CUSPIDAL CHARACTERS AND AUTOMORPHISMS

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Abstract. We investigate the action of outer automorphisms of finite groups of Lie type on their irreducible characters. We obtain a definite result for cuspidal characters. As an application we verify the inductive McKay condition for some further infinite families of simple groups at certain primes.

1. Introduction

An important open problem in the ordinary representation theory of finite groups of Lie type is to determine the action of outer automorphisms on the set of their irreducible characters, and more generally to determine the irreducible character degrees of the corresponding almost (quasi-)simple groups. While the action of diagonal automorphisms and the corresponding extension problems are well understood by the work of Lusztig [13], based on the fact that such extensions can be studied in the framework of finite reductive groups, much less is known in the case of field and also of graph automorphisms.

The most elusive situation seems to be the one where irreducible characters not stable under diagonal automorphisms are concerned. In [15] we obtained a certain reduction of this problem to the case of cuspidal characters. This is the situation we solve here by applying methods and results from block theory and Deligne–Lusztig theory (see Section 4):

Theorem 1. *Let G be a quasi-simple finite group of Lie type. For any cuspidal character ρ of G there is a semisimple character χ in the rational Lusztig series of ρ having the same stabiliser as ρ in the automorphism group of G.*

Observe that the action on semisimple characters is well-understood by the theory of Gelfand–Graev characters, see [19]. For linear and unitary groups this result was obtained by Cabanes and Späth [6] and we use it in our proof, for symplectic groups it follows from recent work of Cabanes–Späth [7] and of Taylor [20]; for the other types it is new. For the proof we first consider quasi-isolated series, see Sections 2 and 3. Here, we connect ρ to χ either via a sequence of Brauer trees, in which case we also obtain information on maximal extendibility (see Corollary 3.3), or of Deligne–Lusztig characters.

As an application we verify the inductive McKay condition for some series of simple groups of Lie type and suitable primes ℓ (see Section 5):

Theorem 2. *Let q be a prime power and S a finite simple group ${}^2E_6(q)$, $E_7(q)$, $B_n(q)$ or $C_n(q)$. Let $\ell \equiv 3 \pmod 4$ be a prime with $\ell|(q^2-1)$. Then S satisfies the inductive McKay condition at ℓ. In particular, the inductive McKay condition holds for S at $\ell = 3$.*

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This is a contribution to a programme to prove McKay's 1972 conjecture on characters of ℓ'-degree based on its reduction to properties of quasi-simple groups, an approach which has recently led to the completion of the proof in the case when $\ell = 2$ (see [15]).

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2. Cuspidal characters in classical groups

Throughout the paper we fix the following notation. We let $\mathbf{G}$ be a simple simply connected linear algebraic group over an algebraically closed field of characteristic p with a Frobenius map $F : \mathbf{G} \to \mathbf{G}$ inducing an $\mathbb{F}_q$-structure, and we set $G := \mathbf{G}^F$ the finite group of fixed points under F. It is well-known that G then is a finite quasi-simple group in all but finitely many cases (see [16, Thm. 24.17]), and furthermore all but finitely many quasi-simple finite groups of Lie type can be obtained as G/Z for some suitable central subgroup $Z \leq Z(G)$ (the exceptions being the Tits simple group and a few exceptional covering groups, see e.g. [16, Tab. 24.3]). For us, a "finite group of Lie type" is any nearly simple group whose non-abelian composition factor is neither sporadic nor alternating.

Let $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be a regular embedding; thus $\widetilde{\mathbf{G}}$ is a connected reductive group with connected centre and derived subgroup equal to $\mathbf{G}$. For an extension $F : \widetilde{\mathbf{G}} \to \widetilde{\mathbf{G}}$ of the Frobenius map on $\mathbf{G}$ we let $\tilde{G} := \widetilde{\mathbf{G}}^F$. We choose a group $\widetilde{\mathbf{G}}^*$ dual to $\widetilde{\mathbf{G}}$, with corresponding Frobenius map again denoted by F, and an epimorphism $\pi : \widetilde{\mathbf{G}}^* \to \mathbf{G}^*$ dual to the regular embedding $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$, and we write $\tilde{G}^* = \widetilde{\mathbf{G}}^{*F}$ and $G^* = \mathbf{G}^{*F}$ for the F-fixed points. Throughout, for closed F-stable subgroups $\mathbf{H}$ of $\mathbf{G}$, $\widetilde{\mathbf{G}}$, $\mathbf{G}^*, \ldots$ we will write $H := \mathbf{H}^F$ (in roman font) for their group of fixed points.

Let us recall the description of automorphisms of a finite simple group of Lie type: any automorphism of $S = G/Z(G)$ is a product of an inner, a diagonal, a graph and a field automorphism. Here, the *diagonal automorphisms* are those induced by the embedding $G \hookrightarrow \tilde{G}$, the *graph automorphisms* come from symmetries of the Dynkin diagram of $\mathbf{G}$ commuting with F and *field automorphisms* are induced by Frobenius maps on $\mathbf{G}$ defining a structure over some subfield $\mathbb{F}_{q'}$ of $\mathbb{F}_q$ some power of which is F (see e.g. [16, Thm. 24.24]).

2.1. **Cuspidal unipotent characters on Brauer trees.** One crucial tool in our determination of the action of automorphisms is the observation that cuspidal unipotent characters of classical groups lie in blocks of cyclic defect for suitable primes. Lusztig gave a parametrisation of the unipotent characters of groups G of classical type in terms of combinatorial objects called *symbols*. According to this classification a classical group has at most one cuspidal unipotent character, as recalled in Table 1, which is thus in particular fixed by all automorphisms of G. (The parameter d_G occurring in the table will be used in the statement of Lemmas 2.1 and 2.2.)

The symbols parametrising unipotent characters behave very much like partitions; in particular one can define hooks and cohooks, and the degrees of the associated unipotent

G	A_{n-1}	${}^2A_{n-1}$	B_n, C_n	D_n	${}^2D_n$
n	1	$a(a+1)/2$	$a(a+1)$	a^2 even	a^2 odd
label	()	$(a, a-1, \ldots, 1)$	$\binom{0\ldots 2a}{-}$	$\binom{0\ldots 2a-1}{-}$	$\binom{0\ldots 2a-1}{-}$
d_G	–	$2(2a-1)$	$4a$	$2(2a-1)$	$2(2a-1)$

TABLE 1. Cuspidal unipotent characters in classical groups

characters can be given in terms of a combinatorial expression, called the hook formula (see e.g. [18]). Recall that for every prime power q and any integer $d > 2$ with $(q, d) \neq (2, 6)$ there exists a prime dividing $q^d - 1$ but no $q^f - 1$ for $f < d$ called *Zsigmondy (primitive) prime of degree d*. The following is easily checked from the hook formula:

Lemma 2.1. *Let ρ be a cuspidal unipotent character of a quasi-simple group G of classical type. Then ρ is of defect zero for every Zsigmondy prime of odd degree, as well as for those of even degree $d > d_G$.*

The blocks of cyclic defect and their Brauer trees for groups G of classical type have been determined by Fong and Srinivasan [10]: assume that $\ell \neq 2$ is an odd prime and write $d = d_\ell(q)$ for the order of q modulo ℓ. First assume that d is odd. Then a unipotent character of G lies in an ℓ-block of cyclic defect if and only if the associated symbol has at most one d-hook, and two unipotent characters lie in the same ℓ-block if their symbols have the same d-core. If $d = 2d'$ is even, the same statements hold with d replaced by d', "hook" replaced by "cohook" and "core" replaced by "cocore".

Let us write Φ_d for the dth cyclotomic polynomial over $\mathbb{Q}$.

