the description of p-blocks of alternating groups and their covering groups in [28]. First consider B a p-block of $\mathfrak{A}_n$. Then B corresponds to a p-core partition μ of an integer $n - wp$, where w is called the weight of the block. A core and its conjugate parametrise the same p-block. First assume that p is odd. Then there are two blocks of $\mathfrak{S}_n$ above any non self-conjugate block of $\mathfrak{A}_n$, and there is one above each self-conjugate one. By [28, Prop. 12.8],

$$l(B) = \begin{cases} k(p-1, w) & \text{if } B \text{ is not self-conjugate,} \\ \frac{1}{2}(k(p-1, w) + 3k^s(p-1, w)) & \text{else,} \end{cases}$$

where $k(a,w)$ denotes the number of a-tuples of partions of w, and $k^s(a,w)$ is the number of symmetric a-quotients. When $w=0$ then B is of defect zero. For $p=3$ and $w\geq 2$ we have

$$l(B)\geq \min\{k(2,2),\frac{1}{2}(k(2,2)+3k^s(2,2))\}=\min\{5,4\}=4,$$

which is larger than $|\mathrm{Out}(\mathfrak{A}_n)|$ as $n\geq 8$. When $w=1$ and B is not self-conjugate, then $l(B)=2$, hence this gives no example. For B self-conjugate and $p\geq 5$ we find

$$l(B)\geq \min\{k(4,1),\frac{1}{2}(k(4,1)+3k^s(4,1))\}=\min\{4,2\}=2,$$

and equality only holds when $p=5$, $w=1$. But in that case the block B' of $\mathfrak{S}_n$ above B has $l(B')=4$ by [28, Prop. 11.4]. This leaves only case (1).

When $p=2$, note that 2-cores are triangular partitions and thus always self-conjugate. In this case there is a unique 2-block of $\mathfrak{S}_n$ above each 2-block of $\mathfrak{A}_n$. Also, blocks of weight 0 or 1 are of defect 0. By [28, Prop. 12.9] we have

$$l(B)=\begin{cases}p(w) & \text{if } w \text{ is odd,}\\ p(w)+p(w/2) & \text{if } w \text{ is even,}\end{cases}$$

where $p(w)=k(1,w)$ denotes the number of partitions of w. Thus, for $w\geq 2$ we have $l(B)\geq 3>|\mathrm{Out}(\mathfrak{A}_n)|$, and no further example arises.

Now we turn to spin blocks, that is, faithful blocks of the 2-fold covering $G=2.\mathfrak{A}_n$, with $p>2$. Any such block B is covered by a unique p-block B' of $2.\mathfrak{S}_n$, and $l(B)=l^n(B')+2l^s(B')$ by [28, Prop. 13.19], where $l^n(B')$, $l^s(B')$ denote the number of pairs of not self-conjugate Brauer characters in B, respectively the number of self-conjugate ones. A straightforward calculation shows that $l(B)\leq 2$ implies that either $p=3$, $w\leq 2$, or $p=5$, $w=1$, $l(B)=2$. In the latter case the block B' of $2.\mathfrak{S}_n$, $n\geq 8$, above B has $l(B')=4$, and the same holds if $p=3$ and $w=2$. In the case $p=3$ and $w=1$, $l(B)=1$ if the sign of the block B is $+1$, and $l(B)=2$ if it has sign -1. In either case, $\mathrm{IBr}(B)$ is an orbit under $\mathrm{Aut}(G)$. So we reach the conclusion in case (2). □

Note that $2.\mathfrak{A}_6\cong \mathrm{SL}_2(9)$ is a group of Lie type, and the example (3) in the preceding result forms part of an infinite series of cases.

Corollary 2.9. *In the situation of Theorem 2.8 all $\varphi\in\mathrm{IBr}(B)$ are liftable to an $\mathrm{Aut}(G)_\varphi$-invariant character in $\mathrm{Irr}(G)$, and either φ has a unique lift, which is p-rational, or $p=3$ and there are exactly three distinct lifts two of which are algebraically conjugate.*

Note that in all cases of Theorem 2.8, p is prime to $|\mathrm{Out}(G)|$.

Remark 2.10. The generating function for the number of symmetric 3-cores is known (see for example [28, Prop. 9.13]). From this it can be seen that blocks as in conclusion (1) of Theorem 2.8 are rather rare; the first few occur in degrees $n=8,11,19,24$. Similarly, it can be computed from the generating function in [28, Prop. 9.9] that the first instances for case (2) occur for $n=5,8,10,15,18$, with $l(B)=1$ only when $n=18$.

Proposition 2.11. *Let G be an exceptional covering group of a finite simple group of Lie type. Let p be a prime and B a faithful p-block of G, not of central defect, such that $\mathrm{IBr}(B)$ is a single orbit under $\mathrm{Aut}(G)_B$. Then $G=2.G_2(4)$, $p=3$, B has defect 1 and* *is as in Table 2. Every* $\varphi \in \mathrm{IBr}(B)$ *is liftable to a unique* $\mathrm{Aut}(G)_\varphi$*-invariant character in* $\mathrm{Irr}(G)$*, which is* 3*-rational.*

TABLE 2. Blocks in exceptional covering groups

G	p	$\mathrm{IBr}(B)$
$2.G_2(4)$	3	$1800, 1800$
$2.G_2(4)$	3	$3744, 3744$

Proof. As in the proof of Proposition 2.6 this can be checked from the known ordinary character tables. Apart from the cases listed in the table, there exist 5-blocks of $12_1.\mathrm{L}_3(4)$, of $6_2.\mathrm{U}_4(3)$, of $2.{}^2E_6(2)$ and of $6.{}^2E_6(2)$ of defect 1 with two or four irreducible Brauer characters. But from the shapes of their Brauer trees it is immediate that they cannot lead to examples. □

Remark 2.12. For all quasi-simple groups discussed in this section the blocks with one orbit of Brauer characters have defect 1. Note that blocks with cyclic defect group containing just one modular irreducible character have trivial inertial quotient and hence are nilpotent.

3. THE SIMPLE GROUPS OF LIE TYPE

In this section we deal with the groups of Lie type by considering them as subgroups of simple linear algebraic groups. For consistency with the cited literature we prefer to denote our chosen prime by ℓ and denote by p the characteristic of the underlying field.

We consider the following setup: $\mathbf{G}$ is a simple linear algebraic group of simply connected type over an algebraic closure of a finite field $\mathbb{F}_p$, and $F : \mathbf{G} \to \mathbf{G}$ is a Steinberg endomorphism with finite group of fixed points $G = \mathbf{G}^F$. It is well-known that all quasi-simple groups of Lie type, apart from exceptional covering groups as considered in Proposition 2.11, and apart from ${}^2F_4(2)'$ which was already dealt with in Proposition 2.6, can be obtained as central quotients of groups G as above. In particular, any block of a covering group S of a simple group of Lie type is also a block of such a group G.

3.1. **The defining characteristic case.** We first consider the case when $\ell = p$.

Proposition 3.1. *Let S be a covering group of a simple group of Lie type, and $\ell = p$ the defining characteristic of S. Let B be an ℓ-block of S, not of central defect. Then* $\mathrm{IBr}(B)$ *is not a single orbit under* $\mathrm{Aut}(S)_B$.

Proof. All exceptional covering groups S have a center of order divisible by p, so we may assume that S is a non-exceptional covering. In particular, it is a central quotient of a finite reductive group $G = \mathbf{G}^F$, with $\mathbf{G}$ of simply connected type and $F : \mathbf{G} \to \mathbf{G}$ as above. Denote by δ the smallest integer such that F^δ acts trivially on the Weyl group of $\mathbf{G}$ and let q^δ denote the unique eigenvalue of F^δ on the character group of an F-stable maximal torus of $\mathbf{G}$. By a result of Humphreys, the p-blocks of $\mathrm{Irr}(G)$ of non-zero defect are given by $\mathrm{Irr}(G \mid \lambda)$ for λ running over $\mathrm{Irr}(\mathbf{Z}(G))$. Since p divides $|G|$ and G is perfect, the principal p-block of G certainly contains Brauer characters of distinct degrees.

Thus now we may assume that B is a block of G with non-trivial corresponding central character $\lambda \in \mathrm{Irr}(\mathbf{Z}(G))$. Then $\mathrm{IBr}(B)$ is parametrised by q^δ-restricted F-invariant weights of $\mathbf{G}$ whose restriction to $\mathbf{Z}(G)$ is a multiple of λ, and $\mathrm{Aut}(G)$ acts on the characters as it does on the weights. If $\delta = 1$, so F is split, then there are $q^r - 1$ weights different from the Steinberg weight that are q-restricted, where r is the rank of $\mathbf{G}$. Thus, there are at least two not $\mathrm{Aut}(G)$-conjugate weights for every $\lambda \in \mathrm{Irr}(\mathbf{Z}(G))$ if $q^r - 1 > 2|\mathbf{Z}(G)|$. It is easily seen that this inequality holds unless $r = 1$ and $q \leq 3$, in which case $G = \mathrm{SL}_2(q)$ is solvable. A very similar discussion deals with the groups of twisted type. (Note that here we need not consider the very twisted groups, since their center is trivial.) □

3.2. **Non-defining characteristic.** We now turn to the case of non-defining characteristic. We suspect that most of the blocks considered in Theorem 3.4 satisfy the assumptions from Proposition 2.4, but don't see how to prove that. In our considerations we will make use of the following simple observation where for any set of characters X of a finite group G we denote by $X^0 := \{\chi^0 \mid \chi \in X\}$ the *multi-set* of restrictions to the ℓ'-elements of G.

