THE DECOMPOSITION OF LUSZTIG INDUCTION IN CLASSICAL GROUPS

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To Michel, for his vision and his kindness

Abstract. We give a short combinatorial proof of Asai's decomposition formula for Lusztig induction of unipotent characters in groups of classical type, relying solely on the Mackey formula.

1. Introduction

Lusztig induction is a powerful tool in the representation theory of finite reductive groups. A combinatorial formula for the decomposition of Lusztig induction of unipotent characters of classical groups was presented in the 1980's by Fong and Srinivasan [7, (3.1), (3.2)], distilled from a series of papers of Asai [1, 2.8], [2, 1.5] and [3, 2.2.3] running over a total of more than 200 pages. This decomposition formula has come to play a fundamental role in the block theory of finite reductive groups, for example as a crucial ingredient in Broué, Malle and Michel's determination of generic blocks [4], and thus eventually for example also in the proof of one direction of Brauer's height zero conjecture [9].

Asai's proofs are somewhat difficult to follow and seem to be not entirely correct. In this article we sketch a new proof of his result using only the Mackey formula for Lusztig induction; the details of a purely combinatorial nature can easily be filled in and will be presented in full in [8].

Theorem 1 (Asai)**.** *Let $\mathbf{G}$ be simple of type B_n, C_n or D_n with a Frobenius map F not inducing triality. For S a symbol we write ρ_S for the corresponding unipotent character if S is not degenerate, respectively for the sum of the two corresponding characters if S is.*

(a) *Let d be odd and $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T}_d)$ where $\mathbf{T}_d \leq \mathbf{G}$ is an F-stable torus with $|\mathbf{T}_d^F| = q^d - 1$. Then for $\rho_S \in \mathrm{Uch}(\mathbf{L}^F)$ we have*

$$R_{\mathbf{L}}^{\mathbf{G}}(\rho_S) = \sum_{(S',h)} \epsilon_h \, \rho_{S'} \, ,$$

where the sum runs over all symbols S' with a d-hook h such that $S = S' \setminus h$.

(b) *Let $d \geq 1$ and $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T}_d)$ where $\mathbf{T}_d \leq \mathbf{G}$ is an F-stable torus with $|\mathbf{T}_d^F| = q^d + 1$. Then for $\rho_S \in \mathrm{Uch}(\mathbf{L}^F)$ we have*

$$R_{\mathbf{L}}^{\mathbf{G}}(\rho_S) = (-1)^{\delta} \sum_{(S',c)} \epsilon_c \, \rho_{S'} \, ,$$

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where the sum runs over all symbols S' with a d-cohook c such that $S = S' \setminus c$, and $\delta = 0$ for types B_n, C_n, $\delta = 1$ for type D_n.

The notions of symbols, hooks, cohooks, as well as the signs ϵ_h and ϵ_c will be defined in Section 2.

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2. Setting and preliminary considerations

Our general setting will be the following. Let $\mathbf{G}$ be a simple algebraic group of classical type, $F : \mathbf{G} \to \mathbf{G}$ a Frobenius map not inducing triality, and $\mathbf{G}^F$ the group of fixed points, a finite group of classical Lie type. Since we will be concerned solely with the unipotent characters of $\mathbf{G}^F$, and the latter are known to be trivial on the centre and invariant under diagonal automorphisms, the precise isogeny type of $\mathbf{G}$ will not matter, and we can choose, for example, to work with $\mathbf{G}$ one of SO_{2n+1} (type B_n), Sp_{2n} (type C_n) or SO_{2n} (type D_n), so that $\mathbf{G}^F$ is one of $\mathrm{SO}_{2n+1}(q)$, $\mathrm{Sp}_{2n}(q)$, $\mathrm{SO}_{2n}^+(q)$ or $\mathrm{SO}_{2n}^-(q)$, for q an arbitrary prime power.

Following Lusztig the unipotent characters of $\mathbf{G}^F$ as above are parametrised by equivalence classes of symbols of rank n, of odd defect when $\mathbf{G}$ has type B_n or C_n, of defect divisible by 4 when $\mathbf{G}^F = \mathrm{SO}_{2n}^+(q)$, and of defect congruent to 2 modulo 4 when $\mathbf{G}^F = \mathrm{SO}_{2n}^-(q)$ (see, e.g., [8, §4.4]). Here, by convention, a degenerate symbol (that is, a symbol with two equal rows, which can only occur for untwisted type D_n) labels two distinct unipotent character.