Lemma 2.2. *Let ρ be a cuspidal unipotent character of a quasi-simple group G of classical type. Then there exist sequences of unipotent characters $\rho = \rho_1, \ldots, \rho_m = 1_G$ of G and of Zsigmondy primes $\ell_i \neq p$ of either odd degree $d_i > 2$ or even degree $d_i \geq d_G$ such that ρ_i, ρ_{i+1} lie in the same ℓ_i-block of cyclic defect of G, for $i = 1, \ldots, m-1$, except when $G = D_4(2)$.*

Proof. The claim is clear for type A_0 as here the cuspidal character is the trivial character. For G of type ${}^2A_{n-1}$, $n \geq 3$, there exists a cuspidal unipotent character ρ if and only if $n = a(a+1)/2$ for some $a \geq 2$ (see Table 1). This is labelled by the triangular partition $\delta_a = (a, \ldots, 1)$ of n, which has a unique hook of length $2a-1$. By the hook formula (see e.g. [18]) this implies that $|G|/\rho(1)$ is divisible by $\Phi_d(q)$ exactly once, where $d = 2(2a-1)$. So by [10] ρ lies in an ℓ_1-block of cyclic defect for any Zsigmondy prime divisor ℓ_1 of $q^{2a-1}+1$. (Such a prime ℓ_1 exists unless $(q, a) = (2, 2)$, in which case $n = 3$, but ${}^2A_2(2)$ is solvable.) Moreover, the partition $(3a-3, a-3, \ldots, 1)$ has the same $2a-1$-core as δ_a, so the unipotent character ρ_2 labelled by it lies in the same ℓ_1-block as $\rho_1 = \rho$. The latter partition has a unique $4a-4$-hook, and arguing as before, we conclude that ρ_2 lies in the same ℓ_2-block of cyclic defect as the character ρ_3 labelled by $(5a-10, a-5, \ldots, 1)$, for ℓ_2 a Zsigmondy prime divisor of $q^{4a-4}+1$. Continuing inductively we arrive at $\rho_{2a+1} = 1_G$ with label the partition (n).

Groups of type B_n and C_n, $n \geq 2$, have a cuspidal unipotent character ρ if and only if $n = a(a+1)$ for some $a \geq 1$. This is labelled by the symbol $\binom{0\ldots 2a}{-}$, which has a single

$2a$-cohook. So ρ lies in a block of cyclic defect for any Zsigmondy prime divisor ℓ_1 of $q^{2a}+1$ (see again [10]). Furthermore this block also contains the unipotent character ρ_2 labelled by $\binom{0\,\ldots\,2a-2}{0\;4a-1}$. This has a unique $4a-2$-hook, whose removal gives $\binom{0\,\ldots\,2a-2}{0\;1}$, and adding a suitable $4a-2$-hook we find $\binom{0\,\ldots\,2a-3\;\,6a-4}{0\;1}$, the symbol of a unipotent character ρ_3 lying on the same Brauer tree as ρ_2 for primes ℓ_3 dividing $q^{4a-2}-1$. The Zsigmondy exception $a=2$ can be avoided by working with 5-hooks instead of 6-hooks in this step. Continuing this way, after $m=a$ steps we arrive at the symbol $\binom{n}{-}$ labelling $\rho_{a+1}=1_G$.

For groups of type D_n or ${}^2D_n$, $n\geq 4$, a cuspidal unipotent character exists only if $n=a^2$ for some $a\geq 2$. It is labelled by the symbol $\binom{0\,\ldots\,2a-1}{-}$. This has a single $2a-1$-cohook and thus lies in an ℓ_1-block of cyclic defect for ℓ_1 any primitive prime divisor of $q^{2a-1}+1$. (Such a prime ℓ_1 exists unless $(q,a)=(2,2)$, which leads to the stated exception.) This block also contains the character ρ_2 labelled by $\binom{0\,\ldots\,2a-3}{0\;4a-3}$. The latter symbol has a unique $4a-4$-hook; removing this and adding a different one leads to the symbol $\binom{0\,\ldots\,2a-4\;\,6a-7}{0\;1}$ of a unipotent character ρ_3. Again, a straightforward induction completes the proof. $\square$

2.2. **Constituents of Lusztig induction.** Recall from [8, 11.1] that for any F-stable Levi subgroup $\mathbf{L}$ of a parabolic subgroup of $\mathbf{G}$, Lusztig defines a linear map

$$R_{\mathbf{L}}^{\mathbf{G}}:\mathbb{Z}\,\mathrm{Irr}(\mathbf{L}^F)\longrightarrow\mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^F),$$

called *Lusztig induction*. (In fact, this map might depend on the choice of parabolic subgroup containing $\mathbf{L}$, but it does not in the case of unipotent characters, see e.g. [3, Thm. 1.33].) Also recall from [3] that an F-stable torus $\mathbf{T}\leq\mathbf{G}$ is called a *d-torus* (for some $d\geq 1$) if $\mathbf{T}$ is split over $\mathbb{F}_{q^d}$ but no non-trivial F-stable subtorus of $\mathbf{T}$ splits over any smaller field. The centralisers in $\mathbf{G}$ of d-tori are the *d-split Levi subgroups*. They are F-stable Levi subgroups of suitable parabolic subgroups of $\mathbf{G}$.

Lemma 2.3. *Let G be quasi-simple of classical type B_n, C_n, D_n or ${}^2D_n$ and ρ be a cuspidal unipotent character of G. Let $\mathbf{L}\leq\mathbf{G}$ be a 2-split Levi subgroup of $\mathbf{G}$*

- *of twisted type ${}^2A_{n-1}(q).\Phi_2$ in types B_n and C_n; or*
- *of twisted type ${}^2A_{n-2}(q).\Phi_2^2$ in types D_n and ${}^2D_n$.*

There exist sequences of unipotent characters $\rho=\rho_1,\ldots,\rho_m=1_G$ of G and $\psi_1,\ldots,\psi_{m-1}$ of L such that ρ_i,ρ_{i+1} both occur with multiplicity ±1 in $R_{\mathbf{L}}^{\mathbf{G}}(\psi_i)$ for $i=1,\ldots,m-1$.

Here, as in later results and tables, a notation like ${}^2A_{n-1}(q).\Phi_2$ is meant to indicate not the precise group theoretic structure of the finite group, but rather the root system of the underlying algebraic group (not its isogeny type) together with the action of the Frobenius: in our example the underlying group has a root system of type A_{n-1} on which F acts by the non-trivial graph automorphism, and its center is a 1-dimensional 2-torus.

Proof. First consider types B_n and C_n. So by Table 1, $n=a(a+1)$ for some $a\geq 1$, and ρ is parametrised by the symbol $\mathcal{S}=\binom{0\,\ldots\,2a}{-}$. Observe that all unipotent characters of groups of type A are uniform, so that $R_{\mathbf{L}}^{\mathbf{G}}(\rho)$ for any unipotent character ρ of L can be expressed in terms of Deligne–Lusztig characters $R_{\mathbf{T}}^{\mathbf{G}}(1_T)$ for various F-stable maximal tori $\mathbf{T}$ of $\mathbf{G}$. The decomposition of Deligne–Lusztig characters has been determined explicitly by Lusztig, and from this our claim can be checked by direct computation.

To do this, we appeal to [3, Thm. 3.2], which shows that Lusztig induction of unipotent characters from 2-split Levi subgroups is, up to signs, the same as induction in the corresponding relative Weyl groups. Since the symbol $\mathcal{S}$ has exactly n 1-cohooks, its 1-cocore is trivial and so ρ lies in the principal 2-series. The relative Weyl group for the principal 2-Harish-Chandra series of G is $W{=}W(B_n)$, the one for L is its maximal parabolic subgroup $W_L = \mathfrak{S}_n$. Determination of the 2-quotient of the symbol $\mathcal{S}$ shows that it corresponds to the character with label $(\delta_a;\delta_a)$ of W, with $\delta_a = (a,\ldots,1)$ the triangular partition. Let ψ be the unipotent character of L parametrised by the partition $2\delta_a = (2a, 2a-2,\ldots,2)$. The constituents of $\mathrm{Ind}_{W_L}^{W}(2\delta_a)$ are exactly those bipartitions whose parts (including zeroes) can be added up so as to obtain the partition $2\delta_a$, with multiplicity one if this is possible in a unique way. Thus it contains $(\delta_a;\delta_a)$ exactly once, but also $(2a, 2a-4,\ldots; 2a-2, 2a-6,\ldots)$. This in turn is contained once in $\mathrm{Ind}_{W_L}^{W}((4a-2, 4a-10,\ldots))$, and so on. Continuing in this way we reach the symbol $(a(a+1);-)$ which parametrises the trivial character of G.