Lemma 3.2. *Let B be an ℓ-block of a finite group G, and assume that $X \subset \mathbb{Z}\mathrm{Irr}(B)$ is an $\mathrm{Aut}(G)_B$-invariant subset such that X^0 is linearly independent. If $\mathrm{Aut}(G)_B$ has at least two orbits on X, then $\mathrm{IBr}(B)$ is not a single $\mathrm{Aut}(G)_B$-orbit.*

Proof. By assumption, the $A := \mathrm{Aut}(G)_B$-permutation module $U := \mathbb{C}X^0$ is a submodule of the A-permutation module $V := \mathbb{C}\mathrm{IBr}(B)$. Since A is not transitive on the basis X^0 of U, the A-permutation module V must *a fortiori* be intransitive. □

For example this criterion applies if X is an $\mathrm{Aut}(G)_B$-invariant basic set for the block B which is not a single $\mathrm{Aut}(G)_B$-orbit. The second situation we will consider is when $X = \{\chi_1, \chi_2\}$, where χ_1, χ_2 are sums over $\mathrm{Aut}(G)_B$-orbits of irreducible characters in $\mathrm{Irr}(B)$, such that χ_1^0 and χ_2^0 are not multiples of one another.

We will also need the following observations pertaining to Lusztig induction:

Proposition 3.3. *Let $\mathbf{G}$ be connected reductive with Frobenius map F, and ℓ not the defining prime for $\mathbf{G}$.*

(a) *Let $\mathbf{L} \leq \mathbf{G}$ be an F-stable Levi subgroup. If $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$ is ℓ-rational, then so is $R_\mathbf{L}^\mathbf{G}(\lambda)$.*

(b) *Let $s \in \mathbf{G}^{*F}$ be a semisimple ℓ'-element. Then $\mathcal{E}(\mathbf{G}^F, s)$ is $\mathrm{Gal}(\mathbb{Q}_{|\mathbf{G}^F|}/\mathbb{Q}_{|\mathbf{G}^F|_{\ell'}})$-stable.*

Proof. The character formula [7, Prop. 12.2] expresses $R_\mathbf{L}^\mathbf{G}(\lambda)(g)$, $g \in \mathbf{G}^F$, as a linear combination of values $\lambda(l)$, $l \in \mathbf{L}^F$, with integral coefficients (for this note that the 2-variable Green functions are integral valued by [7, Def. 12.1]). Claim (a) follows.

The characters in $\mathcal{E}(\mathbf{G}^F, s)$ are by definition the constituents of the various $R_\mathbf{T}^\mathbf{G}(\theta)$, where (T, θ) lies in the geometric conjugacy class of s. In particular, all such θ have ℓ'-order. The second assertion is then immediate from part (a). □

We will show the following slightly more general result than Theorem 2.1 which is better adapted to our inductive argument:

Theorem 3.4. *Let $\mathbf{G}$ be an F-stable Levi subgroup of a simple algebraic group $\mathbf{H}$ of simply connected type with a Steinberg endomorphism $F : \mathbf{H} \to \mathbf{H}$. Let ℓ be a prime different from the defining characteristic of $\mathbf{H}$. If B is an ℓ-block of $G = \mathbf{G}^F$ such that* $\mathrm{IBr}(B)$ *is one* $\mathrm{Aut}(G)_B$*-orbit, then any $\varphi \in \mathrm{IBr}(B)$ lifts to some ℓ-rational $\chi \in \mathcal{E}(G, \ell')$ such that χ, φ have the same stabiliser in* $\mathrm{Aut}(G)_B$.

The parts (1)–(3) of Theorem 2.1 for the quasi-simple groups $S = \mathbf{H}^F/Z$, with $Z \leq \mathbf{Z}(\mathbf{H}^F)$ and $\ell \neq p$ now follow from this by taking $\mathbf{G} = \mathbf{H}$.

We will give the proof of Theorem 3.4 in several steps. We begin by recalling some known facts about ℓ-blocks of finite reductive groups. Let $\ell \neq p$ and let B be an ℓ-block of $G = \mathbf{G}^F$. By the result of Broué and Michel (see [5, Thm. 9.12]) there exists a semisimple ℓ'-element s in the dual group $G^* = \mathbf{G}^{*F}$ such that $\mathrm{Irr}(B)$ is contained in the union

$$\mathcal{E}_\ell(G, s) = \coprod_t \mathcal{E}(G, st)$$

of Lusztig series, where t runs over a system of representatives of conjugacy classes of ℓ-elements in $C_{G^*}(s)$. We say that B is *quasi-isolated* if the centraliser $C_{\mathbf{G}^*}(s)$ is not contained in any proper F-stable Levi subgroup of $\mathbf{G}^*$.

Let $\mathbf{G}_1^*$ be the minimal F-stable Levi subgroup of $\mathbf{G}^*$ (and hence of $\mathbf{H}^*$) with $C_{\mathbf{G}^*}(s) \leq \mathbf{G}_1^*$. Note that this is uniquely determined by s , since the intersection of any two Levi subgroups containing $C_{\mathbf{G}^*}(s)$ (and hence a maximal torus of $\mathbf{H}^*$) is again a Levi subgroup. Then by construction s is quasi-isolated in $\mathbf{G}_1^*$. Let $\mathbf{G}_1$ be an F-stable Levi subgroup of $\mathbf{G}$ in duality with $\mathbf{G}_1^*$ and set $G_1 := \mathbf{G}_1^F$. In this situation we have:

Lemma 3.5. *Theorem 3.4 holds for a block B in $\mathcal{E}_\ell(G, s)$ if it holds for its Jordan correspondent in $\mathcal{E}_\ell(G_1, s)$.*

Proof. According to the theorem of Bonnafé and Rouquier [5, Thm. 10.1] Lusztig induction $R_{\mathbf{G}_1}^{\mathbf{G}}$ induces a Morita equivalence between the ℓ-blocks in $\mathcal{E}_\ell(G_1, s)$ and those in $\mathcal{E}_\ell(G, s)$, which sends $\mathrm{Irr}(B_1)$ bijectively to $\mathrm{Irr}(B)$ for some ℓ-block B_1 contained in $\mathcal{E}_\ell(G_1, s)$. By the very definition of Lusztig series any automorphism of G either fixes $\mathcal{E}(G, s)$ or sends it to a disjoint series $\mathcal{E}(G, s')$, for $s' \in G^*$ another semisimple ℓ'-element, see also [27, Cor. 2.4]. In particular, any automorphism of G stabilising B will also stabilise $\mathcal{E}(G, s)$. Following the argument of Bonnafé [27, Sect. 2] any $\sigma \in \mathrm{Aut}(G)_B$ induces an automorphism σ^* of $(\mathbf{G}^*)^F$ unique up to inner automorphisms of $(\mathbf{G}^*)^F$. Since σ^* stabilizes the $(\mathbf{G}^*)^F$-class containing s, we may assume σ to stabilise G_1. This implies $\mathrm{Aut}(G)_B = \mathrm{Inn}(G)\mathrm{Aut}(G)_{G_1,B}$.

Now the Lusztig functor $R_{\mathbf{G}_1}^{\mathbf{G}}$ inducing the Bonnafé–Rouquier Morita equivalence between B_1 and B is an isometry by the definition of G_1, and by [7, Rem. 13.28] it is independent of the choice of parabolic subgroup containing $\mathbf{G}_1$ as Levi subgroup. Hence [27, Cor. 2.3] implies that $R_{\mathbf{G}_1}^{\mathbf{G}}$ is $\mathrm{Aut}(G)_{G_1,B}$-equivariant. So if $\mathrm{IBr}(B)$ is one orbit under $\mathrm{Aut}(G)_B = \mathrm{Inn}(G)\mathrm{Aut}(G)_{G_1,B}$, then $\mathrm{IBr}(B_1)$ is one $\mathrm{Aut}(G_1)_{B_1}$-orbit. Thus, if the assertion of Theorem 3.4 holds for B_1, then every $\varphi_1 \in \mathrm{IBr}(B_1)$ lifts to some $\chi_1 \in \mathrm{Irr}(B_1) \cap \mathcal{E}(G_1, \ell')$ with the same stabiliser in $\mathrm{Aut}(G_1)_{B_1}$. This shows that any $\varphi \in \mathrm{IBr}(B)$ lifts to some $\chi \in \mathrm{Irr}(B) \cap \mathcal{E}(G, \ell')$ with the same stabiliser in $\mathrm{Aut}(G)_B$. As this bijection is given by $R_{\mathbf{G}_1}^{\mathbf{G}}$ the ℓ-rationality properties are also preserved by Proposition 3.3(a). $\square$

Lemma 3.6. *Theorem 3.4 holds whenever B lies in a Lusztig series $\mathcal{E}_\ell(G,s)$ such that $\mathcal{E}(G,s)$ is a basic set for $\mathcal{E}_\ell(G,s)$.*

Proof. Let B be as in the assertion. By assumption the set $X := \mathrm{Irr}(B) \cap \mathcal{E}(G,s)$ is a basic set for B, and $\mathrm{Aut}(G)_B$-invariant by the remarks in the proof of Lemma 3.5. Thus, if $\mathrm{Aut}(G)_B$ has more than one orbit on X, then by Lemma 3.2 it necessarily has more than one orbit on $\mathrm{IBr}(B)$. So, if $\mathrm{IBr}(B)$ is a single $\mathrm{Aut}(G)_B$-orbit, then all elements in X are $\mathrm{Aut}(G)_B$-conjugate and in particular have the same degree d, say. But since X is a basic set, then all elements of $\mathrm{IBr}(B)$ must have this degree d, hence $\mathrm{IBr}(B)$ consists of the Brauer characters of the ℓ-modular reductions of the characters in X. In particular, they are all liftable to elements of $\mathcal{E}(G,\ell')$ under preservation of stabiliser in $\mathrm{Aut}(G)_B$, and these lifts have to be ℓ-rational by Proposition 3.3. □

Before we continue, we need to recall an auxiliary result about the existence of ordinary basic sets:

