DECOMPOSITION MATRICES FOR LOW RANK UNITARY GROUPS

OLIVIER DUDAS AND GUNTER MALLE

Abstract. We study the decomposition matrices of the unipotent ℓ-blocks of finite special unitary groups $\mathrm{SU}_n(q)$ for unitary primes ℓ larger than n. Up to few unknown entries, we give a complete solution for $n = 2, \ldots, 10$. We also prove a general result for two-column partitions when ℓ divides $q + 1$. This is achieved using projective modules coming from the ℓ-adic cohomology of Deligne–Lusztig varieties.

1. Introduction

One of the fundamental tasks in the representation theory of finite groups is the determination of all decomposition matrices of the finite simple groups. This paper aims at understanding the decomposition matrices of unipotent blocks of finite unitary groups for unitary primes. This is considered a very hard problem; for example, the 3×3-decomposition matrix for the 3-dimensional unitary groups $\mathrm{SU}_3(q)$ has only been determined in 2002. Recall that for so-called *linear* primes, cuspidal unipotent modules in positive characteristic all occur as ℓ-modular reduction of cuspidal unipotent modules in characteristic zero, and in that case the modular representation theory of unitary groups is controlled by the representation theory of q-Schur algebras, as shown in [17]. No analog approach is known to exist for *unitary* primes, and in fact many new cuspidal modules can arise. The strategy of this paper is to adapt the theory of Deligne and Lusztig to the modular framework, and construct these cuspidal modules and their projective covers via the mod-ℓ cohomology of suitably chosen Deligne–Lusztig varieties.

Let $\mathbf{G}$ be a connected reductive algebraic group defined over $\overline{\mathbb{F}}_q$ with an additional $\mathbb{F}_q$-structure. Given an element w in the Weyl group W of $\mathbf{G}$, we can construct a virtual projective $\mathbf{G}(\mathbb{F}_q)$-module R_w given by the cohomology with compact support of the Deligne–Lusztig variety $\mathrm{X}(w)$. The classification of the irreducible characters (in characteristic zero) of $\mathbf{G}(\mathbb{F}_q)$ given by Lusztig [25] relies, amongst other things, on the following remarkable property of R_w:

> If ρ is a "new" cuspidal unipotent character occurring in R_w, then it occurs only in the middle degree of the cohomology of $\mathrm{X}(w)$ (see [24, Ex. 3.10(c)]).

The modular counterpart of this property is proved in [1, Prop. 8.10]. It gives some control on the multiplicity of projective covers of cuspidal modules in the virtual module R_w, and has already been shown to be a powerful tool to determine decomposition numbers in [6, 20]. In this paper we combine this new ingredient with standard methods involving Harish-Chandra induction from proper Levi subgroups, but also Harish-Chandra restriction from

Date: April 13, 2015.

1991 *Mathematics Subject Classification.* Primary 20C20,20C33; Secondary 20G05.

The second author gratefully acknowledges financial support by ERC Advanced Grant 291512.

suitable groups of larger rank, tensor products of characters and the known fact that the decomposition matrix is uni-triangular to determine, up to only few unknown entries when $n = 10$, the decomposition matrices of unipotent ℓ-blocks of $\mathrm{SU}_n(q)$ for $n \leq 10$ and ℓ a unitary prime, as well as the repartition of the simple modules into ℓ-modular Harish-Chandra series. Thus, parts of our paper can be considered as complementing James's determination [22] of the decomposition matrices of $\mathrm{GL}_n(q)$ for $n \leq 10$, as well as Geck, Hiss and the second authors determination [13] of the decomposition matrices of unitary groups $\mathrm{SU}_n(q)$ for $n \leq 10$ at linear primes.

Theorem A. *Let $\ell > 2n-3$ be a prime number not dividing q. Assume that the order of $-q$ modulo ℓ is odd. Then the decomposition matrices of the unipotent ℓ-blocks of $\mathrm{SU}_n(q)$ with $2 \leq n \leq 10$ are as given in Tables 1–15.*

(See the individual statements for better bounds on ℓ in the various cases.) Note that as expected, the entries we determine depend only on the order of $-q$ modulo ℓ, and not on q and ℓ individually.

In terms of Harish-Chandra theory, the most difficult situation is when the order of $-q$ modulo ℓ is 1, that is when ℓ divides $q+1$, in which case the largest number of cuspidal modules will occur. However, the virtual modules R_w are of particular interest here, and it is to be expected that they provide an alternative parametrization of cuspidal modules in terms of twisted conjugacy classes of the symmetric group. This is achieved here for classes corresponding to certain two-column partitions, and the corresponding part of the decomposition matrix is given as follows (see Theorem 5.9):

Theorem B. *Assume $\ell > n$ and $\ell \mid (q+1)$. Let $b \leq \lfloor n/3 \rfloor + 1$. Then the multiplicity of the unipotent character of $\mathrm{SU}_n(q)$ indexed by 2^b1^{n-2b} in the projective indecomposable module indexed by 2^c1^{n-2c} is given by $\binom{n-c-b}{c-b}$ if $b \leq c$ and 0 otherwise.*

The entries in that lower right-hand corner of the decomposition matrix are precisely those for which the previously known approaches tended not to yield any information at all. This example is also very instructive as it gives strong evidence (see Proposition 6.10) that an analogue of James's row removal rule for $\mathrm{GL}_n(q)$ (see [22, Rule 5.8]) should hold for $\mathrm{SU}_n(q)$ as well, see §5.4, although we do not even have a conjectural explanation for why that should be the case:

Conjecture C. *James's row and column removal rule holds for the decomposition matrices of $\mathrm{SU}_n(q)$.*

We hope to apply our new methods to other exceptional and classical type groups of low rank in a forthcoming paper.

The paper is organized as follows. In Section 2 we set our notation and recall some results from Deligne–Lusztig theory. In Section 3 we investigate a certain order relation on the set of twisted conjugacy classes in symmetric groups. This is then applied, after some preparations in Section 4, to prove in Section 5 the main result on decomposition numbers of $\mathrm{SU}_n(q)$ for 2-column partitions in Theorem 5.9. Finally, in Section 6 we apply the previously developed methods and results to determine most entries of the decomposition matrices of $\mathrm{SU}_n(q)$ for $n \leq 10$ at unitary primes.