Let $S = (X, Y)$ be a symbol. A d-hook of S is an element $d \le x \in X$ with $x - d \notin X$, or an element $x \in Y$ with $x - d \notin Y$. Attached to it is the sign

$$\epsilon_h := (-1)^m \quad \text{where } m := \begin{cases} |\{y \in X \mid x - d < y < x\}| & \text{if } x \in X, \\ |\{y \in Y \mid x - d < y < x\}| & \text{if } x \in Y. \end{cases}$$

A d-cohook of S is an $x \in X$ with $x - d \notin Y$, or $x \in Y$ with $x - d \notin X$. Associated to it is the sign

$$\epsilon_c := (-1)^m \ \text{ where } m := \begin{cases} |\{y \in X \mid y < x\}| + |\{y \in Y \mid y < x - d\}| & \text{if } x \in X, \\ |\{y \in Y \mid y < x\}| + |\{y \in X \mid y < x - d\}| & \text{if } x \in Y. \end{cases}$$

Let us recall a Murnaghan–Nakayama type rule for restriction of irreducible characters in certain wreath products. The complex reflection group denoted $G(b, 1, m)$ is isomorphic to the wreath product of the cyclic group of order b with the symmetric group $\mathfrak{S}_m$. The irreducible characters of $G(b, 1, m)$ are parametrised by b-multi-partitions $\underline{\alpha} = (\alpha_1, \ldots, \alpha_b)$ of m. A hook of $\underline{\alpha}$ will mean a hook of one of the α_i. For h a hook, $f(h)$ denotes its foot length. The following generalised Murnaghan–Nakayama rule was shown by Osima [10, Thm. 7].

Proposition 2. *Let $b, e \geq 1$, $\phi^{\underline{\alpha}} \in \mathrm{Irr}(G(b,1,m))$ be parametrised by the b-multi-partition $\underline{\alpha} \vdash m$, $x \in \mathfrak{S}_d$ an e-cycle and $y \in G(b,1,m-e))$. Then*

$$\phi^{\underline{\alpha}}(xy) = \sum_h (-1)^{f(h)} \phi^{\underline{\alpha}\setminus h}(y)$$

where the sum runs over all e-hooks h of $\underline{\alpha}$ and $\underline{\alpha} \setminus h$ denotes the multi-partition obtained by removing h from $\underline{\alpha}$.

3. The case of odd d

The first observation to make is that part (a) of the theorem can be proved only using Harish-Chandra theory and in particular without relying on the Mackey formula.

So, in this section we assume that d is an odd integer. Consider the d-split Levi subgroup $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}\mathbf{G}'$ of $\mathbf{G}$ where $\mathbf{T}$ is an F-stable torus of $\mathbf{G}$ with $\mathbf{T}^F \cong \mathrm{GL}_1(q^d)$ and $\mathbf{G}'$ is of the same type as $\mathbf{G}$ but of rank $n-d$. Let $\mathbf{L}_1 = \mathrm{GL}_d\,\mathbf{G}'$ be the intermediate 1-split Levi subgroup of $\mathbf{G}$ with $\mathbf{T} \leq \mathrm{GL}_d$. By transitivity of Lusztig restriction, ${}^*R_{\mathbf{L}}^{\mathbf{G}}(\rho) = {}^*R_{\mathbf{L}}^{\mathbf{L}_1}{}^*R_{\mathbf{L}_1}^{\mathbf{G}}(\rho)$ for any class function ρ on $\mathbf{G}^F$.