Lemma 3.7. *Let $\mathbf{G}$ be connected reductive with a Steinberg endomorphism F. Let $s \in \mathbf{G}^{*F}$ be a semisimple ℓ'-element, and $\mathbf{G}_1 \le \mathbf{G}$ an F-stable Levi subgroup such that $\mathbf{G}_1^*$ contains $C_{\mathbf{G}^*}(s)^F$. Assume that ℓ is good for $\mathbf{G}_1$ and prime to the order of $(\mathbf{Z}(\mathbf{G}_1)/\mathbf{Z}^\circ(\mathbf{G}_1))^F$. Then $\mathcal{E}(\mathbf{G}^F,s)$ is a basic set for the blocks in $\mathcal{E}_\ell(\mathbf{G}^F,s)$.*

Proof. By the theorem of Geck and Hiss [5, Thm. 14.4] the assumptions made on $\mathbf{G}_1$ ensure that $\mathcal{E}(\mathbf{G}_1^F,s)$ is a basic set for the blocks in $\mathcal{E}_\ell(\mathbf{G}_1^F,s)$. Now by [5, Thm. 10.1] Lusztig induction $R_{\mathbf{G}_1}^{\mathbf{G}}$ induces a Morita equivalence between the ℓ-blocks in $\mathcal{E}_\ell(\mathbf{G}_1^F,s)$ and those in $\mathcal{E}_\ell(\mathbf{G}^F,s)$, which sends $\mathcal{E}(\mathbf{G}_1^F,s)$ bijectively to $\mathcal{E}(\mathbf{G}^F,s)$. The claim follows. □

Corollary 3.8. *Let $\mathbf{G}$ be simple of type G_2, F_4 or E_6 and $\ell=3$, or of type E_8 and $\ell=5$. Then $\mathcal{E}(\mathbf{G}^F,s)$ is a basic set for the ℓ-blocks in $\mathcal{E}_\ell(\mathbf{G}^F,s)$ unless $s \in \mathbf{G}^{*F}$ is quasi-isolated (and hence of order at most 2 if $\mathbf{G}$ does not have type E_8).*

Proof. In all listed cases, the prime ℓ is good for any proper Levi subgroup of $\mathbf{G}$. Unless $\mathbf{G}$ is of type E_6, $\mathbf{G}$ has trivial center, so all assumptions of Lemma 3.7 are satisfied as soon as $C_{\mathbf{G}^*}(s)^F$ is contained in a proper Levi subgroup of $\mathbf{G}$. In type E_6 it is readily checked that any proper Levi subgroup has connected center. □

Proposition 3.9. *Theorem 3.4 holds when $\mathbf{H}$ is of type A.*

Proof. Note that any prime ℓ is good for $\mathbf{H}$, hence for $\mathbf{G}$. If ℓ does not divide $|\mathbf{Z}(\mathbf{G})^F|$ then $\mathcal{E}(G,s)$ is a basic set for all blocks in $\mathcal{E}_\ell(G,s)$ by Lemma 3.7, and the claim follows from Lemma 3.6.

Now first assume that $\mathbf{H}^F = \mathrm{SL}_n(q)$. As $|\mathbf{Z}(\mathbf{H})^F| = \gcd(n, q-1)$ we may assume that $\ell|(q-1)$. Let B be an ℓ-block of $G = \mathbf{G}^F$ in $\mathcal{E}_\ell(G,s)$. Then by [20, Thm. A(a)] there exists a 1-cuspidal pair $(\mathbf{L},\lambda)$ with a 1-split Levi subgroup $\mathbf{L} \le \mathbf{G}$ such $\mathrm{Irr}(B)$ contains all constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$. Consider a regular embedding $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ and let $\tilde{s} \in \tilde{\mathbf{G}}^{*F}$ be a preimage of s under the induced map $\tilde{\mathbf{G}}^* \to \mathbf{G}^*$ of dual groups. Then $\mathcal{E}(\tilde{G},\tilde{s})$ is a single 1-Harish-Chandra series, for a 1-cuspidal character $\tilde{\lambda}$ lying above λ of the Levi subgroup $\tilde{L}$ of $\tilde{G}$ with $\tilde{L} \cap G = \mathbf{L}^F$. Now let χ_1, χ_2 be constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ below the semisimple and the regular character of $\mathcal{E}(\tilde{G},\tilde{s})$, respectively.

First assume that $\chi_1(1) \neq \chi_2(1)$. As ℓ is good for G, $\mathcal{E}(G,s)^0$ is linearly independent by [5, Thm. 14.6], hence the $\mathrm{Aut}(G)_B$-orbit sums of χ_1, χ_2 satisfy the assumptions of Lemma 3.2, and $\mathrm{IBr}(B)$ cannot be a single $\mathrm{Aut}(G)_B$-orbit. Thus we have $\chi_1(1) = \chi_2(1)$. But these correspond, under Jordan decomposition, to the trivial and the Steinberg character of $C^\circ_{\mathbf{G}^*}(s)^F$, whence $C^\circ_{\mathbf{G}^*}(s)$ is a torus. Then so is $C_{\tilde{\mathbf{G}}^*}(\tilde{s})$, and hence $\mathcal{E}(\tilde{G},\tilde{s}) = \{\tilde{\chi}\}$ has just one element, and since $\tilde{\mathbf{G}}$ has connected center and ℓ is good for $\tilde{\mathbf{G}}$, $\{\tilde{\chi}\}$ is a basic set for the block $\tilde{B}$ of $\tilde{G}$ covering B containing $\tilde{\chi}$. In particular $\mathrm{IBr}(\tilde{B}) = \{\tilde{\chi}^0\}$, and $\tilde{\chi}, \tilde{\chi}^0$ are invariant under the same automorphisms of $\tilde{G}$.

Since $\mathcal{E}(G,s)^0$ is linearly independent (again by [5, Thm. 14.6]), all elements of $\mathrm{Irr}(B)\cap \mathcal{E}(G,s)$ must have the same degree d, say. Now clearly $C_{\mathbf{G}^*}(st) \leq C_{\mathbf{G}^*}(s)$ for all ℓ-elements $t \in C_{\mathbf{G}^*}(s)$, so all characters in $\mathcal{E}_\ell(G,s)$ have degree divisible by d by the Jordan decomposition of characters. Thus all elements of $\mathrm{IBr}(B)$ have degree divisible by d. By what we said before, then certainly $\mathcal{E}(G,s)^0 \subseteq \mathrm{IBr}(B)$. On the other hand, since $\mathcal{E}(\tilde{G},\tilde{s}) = \{\tilde{\chi}\}$ and $\mathrm{IBr}(\tilde{B}) = \{\tilde{\chi}^0\}$, we must actually have $\mathcal{E}(G,s)^0 = \mathrm{IBr}(B)$. Then clearly B satisfies the conclusion of Theorem 3.4.

If $\mathbf{H}^F = \mathrm{SU}_n(q)$, then necessarily ℓ divides $q+1$, and we can argue entirely similarly, with 1-cuspidal and 1-series replaced by 2-cuspidal and 2-series. □

Proposition 3.10. *Theorem 3.4 holds if all components of $\mathbf{G}$ are of classical type A, B, C or D and moreover $\ell \neq 2$ if $\mathbf{G}^F$ has a component of type ${}^3D_4$.*

Proof. By Lemma 3.9 we may assume that $\mathbf{H}$ is not of type A. We claim that $\mathbf{Z}(\mathbf{G})/\mathbf{Z}^\circ(\mathbf{G})$ is of 2-power order. Indeed, if $\mathbf{H}$ is not of type E_6, then $\mathbf{Z}(\mathbf{H})$ has 2-power order and the claim follows. If $\mathbf{H}$ is of type E_6, then it is easily verified that all proper Levi subgroups have connected center (in fact, this is only an issue for those having a factor A_5 or A_2).

Now first assume that $\ell > 2$. Let B be an ℓ-block of G, lying in series $s \in \mathbf{G}^{*F}$. By our previous observation $|\mathbf{Z}(\mathbf{G})^F/\mathbf{Z}^\circ(\mathbf{G})^F|$ is prime to ℓ, and as $\ell \geq 3$ is good for $\mathbf{G}$, Lemma 3.7 applies to B to yield that $\mathrm{Irr}(B)\cap\mathcal{E}(G,s)$ is a basic set for B. Thus, we conclude by Lemma 3.6.

We are left to consider the case $\ell = 2$. Let $s \in G^*$ be a semisimple $2'$-element such that $\mathrm{Irr}(B) \subset \mathcal{E}_\ell(G,s)$. Recall that G has no component of type ${}^3D_4$. Then by a result of Enguehard (see e.g. [20, Lemma 3.3]) we have that $\mathrm{Irr}(B) = \mathcal{E}_\ell(G,s)$. Since $\mathbf{Z}(\mathbf{G})/\mathbf{Z}^\circ(\mathbf{G})$ is a 2-group and s has odd order, the centraliser $C_{\mathbf{G}^*}(s)$ is connected and a Levi subgroup of $\mathbf{G}^*$. Let $\mathbf{G}_1 \leq \mathbf{G}$ be an F-stable Levi subgroup dual to $C_{\mathbf{G}^*}(s)$. By Lemma 3.5 we may pass to the Morita equivalent Jordan corresponding block B_1 in $\mathcal{E}(\mathbf{G}_1^F, s)$, which in turn is Morita equivalent to the principal block of $\mathbf{G}_1^F$ as $s \in \mathbf{Z}(\mathbf{G}_1^{*F})$. The latter obviously satisfies the assertions of Theorem 3.4. □

Proposition 3.11. *Theorem 3.4 holds when $\mathbf{G}^F$ has a component of exceptional type.*

Proof. By Lemma 3.6 in conjunction with Lemma 3.7 we may assume that ℓ is bad for $\mathbf{G}$. Moreover, by Lemma 3.5 the block B is quasi-isolated in G. There is no bad non-defining prime for Suzuki groups. For the Ree groups ${}^2G_2(3^{2n+1})$ the only relevant bad prime is $\ell = 2$, and the only quasi-isolated block is the principal block by Ward [31], so we are done in this case. The only bad prime for ${}^2F_4(2^{2n+1})$ is $\ell = 3$, and $s = 1$ is the only quasi-isolated semisimple $3'$-element. Here the unipotent blocks were determined in [22, Bem. 1] and the tables in loc. cit. show that in each such block of positive defect there are two $\operatorname{Aut}(G)$-invariant unipotent characters with linearly independent restriction to ℓ'-elements, so Lemma 3.2 allows to conclude.