Acknowledgement: We would like to thank Gerhard Hiss for pointing out an error in a previous version of Table 12.

2. Projective modules from Deligne–Lusztig characters

2.1. **Characters and basic sets.** Let ℓ be a prime number and $(K, \mathcal{O}, k)$ be an ℓ-modular system. We assume that it is large enough for all the finite groups encountered. Furthermore, since we will be working with ℓ-adic cohomology we will assume throughout this paper that K is a finite extension of $\mathbb{Q}_\ell$.

Let H be a finite group. A representation of H will always be assumed to be finite-dimensional. Given a simple kH-module N, we shall denote by φ_N (resp. P_N, resp. Ψ_N) its Brauer character (resp. its projective cover, resp. the character of its projective cover). The restriction of an ordinary character χ of H to the set of ℓ'-elements will be denoted by χ^0. It decomposes uniquely on the family of irreducible Brauer characters as $\chi^0 = \sum d_{\chi,\varphi}\varphi$. The coefficients $(d_{\chi,\varphi})_{\chi\in \mathrm{Irr} H, \varphi \in \mathrm{IBr} H}$ form the *decomposition matrix* of H. Equivalently, if $\varphi = \varphi_N$ is the Brauer character of a simple kH-module N, then $\Psi_N = \sum_{\chi \in \mathrm{Irr} H} d_{\chi,\varphi}\chi$ by Brauer reciprocity.

Recall that a *basic set of characters* is a set $\mathcal{B}$ of irreducible characters of H such that $\mathcal{B}^0 = \{\chi^0 \mid \chi \in \mathcal{B}\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}\mathrm{IBr}H$. This means that the restriction of the decomposition matrix to $\mathcal{B}$ is invertible over $\mathbb{Z}$. Let $\langle -, - \rangle$ be the usual inner product on the $\mathbb{C}$-vector space of class functions on H. For simple kH-modules N and M we have $\langle \Psi_N, \varphi_M\rangle = 0$ if N is not isomorphic to M and $\langle \Psi_N, \varphi_N \rangle = 1$. This inner product can be also computed using the basic set: write $\varphi = \sum_{\chi\in\mathcal{B}} a_{\varphi,\chi}\chi^0$. Then for any projective character Ψ, we have $\langle \Psi, \varphi\rangle = \langle \Psi, \sum_{\chi\in\mathcal{B}} a_{\varphi,\chi}\chi\rangle$ since Ψ vanishes outside the set of ℓ'-elements. In other words, $\langle \Psi, \varphi \rangle$ can be computed from the orthogonal projection of Ψ to the subspace of $\mathbb{C}\mathrm{Irr}H$ spanned by the basic set $\mathcal{B}$ and the expression of φ in $\mathcal{B}^0$.

2.2. **Finite reductive groups.** Let $\mathbf{G}$ be a connected reductive linear algebraic group over an algebraically closed field of positive characteristic p, and $F : \mathbf{G} \to \mathbf{G}$ be a Steinberg endomorphism. There exists a positive integer δ such that F^δ defines a split $\mathbb{F}_{q^\delta}$-structure on $\mathbf{G}$ (with $q \in \mathbb{R}_+$), and we will choose δ minimal for this property. We set $G := \mathbf{G}^F$.

Throughout this paper, we shall make the following assumptions on ℓ:

- $\ell \neq p$ (non-defining characteristic),
- ℓ is good for $\mathbf{G}$ and $\ell \nmid |(Z(\mathbf{G})/Z(\mathbf{G})^\circ)^F|$.

In this situation, the unipotent characters lying in a given unipotent ℓ-block of G form a basic set of this block [12, 11]. Consequently, the restriction of the decomposition matrix of the block to the unipotent characters is invertible. In particular every (virtual) unipotent character is a virtual projective character, up to adding and removing some non-unipotent characters. If N is a simple unipotent kG-module and P is a projective kG-module with character Ψ, the multiplicity of P_N as a direct summand of P is given by $\langle \Psi, \varphi_N\rangle = \langle P, N\rangle = \dim \mathrm{Hom}_{kG}(P, N)$. By extension, if P is a virtual unipotent module, we shall denote by $\langle P, N\rangle$ the multiplicity of P_N in any virtual projective module obtained from P by adding or removing non-unipotent modules (this does not depend on the modules added and removed since N is a unipotent module). Note that if we decompose P and N on the basic set of unipotent characters, $\langle P, N\rangle$ coincides with the usual scalar product of characters (see §2.1).

2.3. **Deligne–Lusztig characters.** In this section we recall/prove some general properties of modules related to Deligne–Lusztig induction.

We fix a pair $(\mathbf{T}, \mathbf{B})$ consisting of a maximal torus contained in a Borel subgroup of $\mathbf{G}$, both of which are assumed to be F-stable. We denote by W the Weyl group of $\mathbf{G}$, and by S the set of simple reflections in W associated with $\mathbf{B}$. Then F^δ acts trivially on W. Attached to $w \in W$ is the Deligne–Lusztig variety $\mathrm{X}(w)$, defining the virtual Deligne–Lusztig character

$$R_{\mathbf{T}_{wF}}^{\mathbf{G}}(1) := \sum_{i\in\mathbb{Z}} (-1)^i \big[H^i(\mathrm{X}(w))\big],$$

where $H^i(\mathrm{X}(w))$ denotes the ℓ-adic cohomology with coefficients in $K \supset \mathbb{Q}_\ell$. Following Lusztig [26, 17.2], for any F-stable irreducible character χ of W, one can choose a preferred extension $\widetilde{\chi}$ of χ to the group $W \rtimes \langle F\rangle$ which is trivial on F^δ. The *almost character* associated to $\widetilde{\chi}$ is then the uniform character

$$R_{\widetilde{\chi}} = \frac{1}{|W|} \sum_{w\in W} \widetilde{\chi}(wF) R_{\mathbf{T}_{wF}}^{\mathbf{G}}(1).$$

By [25, Thm. 4.23] the decomposition of almost characters in terms of unipotent characters is known explicitly. Conversely, for $w \in W$ the orthogonality relations for the Deligne–Lusztig characters yield

$$R_{\mathbf{T}_{wF}}^{\mathbf{G}}(1) = \sum_{\chi\in(\operatorname{Irr} W)^F} \widetilde{\chi}(wF) R_{\widetilde{\chi}}.$$

The Frobenius F^δ acts on the Deligne–Lusztig variety $\mathrm{X}(w)$, making the cohomology groups $H^i(\mathrm{X}(w))$ into $G \times \langle F^\delta\rangle$-modules. Digne and Michel have extended in [3] the previous formula to take into account this action, see Theorem 6.1 below. For $\lambda \in k^\times$, we denote by $R_w[\lambda]$ the virtual G-module obtained from R_w by keeping only the characters on which F^δ acts by an eigenvalue congruent to λ modulo ℓ. Recall from §2.2 that it is a virtual projective module, up to adding and removing non-unipotent modules.