Proof of Theorem 1(a). Let $\rho \in \mathrm{Uch}(\mathbf{G}^F)$, labelled by the symbol S. Let $(\mathbf{L}_0, \lambda_0)$ be a Harish-Chandra source of ρ, that is, $\mathbf{L}_0$ is a 1-split Levi subgroup of $\mathbf{G}$ and $\lambda_0 \in \mathrm{Uch}(\mathbf{L}_0^F)$ is cuspidal with $\langle \rho, R_{\mathbf{L}}^{\mathbf{G}}(\lambda_0)\rangle \neq 0$. Then $[\mathbf{L}_0, \mathbf{L}_0]$ is simple of the same classical type as $\mathbf{G}$. If $\mathrm{rnk}([\mathbf{L}_0, \mathbf{L}_0]) > n-d$ then fewer than d 1-hooks can be removed successively from S, and hence S cannot have a d-hook. Moreover, Moreover, $\mathbf{L}_0$ cannot be contained in $\mathbf{L}$, so ${}^*R_{\mathbf{L}}^{\mathbf{G}}(\rho) = 0$ by the Mackey formula for Harish-Chandra induction, in accordance with the claim.

Else, we may assume after conjugation that $\mathbf{L}_0$ is contained in the maximal 1-split Levi subgroup $\mathbf{L}_1$. Let S_0 denote the symbol labelling λ_0 and first assume that S_0 is non-degenerate. Let $W_0 := W_{\mathbf{G}}(\mathbf{L}_0, \lambda_0) \cong W(B_{n-k})$ be the relative Weyl group of $(\mathbf{L}_0, \lambda_0)$ in $\mathbf{G}$, and $W_1 := W_{\mathbf{L}_1}(\mathbf{L}_0, \lambda_0) \cong \mathfrak{S}_d \times W(B_{n-k-d})$ the one in $\mathbf{L}_1$. The irreducible characters of W_0 are parametrised by bipartitions $\underline{\alpha} \vdash n-k$. Let S_0 denote the symbol parametrising λ_0. By [8, Prop. 4.4.29] addition of S_0 defines the natural labelling

$$\mathrm{Irr}(W_0) \to \mathcal{E}(\mathbf{G}^F, (\mathbf{L}_0, \lambda_0)), \quad \phi^{\underline{\alpha}} \mapsto \rho_{\underline{\alpha}},$$

of the Harish-Chandra series above λ_0, with $\rho_{\underline{\alpha}}$ denoting the unipotent character labelled by the symbol $S_0 + \underline{\alpha}$. Any $\psi \in \mathrm{Irr}(W_1)$ is of the form $\psi_1 \boxtimes \psi_2$ with $\psi_1 \in \mathrm{Irr}(\mathfrak{S}_d)$ and $\psi_2 \in \mathrm{Irr}(G(2,1,n-k-d))$, and correspondingly we obtain a parametrisation $\mathrm{Irr}(W_1) \to \mathcal{E}(\mathbf{L}_1^F, (\mathbf{L}_0, \lambda_0))$, $\psi_1 \boxtimes \psi_2 \mapsto \rho_{\psi_1} \boxtimes \rho_{\underline{\alpha}}$, if ψ_2 is labelled by $\underline{\alpha} \vdash n-k-d$.

Let ρ correspond to the character $\phi \in \mathrm{Irr}(W_0)$. By the Howlett–Lehrer Comparison Theorem the decomposition of ${}^*R_{\mathbf{L}_1}^{\mathbf{G}}(\rho)$ is given by

$$ {}^*R_{\mathbf{L}_1}^{\mathbf{G}}(\rho) = \sum_{\psi} a_\psi \rho_\psi \quad \text{where} \quad \phi|_{W_1} = \sum_{\psi \in \mathrm{Irr}(W_1)} a_\psi\, \psi.$$

Now ${}^*R_{\mathbf{L}}^{\mathbf{L}_1}(\rho_{\psi_1} \boxtimes \rho_{\psi_2}) = {}^*R_{\mathbf{T}}^{\mathrm{GL}_d}(\rho_{\psi_1}) \boxtimes \rho_{\psi_2}$, and ${}^*R_{\mathbf{T}}^{\mathrm{GL}_d}(\rho_{\psi_1})$ is non-zero if and only if ψ_1 does not vanish on the d-cycles, that is, if and only if it is parametrised by a hook partition. Thus we may set $a_\psi = 0$ unless $\psi_1(x) \neq 0$ for $x \in \mathfrak{S}_d$ a d-cycle. But then Proposition 2 applied to $W_0 \cong G(2,1,n-k)$ shows that $a_\psi = (-1)^{f(h)}$ if ψ_2 is obtained from ϕ by removing the d-hook h, and $a_\psi = 0$ else. Since removing d-hooks commutes with addition of S_0, our claim follows.