For G of type D_n, by Table 1 a cuspidal unipotent character exists if $n = a^2$ for some even $a \geq 2$, and it is labelled by the symbol $\binom{0\,\ldots\,2a-1}{-}$. Again, this character lies in the principal 2-series of G, and by [3, Thm. 3.2] the decomposition of $R_{\mathbf{L}}^{\mathbf{G}}$ can be computed in the relative Weyl groups $W_L = \mathfrak{S}_{n-1} \leq W = W(D_n)$. Here, ρ corresponds to the character of W with label $(\delta_a;\delta_{a-1})$. Let $W_1 = \mathfrak{S}_n \leq W(D_n)$ be a maximal parabolic subgroup of $W(D_n)$ containing W_L. For $\psi \in \mathrm{Irr}(W_L)$ labelled by $\lambda = (\delta_a + \delta_{a-1}) \setminus \{1\}$, $\mathrm{Ind}_{W_L}^{W_1}(\lambda)$ contains all characters whose label is obtained by adding one box to the Young diagram of λ. Then $\mathrm{Ind}_{W_L}^{W}(\lambda) = \mathrm{Ind}_{W_1}^{W}\mathrm{Ind}_{W_L}^{W_1}(\lambda)$ can be computed as before. It ensues that the multiplicities of the characters labelled by $(\delta_a;\delta_{a-1})$ and by $(2a, 2a-5,\ldots; 2a-3, 2a-7,\ldots)$ in $\mathrm{Ind}_{W_L}^{W}(\lambda)$ are both 1. Again, an easy induction gives the claim. The same type of reasoning applies for ${}^2D_n$ with n an odd square. □

2.3. **Brauer trees and automorphisms.** The following result of Feit on Brauer trees will also provide some information on extendibility. Let ℓ be a prime.

Lemma 2.4. *Let B be an ℓ-block of a finite group H with cyclic defect. Let γ be an automorphism of H fixing some non-exceptional character in B. Then γ fixes every non-exceptional character in B.*

Proof. Assume that $\chi \in \mathrm{Irr}(B)$ is non-exceptional and fixed by γ. Then by [9, Thm. 2.4] all nodes in the Brauer tree of B are fixed by γ, hence in particular all non-exceptional characters in B. □

Corollary 2.5. *Let $N \trianglelefteq H$ be finite groups with H/N solvable and of order prime to ℓ. Let χ, χ' be non-exceptional characters on the same ℓ-Brauer tree for N. Then χ extends to H if and only if χ' does.*

Proof. As H/N is solvable, there is a sequence of subgroups $N = N_1 \trianglelefteq \cdots \trianglelefteq N_r = H$ with N_i/N_{i-1} cyclic of prime order. Assume that χ extends to $\tilde\chi \in \mathrm{Irr}(H)$, and let $\chi_i = \tilde\chi|_{N_i}$, $1 \leq i \leq r$, a system of compatible extensions of χ to N_i.

Assume that we have extended χ' to a character χ_i' of N_i on the same ℓ-Brauer tree as χ_i. Since χ_i extends to H, it is invariant in N_{i+1}, hence by Lemma 2.4 the same is true for χ_i'. So χ_i' also extends to N_{i+1} (as N_{i+1}/N_i is cyclic), and clearly we may choose an extension χ_{i+1}' in the same ℓ-block as χ_{i+1}. So the claim follows by induction. □

Remark 2.6. We can now lay out two types of arguments we will use to relate the stabilisers in $\mathrm{Aut}(G)$ of two characters $\rho, \chi \in \mathrm{Irr}(G)$:

(a) Assume there is a sequence $\rho = \rho_1, \ldots, \rho_m = \chi$ of irreducible characters of G, a sequence of primes $\ell_1, \ldots, \ell_{m-1}$ and a sequence of ℓ_i-blocks $B_1, \ldots, B_{m-1}$ of G with cyclic defect such that $\rho_i, \rho_{i+1} \in \mathrm{Irr}(B_i)$ are non-exceptional for all i. Then by Lemma 2.4 any automorphism fixing $\rho = \rho_1$ also fixes $\rho_m = \chi$ and vice versa.

(b) Similarly, assume there is a Levi subgroup L of G and sequences $\rho = \rho_1, \ldots, \rho_m = \chi$ of irreducible characters of G, $\psi_1, \ldots, \psi_{m-1}$ of L satisfying the conclusion of Lemma 2.3. Assume that γ is an automorphism of G stabilising L and fixing all ρ_i and all ψ_i, and that there are γ-invariant normal subgroups $L' \le L$, $G' \le G$ such that all ρ_i, ψ_i have exactly two constituents upon restriction. Then if γ does not fix the constituents of ρ_1, it cannot fix those of ρ_m and vice versa.

In our arguments we will have to deal not only with unipotent characters. Recall that Lusztig gives a partition $\mathrm{Irr}(G) = \coprod_s \mathcal{E}(G,s)$ of the set of irreducible characters of G into *rational Lusztig series* $\mathcal{E}(G,s)$ indexed by semisimple elements $s \in G^*$ up to conjugacy (see [2, Thm. 11.8]). Moreover, for any such s the Lusztig series $\mathcal{E}(G,s)$ is in bijection with the unipotent characters of $C_{G^*}(s)$, where, as customary, a character of $C_{G^*}(s)$ is called unipotent if its restriction to $C^\circ_{\mathbf{G}^*}(s)^F$ has unipotent constituents (see [13, Prop. 5.1]). We will be particularly interested in semisimple characters. Recall our regular embedding $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ with dual epimorphism $\widetilde{\mathbf{G}}^* \to \mathbf{G}^*$ and let $\tilde{s} \in \widetilde{\mathbf{G}}^{*F}$ be a preimage of s. The *semisimple character* $\tilde{\chi}_s$ in $\mathcal{E}(\tilde{G}, \tilde{s})$ can be defined as an explicit linear combination of Deligne–Lusztig characters (see [2, (15.6)]); the semisimple characters in $\mathcal{E}(G,s)$ are then just the constituents of the restriction of $\tilde{\chi}_s$ to G, see [2, (15.8)].

We will need the following properties of Jordan decomposition:

Lemma 2.7. *In the above situation we have:*

(a) *Jordan decomposition sends semisimple characters to semisimple characters.*

(b) *$\mathcal{E}(G,s)$ contains a cuspidal character if and only if $C_{G^*}(s)$ has a cuspidal unipotent character and moreover $Z^\circ(C_{\mathbf{G}^*}(s))$ and $Z^\circ(\mathbf{G})$ have the same $\mathbb{F}_q$-rank. In this case, Jordan decomposition induces a bijection between cuspidal characters.*

Proof. As explained above the semisimple character $\tilde{\chi}_s \in \mathcal{E}(\tilde{G}, \tilde{s})$ is uniform. Jordan decomposition preserves uniform functions, so $\tilde{\chi}_s$ is sent to the semisimple character in $\mathcal{E}(C_{\tilde{G}^*}(\tilde{s}), 1)$. As the Deligne–Lusztig characters of G are obtained by restriction from those of $\tilde{G}$, the claim in (a) follows from our definition of semisimple characters in $\mathcal{E}(G,s)$.

Part (b) is pointed out for example in [11, Rem. 2.2(1)]. □

2.4. **Cuspidal characters in quasi-isolated series.** We now study the action of automorphisms on cuspidal characters in quasi-isolated series of classical groups.