For $x, y \in W$, we write $x < y$ when $x \neq y$ and x is smaller than y for the Bruhat order. Recall the following result from [6, Prop. 1.5].

Proposition 2.1. *Let $w \in W$ and N be a simple kG-module such that $\langle R_y, N\rangle = 0$ for all $y < w$. Then $\big\langle (-1)^{\ell(w)} R_w[\lambda], N\big\rangle \geq 0$ for all $\lambda \in k^\times$.*

Remark 2.2. If w is a Coxeter element of (W, F) then any $y < w$ lies in a proper F-stable parabolic subgroup. In that case R_y is obtained by Harish-Chandra induction. In particular, if N is cuspidal then the condition $\langle R_y, N\rangle = 0$ will be automatically satisfied for all such y. In other words, when w is a Coxeter element, the projective cover of any cuspidal unipotent kG-module appears with nonnegative multiplicity in $(-1)^{\ell(w)} R_w$.

3. Twisted conjugacy classes

In this section we deduce from [23, 15, 19] some results on the F-conjugacy classes in the symmetric group. The character of the virtual module R_w depends only on the F-conjugacy class of w, and these results will be needed for studying the decomposition matrices of $\mathrm{SU}_n(q)$ from the point of view of §2.3. We keep the notations from the previous section. In particular W is a Weyl group with set of simple reflections S.

3.1. **Ordering the conjugacy classes.** Given $x, y \in W$ and $s \in S$ we write $x \to^s y$ if $y = sxF(s)$ and $\ell(y) \leq \ell(x)$. By transitivity, we write $x \to y$ if there exists a sequence $x = x_0 \to^{s_1} x_1 \cdots \to^{s_r} x_r = y$. If moreover $y \to x$, then $\ell(x) = \ell(y)$ and we write $x \approx y$. Note that $x \approx y$ if and only if x and y are conjugate by a sequence of cyclic shifts (see [15, Remark 2.3]).

Given $w \in W$ we denote by $[w]_F = \{x^{-1}wF(x) \mid x \in W\}$ (or sometimes only by $[w]$) its F-conjugacy class, and by $[W]_F$ the set of all F-conjugacy classes of W. Given $\mathcal{O} \in [W]_F$, the subset of elements with minimal length in $\mathcal{O}$ will be denoted by $\mathcal{O}_{\min}$. We say that $\mathcal{O}$ is *cuspidal* (or *elliptic*) if it has empty intersection with every proper F-stable parabolic subgroup of W. The following result is proven in [16, 15, 18] (see also [19] for a case-free proof).

Theorem 3.1 (Geck–Pfeiffer, Geck–Pfeiffer–Kim, He)**.** *Let $\mathcal{O} \in [W]_F$. Then for all $w \in \mathcal{O}$ there exists $x \in \mathcal{O}_{\min}$ such that $w \to x$.*

Following [18, §4.7], we define a partial order on the set of conjugacy classes by $\mathcal{O}' \leq \mathcal{O}$ if and only if for all $x \in \mathcal{O}_{\min}$ there exists $x' \in \mathcal{O}'_{\min}$ such that $x' \leq x$ in the Bruhat order.

Lemma 3.2. *Let $x, y \in W$ and $s, t \in S$ be such that $y = sxt$ and $\ell(x) = \ell(y)$. If $x' \leq x$ then either $x' \leq y$ or $sx't \leq y$ with $\ell(sx't) \leq \ell(x')$.*

Proof. The result is trivial if $y = x$. Otherwise since $y = sxt$ and $\ell(y) = \ell(x)$ we can assume without loss of generality that $sx > x$ and $xt < x$ (see [21, Lemma 7.2]). If $x' \leq xt$ then $x' \leq xt < sxt = y$ and we are done. Otherwise $x' = zt$ with $z \leq xt$ and $z < x'$. Since $sxt > xt$ we have $sx't = sz \leq sxt = y$ and $\ell(sx't) = \ell(sz) \leq \ell(x')$. ☐

In particular, for $t = F(s)$, we deduce that if $x \to^s y$ and $x' \leq x$ then $x' \leq y$ or $sx'F(s) \leq y$ with $\ell(sx'F(s)) \leq \ell(x')$. Together with the following theorem this yields that when comparing the classes it is enough to look at one specific element of minimal length.

Theorem 3.3 (He-Nie [19])**.** *If $\mathcal{O} \in [W]_F$ is a cuspidal class then every two elements of $\mathcal{O}_{\min}$ are conjugate by a sequence of cyclic shifts.*

Corollary 3.4 (He)**.** *Let $\mathcal{O}$, $\mathcal{O}'$ be two F-conjugacy classes of W. Then the following are equivalent:*

(i) *$\mathcal{O}' \leq \mathcal{O}$, i.e., for all $x \in \mathcal{O}_{\min}$ there exists $x' \in \mathcal{O}'_{\min}$ such that $x' \leq x$, and*
(ii) *there exists $x \in \mathcal{O}_{\min}$ and $x' \in \mathcal{O}'_{\min}$ such that $x' \leq x$.*