If S_0 is degenerate then, as λ_0 is 1-cuspidal, it must be the trivial character of $\mathbf{L}_0^F$. Here, $W_0 \cong W(D_{n-k})$ and $W_1 \cong \mathfrak{S}_d \times W(D_{n-k-d})$. Again, by [8, Prop. 4.4.29] the Harish-Chandra series of $\mathbf{G}^F$ above $(\mathbf{L}_0, \lambda_0)$ is naturally labelled by $\mathrm{Irr}(W_0)$, and the one of $\mathbf{L}_1^F$ by $\mathrm{Irr}(W_1)$. The precisely same argument as before then gives the result, noting that Theorem 1 only makes a claim about sums of characters labelled by degenerate symbols. □

4. Almost characters

Part (b) requires more work. It ultimately relies on the fact that unipotent characters of classical groups, and even sums of two such, are determined by their uniform projections.

Theorem 3. *Let $\mathbf{G}$ be simple of classical type and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius map. Then:*
(a) *The unipotent characters of $\mathbf{G}^F$ are uniquely determined by their uniform projections.*
(b) *Let $\mathcal{U} \subset \mathrm{Uch}(\mathbf{G}^F)$ be a family with $|\mathcal{U}| > 4$. Then the various sums of two distinct unipotent characters from $\mathcal{U}$ are uniquely determined by their uniform projections.*

Proof. Since the characteristic function of the identity element is uniform, in order to prove (a) it suffices to see that unipotent characters are uniquely determined in their family by their degrees. Let ρ be labelled by a symbol $S = (X, Y)$. The degree formula (see [8, Prop. 4.4.15]) shows that it would be enough to see that

$$\prod_{\{i<j\}\subseteq X} (q^j - q^i) \prod_{\{i<j\}\subseteq Y} (q^j - q^i) \prod_{(i,j)\in X\times Y} (q^i + q^j)$$

distinguishes characters in a family. Now inside a family the Fourier matrix as well as the set of columns indexed by almost characters in the family do not depend on the exact values of the multi-set $X \cup Y$ of entries of the symbols in the family, only on their number. Thus, it suffices to show the above assertion in just one family with a given Fourier, for which we may choose these entries arbitrarily. Since the powers of q are Zariski dense in the integers, it thus suffices to show that the polynomial

$$\prod_{\{i<j\}\subseteq X} (Z_j - Z_i) \prod_{\{i<j\}\subseteq Y} (Z_j - Z_i) \prod_{(i,j)\in X\times Y} (Z_i + Z_j) \qquad (*)$$

with indeterminates Z_i, determines $\{X, Y\}$. But this is evident.

For (b) set $I = X \cup Y$ and rewrite the polynomial in (*) as

$$f(X, Y) := \prod_{\{i<j\}\subseteq I} (Z_j - \epsilon_{ij} Z_i)$$

with $\epsilon_{ij} := -1$ if $(i, j) \in X \times Y \cup Y \times X$ and $\epsilon_{ij} := 1$ otherwise. For $S' = (X', Y')$ the symbol of a second unipotent character in the family we have

$$f(X', Y') = \prod_{\{i<j\}\subseteq I} (Z_j - \epsilon'_{ij} Z_i)$$

with ϵ'_{ij} defined accordingly. We claim that $f(X, Y) + f(X', Y')$ uniquely determines the set $\{\{X, Y\}, \{X', Y'\}\}$. If $k, l \in I$ with $\epsilon_{kl} = \epsilon'_{kl}$ then the specialisation of both $f(X, Y)$

and $f(X',Y')$ and hence of their sum at $Z_k = \epsilon_{kl}Z_l$ vanishes. On the other hand, if $\epsilon_{kl} \neq \epsilon'_{kl}$ then the specialisation of $f(X,Y) + f(X',Y')$ at $Z_k = \pm\epsilon_{kl}Z_l$ gives

$$f(X',Y')|_{Z_k=\epsilon_{kl}Z_l} \neq 0, \quad \text{respectively} \quad f(X,Y)|_{Z_k=-\epsilon_{kl}Z_l} \neq 0.$$

From these we can determine the sets X, Y and X', Y' up to the positions of k, l. Observe that $|I| \geq 5$ since $|\mathcal{U}| > 4$; but then if $S \neq S'$ there are at least three distinct $k < l$ with $\epsilon_{kl} \neq \epsilon'_{kl}$, so the above allows us to unambiguously retrieve $\{\{X,Y\},\{X',Y'\}\}$. □

While part (a) (if not with this proof) is well-known [5, Prop. 6.3], part (b) seems to be new.