Let $s \in G^*$ be semisimple. Recall that s is *quasi-isolated in* $\mathbf{G}^*$ if $C_{\mathbf{G}^*}(s)$ is not contained in any proper F-stable Levi subgroup of $\mathbf{G}^*$. If $C^\circ_{\mathbf{G}^*}(s)^F$ is a product of classical groups, then it has a unique cuspidal unipotent character (see Table 1), so the cuspidal characters in $\mathcal{E}(G,s)$ form a single orbit under diagonal automorphisms. In particular, $\mathcal{E}(G,s)$ can contain cuspidal characters not fixed by some automorphism of G stabilising $\mathcal{E}(G,s)$ only if $C_{\mathbf{G}^*}(s)$ is not connected.

Theorem 2.8. *Let G be quasi-simple of classical type B, C, D or ${}^2D$, let $s \in G^*$ be quasi-isolated and $\rho \in \mathcal{E}(G,s)$ cuspidal. Then there is a semisimple character $\chi \in \mathcal{E}(G,s)$ with the same stabiliser in* $\mathrm{Aut}(G)$ *as ρ.*

Proof. Let $\gamma \in \mathrm{Aut}(G)$. If γ does not stabilise $\mathcal{E}(G,s)$, then it lies neither in the stabiliser of ρ nor of any semisimple character in $\mathcal{E}(G,s)$. So we may assume that $\mathcal{E}(G,s)$ is γ-stable. Moreover, if $C_{\mathbf{G}^*}(s)^F = C^\circ_{\mathbf{G}^*}(s)^F$ then $\mathcal{E}(G,s)$ contains a unique cuspidal and a unique semisimple character, and again we are done. Thus, as $|C_{\mathbf{G}^*}(s) : C^\circ_{\mathbf{G}^*}(s)|$ divides $|Z(\mathbf{G})|$, which is a 2-power and prime to p, we have in particular that q is odd. We discuss the remaining possibilities case-by-case.

The classes of quasi-isolated elements in $\mathbf{G}^*$ were classified by Bonnafé [1, Tab. 2]. The various rational types are worked out in Table 2. Here $A(s) := C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$ denotes the group of components of the centraliser, $o(s)$ is the order of s and the structure of the abelian group $A(s)$ is indicated by giving the orders with multiplicities of its cyclic factors.

Our strategy of proof is as follows. In each case, $[C^\circ_{\mathbf{G}^*}(s), C^\circ_{\mathbf{G}^*}(s)]^F$ is a product of classical groups $G_1 \cdots G_r$. By Lemma 2.7(b), the Lusztig series $\mathcal{E}(G,s)$ contains a cuspidal character only if each of these factors G_i has a cuspidal unipotent character ρ_i. If all factors are non-isomorphic, then for the factor of largest rank, say G_1, take a Zsigmondy prime ℓ as in Lemma 2.2. (Note that the exception $D_4(2)$ does not occur here as q is odd.) Then ρ_1 lies in an ℓ-block of cyclic defect, while all the other ρ_i are of ℓ-defect 0. As Jordan decomposition preserves blocks with cyclic defect and their Brauer trees (see [10]), the Jordan correspondent ρ of $\rho_1 \otimes \cdots \otimes \rho_r$ then also lies in a block of cyclic defect. So we conclude with Remark 2.6(a) using the sequence of characters from Lemma 2.2.

For example, in $G = \mathrm{Sp}_{2n}(q)$ centralisers of quasi-isolated elements s in $G^* = \mathrm{SO}_{2n+1}(q)$ with $|C_{\mathbf{G}^*}(s) : C^\circ_{\mathbf{G}^*}(s)| = 2$ have types ${}^\delta D_d(q) B_{n-d}(q)$ for $1 \le d \le n$ and $\delta \in \{\pm\}$. These possess a cuspidal unipotent character $\rho_d \otimes \rho_{n-d}$ only if $d = a^2$ and $n - d = b(b+1)$ for some $a, b \ge 1$. Now for $a > b$ the cuspidal unipotent character ρ_d of ${}^\delta D_d(q)$ lies in a Brauer tree for primitive prime divisors ℓ of $q^{2a-1} + 1$, while ρ_{n-d} is of ℓ-defect zero in $B_{n-d}(q)$; and for $a \le b$ we have that ρ_{n-d} lies in a Brauer tree for primitive prime divisors ℓ of $q^{2b} + 1$, and ρ_d is of ℓ-defect zero.

Thus we are only left with those cases when $C_{G^*}(s)$ has two isomorphic quasi-simple factors, and these are the factors of largest rank. The corresponding lines are marked (1)–(5) in the last column of Table 2. We discuss them individually.

Cases 2, 4 and 5: Note that here cuspidal characters only arise when $\epsilon = -$, so when $q \equiv 3 \pmod 4$. But then q is not a square and so field automorphisms have odd order. In Case 2 the outer automorphism group of G is the direct product of the diagonal automorphism group A of order 2 with an odd order group of field automorphisms. So the latter must act trivially on all A-orbits in $\mathcal{E}(G,s)$. The cuspidal characters as well as the semisimple characters both lie in A-orbits of length 2, so we are done.

Table 2. Disconnected centralisers of quasi-isolated elements in classical groups $\mathbf{G}^*$

$\mathbf{G}^{*F}$	$o(s)$	$C^\circ_{\mathbf{G}^*}(s)^F$		$A(s)^F$	conditions	$\mathcal{E}_{\text{cusp}}(\mathbf{G}^F,s)\neq\emptyset$ if	
$B_n(q)$	2	$B_{n-d}(q).{}^\delta D_d(q)$		2	$1\le d\le n$	$n-d=2\triangle$, $d=\square$	
$C_n(q)$	2	$C_{n/2}(q)^2$		2	n even	$n=4\triangle$	1)
$(n\ge 3)$	2	$C_{n/2}(q^2)$		2	n even	$n=4\triangle$	
	2	${}^\delta A_{n-1}(q).(q-\delta 1)$		2		$\delta=-$, $n=\triangle$	
	4	$C_d(q)^2.{}^\epsilon A_{n-2d-1}(q).(q-\epsilon 1)$		2	$1\le d<\frac{n}{2}$	$d=2\triangle$, $\epsilon{=}-$, $n-2d{=}\triangle$	2)
	4	$C_d(q^2).{}^{-\epsilon}A_{n-2d-1}(q).(q+\epsilon 1)$		2	$1\le d<\frac{n}{2}$	$d=2\triangle$, $\epsilon{=}+$, $n-2d{=}\triangle$	
$D_n(q)$	2	${}^\delta D_d(q).{}^\delta D_{n-d}(q)$		2	$1\le d<\frac{n}{2}$	$d=\square$, $n-d=\square$	
$(n\ge 4)$	2	${}^\delta D_{n/2}(q)^2$		2^2	n even	$n=2\square$	3)
	2	$D_{n/2}(q^2)$	$(2\times)$	2^2	n even	$n=2\square$	
	2	${}^\delta A_{n-1}(q).(q-\delta 1)$	$(2\times)$	2	n even	$\delta=-$, $n=\triangle$	
	4	${}^\delta D_d(q)^2.{}^\epsilon A_{n-2d-1}(q).(q-\epsilon 1)$		2^2	n even, $1\le d<\frac{n}{2}$	$d=\square$, $\epsilon{=}-$, $n-2d=\triangle$	4)
	4	$D_d(q^2).{}^{-\epsilon}A_{n-2d-1}(q).(q+\epsilon 1)$	$(2\times)$	2^2	n even, $1\le d<\frac{n}{2}$	$d=\square$, $\epsilon{=}+$, $n-2d=\triangle$	
	4	${}^\delta D_d(q)^2.A_{n-2d-1}(q).(q-1)$		4	n odd, $1\le d<\frac{n}{2}$, $\epsilon{=}+$	never	
	4	${}^2D_d(q^2).{}^2A_{n-2d-1}(q).(q+1)$	$(2\times)$	4	n odd, $1\le d<\frac{n}{2}$, $\epsilon{=}+$	$d=\square$, $n-2d=\triangle$	
	4	$D_d(q).{}^2D_d(q).{}^2A_{n-2d-1}(q).(q+1)$		2	n odd, $1\le d<\frac{n}{2}$, $\epsilon{=}-$	$d=1$, $n-2=\triangle$	
	4	$D_d(q^2).A_{n-2d-1}(q).(q-1)$		2	n odd, $1\le d<\frac{n}{2}$, $\epsilon{=}-$	never	
${}^2D_n(q)$	2	${}^\delta D_d(q).{}^{-\delta}D_{n-d}(q)$		2	$1\le d<n/2$	$d=\square$, $n-d=\square$	
$(n\ge 4)$	2	$D_{n/2}(q).{}^2D_{n/2}(q)$		2	n even	never	
	2	${}^2D_{n/2}(q^2)$		2	n even	$n=2\square$	
	4	$D_d(q).{}^2D_d(q).{}^\epsilon A_{n-2d-1}(q).(q-\epsilon 1)$		2	n even, $1\le d<\frac{n}{2}$	$d=1$, $\epsilon{=}-$, $n-2=\triangle$	
	4	${}^2D_d(q^2).{}^{-\epsilon}A_{n-2d-1}(q).(q+\epsilon 1)$		2	n even, $1\le d<\frac{n}{2}$	$d=\square$, $\epsilon{=}+$, $n-2d=\triangle$	
	4	$D_d(q).{}^2D_d(q).A_{n-2d-1}(q).(q-1)$		2	n odd, $1\le d<\frac{n}{2}$, $\epsilon{=}+$	never	
	4	$D_d(q^2).{}^2A_{n-2d-1}(q).(q+1)$		2	n odd, $1\le d<\frac{n}{2}$, $\epsilon{=}+$	$d=\square$, $n-2d=\triangle$	
	4	${}^\delta D_d(q)^2.{}^2A_{n-2d-1}(q).(q+1)$		4	n odd, $1\le d<\frac{n}{2}$, $\epsilon{=}-$	$d=\square$, $n-2d=\triangle$	5)
	4	${}^2D_d(q^2).A_{n-2d-1}(q).(q-1)$	$(2\times)$	4	n odd, $1\le d<\frac{n}{2}$, $\epsilon{=}-$	never	