Proof. It is enough to show that (ii) implies (i). Let $I \subset S$ (resp. I') be the *saturated* support of x (resp. x'), *i.e.*, the union of F-orbits of simple reflections in the support of x. Let $y \in \mathcal{O}_{\min}$ be another element of minimal length in the class of x and let J be its saturated support. Choose $w \in W$ such that $w^{-1}xF(w) = y$ and let z be the unique I-reduced-J element of W_IwW_J, where W_I denotes the standard parabolic subgroup of W generated by I. Let us decompose w as $w = azb$ with $a \in W_I$, $b \in W_J$ and az reduced-J. We set $\widetilde{x} = a^{-1}xF(a) \in W_I$ and $\widetilde{y} = byF(b)^{-1} \in W_J$. Since az is reduced-J, we have $\ell(x) = \ell(az\widetilde{y}F(az)^{-1}) \geq \ell(\widetilde{y})$, which forces $\widetilde{y} \in \mathcal{O}_{\min}$. Furthermore, since z^{-1} is reduced-I, we have $\ell(\widetilde{y}) = \ell(z^{-1}\widetilde{x}F(z)) \geq \ell(\widetilde{x})$ which forces $\widetilde{x} \in \mathcal{O}_{\min}$. By Theorem 3.3, we deduce that $\widetilde{x} \approx x$ (resp. $\widetilde{y} \approx y$) are conjugate by cyclic shifts. Note that since $z\widetilde{y} = \widetilde{x}F(z)$ and z is the unique element of minimal length in W_IzW_J we have $z = F(z)$.

Now, by successive applications of Lemma 3.2, we can find $\widetilde{x}'$ such that $\widetilde{x}' \leq \widetilde{x}$ and $\widetilde{x}' \approx x'$. Since z is I-reduced and $\widetilde{x}', \widetilde{x} \in W_I$, we have $\widetilde{x}'F(z) \leq \widetilde{x}F(z) = z\widetilde{y}$. With z being also reduced-J, we deduce that $\widetilde{x}'F(z) = z'\widetilde{y}'$ for some $\widetilde{y}' \leq \widetilde{y}$ and $z' \leq z$. But z is the unique element of minimal length in W_IzW_J, therefore we must have $z' = z$ and $\widetilde{y}' = z^{-1}\widetilde{x}'F(z)$ with $\ell(\widetilde{y}') = \ell(\widetilde{x}')$. We conclude using again Lemma 3.2 to find $y' \in \mathcal{O}'_{\min}$ such that $y' \approx \widetilde{y}'$ and $y' \leq y$. □

We have seen in §2.3 the importance of considering a module R_w when $w \in W$ is minimal for the property that a given projective indecomposable module appears as a constituent of R_w. Proposition 2.1 can be rephrased as follows:

Proposition 3.5. *Let N be a simple unipotent kG-module such that $\langle R_w, N\rangle \neq 0$ for some $w \in W$. Then there exists an F-conjugacy class $\mathcal{O} \in [W]_F$ such that:*

(1) *$\mathcal{O} \leq [w]_F$,*
(2) *$\langle (-1)^{\ell(x)}R_x, N\rangle > 0$ for all $x \in \mathcal{O}$, and*
(3) *$\langle R_x, N\rangle = 0$ whenever $[x]_F < \mathcal{O}$.*

Proof. Since R_w depends on $[w]_F$ only we can assume that w has minimal length in its conjugacy class. Take $z \leq w \in W$ to be minimal for the Bruhat order and for the property that $\langle R_z, N\rangle \neq 0$, and let us consider $\mathcal{O} = [z]_F$. The assertions of the proposition will follow from Proposition 2.1 if we can prove that $z \in \mathcal{O}_{\min}$.

By minimality of z, every $z' < z$ satisfies $\langle R_{z'}, N\rangle = 0$. Let $z \to^s y$ with $\ell(y) = \ell(z)$. Then by Lemma 3.2, the relation $y' < y$ forces $y' < z$ or $sy'F(s) < z$, and therefore $\langle R_{y'}, N\rangle = 0$ since $R_{sy'F(s)} = R_{y'}$. Now if z is not of minimal length in $\mathcal{O}$ then by Theorem 3.1 there exists $z \to x$ with $\ell(x) = \ell(z)$ and $t \in S$ such that $txF(t) < x$. This would force $\langle R_z, N\rangle = \langle R_{txF(t)}, N\rangle = 0$, hence a contradiction. □

Remark 3.6. It is not clear whether $\mathcal{O}$ as in Proposition 3.5 is unique, except when $\ell \nmid |G|$ (see [5, Prop. 3.3.21]).

3.2. **The case of the symmetric group.** From now on we shall assume that $\mathbf{G} = \mathrm{SL}_n(\overline{\mathbb{F}}_q)$, so that W can be identified with the symmetric group on n letters $\mathfrak{S}_n$. We write $s_i = (i, i+1)$ for $i = 1, \ldots, n-1$ and $S = \{s_1, \ldots, s_{n-1}\}$. The twisted Frobenius structure on $\mathrm{SL}_n(\overline{\mathbb{F}}_q)$ induces an automorphism of the Coxeter system (W, S), given by $F(w) = w_0ww_0$, where w_0 is the longest element of W. The map $w \longmapsto ww_0$ induces a bijection between F-conjugacy classes and usual conjugacy classes of $\mathfrak{S}_n$, which in turn are parametrized by partitions of n. The F-conjugacy class corresponding to the partition $\lambda \vdash n$ will be denote by $\mathcal{O}_\lambda$. Under this parametrization, cuspidal classes correspond to partitions of n with only odd terms (see for example [18, §7.14]).

One should be able to describe the partial ordering on F-conjugacy classes introduced above in terms of the partitions. It is likely that when restricted to the cuspidal conjugacy classes, it coincides with the dominance order on partitions. The following conjecture has been checked by computer for $n \leq 10$.

Conjecture 3.7. *Let λ, μ be two partitions having only odd parts, and $\mathcal{O}_\lambda$, $\mathcal{O}_\mu$ be the corresponding cuspidal classes. Then $\mathcal{O}_\lambda \leq \mathcal{O}_\mu$ if and only if $\mu \trianglelefteq \lambda$.*

Remark 3.8. Lusztig defined in [27] a map from the set of (usual) conjugacy classes of any Weyl group W to the set of unipotent classes of the corresponding split reductive group $\mathbf{G}$, and then generalized the construction to twisted conjugacy classes in [28]. Computations in small-rank classical groups and exceptional groups give evidence for this map to preserve the order, when restricted to cuspidal classes. Conjecture 3.7 predicts that this property holds for groups of type ${}^2A_n$.