5. Norm and uniform projection of $R_{\mathbf{L}}^{\mathbf{G}}(\rho)$

For simplicity we will assume that we are in the setting of part (b) of the theorem, but similar arguments apply to part (a).

Let $d \geq 1$. We write $a_d(S)$ for the number of d-cohooks that can be added to a symbol S, and $r_d(S)$ for the number of d-cohooks that can be removed from S. We reinterpret these numbers in terms of an abacus diagram. The two rows of $S = (X_0, X_1)$ can naturally be encoded in a $2d$-runner abacus diagram A as follows: the ith runner of A has a bead at position j if $X_{i+j \pmod 2}$ has an entry $\lfloor i/2 \rfloor + dj$. Adding a d-cohook to S corresponds to moving one bead of A one position down. Thus $a_d(S)$ equals the number of beads in A for which the next position downwards is empty. An easy discussion now shows that

$$a_d(S) = 2d + r_d(S).$$

Proposition 4. *Let $\mathbf{L} \leq \mathbf{G}$ be a $2d$-split Levi subgroup and $\rho \in \mathrm{Uch}(\mathbf{L}^F)$ parametrised by the symbol S. Then $||R_{\mathbf{L}}^{\mathbf{G}}(\rho)||^2 = a_d(S)$ is as claimed by Asai's formula.*

Proof. Assume $\mathbf{G}$ is of type B_n or C_n. We apply the Mackey formula for d-split Levi subgroups (see [4, Thm. 1.35(2)]). Let $\mathbf{M}$ be a minimal d-split Levi subgroup contained in $\mathbf{L}$. Then

$$\langle R_{\mathbf{L}}^{\mathbf{G}}(\rho), R_{\mathbf{L}}^{\mathbf{G}}(\rho)\rangle = \langle \rho, {}^*R_{\mathbf{L}}^{\mathbf{G}}(R_{\mathbf{L}}^{\mathbf{G}}(\rho))\rangle = \langle \rho, \sum_w R_{\mathbf{L}\cap {}^w\mathbf{L}}^{\mathbf{L}}({}^*R_{\mathbf{L}\cap {}^w\mathbf{L}}^{{}^w\mathbf{L}}({}^w\rho))\rangle$$

where the sum is over $W_{\mathbf{L}}(\mathbf{M})$–$W_{\mathbf{L}}(\mathbf{M})$ double coset representatives w in $W_{\mathbf{G}}(\mathbf{M})$. Here, $W_{\mathbf{G}}(\mathbf{M})$ is the complex reflection group $G(2d,1,m)$, where $dm = \mathrm{rnk}(\mathbf{G}) - \mathrm{rnk}(\mathbf{M})$, and $W_{\mathbf{L}}(\mathbf{M})$ is its maximal parabolic subgroup $G(2d,1,m-1)$. If $m = 1$ then $G(2d,1,1)$ is cyclic of order $2d$ and there are $2d$ double coset representatives for the trivial group in $G(2d,1,1)$, each with ${}^w\mathbf{L} = \mathbf{L}$, leading to $\langle R_{\mathbf{L}}^{\mathbf{G}}(\rho), R_{\mathbf{L}}^{\mathbf{G}}(\rho)\rangle = 2d$.

Now assume that $m \geq 2$. It is easy to see that then there are exactly $2d+1$ double coset representatives $w_0, \ldots, w_{2d}$, with $W_{\mathbf{L}}(\mathbf{M}) \cap {}^{w_i}W_{\mathbf{L}}(\mathbf{M}) = W_{\mathbf{L}}(\mathbf{M})$ for $i = 1, \ldots, 2d$ and $W_{\mathbf{L}}(\mathbf{M}) \cap {}^{w_0}W_{\mathbf{L}}(\mathbf{M}) = G(2d,1,m-2)$, so $\mathbf{L}_0 := \mathbf{L} \cap {}^{w_0}\mathbf{L}$ is a maximal $2d$-split Levi subgroup of $\mathbf{L}$. Thus we find

$$\langle R_{\mathbf{L}}^{\mathbf{G}}(\rho), R_{\mathbf{L}}^{\mathbf{G}}(\rho)\rangle = 2d\langle \rho, \rho\rangle + \langle {}^*R_{\mathbf{L}_0}^{\mathbf{L}}(\rho), {}^*R_{\mathbf{L}_0}^{\mathbf{L}}(\rho)\rangle = 2d + r_d(S)$$

by induction, which is our claim by the considerations prior to this proposition.

