

On Principal Indecomposable Degrees and Sylow Subgroups

Gunter Malle and Gabriel Navarro

Abstract. We conjectured in [3] that a Sylow p -subgroup P of a finite group G is normal if and only if whenever p does not divide the multiplicity of $\chi \in \text{Irr}(G)$ in the permutation character $(1_P)^G$, then p does not divide the degree $\chi(1)$. In this note, we prove an analogue of this for p -Brauer characters.

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1. Introduction

In [3] we proved that if G is a finite group and $P \in \text{Syl}_p(G)$, then $P \trianglelefteq G$ if and only if p does not divide the degrees of the irreducible constituents of the permutation character $(1_P)^G$. Furthermore, we conjectured that this happens if and only if the irreducible constituents of $(1_P)^G$ with multiplicity not divisible by p had degree not divisible by p . This has turned out to be a difficult problem for symmetric groups (in the cases where $p = 2$ or $p = 3$). In this short note, we aim for an analogous result on modular characters. By using the induction formula and [5, Thm. 2.13], notice that the permutation p -Brauer character $((1_P)^G)^0$ decomposes as

$$((1_P)^G)^0 = \sum_{\varphi \in \text{IBr}(G)} \frac{\Phi_\varphi(1)}{|G|_p} \varphi,$$

where Φ_φ is the projective indecomposable character associated with the irreducible Brauer character φ .

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We consider the following modular analogue of our conjecture in [3]:

Theorem A. *Let p be a prime, let G be a finite group, and let $P \in \text{Syl}_p(G)$. Then $P \trianglelefteq G$ if and only if whenever $\varphi \in \text{IBr}(G)$ is such that $\Phi_\varphi(1)/|P|$ is not divisible by p , then $\varphi(1)$ is not divisible by p .*

This generalizes the well-known result of G. Michler [4, Thm. 5.5] that G has a normal Sylow p -subgroup if and only if all irreducible Brauer characters of G have degree prime to p .

Our proof (as in [4]) relies on the Classification of Finite Simple Groups for p odd only.

2. Proof of Theorem A

We use the notation in [5]. Let p be a fixed prime. We choose a maximal ideal of the ring of algebraic integers in \mathbb{C} containing p , with respect to which we calculate a set $\text{IBr}(G)$ of irreducible (p -)Brauer characters for every finite group G . If $N \trianglelefteq G$ and $\theta \in \text{IBr}(N)$, then $\text{IBr}(G|\theta)$ is the set of irreducible constituents of the induced character θ^G , which, by [5, Cor. 8.7], is the set of irreducible Brauer characters $\varphi \in \text{IBr}(G)$ such that θ is a constituent of the restriction φ_N . For $\varphi \in \text{IBr}(G)$, we write $c_\varphi = \Phi_\varphi(1)/|G|_p$. (This is an integer by Dickson's theorem [5, Cor. 2.14].)

We start with the following observation:

Proposition 2.1. *Let B be a p -block of a finite group G of non-maximal defect. Then B contains an irreducible Brauer character θ of degree divisible by p but with c_θ not divisible by p .*

Proof. By a result of Brauer [5, Thm. 3.28] the p -part of the dimension of B is precisely p^{2a-d} , where p^a is the order of a Sylow p -subgroup of G , and d is the defect of B . Now

$$\dim B = \sum_{\theta \in \text{IBr}(B)} \theta(1)\Phi_\theta(1)$$

and all $\theta(1)$ are divisible by p^{a-d} , as all $\chi(1)$ for $\chi \in \text{Irr}(B)$, are divisible by p^{a-d} . It follows that there must be some $\theta \in \text{IBr}(B)$ with $\Phi_\theta(1)/p^a$ not divisible by p . \square

We shall need the first two parts of the following lemma. We keep part (c) for future use.

Lemma 2.2. *Let $N \trianglelefteq G$, and let $\theta \in \text{IBr}(N)$. Write*

$$\theta^G = \sum_{\varphi \in \text{IBr}(G|\theta)} a_\varphi \varphi,$$

where a_φ are non-negative integers. Let T be the stabilizer of θ in G , and let $t = |G : T|$. Then for all $\varphi \in \text{IBr}(G|\theta)$ we have:

(a)

$$a_\varphi t c_\theta = c_\varphi |G : N|_p;$$

- (b) If G/N is a p -group, then $c_\varphi = c_\theta$; and
- (c) c_θ divides c_φ and thus, $|T : N|_p$ divides a_φ .

Proof. By [5, Cor. 8.8], we have $\Phi_\varphi(1) = a_\varphi t \Phi_\theta(1)$. Hence, (a) is clear.

To prove (b), notice that in this case $\text{IBr}(G|\theta) = \{\varphi\}$ and $\varphi(1) = t\theta(1)$, by [5, Thm. 8.11]. Now, $\theta^G = a_\varphi \varphi$ (using [5, Cor. 8.7]) and by part (a), we have $c_\theta = c_\varphi$.