Kim [23] gives an explicit representative of $\mathcal{O}_\lambda$ which has minimal length in $\mathcal{O}_\lambda$. The element is given as a permutation but we will need a reduced expression of it, that is, in terms of the simple reflections s_i. For $I \subset S$, we shall denote by w_I the longest element of the parabolic subgroup W_I.

Lemma 3.9. *Let $r \geq 0$ be an integer. Then*

$$\begin{aligned}\sigma_{2r+1} &:= (1, n, 2, n-1, 3, \ldots, n-r+1, r+1) = s_1 \cdots s_r w_0 w_{\{r+1,\ldots,n-1-r\}},\\ \sigma_{2r} &:= (1, n, 2, n-1, 3, \ldots, n-r+1) = s_1 \cdots s_{r-1} w_0 w_{\{r+1,\ldots,n-1-r\}}.\end{aligned}$$

If $I = \{i, i+1, \ldots, j\}$ is a set of consecutive integers, we will denote by σ_{2r+1}^I (resp. σ_{2r}^I) the analogue of σ_{2r+1} (resp. σ_{2r}) viewed as an element of the parabolic subgroup $W_{\{i,\ldots,j\}}$.

Recall from [23] that a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of n is said to be *maximal* if there exists $c \in \{0, \ldots, m\}$ such that

- $\lambda_1, \ldots, \lambda_c$ are even integers (in any ordering), and
- $\lambda_{c+1}, \ldots, \lambda_m$ is a decreasing sequence of odd integers.

In particular, any partition with only odd parts is a maximal composition. For such a composition, we can consider $I_j = \{\lfloor \frac{\lambda_1+\cdots+\lambda_{j-1}}{2} \rfloor + 1, \ldots, n-1-\lceil \frac{\lambda_1+\cdots+\lambda_{j-1}}{2} \rceil\}$, $1 \leq j \leq m$, and we set

$$\sigma_\lambda := \sigma_{\lambda_1}^{I_1} \cdots \sigma_{\lambda_c}^{I_c} \sigma_{\lambda_{c+1}}^{I_{c+1}} F(\sigma_{\lambda_{c+2}}^{I_{c+2}}) \cdots F^{m-c-1}(\sigma_{\lambda_m}^{I_m}).$$

Theorem 3.10 (Kim [23, Thm. 2.1])**.** *For any maximal composition λ of n, the element $\sigma_\lambda w_0$ is an element of $\mathcal{O}_\lambda$ of minimal length.*

From the expression of σ_{2r+1} one can prove the following result towards Conjecture 3.7:

Proposition 3.11. *Let $m \leq n/3$. Assume $\lambda = 3^b 1^{n-3b}$ and let μ be a partition of n into odd parts. Then $\mathcal{O}_\lambda \leq \mathcal{O}_\mu$ if and only if $\mu \trianglelefteq \lambda$.*

Proof. By [15, Lemma 3.2] the lengths of the cycles σ_{λ_i} add up to the length of σ_λ. Therefore if $\mu = 3^l 1^{n-3l}$ with $l \leq b$ then σ_μ is a left divisor of σ_λ and therefore in particular we have $\mathcal{O}_\lambda \leq \mathcal{O}_\mu$.

Conversely, if $\mathcal{O}_\lambda \leq \mathcal{O}_\mu$ then the following lemma forces μ to be of the form $3^l 1^{n-3l}$, and by the previous argument we have necessarily $l \leq b$, otherwise $\mathcal{O}_\mu$ would be strictly smaller than $\mathcal{O}_\lambda$. ∎

Lemma 3.12. *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ and $\mu = (\mu_1 \geq \mu_2 \geq \ldots)$ be two partitions with odd parts. Then $\mathcal{O}_\lambda \leq \mathcal{O}_\mu$ forces $\mu_1 \leq \lambda_1$.*

Proof. Let $y \in W$ be such that $yw_0 \in \mathcal{O}_\lambda$ and $yw_0 \leq \sigma_\mu w_0$. Let $r = (\mu_1 - 1)/2$. From Lemma 3.9 and the definition of σ_μ, we have

$$\sigma_\mu w_0 = (s_1 \cdots s_r w_{\{r+1,\ldots,n-r-1\}} w_0) \cdot \Big(\prod_{j=2}^{m} F^{j-1}(\sigma_{\mu_j}^{I_j}) \Big) \cdot w_0,$$

where each $\sigma_{\mu_j}^{I_j}$ is an element of W_{I_j} with $I_j = \{\lfloor \frac{\mu_1+\cdots+\mu_{j-1}}{2} \rfloor + 1, \ldots, n-1-\lceil \frac{\mu_1+\cdots+\mu_{j-1}}{2} \rceil\}$. For $j \geq 2$, we have $\mu_1+\cdots+\mu_{j-1} \geq \mu_1 = 2r+1$, and hence $\sigma_\mu w_0 \in s_1 \cdots s_r W_{\{r+1,\ldots,n-r-1\}}$. Now, since $yw_0 \leq \sigma_\mu w_0$ and yw_0 does not lie in any proper F-stable parabolic subgroup of W, we deduce that $yw_0 = s_1 \cdots s_r w$ for some $w \in W_{\{r+1,\ldots,n-r-1\}}$. Since $w(i) = i$ for $i \notin \{r+1, \ldots, n-r\}$, then y contains a cycle of the form $(1, n, 2, n-1, \ldots, n-r+1, r+1, y(r+1), \ldots)$ which has length at least $2r+1 = \mu_1$. This forces $\lambda_1 \geq \mu_1$. $\square$

Remark 3.13. The proof of the previous lemma does not use the fact that yw_0 is of minimal length in its conjugacy class. Note however that the assumption that it is cuspidal is crucial. Indeed, the order on non-cuspidal classes differs from the dominance order in general (see for example the following lemma).

Proposition 3.14. *Assume that $\lambda = 3^b21^{n-2-3b}$. If $\mathcal{O}_\lambda \leq \mathcal{O}_\mu$, then μ is one of the following partitions:*

(1) 3^c21^{n-2-3c} *with $c \leq b$,*
(2) 53^c1^{n-5-3c} *with $c \leq b-1$, or*
(3) 3^c1^{n-3c}.