We are left to consider the exceptional groups of types G_2, ${}^3D_4$, F_4, E_6, ${}^2E_6$, E_7 and E_8. Assume first that $\mathbf{G} = \mathbf{H}$. For these groups the unipotent blocks at bad primes have been determined by Hiss–Shamash, Deriziotis–Michler and Enguehard (see the tables in [8]), while their other quasi-isolated blocks can be found in Kessar–Malle [19, Tables 1–8]. It is easily checked from these sources that each such block of non-zero defect contains two characters χ_1, χ_2 with different a-values $a_i = a(\chi_i)$ (in the sense of [10, §3B]), with $a_1 < a_2$ say. It then follows by [10, Thm. 3.7] that there exists an F-stable unipotent conjugacy class $\mathbf{C} \subset \mathbf{G}$ such that the variety of Borel subgroups of $\mathbf{G}$ containing a given $u \in \mathbf{C}$ has dimension a_1 with the following property: the average value of χ_2 on $\mathbf{C}^F$ is zero, while it is non-zero for χ_1. Since the unipotent classes of $\mathbf{G}$ are invariant under all automorphisms of $\mathbf{G}$, and thus their F-fixed points $\mathbf{C}^F$ are invariant under all automorphisms of G, the orbit sums ψ_1, ψ_2 of χ_1 and χ_2 under $\operatorname{Aut}(G)_B$ still have the same vanishing respectively non-vanishing property on $\mathbf{C}^F$. In particular ψ_1^0, ψ_2^0 are linearly independent. Thus, by Lemma 3.2 there are at least two orbits of $\operatorname{Aut}(G)_B$ on $\operatorname{IBr}(B)$. (For example, in many cases, we can take for χ_1 the semisimple character and for χ_2 the regular character in $\mathcal{E}(G, s)$.)

Finally assume that $[\mathbf{G}, \mathbf{G}]$ is not simple, but has some exceptional component or a component with F-fixed points of type ${}^3D_4$. In the latter case, since ${}^3D_4(q)$ has trivial Schur multiplier, G splits into a direct product, and since the claim holds for the factors, it clearly also holds for G. The only possibility in the former case is that $\mathbf{G}$ has type $E_6 + A_1$ inside E_8, and $\ell \in \{2, 3\}$. Note that then $\mathbf{G}$ has connected center, so the quasi-isolated elements are just the isolated elements in the component of type E_6. For those we had already argued that all ℓ-blocks B not of defect zero contain at least two $\operatorname{Aut}(G)_B$-orbits on $\operatorname{Irr}(B)$ with linearly independent restrictions to ℓ'-classes, so we can conclude as before. $\square$

Together, Propositions 3.9–3.11 cover all cases and thus the proof of Theorem 3.4 is complete. Note that here we find examples of blocks of arbitrarily high defect. Indeed, if $s \in G^*$ is a regular semisimple ℓ'-element lying in the unique maximal torus T^* then the block containing the corresponding Deligne–Lusztig character is covered by a nilpotent block of $\tilde{G}$ and has defect at least $|T^*|_\ell$, which can be an arbitarily high power of ℓ.

3.3. **Extendibility.** We now turn to the proof of Theorem 2.1(4) in our situation and assume in the following that $\ell = 2$. Let $\mathbf{G}$ be simple, simply connected, and $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ a regular embedding, so $\tilde{\mathbf{G}}$ has connected center and $\tilde{\mathbf{G}} = \mathbf{Z}(\tilde{\mathbf{G}})\mathbf{G}$, with compatible Frobenius map F. Recall that the automorphism group of $G := \mathbf{G}^F$ has a subgroup $\operatorname{Diag}(G)$ consisting of automorphism induced by elements of $\tilde{G} := \tilde{\mathbf{G}}^F$

Proposition 3.12. *Let G, B, χ be as in Theorem 3.4, $\ell = 2$, and let P be a Sylow 2-subgroup of $\operatorname{Aut}(G)_\chi$. If the Sylow 2-subgroups of $\operatorname{Aut}(G)_\chi/\operatorname{Diag}(G)_\chi$ are cyclic (which is the case whenever $G \notin \{\operatorname{SL}_n(q), D_n(q), E_6(q)\}$), then χ extends to some $\tilde{\chi} \in \operatorname{Irr}(G \rtimes P)$ such that $\ker(\chi_{\mathbf{C}_{G\rtimes P}(G)})$ has odd index in $\mathbf{C}_{G\rtimes P}(G)$.*

Proof. Let A be a Sylow 2-subgroup of $\tilde{\mathbf{G}}^F$. Now by [4, Prop. 1.3], restriction induces a bijection between $\mathcal{E}(GA, 2')$ and $\mathcal{E}(G, 2')$. Hence χ^{GA} has a unique constituent $\tilde{\chi}$ in

$\mathcal{E}(GA, 2')$, which then must be $\mathrm{Aut}(G)_\chi$-stable as well, with respect to the natural action of $\mathrm{Aut}(G)$ on GA. By construction $\ker(\tilde\chi)$ contains the Sylow 2-subgroup of $\mathbf{Z}(GA)$. Any Sylow 2-subgroup D of $\mathrm{Aut}(G)_\chi/\mathrm{Inn}(GA)$ is isomorphic to a group generated by field and graph automorphisms of G that act faithfully on GA. By our assumption on $\mathrm{Aut}(G)_\chi/\mathrm{Diag}(G)_\chi$, D is cyclic. Hence $\tilde\chi$ extends further to $(GA)D$. By our construction $\mathbf{C}_{(GA)D}(G)$ is contained in $\mathbf{Z}(\tilde G)$ and the 2-part $\tilde Z$ of $\mathbf{Z}(\tilde G)$ is contained in $\ker(\tilde\chi)$.

In order to now prove the extendibility of χ to $G \rtimes P$ we apply Lemma 2.2: Let $\overline{G} := G/\mathbf{Z}(G)_2$, where $\mathbf{Z}(G)_2$ is the Sylow 2-subgroup of $\mathbf{Z}(G)$. Then the above implies that χ seen as a character of $\overline{G}$ extends to $(GA)D/\tilde Z$. Now the group $\mathbf{C}_{(GA)D/\tilde Z}(\overline{G})$ is an $2'$-group, and Lemma 2.2 proves that χ as a character of $\overline{G}$ extends to $\overline{G} \rtimes P$. Since $\mathrm{Aut}(G) = \mathrm{Aut}(\overline{G})$, P naturally acts on G and this implies the extendibility part of the statement.

Note that since D is isomorphic to a group generated by field and graph automorphisms, D is cyclic if $G \notin \{\mathrm{SL}_n(q), D_n(q), E_6(q)\}$. □

It remains to deal with the small prime cases left open in Proposition 3.12.

Proposition 3.13. *Let G, B, χ be as in Theorem 3.4. Assume that $\ell = 2$ and $G = \mathrm{SL}_n(q)$ for some $n \geq 3$ and some odd prime power q. Let P be a Sylow 2-subgroup of $\mathrm{Aut}(G)_\chi$. Then χ extends to some character $\tilde\chi$ of $G \rtimes P$ such that $\ker(\tilde\chi|_{\mathbf{C}_{G\rtimes P}(G)})$ has odd index in $\mathbf{C}_{G\rtimes P}(G)$.*

Proof. By Proposition 3.12 we may assume that $\mathrm{Aut}(G)_\chi/\mathrm{Diag}(G)_\chi$ is not cyclic. Hence we can assume that the $\tilde{\mathbf{G}}^F$-orbit of χ is invariant under the graph automorphism γ and a field automorphism of even order. Let D_2 be a Sylow 2-subgroup of the stabiliser of this orbit in the group generated by field automorphisms and γ. Let F_0 be a field automorphism such that $D_2 = \langle F_0, \gamma\rangle$. According to [6, Thm. 4.1] some character in the $\tilde{\mathbf{G}}^F$-orbit of χ is D_2-invariant. We may assume that χ is D_2-invariant. The character $\tilde\chi \in \mathcal{E}(GA, 2')$ from Proposition 3.12 extends to $\tilde{\mathbf{G}}^F_\chi$ and is D_2-invariant. Since the quotient $\tilde{\mathbf{G}}^F_\chi/GA$ has odd order there exists a D_2-invariant extension $\widehat\chi$ of $\tilde\chi$ to $\tilde{\mathbf{G}}^F_\chi$ according to (some easy application of) [14, Lemma 13.8]. Now $\widehat\chi^{\tilde{\mathbf{G}}^F}$ extends to $\tilde{\mathbf{G}}^F D_2$, since by [2, Lemma 4.3.2] there exists an extension ψ of $\widehat\chi^{\tilde{\mathbf{G}}^F}$ to $\tilde{\mathbf{G}}^F\langle F_0\rangle$ such that $\psi(F_0) \neq 0$. Accordingly $\psi^\gamma = \psi$ and ψ extends to $\tilde{\mathbf{G}}^F\langle F_0, \gamma\rangle$. By considering the degrees this proves that χ extends to some $\tilde\chi \in \mathrm{Irr}((\tilde{\mathbf{G}}^F D_2)_\chi)$ such that $\ker(\tilde\chi)$ contains a Sylow 2-subgroup $\tilde Z$ of $\mathbf{C}_{(\tilde{\mathbf{G}}^F D_2)_\chi}(G)$.

Now applying Lemma 2.2 as in the proof of Proposition 3.12 implies the extensibility part of the statement. □

Alternatively, the previous result could be proved by using Deligne–Lusztig characters for disconnected groups as introduced by Digne–Michel.