Here $\epsilon\in\{\pm\}$ is such that $q\equiv\epsilon 1\ (4)$, $\delta\in\{\pm\}$, ${}^\pm D_1(q)$ denotes a torus of order $q\mp 1$, $\square$ a square, $\triangle$ a triangular number.

In Case 5 a Sylow 2-subgroup A of the outer automorphism group of G consists of the diagonal automorphism group A_0 of order 4 extended by the cyclic group of graph-field automorphisms. This has as quotient a dihedral group of order 8, as the graph-field automorphism of $\mathrm{Spin}_{2n}^-(q)$ centralises a subgroup $\mathrm{Spin}_{2n-1}(q)$ and so acts non-trivially on the centre.

Clearly automorphisms of odd order must act trivially on any A_0-orbit in $C_G(s)$ which they fix. Now let $\gamma \in A$ be an automorphism of 2-power order moving a cuspidal character ρ. As there is just one class of elements s with the relevant centraliser, the image of ρ must lie in the A_0-orbit R of ρ, and A acts faithfully on R. The same then holds for the A-orbit of semisimple characters in $\mathcal{E}(G, s)$. In particular γ also moves some semisimple character and vice versa.

The same argument applies to Case 4: again a Sylow 2-subgroup of the outer automorphism group has a dihedral quotient as the graph automorphism interchanges the two half-spin groups.

Case 1: Here, cuspidal characters occur if $n = 2r = 2a(a+1)$. Let $\mathbf{L} \leq \mathbf{G}$ be an F-stable Levi subgroup of twisted type ${}^2A_{n-1}(q).\Phi_2$ containing a Sylow 2-torus of $C_{\mathbf{G}^*}(s)$, with dual $\mathbf{L}^* \leq \mathbf{G}^*$. Then $C_{\mathbf{L}^*}(s)$ is disconnected of type $C_{\mathbf{L}^*}(s)^F = ({}^2A_{r-1}(q).\Phi_2)\wr 2$. Let $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be our regular embedding, and $\widetilde{\mathbf{L}} = \mathbf{L}Z(\widetilde{\mathbf{G}})$. Let $\tilde{s}$ be an F-stable preimage of s in $\widetilde{\mathbf{G}}^*$. Then $C_{\widetilde{\mathbf{G}}^*}(\tilde{s}) = \widetilde{\mathbf{G}}_1^2$ is connected with $\widetilde{\mathbf{G}}_1^F$ of type $C_r(q)$, and $C_{\widetilde{\mathbf{L}}^*}(\tilde{s}) = \widetilde{\mathbf{L}}_1^2$ is connected with $\widetilde{\mathbf{L}}_1^F$ of type ${}^2A_{r-1}(q).\Phi_2$. By Lemma 2.3(1) there is a chain of unipotent characters $\psi_{1,i}$ of $\tilde{L}_1$ and of unipotent characters $\rho_1 = \rho_{1,1}, \ldots, \rho_{1,m} = \chi_1$ of $\tilde{G}_1$ connecting the cuspidal unipotent character ρ_1 to the semisimple character $\chi_1 = 1_{G_1}$ such that $R_{\widetilde{\mathbf{L}}_1}^{\widetilde{\mathbf{G}}_1}(\psi_{1,i})$ contains $\rho_{1,i}, \rho_{1,i+1}$ exactly once. Jordan decomposition maps $\mathcal{E}(\tilde{G}, \tilde{s})$ to $\mathcal{E}(\tilde{G}_1, 1) \times \mathcal{E}(\tilde{G}_1, 1)$, and it maps $\mathcal{E}(\tilde{L}, \tilde{s})$ to $\mathcal{E}(\tilde{L}_1, 1) \times \mathcal{E}(\tilde{L}_1, 1)$. As Lusztig induction of unipotent characters commutes with products, this implies that $R_{\widetilde{\mathbf{L}}_1^2}^{\widetilde{\mathbf{G}}_1^2}(\psi_{1,i}^{\otimes 2})$ contains $\rho_{1,i}^{\otimes 2}$ and $\rho_{1,i+1}^{\otimes 2}$ exactly once.

Let $\tilde{\rho}_i \in \mathrm{Irr}(\tilde{G})$ correspond to $\rho_{1,i}^{\otimes 2}$ under Jordan decomposition, and $\tilde{\psi}_i \in \mathrm{Irr}(\tilde{L})$ correspond to $\psi_{1,i}^{\otimes 2}$. As $\widetilde{\mathbf{L}}$ is of type A, all of its unipotent characters are uniform. Now Jordan decomposition commutes with Deligne–Lusztig induction, so $R_{\widetilde{\mathbf{L}}}^{\widetilde{\mathbf{G}}}(\tilde{\psi}_i)$ contains $\tilde{\rho}_i$ and $\tilde{\rho}_{i+1}$ exactly once. By their description above, the restriction of $\tilde{\psi}_i$ to L splits into two constituents ψ_i, ψ_i', and the restriction of $\tilde{\rho}_i$ to G splits into two constituents ρ_i, ρ_i'. Let γ be an automorphism of G, then we may choose L to be γ-stable, and all characters in the Lusztig series $\mathcal{E}(\widetilde{\mathbf{G}}, \tilde{s})$ and $\mathcal{E}(\widetilde{\mathbf{L}}, \tilde{s})$ are γ-stable. We are thus in the situation of Remark 2.6(b) and may conclude.

Case 3: The argument here is very similar, using a Levi subgroup $\mathbf{L} \leq \mathbf{G}$ of type $D_2(q).{}^2A_{n-3}(q).\Phi_2$ with $C_{\mathbf{L}^*}(s)$ disconnected of type ${}^2A_{r-2}(q)^2.\Phi_2^4.2^2$ where $n = 2r = 2a^2$, and applying Lemma 2.3(2). Observe that $n \neq 4$ so that there are no triality automorphisms and L can again be chosen γ-invariant. □

Remark 2.9. By [6] the conclusion of Theorem 2.8 also holds for $\mathbf{G}$ of type A_n. For type C_n our result also follows from the recent work of Cabanes–Späth [7] and of Taylor [20].