Conversely, if μ is one of the partitions in (1) or (2) then $\mathcal{O}_\lambda \leq \mathcal{O}_\mu$.

Proof. Since the partition $\lambda = 3^b21^{n-2-3b}$ contains an even number, the corresponding class $\mathcal{O}_\lambda$ cannot be cuspidal. Therefore if $w \in \mathcal{O}_\lambda$ has minimal length then there exists a proper F-stable subset J of the set of simple reflections such that $w \in W_J$ and w is minimal in its F-conjugacy class in W_J (see [18, Lemma 7.3 and Th. 7.5]). Since ww_J and w_Jw_0 are two permutations with disjoint support, then ww_J must be of cycle type 3^b1^{n-2-3b} and w_Jw_0 a single transposition, which forces $J = \{s_2, \ldots, s_{n-2}\}$ and $w_Jw_0 = (1, n)$.

Assume now that $w \leq \sigma_\mu w_0$, with μ a maximal composition of n. Then the saturated support of $\sigma_\mu w_0$ contains J. From Lemma 3.9 we observe that $\sigma_\mu w_0 \in W_J$ if and only if $\mu_1 = 2$. In that case $\sigma_\mu w_0 w_J$ is a permutation of type $(\mu_2, \mu_3, \ldots)$. Moreover, since the saturated support of $\sigma_\mu w_0$ is exactly J and since it has minimal length then $(\mu_2, \mu_3, \ldots)$ is actually a partition of n with odd terms and we deduce from Proposition 3.11 that $\mu = 3^c21^{n-2-3c}$ with $c \leq b$. Conversely, if $\mu = 3^c21^{n-2-3c}$ with $c \leq b$ then σ_μ is a left divisor of σ_λ, so that $\mathcal{O}_\lambda \leq \mathcal{O}_\mu$ (as maximal compositions of n, μ is a truncation of λ).

If the saturated support of $\sigma_\mu w_0$ is the set of all simple reflections, then $\mathcal{O}_\mu$ is a cuspidal conjugacy class. If $\mu \neq 1^n$ we consider $s_1\sigma_\mu w_0 \in W_J$. Then $s_1\sigma_\mu w_0 w_J$ is a permutation of type $(\mu_1 - 2, \mu_2, \mu_3, \ldots)$. Let $r = (\mu_1 - 1)/2$ and $t = (\mu_2 - 1)/2 \leq r$. The product of the two first factors of σ_μ, namely of σ_{2r+1} and $F\big(\sigma_{2t+1}^{\{r+1,\ldots,n-r-2\}}\big)$ is given by

$$\begin{aligned} & s_1\cdots s_r w_0 w_{\{r+1,\ldots,n-r-1\}} F\big(s_{r+1}\cdots s_{r+t} w_{\{r+1,\ldots,n-r-2\}} w_{\{r+t+1,\ldots,n-r-t-2\}}\big) \\ = \ & (s_1\cdots s_r)\cdot(s_{n-r-1}\cdots s_{n-r-t})\cdot(s_{r+1}\cdots s_{n-r-1}) w_{\{r+t+1,\ldots,n-r-t-2\}} w_0 \\ = & (s_1\cdots s_{n-r-1})\cdot(s_{n-r-2}\cdots s_{n-r-t-2}) w_{\{r+t+1,\ldots,n-r-t-2\}} w_0. \end{aligned}$$

Therefore $s_1\sigma_\mu w_0 \in (s_2\cdots s_{n-r-1})\cdot(s_{n-r-2}\cdots s_{n-r-t-1})W_{\{r+t+1,n-r-t-2\}}$. As $w\leq s_1\sigma_\mu w_0$ is cuspidal in W_J, we deduce that $w = s_2\cdots s_r yz$ with $y\leq s_{r+1}\cdots s_{n-r-1}\cdots s_{n-r-t-1}$ and $z\in W_{\{r+t+1,n-r-t-2\}}$. As in the proof of Lemma 3.12 we see that ww_J contains the cycle $(2,n-1,3,n-2,\ldots,r+1,w(n-r),\ldots)$. Since by definition ww_J is a product of 3-cycles we must have $t\leq r\leq 2$, and we are left with two possibilities: if $r=1$ then $\mu = 3^c1^{n-3c}$; if $r=2$ we must have $w(n-r)=2$ otherwise ww_J would contain a cycle of length at least 4. This forces $w = s_2\cdots s_{n-3}y'z$ with $y'\leq s_{n-4}\cdots s_{n-3-t}$ and $z\in W_{\{t+3,n-4-t\}}$. Now $t=2$ is impossible, otherwise ww_J would contain either the transposition $(n-2,4)$ (if $y'\leq s_{n-5}$) or the cycle $(n-2,4,n-3,5,w(n-4),\ldots)$ of length at least 4. Therefore $t\leq 1$ and $\mu = 53^c1^{n-5-3c}$. In that case $s_1\sigma_\mu w_0 w_J$ is an element of minimal length in the cuspidal conjugacy class of W_J corresponding to the partition $3^{c+1}1^{n-5-3c}$, therefore we must have $\ell+1\leq b$ by Proposition 3.11. Conversely, for such a partition μ one has $s_1\sigma_\mu w_0 w_J = \sigma_{3^{c+1}1^{n-5-3c}} \leq \sigma_{3^b1^{n-2-3b}} = \sigma_\lambda w_0 w_J$ so that $\mathcal{O}_\lambda \leq \mathcal{O}_\mu$. □

Remark 3.15. It is likely that if $\lambda = 3^b21^{n-2-3b}$ and $\mu = 3^c1^{n-3c}$ then $\mathcal{O}_\lambda\leq\mathcal{O}_\mu$ if and only of $c\leq b+1$. This has been checked with Chevie for $n\leq 18$.