Proposition 3.14. *Let G, B, χ be as in Theorem 3.4. Assume that $\ell = 2$ and $G \in \{D_n(q), E_6(q)\}$. Let P be a Sylow 2-subgroup of $\mathrm{Aut}(G)_\chi$. Then χ extends to some $\tilde\chi \in \mathrm{Irr}(G \rtimes P)$ such that $\ker(\tilde\chi|_{\mathbf{C}_{G\rtimes P}(G)})$ has odd index in $\mathbf{C}_{G\rtimes P}(G)$.*

Proof. First assume that $\mathbf{G}$ has type D_n with untwisted Frobenius map. Let $s \in G^*$ be a semisimple $2'$-element with $\mathrm{Irr}(B) \subseteq \mathcal{E}_2(G,s)$. Then as shown in the proof of Proposition 3.10, B is Morita equivalent to the principal block B_1 of $\mathbf{G}_1$, where $\mathbf{G}_1$ is dual to the Levi subgroup $C_{\mathbf{G}^*}(s)$, and $\mathcal{E}(\mathbf{G}_1^F, 1) \subseteq \mathrm{Irr}(B_1)$. Thus, B_1 contains all unipotent characters of $\mathbf{G}_1^F$, hence it will satisfy our assumptions only if the trivial character is the only unipotent character of $\mathbf{G}_1^F$, hence if $\mathbf{G}_1$ is a (maximal) torus. Since $\mathbf{G}_1$ is abelian, every block of $\mathbf{G}_1^F$ is nilpotent. Accordingly the block B is nilpotent and the statement follows from Proposition 2.3.

Now let $\mathbf{G}$ be of type E_6 with untwisted Frobenius map and assume that Proposition 3.12 does not apply. This forces $\mathrm{Aut}(G)_B$ to contain graph and field automorphisms of order 2. Let $s \in G^*$ be a semisimple $2'$-element with $\mathrm{Irr}(B) \subseteq \mathcal{E}_2(G,s)$. If $C_{\mathbf{G}^*}(s)$ is contained in a proper F-stable Levi subgroup $\mathbf{G}_1^*$, then all components of $\mathbf{G}_1$ are of classical type, the center of $\mathbf{G}_1$ is connected, and thus $\mathrm{Irr}(B) = \mathcal{E}_2(G,s)$. We see that then our assumptions imply that $C_{\mathbf{G}^*}(s)$ is a torus. This implies again that B is nilpotent, and the statement follows from Proposition 2.3.

Thus we may assume that s is quasi-isolated, hence as in [19, Tab. 3]. Inspection shows that unless B is of defect 0 there always exist characters with distinct a-values in $\mathrm{Irr}(B) \cap \mathcal{E}(G,s)$, and hence B does not satisfy our assumptions. □

This completes the proof of Theorem 2.1.

4. p-rational lifts

From now we (again) consider p-blocks or p-Brauer characters for some odd prime p. In this section we show in Theorem 4.5 that for an odd prime p in some situations p-rationality is preserved by isomorphisms of character triples. This ensures that in the proof of the Main Theorem the lift can be chosen p-rational.

Recall that $\chi \in \mathrm{Irr}(G)$ is *p-rational* if the values of χ lie in some cyclotomic field $\mathbb{Q}_m := \mathbb{Q}(\zeta)$, for some mth root of unity ζ of order not divisible by p. By elementary character theory, this happens if the values of χ lie in $\mathbb{Q}_{|G|_{p'}}$. If χ is p-rational, it is non-trivial to prove that χ can be afforded by an absolutely irreducible $\mathbb{Q}_m$-representation (see [14, Thm. (10.13)].)

If $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$ is p-rational it is not true in general that there exists a p-rational irreducible character $\chi \in \mathrm{Irr}(G)$ over θ, even if G/N is a p'-group. We start with the following.

Theorem 4.1. *Suppose that $N \lhd G$ with G/N a p'-group for p odd. Assume that $\theta \in \mathrm{Irr}(N)$ is p-rational such that $\theta^0 \in \mathrm{IBr}(N)$. Then every $\chi \in \mathrm{Irr}(G)$ over θ is p-rational.*

Proof. We may assume that θ is G-invariant. Let $\chi \in \mathrm{Irr}(G|\theta)$ and $x \in G$. Let $m = |G|_{p'}$. We want to show that $\chi(x) \in \mathbb{Q}_m$. We may assume that G/N is generated by Nx. Hence $\chi_N = \theta$. Now let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_m)$. We have that $\chi^\sigma = \lambda\chi$, for some $\lambda \in \mathrm{Irr}(G/N)$. Now, $(\chi_N)^0 = \theta^0 \in \mathrm{IBr}(N)$. Thus $\chi^0 = \varphi \in \mathrm{IBr}(G)$, and $\varphi_N = \theta^0$. Now, $\varphi^\sigma = \varphi$. Thus $(\chi^\sigma)^0 = \chi^0$. Hence $\lambda^0\chi^0 = \chi^0$. By Gallagher's Lemma for Brauer characters, $\lambda^0 = 1$, and therefore $\lambda = 1$. □

Corollary 4.2. *Suppose that G/N has a normal Sylow p-subgroup P/N. Let $\theta \in \mathrm{Irr}(N)$ be p-rational such that $\theta^0 \in \mathrm{IBr}(N)$. Then there exists a p-rational $\chi \in \mathrm{Irr}(G|\theta)$.*

Proof. We may assume that θ is G-invariant. Now, θ has a canonical p-rational extension $\hat{\theta} \in \mathrm{Irr}(P|\theta)$ by [14, Thm. (6.30)]. Since it extends, it lifts an irreducible Brauer character. Now apply Theorem 4.1 to $\hat{\theta}$. □

Corollary 4.3. *Let N be a normal subgroup of G with G/N cyclic and let $\theta \in \mathrm{Irr}(N)$ be a G-invariant p-rational character of N with $\theta^0 \in \mathrm{IBr}(N)$. Then there exists a p-rational character $\chi \in \mathrm{Irr}(G)$ extending θ.*

Proof. This follows from Corollary 4.2. □

In the following we apply a standard construction of *ordinary-modular character triples* as developed in [24] taking additionally into account ordinary and modular characters at the same time as well as fields of values. If $N \triangleleft G$ and $\theta \in \mathrm{Irr}(N)$ is G-invariant, then we say that (G, N, θ) is a *character triple*. The classical theory of projective representations allows us to replace (G, N, θ) by another triple (Γ, M, ψ) in which M is in the center of Γ, and the character theory of G over θ is *analogous* to the character theory of Γ over ψ. (See Chapter 11 of [14] for details.) Furthermore, if $\theta^0 \in \mathrm{IBr}(N)$, then this replacement can be done in such a way that both the ordinary and the modular character theory are preserved. (See Problem (8.12) of [24].)

Recall that for the definition of irreducible Brauer characters a maximal ideal M of the ring of algebraic integers $\mathbf{R}$ in $\mathbb{C}$ has been chosen, with respect to which Brauer characters are calculated. As in [24, Sect. 2], let $F = \mathbf{R}/M$ be an algebraically closed field of characteristic p, and ${}^* : \mathbf{R} \to F$ be the canonical homomorphism. Note that * can be extended to the localisation S of $\mathbf{R}$ at M, and therefore induces a map $\mathrm{Mat}_n(S) \to \mathrm{Mat}_n(F)$, denoted again by * .

Lemma 4.4. *Suppose that $N \triangleleft G$ and let $\theta \in \mathrm{Irr}(N)$ be a G-invariant p-rational character with $\theta^0 \in \mathrm{IBr}(N)$, where p is odd. Let $\mathbb{F} = \mathbb{Q}_m$, where $m = |G|_{p'}$. Then there exists a projective representation $\mathcal{X}$ of G with entries in $\mathbb{F} \cap S$ and with factor set α satisfying the following conditions:*

(1) *$\mathcal{X}_N$ is an ordinary representation of N affording θ;*
(2) *$\mathcal{X}(gn) = \mathcal{X}(g)\mathcal{X}(n)$, $\mathcal{X}(ng) = \mathcal{X}(n)\mathcal{X}(g)$ for $g \in G$ and $n \in N$;*
(3) *$\alpha(g, h) \in \mathbb{F}$; and*
(4) *$\mathcal{X}^*$ defined as $\mathcal{X}^*(g) = \mathcal{X}(g)^*$ for $g \in G$, is an F-projective representation such that $(\mathcal{X}^*)_N$ affords the Brauer character θ^0.*

Proof. By [14, Thm. (10.13)] and Problem (2.12) of [24] we know that θ can be afforded by an absolutely irreducible $\mathbb{F}N$-representation $\mathcal{Y}$ with entries in $\mathbb{F} \cap S$. Hence $\mathcal{Y}^*$ affords θ^0. For each $\bar{g} = gN \in G/N$, by using Corollary 4.3, there is a p-rational character ψ of $N\langle g\rangle = H$ extending θ. Again ψ can be afforded by an absolutely irreducible $\mathbb{F}H$-representation $\mathcal{Y}_1$ with entries in $\mathbb{F} \cap S$. Notice that $\mathcal{Y}_1^*$ is irreducible since it affords the irreducible Brauer character ψ^0 (which extends θ^0). We would like to choose $\mathcal{Y}_1$ such that it extends $\mathcal{Y}$. We know that $(\mathcal{Y}_1)_N$ and $\mathcal{Y}_N$ are $\mathbb{F}$-similar. By using [24, Lemma (2.5)] there exists a matrix P with entries in $\mathbb{F} \cap S$ such that $(\mathcal{Y}_1)_N = P^{-1}\mathcal{Y}P$, and $P^* \neq 0$. Thus $P^*((\mathcal{Y}_1)^*)_N = \mathcal{Y}^*P^*$, and so by Schur's Lemma P^* is invertible. Therefore P^{-1} has entries in $\mathbb{F} \cap S$. Hence, $\mathcal{Y}_{\bar{g}} := P\mathcal{Y}_1P^{-1}$ is an $\mathbb{F}$-representation of $\langle N, g\rangle$ with entries in

$\mathbb{F} \cap S$ extending $\mathcal{Y}$. We define

$$\mathcal{X}(g) = \mathcal{Y}_{\bar{g}}(g) \qquad \text{for } g \in G.$$

By [12, Lemma (2.2)], note that $\mathcal{X}$ is a projective representation of G, with factor set α, say. Also, $\mathcal{X}(gn) = \mathcal{X}(g)\mathcal{Y}(n)$, $\mathcal{X}(ng) = \mathcal{Y}(n)\mathcal{X}(g)$ and $\mathcal{X}(n) = \mathcal{Y}(n)$ for $n \in N$ and $g \in G$. Furthermore the factor set α of $\mathcal{Y}$ satisfies

$$\alpha(g,h) \in \mathbb{F} \quad \text{for all } g,h \in G$$

by definition. Condition (4) now easily follows from the definition of $\mathcal{X}$. $\square$

Suppose that (G,N,θ) is an ordinary character triple with $\theta^0 \in \mathrm{IBr}(N)$. Then we say that (G,N,θ) is an *ordinary-modular* character triple. The precise definition of isomorphism of ordinary-modular character triples is given in Problem (8.10) of [24].