The above arguments remain valid for $\mathbf{G}$ of type D_n, except when ρ lies in the $2d$-Harish-Chandra series above a $2d$-cuspidal unipotent character parametrised by a degenerate symbol. This case can be handled by an easy adaptation of the previous argument. □

We consider the $2d$-split Levi subgroup $\mathbf{L} = \mathbf{T}\mathbf{G}' = C_{\mathbf{G}}(\mathbf{T})$ of $\mathbf{G}$ where $\mathbf{T}$ is an F-stable torus with $\mathbf{T}^F \cong \mathrm{GU}_1(q^d)$ and $\mathbf{G}'$ is of the same type as $\mathbf{G}$ of rank $n-d$. Let $\mathbf{L}_1 = \mathrm{GL}_d\,\mathbf{G}'$ be the intermediate F-stable Levi subgroup of $\mathbf{G}$ with $\mathbf{T} \leq \mathrm{GL}_d$ and $\mathrm{GL}_d^F = \mathrm{GU}_d(q)$. We first determine Lusztig restriction on uniform almost characters:

Lemma 5. *In the above setting, let $\phi \in \mathrm{Irr}(\mathbf{W})^F$ be parametrised by $\underline{\alpha}$ and R_ϕ the associated unipotent almost character of $\mathbf{G}^F$. Then*

$$^*R_{\mathbf{L}}^{\mathbf{G}}(R_\phi) = \sum_h (-1)^{f(h)} R_{\phi^{\underline{\alpha}\setminus h}},$$

where the sum runs over all d-hooks h of $\underline{\alpha}$.

Proof. The Levi subgroup $\mathbf{L}_1$ has Weyl group $\mathbf{W}_1 \cong \mathfrak{S}_d \times G(2,1,n-d)$ and $\mathbf{L}$ has Weyl group $\mathbf{W}_{\mathbf{L}} \cong G(2,1,n-d)$. Then $^*R_{\mathbf{L}_1}^{\mathbf{G}}(R_\phi) = R_{\phi_1}$, with $\phi_1 = \phi|_{\mathbf{W}_1}$, and by transitivity of Lusztig induction

$$^*R_{\mathbf{L}}^{\mathbf{G}}(R_\phi) = {}^*R_{\mathbf{L}}^{\mathbf{L}_1}\,{}^*R_{\mathbf{L}_1}^{\mathbf{G}}(R_\phi) = {}^*R_{\mathbf{L}}^{\mathbf{L}_1}(R_{\phi_1}) = R_{\phi'},$$

with ϕ' the restriction of ϕ_1 to the product $C \times \mathbf{W}_1$, with C the class of d-cycles in $\mathfrak{S}_d$. Applying Proposition 2 we thus find

$$\phi' = \sum_h (-1)^{f(h)} \phi^{\underline{\alpha}\setminus h},$$

where the sum runs over all d-hooks h of $\underline{\alpha}$, from which the claim follows. □

Proposition 6. *The uniform projection of Theorem 1(b) holds.*

Proof. Let $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ be parametrised by the symbol S. We write $T \sim S$ if T is the symbol of some $\phi \in \mathrm{Irr}(\mathbf{W})$ lying in the family of ρ. As uniform projection commutes with Lusztig restriction,

$$\pi_{\mathrm{un}}({}^*R_{\mathbf{L}}^{\mathbf{G}}(\rho)) = {}^*R_{\mathbf{L}}^{\mathbf{G}}(\pi_{\mathrm{un}}(\rho)) = \sum_{T\sim S} \langle S, T\rangle\, {}^*R_{\mathbf{L}}^{\mathbf{G}}(R_T)$$

where R_T is the uniform almost character labelled by T and $\langle S,T\rangle$ is the Fourier coefficient. By Lemma 5 this equals