To prove (c), using the Clifford correspondence for Brauer characters [5, Thm. 8.9], we may assume that $T = G$. We have $(\Phi_\varphi)_N = a_\varphi \Phi_\theta$, again by [5, Cor. 8.8]. Let $H = PN$, where $P \in \text{Syl}_p(G)$. By Problem 8.7 of [5], we have that $(\Phi_\varphi)_H$ is a positive of sum Φ_τ , for suitable $\tau \in \text{IBr}(H)$. Using [5, Cor. 8.8] (and the fact that $\{\Phi_\mu \mid \mu \in \text{IBr}(N)\}$ is linearly independent [5, Thm. 2.13]), we conclude that every such τ necessarily lies over θ . However, there is a unique $\hat{\theta} \in \text{IBr}(H)$ over θ (by [5, Thm. 8.11]). We conclude that

$$\Phi_\varphi = v\Phi_{\hat{\theta}}$$

for some integer v . Then

$$c_\varphi = v c_{\hat{\theta}} = v c_\theta,$$

using part (b). The last part now follows from (a). □

We are now ready to prove our main result.

Theorem 2.3. *Let p be a prime, let G be a finite group, and let $P \in \text{Syl}_p(G)$. Then $P \trianglelefteq G$ if and only if whenever $\Phi_\varphi(1)/|P|$ is not divisible by p , then $\varphi(1)$ is not divisible by p .*

Proof. If $P \trianglelefteq G$, then p does not divide $\varphi(1)$ for all $\varphi \in \text{IBr}(G)$, since $P \subseteq \ker \varphi$ ([5, Lemma 2.32]) and $\text{IBr}(G/P) = \text{Irr}(G/P)$ (by [5, Thm. 2.12]).

To prove the converse in the case $p = 2$, we borrow an argument in [2]. First, recall that $\varphi(1)$ is even if $1 \neq \varphi \in \text{IBr}(G)$ is real-valued ([5, Thm. 2.30]) and c_1 is odd ([5, Cor. 3.34]). By hypothesis, and using that $c_\varphi = c_{\bar{\varphi}}$ since $(1_P)^G$ is real-valued, we can write

$$((1_P)^G)^0 = c_1 1 + 2\Delta + \sum_{\varphi \in \Lambda} c_\varphi (\varphi + \bar{\varphi}),$$

where Δ is a Brauer character of G , $\bar{\varphi}$ is the complex conjugate of φ , and Λ is a subset of non-trivial non-real-valued characters in $\text{IBr}(G)$. If P is not normal in G , then there exists a real element $1 \neq x \in G$ of odd order ([1, Prop. 6.4]). Then $0 = (1_P)^G(x)$, and we conclude that $-c_1 = 2\alpha$, for some algebraic integer α . This is impossible since c_1 is odd.

To prove the converse for p odd, we argue by induction on $|G|$. By Proposition 2.1, all blocks of G have maximal defect. Let N be a proper non-trivial normal subgroup of G .

First, we claim that p divides $|G/N|$. Assume the contrary. Let $\theta \in \text{IBr}(N)$ be such that p does not divide c_θ . Let T be the stabilizer of θ in G , and let $t = |G : T|$. Let $\varphi \in \text{IBr}(G)$ be over θ . By [5, Cor. 8.7], we have that $\varphi(1) = et\theta(1)$, where e is the multiplicity of θ in φ_N . Also by [5, Cor. 8.7], this is the multiplicity of φ in θ^G (what we called a_φ in Lemma 2.2). By

Lemma 2.2(a), we have $c_\varphi = \frac{\varphi(1)}{\theta(1)}c_\theta$. We conclude that c_φ is not divisible by p , (using Dade's Theorem 8.30 of [5]). By hypothesis, we conclude that $\varphi(1)$ is not divisible by p , and therefore neither is $\theta(1)$. Hence N has a normal Sylow p -subgroup, and we are done in this case.

Suppose now that G/N is a p -group. Let $\theta \in \text{IBr}(N)$ such that p does not divide c_θ , and let $\varphi \in \text{IBr}(G|\theta)$. By Lemma 2.2(b), we have $c_\theta = c_\varphi$. Using the hypothesis, we conclude that p does not divide $\varphi(1)$, and therefore p does not divide $\theta(1)$. Therefore, N has a normal Sylow p -subgroup, by induction.

Suppose now that $c_{1_N} = 1$. (By Fong's dimensional formula [5, Cor. 10.14] this happens, for instance, if N is p -solvable.) By [6, Lemma 2.6], if $\bar{\varphi} \in \text{IBr}(G/N)$ is the inflation of $\varphi \in \text{IBr}(G)$, then $c_\varphi = c_{\bar{\varphi}}$, and by induction, G/N has a normal Sylow p -subgroup. By the two previous paragraphs, we deduce that $G = N$. In particular, G does not have proper non-trivial normal p -solvable subgroups.

Suppose now that N is a non-abelian minimal normal subgroup of G . Let $S \trianglelefteq N$ be simple, so that $N = S^{x_1} \times \cdots \times S^{x_k}$, for some $x_i \in G$. By [4, Thm. 5.3], there exists a block b of S that does not have maximal defect. Then the block $e = b^{x_1} \times \cdots \times b^{x_k}$ does not have maximal defect. However, if B is any block of G covering e , then B has maximal defect, and therefore so does e by Knörr's Theorem 9.26 of [5]. \square

A possible variation of Theorem A would be to consider $\text{Fix}_P(\varphi)$ instead of c_φ , where $\text{Fix}_P(\varphi)$ is the dimension of the fixed points of P in any G -module affording the Brauer character φ .

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Gunter Malle

FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany.

e-mail: malle@mathematik.uni-kl.de

Gabriel Navarro
Departament d'Àlgebra, Universitat de València, 46100 Burjassot, València, Spain.
e-mail: gabriel@uv.es