3. Cuspidal characters in exceptional groups

We now turn to groups of exceptional type. Here, there exist Lusztig series containing more than one cuspidal character even for groups with connected centre, which makes the situation somewhat more involved.

3.1. **Automorphisms in type E_6.** We start by considering $\mathbf{G}$ simple simply connected of type E_6. Then $\mathbf{G}$ has a graph automorphism of order 2 which we may choose to commute with F.

The interesting case occurs when there exist non-trivial diagonal automorphisms of G. If F is untwisted, so $G = \mathbf{G}^F = E_6(q)_{\mathrm{sc}}$, this happens when $q = p^f \equiv 1 \pmod 3$. Then $\mathrm{Out}(G) \cong \mathfrak{S}_3 \times C_f$, with the first factor inducing diagonal and graph automorphisms, the second the field automorphisms if $p \equiv 1 \pmod 3$, respectively the product of the field automorphism by the graph automorphism if $p \equiv 2 \pmod 3$. If F is twisted, so $G = {}^2E_6(q)_{\mathrm{sc}}$, we have non-trivial diagonal automorphisms for $q = p^f \equiv -1 \pmod 3$. Then q is not a square, so f is odd, and again $\mathrm{Out}(G) \cong \mathfrak{S}_3 \times C_f$, with the symmetric group $\mathfrak{S}_3$ inducing the diagonal automorphisms and the graph-field automorphism γ, and the cyclic group C_f inducing the field automorphisms. We need the following elementary observation.

Lemma 3.1. *The group $\mathfrak{S}_3 \times C_f$ has a unique action up to permutation equivalence on a set of three elements in such a way that the elements of order 3 in the first factor act non-trivially.*

Proof. If the elements of order 3 in the $\mathfrak{S}_3$-factor act non-trivially, then the whole $\mathfrak{S}_3$-factor must act faithfully, via its natural permutation representation. As the cyclic factor C_f must centralise the $\mathfrak{S}_3$-factor in this action, it can only act trivially. So the action is unique up to permutation equivalence. □

Thus we see that if $X \in \mathrm{Irr}(G)$ is an orbit (of length 3) under diagonal automorphisms which is stable under the graph respectively graph-field automorphism, then the action of $\mathrm{Aut}(G)_X$ on X is uniquely determined.

3.2. **Cuspidal characters in quasi-isolated series.** Next, we consider cuspidal characters in quasi-isolated series in exceptional type groups. Let $\mathbf{G}$ be simple simply connected with dual $\mathbf{G}^*$ such that $G = \mathbf{G}^F$ is of exceptional type, and let $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be a regular embedding.

Theorem 3.2. *Let G be quasi-simple of exceptional type, $s \in G^*$ quasi-isolated and $\rho \in \mathcal{E}(G,s)$ cuspidal. Then there is a semisimple character $\chi \in \mathcal{E}(G,s)$ with the same stabiliser as ρ in* $\mathrm{Aut}(G)$.

Proof. The quasi-isolated elements $s \in G^*$ were classified by Bonnafé [1]. We deal with the various possibilities case-by-case. First consider elements with connected centraliser $\mathbf{C} = C_{\mathbf{G}^*}(s)$. Here we claim that cuspidal characters in $\mathcal{E}(G,s)$ are fixed by all automorphisms. If $s = 1$, so $\rho \in \mathcal{E}(G,1)$ is cuspidal unipotent then by results of Lusztig ρ is invariant under all automorphisms of G, see e.g. [14, Thm. 2.5], as is the semisimple character $1_G \in \mathcal{E}(G,1)$. Next assume that $s \neq 1$. If $\mathbf{C}$ has only components of classical type and $\mathbf{C}^F$ does not involve ${}^3D_4$, then each such component has at most one cuspidal unipotent character and one semisimple character in any Lusztig series, and hence by Lemma 2.7(a) the same is true for $\mathcal{E}(G,s)$. The claim then follows trivially. Connected centralisers of isolated elements $s \neq 1$ with an exceptional component are of type E_6 in E_7, or of types $E_6.A_2$ or $E_7.A_1$ in type E_8. Now, the cuspidal unipotent characters of $E_6(q)$ lie on a Brauer tree for Zsigmondy primes dividing Φ_9 together with the trivial character. Thus they lie on such a Brauer tree together with the semisimple character if they occur in quasi-isolated series of type E_6 or $E_6.A_2$. Our claim thus follows from Lemma 2.6(a). Similarly two of the three cuspidal unipotent characters of ${}^2E_6(q)$ lie on the Φ_{18}-Brauer tree, and the third one is uniquely determined by its degree; and the two cuspidal unipotent characters of $E_7(q)$ lie on a Φ_{18}-Brauer tree and we conclude as before.

Thus we may now assume that $C_{\mathbf{G}^*}(s)$ is not connected. As centralisers of semisimple elements are connected in a group whose dual has connected centre, we are in one of three situations: $q \equiv 1 \pmod 3$ for $G = E_6(q)_{\mathrm{sc}}$, $q \equiv -1 \pmod 3$ for $G = {}^2E_6(q)_{\mathrm{sc}}$, or q is odd for $G = E_7(q)_{\mathrm{sc}}$. The various rational forms of the occurring types of disconnected centralisers $C_{\mathbf{G}^*}(s)$ with cuspidal unipotent characters can be computed using Chevie; they are collected in Table 3, depending on certain congruence conditions on q. The column labelled "$|\mathcal{E}(G,s)|$" gives the number of regular orbits (of length 3 or 2) under the group of diagonal automorphisms and of orbits of length 1 respectively, and similar the last column gives the same information for the subset of cuspidal characters. Note that we only need to concern ourselves with the regular orbits under diagonal automorphisms, and that the semisimple characters always form a regular orbit.

$\mathbf{G}^*$	$C_{\mathbf{G}^*}(s)^F$		q	$\lvert\mathcal{E}(G,s)\rvert$	$\lvert\mathcal{E}_{\mathrm{cusp}}(G,s)\rvert$
$E_6(q)$	${}^3D_4(q).\Phi_3.3$	$(2\times)$	$\equiv 1\ (3)$	8×3	2×3
${}^2E_6(q)$	${}^2A_2(q)^3.3$		$\equiv 2\ (3)$	$3\times 3 + 8\times 1$	1×3
	${}^2A_2(q^3).3$	$(2\times)$	$\equiv 2\ (3)$	3×3	1×3
	$D_4(q).\Phi_2^2.3$		$\equiv 2\ (3)$	$8\times 3 + 2\times 1$	1×3
	${}^3D_4(q).\Phi_6.3$	$(2\times)$	$\equiv 2\ (3)$	8×3	2×3
$E_7(q)$	${}^2E_6(q).\Phi_2.2$		$\equiv 1\ (2)$	30×2	3×2
	$D_4(q).A_1(q)^2.\Phi_2.2$		$\equiv 3\ (4)$	$20\times 2 + 18\times 1$	1×2
	${}^2A_2(q)^3.\Phi_2.2$		$\equiv 5\ (6)$	$9\times 2 + 9\times 1$	1×2

Table 3. Cuspidal characters in Lusztig series of quasi-isolated elements with disconnected centralisers in exceptional types

We consider these in turn, starting with $G = E_6(q)_{\mathrm{sc}}$. It can be checked by direct computation in Chevie (see [17]) that the two classes of quasi-isolated elements are invariant under the graph automorphism. In particular the corresponding Lusztig series must be invariant under the graph automorphism of G. Moreover, the $\tilde{G}$-orbits of cuspidal characters in those series are invariant by degree reasons. So our claim follows from Lemma 3.1. The situation is entirely similar for the six quasi-isolated series in ${}^2E_6(q)_{\mathrm{sc}}$.