Theorem 4.5. *Suppose that (G,N,θ) is a character triple where θ is a p-rational character, with p odd, and $\theta^0 \in \mathrm{IBr}(N)$. Then there exists an isomorphic ordinary-modular character triple (Γ, M, ξ) satisfying the following conditions.*

(a) *M is a central p'-subgroup of Γ.*

(b) *If $N \subseteq U \subseteq G$ and $\chi \in \mathrm{Irr}(U \mid \theta)$, then χ is p-rational if and only if its correspondent $\tilde{\chi}$ is p-rational.*

Proof. We follow the proof of [24, Thm. (8.28)]. Let $\mathcal{X}$ be the projective representation of G with factor set α determined by Corollary 4.4. Then $\mathcal{X}$ is a projective representation of G associated with θ satisfying conditions (a)–(c) of Theorem (11.2) of [14]. In particular, $\alpha(g,n) = \alpha(n,g) = 1$ for $n \in N$ and $g \in G$ and $\alpha(gn,hm) = \alpha(g,h)$ for $g,h \in G$ and $n,m \in N$ (see pages 178 and 179 of [14]).

Now, let E be the group of $|G|_{p'}$-th roots of unity contained in $\mathbb{F}$. Notice that E is a p'-group. Let $\tilde{G} = \{(g,\epsilon) \mid g \in G, \epsilon \in E\}$ with multiplication

$$(g_1,\epsilon_1)(g_2,\epsilon_2) = (g_1g_2, \alpha(g_1,g_2)\epsilon_1\epsilon_2)\,.$$

The fact that α is in $\mathbf{Z}^2(G,\mathbb{C})$ makes this multiplication associative. Since $\alpha(g,1) = \alpha(1,g) = \alpha(1,1) = 1$ and $\alpha(g,g^{-1}) = \alpha(g^{-1},g)$ for $g \in G$ (by using that α is a factor set), it easily follows that $(1,1)$ is the identity of $\tilde{G}$ and that $(g^{-1}, \alpha(g,g^{-1})^{-1}\epsilon^{-1})$ is the inverse of (g,ϵ) for $g \in G$ and $\epsilon \in E$. Hence, $\tilde{G}$ is a finite group.

Let $\tilde{N} = \{(n,\epsilon) \mid n \in N, \epsilon \in E\} \subseteq \tilde{G}$. Note that

$$(n_1,\epsilon_1)(n_2,\epsilon_2) = (n_1n_2, \epsilon_1\epsilon_2)$$

since $\alpha(n_1,n_2) = 1$ for $n_1,n_2 \in N$. Hence, $\tilde{N} = N \times E$ is a subgroup of $\tilde{G}$. To make the notation easier we identify E with $1 \times E$ and N with $N \times 1$ whenever we find it necessary. If $n \in N$ and $\delta \in E$, then, by using the fact that α is constant on cosets modulo N, we easily check that

$$(g,\epsilon)(n,\delta)(g,\epsilon)^{-1} = (gng^{-1},\delta) \qquad \text{for } (g,\epsilon) \in \tilde{G}.$$

Thus E is a subgroup of $\mathbf{Z}(\tilde{G})$ and N and $\tilde{N}$ are normal subgroups of $\tilde{G}$. Define

$$\tilde{\mathcal{X}}(g,\epsilon) := \epsilon\mathcal{X}(g)\,.$$

It is clear that $\tilde{\mathcal{X}}$ is an $\mathbb{F}$-representation of $\tilde{G}$. Call τ its p-rational character. Note that

$$\tilde{\mathcal{X}}(n, \epsilon) = \epsilon\mathcal{Y}(n) \qquad \text{for } n \in N,$$

and so

$$\tau(n, \epsilon) = \epsilon\theta(n)\,.$$

Now, define $\tilde{\theta} = \theta \times 1_E \in \mathrm{Irr}(\tilde{N})$ and observe that $\tilde{\theta}_N = \theta = \tau_N \in \mathrm{Irr}(N)$. In particular, $\tau \in \mathrm{Irr}(\tilde{G})$ and $\tau^0 \in \mathrm{IBr}(\tilde{G})$. Also, note that $\tilde{\theta}$ is $\tilde{G}$-invariant because θ is G-invariant.

The map $\tilde{G} \to G$, $(g, \epsilon) \mapsto g$, is an epimorphism with kernel E. Since $E \subseteq \ker\tilde{\theta}$, by [14, Lemma (11.26)], we have that $(\tilde{G}, \tilde{N}, \tilde{\theta})$ and (G, N, θ) are isomorphic character triples. Moreover, this isomorphism preserves p-rational characters.

Now, define $\lambda \in \mathrm{Irr}(\tilde{N})$ by $\lambda(n, \epsilon) = \epsilon^{-1}$, so that λ is a linear character of $\tilde{N}$ with $\ker\lambda = N$. Observe that λ is $\tilde{G}$-invariant. Also,

$$\tau_{\tilde{N}} = \lambda^{-1}\tilde{\theta}\,.$$

By the remark after Lemma (11.27) of [14], we have that $(\tilde{G}, \tilde{N}, \lambda)$ and $(\tilde{G}, \tilde{N}, \tilde{\theta})$ are isomorphic character triples. The bijection $\mathrm{Irr}(\tilde{G}|\lambda) \to \mathrm{Irr}(\tilde{G}|\tilde{\theta})$ is given by $\beta \mapsto \beta\tau$. Furthermore, if $N \subseteq U \subseteq G$ and $\chi \in \mathrm{Irr}(U \mid \theta)$, then the character corresponding to χ under the concatenation of the two character triple isomorphisms is a character $\beta \in \mathrm{Irr}(\tilde{U}|\lambda)$ where $U = \tilde{Z}/E$ and $\beta\tau_U$ is a lift of χ. (The corresponding facts about Brauer characters are precisely quoted in the proof of [24, Thm. (8.28)].)

Now, if β is p-rational, we have that $\beta\tau_U$ is p-rational. If $\beta\tau_U$ is p-rational, and $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{|G|_{p'}})$, then β and β^σ have the same image under the bijection $\beta \mapsto \beta\tau_U$. Thus $\beta = \beta^\sigma$, and β is p-rational. Now, $(\tilde{G}, \tilde{N}, \lambda)$ and $(\tilde{G}/N, \tilde{N}/N, \lambda)$ are isomorphic by [14, Lemma (11.26)] with an isomorphism which preserves p-rationality. Now, write $\Gamma = \tilde{G}/N$, $M = \tilde{N}/N$ and $\xi = \lambda$, and observe that M is a central p'-subgroup of Γ. □

5. Proof of the main theorem

In this section we prove the Main Theorem, based on Theorem 2.1 for quasi-simple groups, shown in Sections 2 and 3. Moreover we apply results of Section 4 for the construction of a p-rational lift.

Let G be a finite group and p be a prime. We first derive some direct consequences of Theorem 2.1.

Corollary 5.1. *Suppose that E is a perfect group such that $E/\mathbf{Z}(E)$ is a direct product of simple groups. Let $A = \mathrm{Aut}(E)$. Let p be a prime, and assume that $|\mathbf{Z}(E)|$ is prime to p. Let b be a p-block of E. If $\mathrm{IBr}(b)$ consists of A-conjugates of φ, then there exists a $\chi \in \mathrm{Irr}(E)$ such that $\chi^0 = \varphi$, the stabilisers $I_A(\chi) = I_A(\varphi) =: I$ coincide, and χ extends to some $\tilde{\chi} \in \mathrm{Irr}(E \rtimes P)$ for some $P \in \mathrm{Syl}_p(I)$ such that $\ker(\tilde{\chi})$ contains a Sylow p-subgroup of $\mathbf{C}_{E\rtimes P}(E)$. If $p \neq 2$ then χ can be chosen to be p-rational in addition.*

Proof. Write $Z = \mathbf{Z}(E)$. By the *universal p'-cover* of a perfect group S we mean here the quotient of a universal covering group $\widehat{S}$ of S by the Sylow p-subgroup of $\mathbb{Z}(\widehat{S})$. By the theory of covering groups, there are non-isomorphic simple groups $S_1, \ldots, S_t$ with universal p'-covers $\hat{S}_i$, integers $n_i \geq 1$ and an epimorphism

$$\pi : G := \hat{S}_1^{\,n_1} \times \cdots \times \hat{S}_t^{\,n_t} \to E,$$

with $K = \ker\pi \leq \mathbf{Z}(G)$, and $\pi(\mathbf{Z}(G)) = \mathbf{Z}(E)$. Via π the group $\mathrm{Aut}(E)$ can be identified with a subgroup of $\mathrm{Aut}(G/\mathbf{Z}(G)) = \mathrm{Aut}(G)$. Hence, we may work in G, and assume that $G = E$.