4. General results for $\mathrm{SU}_n(q)$

We now consider decomposition numbers of special unitary groups $\mathrm{SU}_n(q)$ for unitary primes ℓ. Recall that a prime ℓ is *unitary* for $\mathrm{SU}_n(q)$ if the order $d_\ell(-q)$ of $-q$ modulo ℓ is odd. In the case of linear primes, it is known that the unipotent part of the decomposition matrix of $\mathrm{SU}_n(q)$ is the same as that of the so-called q-Schur algebra, which gives an easy way to compute it (see [17]). No analogous approach is known for unitary primes.

Throughout, we will assume that $\ell > n$, for the following reasons. It is known that the decomposition matrix for $\ell\leq n$ will in general be different, since

(a) the decomposition matrices for Hecke algebras do change when ℓ divides the order of the Weyl group,
(b) certain ℓ-elements which force relations on decomposition numbers do not exist, and
(c) for $\ell|n$, the unipotent characters will no longer form a basic set for the unipotent blocks of $\mathrm{SU}_n(q)$, so that even the indexing sets for the decomposition matrix change.

Under our assumption on ℓ it turns out that in all examples the ℓ-modular decomposition matrix of the unipotent characters only depends on the order $d_\ell(-q)$ of $-q$ modulo ℓ, but not on q and ℓ individually.

4.1. **Unipotent characters of unitary groups.**

Recall that the set of unipotent characters of a finite reductive group depends only on its isogeny class. Here we have the following stronger statement:

Proposition 4.1. *Let $\ell > n$. Then the unipotent characters form a basic set for the unipotent blocks of* $\mathrm{SU}_n(q)$, $\mathrm{GU}_n(q)$ *and* $\mathrm{PGU}_n(q)$, *and the corresponding square part of the ℓ-modular decomposition matrix is the same for all three groups for any fixed n and q.*

Proof. The unipotent characters form a basic set for the unipotent blocks in all three cases by [11, Thm. A]. Since unipotent characters restrict irreducibly from $\mathrm{GU}_n(q)$ to $\mathrm{SU}_n(q)$, the statement on decomposition matrices for that pair follows. Moreover, unipotent characters have the center of $\mathrm{GU}_n(q)$ in their kernel, so can be considered as characters of $\mathrm{PGU}_n(q)$ as well. This completes the proof. □

We thus may and will switch freely between $\mathrm{SU}_n(q)$ and $\mathrm{GU}_n(q)$ in our proofs.

Recall that the irreducible unipotent characters (resp. the irreducible characters of the symmetric group $\mathfrak{S}_n$) are parametrized by partitions $\mu \vdash n$. We shall denote the corresponding character by ρ_μ (resp. χ_μ). The valuation $a(\mu)$ (resp. the degree $A(\mu)$) of the polynomial degree of ρ_μ can be explicitly computed in terms of μ (see [25, §4.4]).

The Frobenius endomorphism F acts on the Weyl group W by $F(w) = w_0ww_0$, where w_0 is the longest element of W. In particular, every irreducible character of W is stable by F. Following [26, 17.2], one can choose a preferred extension $\widetilde{\chi}_\mu$ of χ_μ to the group $W \rtimes \langle F\rangle$ which is trivial on F^δ. It is defined by the property that $\widetilde{\chi}_\mu(wF) = (-1)^{a(\mu)}\chi_\mu(ww_0)$. Up to a sign, the almost character corresponding to $\widetilde{\chi}_\mu$ is the unipotent character ρ_μ.

Lemma 4.2. *We have* $R_{\widetilde{\chi}_\mu} = (-1)^{a(\mu)+A(\mu)}\rho_\mu$.

Proof. We compare the value at the identity element on both sides. For this, let $f \mapsto f^{(-)}$ denote the evaluation at $-q$ on $\mathbb{Q}[q]$. We have $R^{\mathbf{G}}_{\mathbf{T}_{wF}}(1)(1) = R^{\mathbf{G}}_{\mathbf{T}_{ww_0}}(1)(1)^{(-)}$, and also $\widetilde{\chi}_\mu(wF) = (-1)^{a(\mu)}\chi_\mu(ww_0)$ (see [26, 17.2]). Thus

$$R_{\widetilde{\chi}_\mu}(1) = \frac{(-1)^{a(\mu)}}{|W|}\left(\sum_{w\in W}\chi_\mu(ww_0)R^{\mathbf{G}}_{\mathbf{T}_{ww_0}}(1)(1)\right)^{(-)} = (-1)^{a(\mu)}\psi_\mu(1)^{(-)},$$

where ψ_μ is the unipotent character of $\mathrm{GL}_n(q)$ indexed by μ. The claim follows. □

In particular, every unipotent character is uniform (*i.e.*, a linear combination of Deligne–Lusztig characters R_w).

4.2. **ℓ-reduction of characters.** It was conjectured by Geck [9] in general and shown for $\mathrm{GU}_n(q)$ in [10] that the ℓ-modular reduction of an ordinary cuspidal unipotent character is irreducible; for unitary groups we have the following stronger statement:

Proposition 4.3. *Let ρ be a unipotent character of the unitary group* $\mathrm{SU}_n(q)$ *which has minimal a-value in its (ordinary) Harish-Chandra series. Then the ℓ-modular reduction of ρ is irreducible.*

Proof. Let μ be a partition of n and $\mu^\star$ be the conjugate partition. Let C_μ be the unipotent class of $G = \mathrm{SU}_n(q)$ with Jordan form $\mu^\star$. Recall from [13, §6.4] that the smallest split Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ such that $\mathbf{L}\cap C_\mu \neq \emptyset$ is called the *Gelfand–Graev vertex* of C_μ. By [13, Prop. 5.5], one can choose $u \in \mathbf{L}\cap C_\mu$ such that if Γ_u is the generalized Gelfand–Graev representation of $\mathbf{L}^F$ associated with u then

- ρ_μ occurs with multiplicity one in the character of $R^{\mathbf{G}}_{\mathbf{L}}(\Gamma_u)$, and

- if ρ is an irreducible constituent of $R_{\mathbf{L}}^{\mathbf{G}}(\Gamma_u)$ then there exists a unipotent element $x \in G$ such that $\rho(x) \neq 0$ and $C_\mu \subset \overline{\mathbf{G} \cdot x}$. In particular, if $\rho = \rho_\lambda$ is a unipotent character, this forces $\lambda \trianglelefteq \mu$.