In $G = E_7(q)_{\mathrm{sc}}$, in the second case the cuspidal and the semisimple characters are contained in Brauer trees for Zsigmondy primes for Φ_6 (note that $q \neq 2$ as q is odd). By the Bonnafé–Rouquier Morita equivalence these characters are non-exceptional, so Remark 2.6 applies.

In the first case, let $\mathbf{L}^* \leq \mathbf{G}^*$ be a Levi subgroup of type $A_2(q).A_1(q^3).\Phi_3$ belonging to the nodes $1, 2, 3, 5, 7$ of the Dynkin diagram in the standard Bourbaki numbering used for example in Chevie. A Chevie-calculation shows that its dual Levi subgroup $\mathbf{L} \leq \mathbf{G}$ has disconnected centre, and $C_{\mathbf{L}^*}(s)$ is of type $A_2(q).\Phi_2\Phi_3\Phi_6.2$. Application of [3, Thm. 3.2] gives that $R_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}(\psi)$ contains the cuspidal character ρ with label ${}^2E_6[1]$ as well as the semisimple character in $\mathcal{E}(\tilde{G}, \tilde{s})$ exactly once, where ψ is the semisimple character in $\mathcal{E}(\tilde{L}, \tilde{s})$. So the claim holds for ρ by Remark 2.6(b). The two other cuspidal characters in this series lie on a Brauer tree for Zsigmondy primes dividing Φ_{18} and we can use Lemma 2.4.

Finally, for the last case, take a 6-split Levi subgroup $\mathbf{L}^* \leq \mathbf{G}^*$ for the nodes $2, 5, 7$, of type $A_1(q^3).\Phi_6^2$, whose dual again has disconnected centre, and with $C_{\mathbf{L}^*}(s)$ of type $(q^3+1)\Phi_6^2.2$. Then $R_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}(\psi)$ contains the cuspidal character in $\mathcal{E}(\tilde{G}, \tilde{s})$ as well as the semisimple character exactly once and we conclude as before. □

Corollary 3.3. *Let ρ be a cuspidal character of a quasi-simple group G of Lie type in a quasi-isolated series and assume we are neither in cases (3) or (4) of Table 2 nor in the first case of Table 3. Then some $\tilde{G}$-conjugate of ρ extends to its inertia group in the extension $\hat{G}$ of G by graph and field automorphisms. In particular ρ satisfies part (ii) of the inductive McKay condition from* [15, Thm. 2.1].

Proof. For groups of types A_n and ${}^2A_n$ this has been shown by Cabanes–Späth [6]. For the other types, by the proofs of Theorem 2.8 and 3.2 we have connected ρ via a sequence of Brauer trees or of Deligne–Lusztig characters to a semisimple character χ in the same Lusztig series. According to [19, Prop. 3.4(c)] there exists a semisimple character χ' in the $\tilde{G}$-orbit of χ that satisfies the cited condition (ii) and thus in particular extends to its inertia group I in $\hat{G}$. Thus, the corresponding $\tilde{G}$-conjugate ρ' of ρ also has inertia group I.

If ρ, χ (and hence ρ', χ') are connected via Brauer trees, ρ' extends to I by Corollary 2.5. In the other cases not excluded in the statement, all Sylow subgroups of $\hat{G}/G$ are cyclic, and so ρ' also extends by elementary character theory. This yields part (ii) of the inductive McKay condition. □

4. Proof of Theorem 1

We are now ready to prove our main result. Here, for finite groups $U \trianglelefteq V$ with characters $\chi \in \mathrm{Irr}(V)$, $\psi \in \mathrm{Irr}(U)$ we write $\mathrm{Irr}(U \mid \chi)$ for the constituents of $\chi|_U$, and $\mathrm{Irr}(V \mid \psi)$ for those of ψ^V.

Lemma 4.1. *Let $\mathbf{G}$ be a connected reductive group with Frobenius map F and $\mathbf{G}_0 \trianglelefteq \mathbf{G}$ a closed F-stable normal subgroup such that $[\mathbf{G}, \mathbf{G}] = [\mathbf{G}_0, \mathbf{G}_0]$. Let $\chi_0 \in \mathrm{Irr}(\mathbf{G}_0^F)$ and $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ be semisimple characters. Then:*

(a) *All characters in $\mathrm{Irr}(\mathbf{G}^F \mid \chi_0)$ and in $\mathrm{Irr}(\mathbf{G}_0^F \mid \chi)$ are semisimple.*

(b) *Let $\pi : \mathbf{G}^* \to \mathbf{G}_0^*$ be the dual epimorphism and $s_0 \in \mathbf{G}_0^{*F}$ such that $\chi_0 \in \mathcal{E}(\mathbf{G}_0^F, s_0)$. Then $|\mathcal{E}(\mathbf{G}^F, s) \cap \mathrm{Irr}(\mathbf{G}^F \mid \chi_0)| = 1$ for all $s \in \mathbf{G}^{*F}$ with $\pi(s) = s_0$.*

Proof. Choose a regular embedding $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$, then $\mathbf{G}_0 \hookrightarrow \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ is also regular. The first statement now follow immediately from the definition of semisimple characters as the

Alvis–Curtis duals of regular characters, see [2, §15A]. For part (b) set $G := \mathbf{G}^F$, $G_0 := \mathbf{G}_0^F$, and $A_G(s) := (C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s))^F$. By Lusztig's result [2, Thm. 11.12] the induction $\mathrm{Ind}_{G_0}^G(\chi_0)$ is multiplicity-free and by [2, Prop. 15.13] it has exactly $|G : G_0|/|A_{G_0}(s_0) : A_G(s)|$ constituents. On the other hand, by direct counting this is exactly the number of semisimple classes of $\mathbf{G}^{*F}$ lying above the class of s_0. By [2, Prop. 11.7] and part (a) any corresponding Lusztig series will contain a character from $\mathrm{Irr}(\mathbf{G}^F \mid \chi_0)$. The pigeonhole principle now shows that each such series will contain exactly one such character. □

Proof of Theorem 1. Let $\mathbf{G}$ be simple, simply connected and $F : \mathbf{G} \to \mathbf{G}$ such that $G = \mathbf{G}^F$ is quasi-simple. Let $\rho \in \mathrm{Irr}(G)$ be cuspidal and $s \in G^*$ such that $\rho \in \mathcal{E}(G, s)$. Let $\mathbf{L}^* \leq \mathbf{G}^*$ be a minimal F-stable Levi subgroup of $\mathbf{G}^*$ containing $C_{\mathbf{G}^*}(s)$ with dual $\mathbf{L}$. According to [20, Thm. 9.5] the Jordan decomposition between $\mathcal{E}(G, s)$ and $\mathcal{E}(L, s)$ induced by $R_{\mathbf{L}}^{\mathbf{G}}$ commutes with any $\gamma \in \mathrm{Aut}(G)_s$, and by Lemma 2.7 it sends cuspidal characters to cuspidal characters and semisimple characters to semisimple ones. Here $\mathrm{Aut}(G)_s$ denotes the stabiliser in $\mathrm{Aut}(G)$ of $\mathcal{E}(G, s)$. Note that by construction s is quasi-isolated in $\mathbf{L}^*$. Thus we have reduced our question to the corresponding one for quasi-isolated series in L.

Let $\mathbf{L}_0 := [\mathbf{L}, \mathbf{L}]$ and let $s_0 \in \mathbf{L}_0^*$ be the image of s under the natural epimorphism $\mathbf{L}^* \to \mathbf{L}_0^*$ induced by the embedding $\mathbf{L}_0 \leq \mathbf{L}$. Clearly s_0 is quasi-isolated in $\mathbf{L}_0^*$. As $\mathbf{G}$ is of simply connected type, so is $\mathbf{L}_0$ (see [16, Prop. 12.14]), hence a direct product $\mathbf{L}_1 \cdots \mathbf{L}_r$ of F-orbits $\mathbf{L}_i$, $1 \leq i \leq r$, of simple components of $\mathbf{L}_0$. Correspondingly, we have $\mathbf{L}_0^F = L_1 \cdots L_r$ with quasi-simple finite groups of Lie type L_i. Any irreducible character χ of L_0 is an outer tensor product of irreducible characters χ_i of the L_i, which are cuspidal, respectively semisimple, if and only if χ is. Moreover, if $\chi \in \mathcal{E}(L_0, s_0)$ then $\chi_i \in \mathcal{E}(L_i, s_i)$, with s_i the image of s_0 under the epimorphism $\mathbf{L}_0^* \to \mathbf{L}_i^*$ induced by the embedding $\mathbf{L}_i \hookrightarrow \mathbf{L}_0$. Again, the s_i are quasi-isolated in $\mathbf{L}_i^*$. Now for cuspidal characters in quasi-isolated series of quasi-simple groups, our claim holds: For groups of types A_n and ${}^2A_n$ it follows from [6], for the other groups of classical type it is contained in Theorem 2.8 and for groups of exceptional type all relevant cases have been dealt with in Theorem 3.2.