Now b is a p-block of G, and therefore there are blocks $b_{i,j}$ ($1 \leq j \leq n_i$) of $\hat{S}_i$, and natural bijections $\mathrm{Irr}(b_{1,1})\times\cdots\times\mathrm{Irr}(b_{t,n_t}) \to \mathrm{Irr}(b)$ and $\mathrm{IBr}(b_{1,1})\times\cdots\times\mathrm{IBr}(b_{t,n_t}) \to \mathrm{IBr}(b)$. Without loss of generality we may assume that any two $\mathrm{Aut}(\hat{S}_i)$-conjugate blocks $b_{i,j}$ and $b_{i,j'}$ are equal. Now, $\mathrm{Aut}(G) = \mathrm{Aut}(\hat{S}_1) \wr \mathfrak{S}_{n_1} \times \cdots \times \mathrm{Aut}(\hat{S}_t) \wr \mathfrak{S}_{n_t}$, and our assumption forces that all elements of $\mathrm{IBr}(b_{i,j})$ are $\mathrm{Aut}(\hat{S}_i)_{b_{i,j}}$-conjugate. By Theorem 2.1 there exist $\varphi_{i,j} \in \mathrm{IBr}(b_{i,j})$ and $\chi_{i,j} \in \mathrm{Irr}(b_{i,j})$ such that $(\chi_{i,j})^0 = \varphi_{i,j}$, and $I_{\mathrm{Aut}(\hat{S}_i)}(\chi_{i,j}) = I_{\mathrm{Aut}(\hat{S}_i)}(\varphi_{i,j})$. Then

$$\chi := \chi_{1,1} \times \cdots \times \chi_{t,n_t} \in \mathrm{Irr}(G)$$

is a lift of

$$\varphi_{1,1} \times \cdots \times \varphi_{t,n_t} \in \mathrm{IBr}(b)\,.$$

If $p \neq 2$ the characters $\chi_{i,j}$ and hence χ are p-rational and the claim follows from [14, Thm. (6.30)].

The following considerations prove the statement for $p = 2$. Let $P_{i,j}$ be a Sylow p-subgroup of $I_{\mathrm{Aut}(\hat{S}_i)}(\chi_{i,j})$. By Theorem 2.1(4) the character $\chi_{i,j}$ extends to some $\tilde{\chi}_{i,j} \in \mathrm{Irr}(\hat{S}_i \rtimes P_{i,j})$ such that $p \nmid |\mathbf{C}_{\hat{S}_i \rtimes P_{i,j}}(\hat{S}_i) : \ker(\tilde{\chi}_{i,j}|_{\mathbf{C}_{\hat{S}_i \rtimes P_{i,j}}(\hat{S}_i)})|$.

Without loss of generality we may assume that whenever two characters $\varphi_{i,j}$ and $\varphi_{i,j'}$ seen as characters of $\hat{S}_i$ are $\mathrm{Aut}(\hat{S}_i)$-conjugate they are equal, and that the same applies to the characters $\chi_{i,j}$ and $\chi_{i,j'}$. This implies that $\mathrm{Aut}(G)_\chi$ is of the form

$$\mathrm{Aut}(G)_\chi = \Big(\mathrm{Aut}(\hat{S}_1)_{\chi_{1,1}} \times \cdots \times \mathrm{Aut}(\hat{S}_t)_{\chi_{t,n_t}}\Big) \rtimes \mathcal{S},$$

where $\mathcal{S}$ is a direct product of symmetric group permuting some isomorphic factors of G. A Sylow p-subgroup of $G \rtimes \mathrm{Aut}(G)_\chi$ is contained in

$$A := \Big((\hat{S}_1 \rtimes P_{1,1}) \times \cdots \times (\hat{S}_t \rtimes P_{t,n_t})\Big) \rtimes \mathcal{S}.$$

Following [11, Thm. 25.6] the character

$$\tilde{\chi}_{1,1} \times \cdots \times \tilde{\chi}_{t,n_t} \in \mathrm{Irr}((\hat{S}_1 \rtimes P_{1,1}) \times \cdots \times (\hat{S}_t \rtimes P_{t,n_t}))$$

has an extension $\tilde{\chi}$ to A with $p \nmid |\mathbf{C}_{G\rtimes\mathrm{Aut}(G)_\chi}(G) : \ker(\tilde{\chi}|_{\mathbf{C}_{G\rtimes\mathrm{Aut}(G)_\chi}(G)})|$ by construction. □

Corollary 5.2. *Let G be a finite group and $E \lhd G$ be perfect with $E/\mathbf{Z}(E)$ a direct product of simple groups. Assume that $\mathbf{C}_G(E)$ has order prime to p. Let b be a p-block of E, and assume that $\mathrm{IBr}(b)$ consists of G-conjugates of φ. Then there exists some $\chi \in \mathrm{Irr}(E)$ such that $\chi^0 = \varphi$, the stabilisers $I_G(\chi) = I_G(\varphi) = I$ coincide, and χ extends to EP for $P \in \mathrm{Syl}_p(I)$. Moreover, if $p \neq 2$ the character χ can be chosen to be p-rational.*

Proof. We have that $G/\mathbf{C}_G(E)$ is isomorphic to a subgroup C of $A = \mathrm{Aut}(E)$. Now, $\mathrm{IBr}(b)$ consists of A-conjugates of φ. Therefore by Corollary 5.1, there exists $\chi \in \mathrm{Irr}(E)$ such that $\chi^0 = \varphi$, $I_A(\chi) = I_A(\varphi)$ and χ extends to some $\tilde{\chi} \in \mathrm{Irr}(E \rtimes P)$ for $P \in \mathrm{Syl}_p(I)$ such that $\ker(\tilde{\chi})$ contains a Sylow p-subgroup of $\mathbf{C}_{E\rtimes P}(E)$. Hence χ and φ have the same stabiliser in C and $I_G(\chi) = I_G(\varphi)$. By Lemma 2.2 we have that χ extends to EP. If $p \neq 2$ then χ can be chosen to be p-rational by Corollary 5.1. □

Lemma 5.3. *Suppose that $E \leq L$ are normal in G, with L/E solvable. Suppose that $\chi \in \mathrm{Irr}(E)$ is G-invariant, and that $\chi^0 = \varphi \in \mathrm{IBr}(E)$. Assume that χ extends to P, where P/E is a Sylow p-subgroup of G/E. Suppose that $\eta \in \mathrm{IBr}(L|\varphi)$ is G-invariant. Then there exists $\delta \in \mathrm{Irr}(L|\chi)$ such that $\delta^0 = \eta$, δ is G-invariant, and δ extends to Q, where $Q/L \in \mathrm{Syl}_p(G/L)$. Furthermore, if p is odd and χ is p-rational, then δ can be chosen to be p-rational, too.*

Proof. By Problem (8.12) of [24], we may assume that $E \leq \mathbf{Z}(G)$. Using that χ extends to a Sylow p-subgroup of G, we may assume that E is a p'-group. (Use the ideas of Problem (8.13) of [24].) We have now that L is p-solvable. In [15], Isaacs constructed a canonical subset $B_{p'}(L) \subseteq \mathrm{Irr}(L)$ with certain remarkable properties. Among others, restriction to p-regular elements of L provides a canonical bijection $B_{p'}(L) \to \mathrm{IBr}(L)$ (by [15, Cor. (10.3)]). Also, by definition, characters in $B_{p'}(L)$ are p-rational. Now, let $\delta \in \mathrm{Irr}(L)$ be the character in $B_{p'}(L)$ with $\delta^0 = \eta$. Since E is a p'-group, we have that δ lies over χ. Since η is G-invariant, then δ is G-invariant (again using [15, Cor. (10.3)]). Since $\delta \in B_{p'}(L)$, then δ extends to Q by [14, Cor. (6.3)]. If p is odd and χ is p-rational, then we use the same argument as before, but applying Theorem 4.5. ☐

We are now ready to prove the main theorem, which we restate:

Theorem 5.4. *Let G be a finite group and let p be a prime. Assume that B is a p-block of G with $\mathrm{IBr}(B) = \{\varphi\}$. Then there exists a character $\chi \in \mathrm{Irr}(G)$ with $\chi^0 = \varphi$ and χ is p-rational if $p \neq 2$.*

Proof. We argue by induction on $|G : \mathbf{Z}(G)|$. We may assume that $\mathbf{O}_p(G) = 1$ since $\mathbf{O}_p(G) \subseteq \ker(\varphi)$. By the Fong–Reynolds Theorem [24, Thm. (9.14)]), we may assume that B is quasi-primitive, i.e. every block of every normal subgroup of G covered by B is G-invariant.

In particular, when $N \lhd G$ and B covers $\{\theta\}$, where $\theta \in \mathrm{Irr}(N)$ has defect zero, then θ is G-invariant. According to Theorem 4.5 we may assume that N is a central p'-subgroup of G after using ordinary-modular p-rational triples. Thus, if S is the largest solvable normal subgroup of G, we have that the Fitting subgroup $\mathbf{F}(S)$ is central, and therefore $S = Z := \mathbf{Z}(G)$.

Let b be the block of $E = \mathbf{E}(G)$, the group of components of G, covered by B. Then $\mathrm{IBr}(b)$ consists of G-conjugates of some $\theta \in \mathrm{IBr}(b)$. Let $Z_0 := \mathbf{Z}(E) = Z \cap E$.