$$\sum_{T\sim S} \langle S, T\rangle \sum_h (-1)^{f(h)} R_{T\setminus h} \tag{1}$$

with the inner sum ranging over all d-hooks h of T. By our claim, this should equal

$$\pi_{\mathrm{un}}((-1)^\delta \sum_c \epsilon_c \rho_{S\setminus c}) = (-1)^\delta \sum_c \epsilon_c \pi_{\mathrm{un}}(\rho_{S\setminus c}) = (-1)^\delta \sum_c \epsilon_c \sum_{T\sim S\setminus c} \langle S\setminus c, T\rangle R_T, \tag{2}$$

where the outer sum runs over d-cohooks c of S. In order to prove this, we will compare coefficients according to the d-cohooks of S at an entry x of S.

Let us first assume that $x-d$ is not an entry of S. Then any symbol $T \sim S$ also has a d-hook h at x, and we need to show that

$$(-1)^{f(h)} \langle S, T\rangle = (-1)^\delta \epsilon_c \langle S\setminus c, T\setminus h\rangle$$

for all $T \sim S$. This is straightforward from the explicit description of Fourier matrices.

Now assume that $x-d$ is an entry of S in the opposite row to x. Then there is no d-cohook in S at x and (2) gives no contribution. We get a contribution to (1) for all $T \sim S$ for which $x, x-d$ lie in different rows. But these come in pairs, according to whether x is in the first row or the second row, and the corresponding terms in the sum, having opposite sign, cancel.

Finally, assume that $x-d$ lies in the same row of S as x. Then the contribution to (2) is the sum over all $T \sim S \setminus c$. Note that here the size of the family of $S \setminus c$ is smaller than that of S, and the coefficients in the Fourier matrix have twice the absolute value. We get a contribution to (1) for all $T \sim S$ that have $x, x-d$ in distinct rows. Again, these come in pairs, and this time the two symbols give contributions with the same sign, thus making up for the doubled Fourier coefficients in (2). The proof is complete. □

6. Completion of the proof

Proof of Theorem 1(b). It suffices to compare the projection of $R_{\mathbf{L}}^{\mathbf{G}}(\rho)$ and of Asai's formula to any family $\mathcal{U} \subset \mathrm{Uch}(\mathbf{G}^F)$. By Proposition 6 the uniform projection of $R_{\mathbf{L}}^{\mathbf{G}}(\rho)$ and of Asai's formula agree. In Asai's formula, the symbols parametrising the constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\rho)$ are obtained from the symbol of ρ by adding a d-cohook, that is, by increasing one of the entries by d and moving it to the other row. It is obvious that at most two of the resulting symbols can share the same multi-set of entries, and in the latter case, this multi-set has more distinct elements than the one for ρ, so its associated family in $\mathrm{Uch}(\mathbf{G}^F)$ is bigger.

On the other hand, since all unipotent characters have the same positive multiplicity in the special uniform almost character of a family, we can read off from the uniform projection whether the number of constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\rho)$ is one character or the sum or the difference of two of them, and the result will be the same as for Asai's formula. Now by Proposition 4 the norm of $R_{\mathbf{L}}^{\mathbf{G}}(\rho)$ agrees with the one in the formula of Asai, so we conclude that the projection to any family of either have the same number of constituents, either zero, one or two.

Thus, the projection f of $R_{\mathbf{L}}^{\mathbf{G}}(\rho)$ onto $\mathcal{U}$ must in fact be a linear combination of at most two unipotent characters. We are thus left to show that f is already determined by its uniform projection $\pi_{\mathrm{un}}^{\mathbf{G}}(f)$ and its norm being ≤ 2. If f has norm one then by Theorem 3(a) there is exactly one unipotent character having this uniform projection, and we are done. If f has norm 2 and $\mathcal{U}$ has more than four elements, our claim follows by Theorem 3(b). Finally assume that $|\mathcal{U}| = 4$. Then by what we said above, ρ lies in a 1-element family of $\mathrm{Uch}(\mathbf{L}^F)$ and hence is uniform, whence so is $R_{\mathbf{L}}^{\mathbf{G}}(\rho)$. In particular it is orthogonal to the space of non-uniform functions and we conclude again. □

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