It remains to deduce the claim for $\mathbf{L}$ from the one for $\mathbf{L}_0$. First, for $\chi \in \mathcal{E}(L, s)$ and $\chi_0 \in \mathrm{Irr}(L_0 \mid \chi)$, we have by Lemma 4.1(a) that χ is semisimple if and only χ_0 is, and by the definition of cuspidality, χ is cuspidal if and only if χ_0 is. Furthermore, for χ semisimple we have $\mathrm{Irr}(L \mid \chi_0) \cap \mathcal{E}(L, s) = \{\chi\}$ by Lemma 4.1(b). Thus χ is uniquely determined by χ_0 (given its Lusztig series). Since our claim holds for $\mathbf{L}_0$, any cuspidal character $\chi_0 \in \mathcal{E}(L_0, s_0)$ has the same stabiliser in L as the semisimple characters in this series, and so again $\mathrm{Irr}(L \mid \chi_0) \cap \mathcal{E}(L, s) = \{\chi\}$ and χ is uniquely determined by χ_0 and s. Now note that L_0 is invariant under all automorphisms of L. But then our claim for the Lusztig series $\mathcal{E}(L, s)$ follows from the corresponding one for the Lusztig series $\mathcal{E}(L_0, s_0)$ of L_0. □

Example 4.2. Let $F' : \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism commuting with F and such that $Z(G) \subset \mathbf{G}^{F'}$. Let $s \in G^*$ such that $\mathcal{E}(G, s)$ is stable under the field or graph-field automorphism σ of G induced by F'. Then any cuspidal character $\rho \in \mathcal{E}(G, s)$ is σ-invariant. Indeed, by [4, Thm. 3.5] any such σ fixes all regular characters in $\mathcal{E}(G, s)$, hence by [20, Thm. 9.5] the (Alvis–Curtis dual) semisimple characters, hence ρ by Theorem 1.

5. On the inductive McKay condition

We apply our previous result to verify the inductive McKay condition for several series of simple groups at suitable primes. Let still $\mathbf{G}$ be simple of simply connected type, with regular embedding $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ and dual epimorphism $\pi : \widetilde{\mathbf{G}}^* \to \mathbf{G}^*$. By definition $\tilde{G}$ induces the full group of diagonal automorphisms of G. Let D be the group of automorphisms of $\tilde{G}$ induced by inner, graph and field automorphisms of G. We investigate the following property of characters $\tilde{\chi} \in \mathrm{Irr}(\tilde{G})$:

$$\textit{there exists } \chi \in \mathrm{Irr}(G \mid \tilde{\chi}) \textit{ which is } D_{\mathrm{Irr}(G|\tilde{\chi})}\textit{-stable;} \tag{$\dagger$}$$

here $D_{\mathrm{Irr}(G|\tilde{\chi})}$ denotes the stabiliser in D of the set of irreducible characters of G below $\tilde{\chi}$. The following is known:

Lemma 5.1. *Regular and semisimple characters of $\tilde{G}$ satisfy* ($\dagger$).

Proof. First let $\tilde{\chi} \in \mathrm{Irr}(\tilde{G})$ be a semisimple character. Then the claim is just [19, Prop. 3.4(c)]. Regular characters are the images of semisimple characters under the Alvis–Curtis duality (see [8, 14.39]), whose construction commutes with automorphism, so we conclude by the previous consideration. □

In the next result, $\mathrm{Irr}_{\ell'}(G)$ denotes the set of irreducible characters of G of ℓ'-degree.

Proposition 5.2. *Let $G = {}^2E_6(q)_{\mathrm{sc}}$ for a prime power q and $\ell|(q+1)$ be a prime. Then any $\tilde{\chi} \in \mathrm{Irr}(\tilde{G} \mid \mathrm{Irr}_{\ell'}(G))$ satisfies* ($\dagger$).

$C_{G^*}(s)$	q	$\lvert\mathcal{E}(G,s)\rvert$
${}^2A_2(q^3).3$ $(2\times)$	$\equiv 2\ (3)$	3×3
$D_4(q).\Phi_2^2.3$	$\equiv 2\ (3)$	$8\times 3 + 2\times 1$
$A_1(q)^4.\Phi_2^2.3$	$\equiv 5\ (6)$	$4\times 3 + 4\times 1$
$A_1(q)^3.\Phi_2^3.3$	$\equiv 2\ (3)$	$2\times 3 + 2\times 1$
$A_1(q).\Phi_2^5.3$	$\equiv 2\ (3)$	2×3
$\Phi_2^6.3$	$\equiv 2\ (3)$	1×3

Table 4. Some ℓ'-series in ${}^2E_6(q)_{\mathrm{sc}}$, $\ell|(q+1)$

Proof. Let $\tilde{s} \in \tilde{G}^*$ be semisimple such that $\tilde{\chi} \in \mathcal{E}(\tilde{G}, \tilde{s})$, and let $s = \pi(\tilde{s})$. If $\tilde{\chi}$ restricts irreducibly to G, the claim holds trivially. So we may assume $C_{\mathbf{G}^*}(s)$ is not connected and hence that $3|(q+1)$. From the list of character degrees of ${}^2E_6(q)_{\mathrm{sc}}$ provided by Lübeck [12], we obtain the Table 4 of conjugacy classes of semisimple elements $s \in G^*$ with $C_{\mathbf{G}^*}(s)^F \neq C^\circ_{\mathbf{G}^*}(s)^F$ and such that $\mathcal{E}(G,s)$ contains characters of ℓ'-degree. As in Table 3 we also give the number of $\tilde{G}$-orbits in $\mathcal{E}(G,s)$ (this is implicit in the data in [12]).

In the first three entries of Table 4 the element s is quasi-isolated, and the claim follows with Lemma 3.1 as in the proof of Theorem 3.2. In the other three cases the non-invariant characters are either regular or semisimple, so they all satisfy ($\dagger$) by Lemma 5.1. □

Thus we can prove our second main result:

Proof of Theorem 2. If $\ell|(q-1)$ the claim is contained in [15, Thm. 6.4(a)]. So now assume that $q \equiv -1 \pmod{\ell}$. For $S = {}^2E_6(q)$ all $\tilde\chi \in \mathrm{Irr}(\tilde G \mid \mathrm{Irr}_{\ell'}(G))$ satisfy property (†) by Proposition 5.2. As moreover here $D/\mathrm{Inn}(G)$ is cyclic, all such characters $\tilde\chi$ satisfy condition (ii)(1) in [15, Thm. 2.1] and thus the result follows from [15, Thm. 6.4(c)]. For the other families of groups, we have that $q \equiv -1 \pmod{\ell}$ with $\ell \equiv 3 \pmod 4$, so q is not a square. Thus S has no even order field automorphisms, whence $\mathrm{Out}(G)$ is cyclic. The claim then again follows from [15, Thm. 6.4(c)]. □

It was shown in [5, Cor. 7.3] that the inductive McKay condition holds for ${}^2E_6(q)$ at all primes $\ell \geq 5$ if $q \not\equiv -1 \pmod 3$, as well as for $E_7(q)$, $B_n(q)$ (and so $C_n(q)$) if q is even.

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