Write $E/Z_0 = E_1/Z_0 \times \cdots \times E_t/Z_0$, where E_i/Z_0 is the direct product of G-conjugates of distinct simple groups S_i/Z_0 and $E_i \lhd G$. Recall that $[E_i, E_j] = 1$ if $i \neq j$. Let D be a defect group of b. We claim that $E\mathbf{C}_G(D)/E$ is solvable. Write $DZ_0/Z_0 = D_1Z_0/Z_0 \times \cdots \times D_tZ_0/Z_0$, where $D_i = D \cap E_i$. We have that D_i is a defect group of the G-invariant block b_i of $E_i \lhd G$ covered by b. By our assumption, we have that $D_i > 1$ for every i, since B does not cover defect zero blocks of non-central normal subgroups. Also, $G = E_i\mathbf{N}_G(D_i)$ by the Frattini argument. Let U/Z_0 be a simple factor of E_i/Z_0, so that

$$E_i/Z_0 = (U/Z_0)^{x_1} \times \cdots \times (U/Z_0)^{x_t}\,,$$

for some $x_i \in \mathbf{N}_G(D_i)$. Now, it is well-known that

$$D_i = (D_i \cap U)^{x_1} \cdots (D_i \cap U)^{x_r}$$

and we see that $D_i \cap U > 1$. Since $x \in C := \mathbf{C}_G(D) \leq \mathbf{C}_G(D_i)$ centralises $D_i \cap U > 1$, it follows that x normalises U (since G transitively permutes the simple factors of E_i/Z_0). We know that $\mathbf{N}_G(U)/U\mathbf{C}_G(U)$ as subgroup of an outer automorphism group of a non-abelian simple group is solvable (Schreier's conjecture). Hence $U\mathbf{C}_G(U) \cap C \lhd C$ and its quotient is solvable. Since this is true for all components U of G, we have that

$$C/(C \cap \bigcap_U U\mathbf{C}_G(U))$$

is solvable, where U runs over the components of G. Notice that

$$\bigcap_U U\mathbf{C}_G(U) = \mathbf{F}^*(G)\,,$$

as every component commutes with every other component and with every normal solvable subgroup.

Now, let $M = EC = \mathbf{F}^*(G)C \lhd G$, where M/E is solvable. (Recall that $G = E\mathbf{N}_G(D)$ by the Frattini argument.) Let e be the unique block of M covered by B. Let $\eta \in \mathrm{IBr}(M)$ under φ, and let $\theta \in \mathrm{IBr}(b)$ under η. Necessarily $\eta \in \mathrm{IBr}(e)$. Now, if Q is a defect group of e, then we may assume that Q contains D, and therefore $\mathbf{C}_G(Q) \leq \mathbf{C}_G(D) \leq M$. Therefore, B is the only block of G covering e by [26, Lemma 3.1].

Now, let T be the stabiliser of η in G, and let I be the stabiliser of θ in G. By the Frattini argument, we have that $T = M(I \cap T)$. Let $\nu \in \mathrm{IBr}(T|\eta)$ be the Clifford correspondent of φ over η. Now, if $\delta \in \mathrm{IBr}(T|\eta)$, we have that $\delta^G \in \mathrm{IBr}(G)$. Notice that δ^G lies in a block that covers e by [24, Thm. (9.2)], but this must be B. Hence $\delta^G = \varphi$ by hypothesis, and we have that $\delta = \nu$, by the uniqueness in the Clifford correspondence. Hence, η is fully ramified in T/M. Now, let $\nu' \in \mathrm{IBr}(I \cap T|\theta)$ and $\eta' \in \mathrm{IBr}(I \cap M|\theta)$ be the Clifford correspondents of ν and η over θ, respectively. Then,

$$(\nu(1)/\eta(1))\eta = \nu_M = ((\nu')^T)_M = ((\nu')_{I \cap M})^M\,,$$

and we deduce that ν' is fully ramified with respect to η'.

By hypothesis, all elements in $\mathrm{IBr}(b)$ are G-conjugate. Thus, by Corollary 5.2, there are $\theta_1 \in \mathrm{IBr}(b)$ and $\chi \in \mathrm{Irr}(b)$ such that $\chi^0 = \theta_1$, with same stabiliser I_1 in G, and such that χ extends to a Sylow p-subgroup P/E of I_1/E. If $p \neq 2$, χ is in addition p-rational. Since all elements of $\mathrm{IBr}(b)$ are G-conjugate, we may assume that $\chi^0 = \theta$, that χ is I-invariant, and that χ extends to P, where $P/E \in \mathrm{Syl}_p(I/E)$.

Now, by Lemma 5.3, there exists $\eta_1 \in \mathrm{Irr}(I \cap M)$ such that $(\eta_1)^0 = \eta'$, η_1 is $I \cap T$-invariant, and η_1 extends to Q, where $Q/(I \cap M) \in \mathrm{Syl}_p((I \cap T)/(I \cap M))$. Now, as in the proof of Lemma 5.3, there exists a group X with a central p'-subgroup Y, with a linear character $\lambda \in \mathrm{Irr}(Y)$ such that the triples $(I \cap T, I \cap M, \eta_1)$ and (X, Y, λ) are isomorphic as ordinary-modular triples. If $p \neq 2$ the isomorphism of character triples has the properties from Theorem 4.5. Now, notice that if $\lambda^X = f\delta$, with $f \geq 1$ and $\delta \in \mathrm{IBr}(X)$, and B' is the block of δ, then $\mathrm{IBr}(B') = \{\delta\}$ (see, for instance, the proof of [25, Thm. 2.1]). Since $|X : \mathbf{Z}(X)| < |G : \mathbf{Z}(G)|$, we have that δ is liftable by induction and liftable to a p-rational character in case of $p \neq 2$. Hence, we have that ν' is liftable, even to a p-rational character for $p \neq 2$. Thus, there exists $\rho \in \mathrm{Irr}(I \cap T)$ such that $\rho^0 = \nu'$. Now, $(\nu')^T = \nu$ and thus $\nu^G = \varphi$, so $\nu'^G = \varphi$, and $(\rho^G)^0 = \nu'^G = \varphi$. We therefore deduce that ρ^G is a lift of φ and p-rational if $p \neq 2$. $\square$

References

[1] D. Benson, *Representations and Cohomology.* Volume 1, Cambridge University Press, Cambridge, 1995.

[2] C. Bonnafé, On a theorem of Shintani. J. Algebra **218** (1999), 229–245.

[3] M. Broué, L. Puig, A Frobenius theorem for blocks. Invent. Math. **56** (1980), 117–128.

[4] M. Cabanes, M. Enguehard, On blocks of finite reductive groups and twisted induction. Adv. Math. **145** (1999), 189–229.

[5] M. Cabanes, M. Enguehard, *Representation Theory of Finite Reductive Groups.* Volume 1 of New Mathematical Monographs, Cambridge University Press, Cambridge, 2004.

[6] M. Cabanes, B. Späth, Equivariant character correspondences and inductive McKay condition for type A. J. Reine Angew. Math., to appear. DOI: 10.1515/crelle-2014-0104

[7] F. Digne, J. Michel, *Representations of Finite Groups of Lie Type.* London Math. Soc. Student Texts 21. Cambridge University Press, Cambridge, 1991.

[8] M. Enguehard, Sur les l-blocs unipotents des groupes réductifs finis quand l est mauvais. J. Algebra **230** (2000), 334–377.

[9] M. Geck, G. Hiss, Basic sets of Brauer characters of finite groups of Lie type. J. Reine Angew. Math. **418** (1991) 173–188.

[10] M. Geck, G. Malle, On the existence of a unipotent support for the irreducible characters of a finite group of Lie type. Trans. Amer. Math. Soc. **352** (2000), 429–456.

[11] B. Huppert, *Character Theory of Finite Groups.* De Gruyter, Berlin, 1998.

[12] M. Isaacs, Characters of solvable and symplectic groups. Amer. J. Math **95** (1973), 594–635.

[13] M. Isaacs, Lifting Brauer characters of p-solvable groups, Pacific J. Math. **53** (1974), 171–188.

[14] M. Isaacs, *Character Theory of Finite Groups.* Academic Press, 1976.

[15] M. Isaacs, Characters of π-separable groups. J. Algebra **86** (1984), 98–128.

[16] M. Isaacs, G. Malle, G. Navarro, A reduction theorem for the McKay conjecture. Invent. Math. **170** (2007), 33–101.

[17] R. Kessar, On blocks stably equivalent to a quantum complete intersection of dimension 9 in characteristic 3 and a case of the Abelian defect group conjecture. J. London Math. Soc. **85** (2012), 491–510.

[18] R. Kessar, M. Linckelmann, On stable equivalences and blocks with one simple module. J. Algebra **323** (2010), 1607–1621.

[19] R. Kessar, G. Malle, Quasi-isolated blocks and Brauer's height zero conjecture. Annals of Math. **178** (2013), 321–384.

[20] R. Kessar, G. Malle, Lusztig induction and ℓ-blocks of finite reductive groups. Pacific J. Math, to appear, 2015.

[21] B. Külshammer, L. Puig, Extensions of nilpotent blocks. Invent. Math. **102** (1990), 17–71.

[22] G. Malle, Die unipotenten Charaktere von ${}^2F_4(q^2)$. Comm. Algebra **18** (1990), 2361–2381.

[23] G. Malle, G. Navarro, Blocks with equal height zero degrees. Trans. Amer. Math. Soc. **363** (2011), 6647–6669.

[24] G. Navarro, *Blocks and Characters of Finite Groups.* Cambridge University Press, Cambridge, 1998.

[25] G. Navarro, B. Späth, P. H. Tiep, On fully ramified Brauer characters. Adv. Math. **257** (2014), 248–265.

[26] G. Navarro, P. H. Tiep, Brauer's height zero conjecture for the 2-blocks of maximal defect. J. Reine Angew. Math. **669** (2012), 225–247.

[27] G. Navarro, P.H. Tiep, A. Turull, Brauer characters with cyclotomic field of values. J. Pure Appl. Algebra **212** (2008), 628–635.

[28] J. B. Olsson, *Combinatorics and Representations of Finite Groups.* Universität Essen, Fachbereich Mathematik, Essen, 1993.

[29] L. Puig, *On the Local Structure of Morita and Rickard Equivalences Between Brauer Blocks.* Progress in Mathematics, 178. Birkhäuser Verlag, Basel, 1999.

[30] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*; 2004. http://www.gap-system.org.

[31] H. N. Ward, On Ree's series of simple groups. Trans. Amer. Math. Soc. **121** (1966), 62–89.

FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany.
E-mail address: malle@mathematik.uni-kl.de

Departament d'Àlgebra, Universitat de València, 46100 Burjassot, València, Spain.
E-mail address: gabriel@uv.es

FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany.
E-mail address: spaeth@mathematik.uni-kl.de