If P_μ denotes the unique indecomposable summand of $R_{\mathbf{L}}^{\mathbf{G}}(\Gamma_u)$ which involves ρ_μ in its character then the map $\mu \longmapsto P_\mu$ gives a bijection between the partitions of n and the isomorphism classes of projective indecomposable modules lying in the sum of the unipotent blocks of $\mathrm{SU}_n(q)$.

Let (L, η) denote a Harish-Chandra source of ρ, that is, L is a split Levi subgroup of G with a cuspidal unipotent character η such that ρ occurs in $R_{\mathbf{L}}^{\mathbf{G}}(\eta)$. Thus η is parametrized by a triangular partition $\lambda = (d, d-1, \ldots, 3, 2, 1)$, and ρ, having minimal a-value in the (L, η)-series, is parametrized by $\lambda' = (d+m, d-1, \ldots, 3, 2, 1)$, where $m = n - |\lambda| = n - d(d+1)/2$ is even.

Let μ be a partition of n different from λ' and such that $\lambda' \trianglelefteq \mu$. The Gelfand–Graev vertex of C_μ is obtained as follows: we write $\mu = \widetilde{\mu} + 2\nu$ where the dual partition of $\widetilde{\mu}$ has distinct terms. Then the Gelfand–Graev vertex of C_μ has type

$$^2A_{|\widetilde{\mu}|-1}(q) \times A_{\nu_1-1}(q^2) \times A_{\nu_2-1}(q^2) \times \cdots \times A_{\nu_r-1}(q^2).$$

Now since $\lambda' \trianglelefteq \mu$ and $\mu^\star$ is the concatenation of $(\widetilde{\mu}^\star, \nu^\star, \nu^\star)$ we deduce that $\widetilde{\mu}^\star \trianglelefteq \lambda'^\star$. In particular, the largest term in $\widetilde{\mu}^\star$ is less than d and since $\widetilde{\mu}^\star$ has distinct terms we must have $|\widetilde{\mu}^\star| = |\widetilde{\mu}| \leq d(d+1)/2$ with equality if and only if $\widetilde{\mu}^\star = \widetilde{\mu} = \lambda$. In that case the size of ν is exactly $m/2$ and $\lambda'_1 \leq \mu_1 = \widetilde{\mu}_1 + 2\nu_1$ forces $\nu_1 = m/2$ and therefore $\mu = \lambda$. Since this is impossible, it proves that the Gelfand–Graev vertex of C_μ cannot contain L or any of its G-conjugates. Consequently, $R_{\mathbf{L}}^{\mathbf{G}}(\Gamma_u)$ (and hence P_μ) cannot have any constituent lying in the Harish-Chandra series of (L, η) and in particular the decomposition number $d_{\lambda',\mu}$ must be zero. $\square$

5. The special case $\ell \mid (q+1)$

Throughout this section we will assume that $q \equiv -1 \pmod{\ell}$ with $\ell > n$.

5.1. **Non-unipotent characters.** Let $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^m)$ be a multipartition of n, that is, an m-tuple of partitions λ^i of size n_i such that $\sum n_i = n$. We assume here that $\mathbf{G} = \mathrm{GL}_n$, with $G := \mathbf{G}^F = \mathrm{GU}_n(q)$, so that one can identify $(\mathbf{G}, F)$ with its Langlands dual $(\mathbf{G}^*, F^*)$. Note also that centralizers of semisimple elements of $\mathbf{G}$ are automatically connected. Recall that $\mathbf{T}$ denotes a maximally split torus of $(\mathbf{G}, F)$. Since $\ell > n$, there exists a semisimple ℓ-element of $\mathbf{T}^{w_0 F}$ such that $(C_{\mathbf{G}}(s), w_0 F)$ is a connected reductive group of semisimple type $^2A_{n_1-1}(q) \times \cdots \times {}^2A_{n_m-1}(q)$ (by taking m distinct eigenvalues for s with respective multiplicities $n_1, \ldots, n_m$). The Weyl group $W(s)$ of $C_{\mathbf{G}}(s)$ is just the Young subgroup $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_m}$ of $\mathfrak{S}_n$. Let w_s be the longest element of $W(s)$ and let F_s be the Frobenius endomorphism of $C_{\mathbf{G}}(s)$ induced by $w_s w_0 F$. Then $\mathbf{T}$ is a maximally split torus of $(C_{\mathbf{G}}(s), F_s)$ and we can consider Deligne–Lusztig characters $R_{\mathbf{T}_{wF_s}}^{C_{\mathbf{G}}(s)}(1)$ for various $w \in W(s)$. Let $\rho_{\boldsymbol{\lambda}}^s$ be the unipotent character of $C_{\mathbf{G}}(s)^{F_s}$ corresponding to $\boldsymbol{\lambda}$. As a uniform function it can be written by Lemma 4.2 as

$$\begin{aligned}\rho^s_{\boldsymbol\lambda} &= (-1)^{a(\boldsymbol\lambda)+A(\boldsymbol\lambda)} R_{\widetilde\chi_{\boldsymbol\lambda}} \\ &= (-1)^{a(\boldsymbol\lambda)+A(\boldsymbol\lambda)} \frac{1}{|W(s)|} \sum_{w\in W(s)} \widetilde\chi_{\boldsymbol\lambda}(wF_s) R^{C_{\mathbf G}(s)}_{\mathbf T_{wF_s}}(1) \\ &= (-1)^{A(\boldsymbol\lambda)} \frac{1}{|W(s)|} \sum_{w\in W(s)} \chi_{\boldsymbol\lambda}(ww_s) R^{C_{\mathbf G}(s)}_{\mathbf T_{wF_s}}(1).\end{aligned}$$

We denote by $\rho_{\boldsymbol\lambda}$ the irreducible character of G corresponding to $\rho^s_{\boldsymbol\lambda}$ via the Jordan decomposition. Note that by [4, Thm. 13.23] the virtual character $R^{C_{\mathbf G}(s)}_{\mathbf T_{wF_s}}(1)$ corresponds to $(-1)^{\ell(w_sw_0)} R^{\mathbf G}_{\mathbf T_{ww_sw_0F}}(\theta)$ where θ is an ℓ-character of $\mathbf T^{wF_s}$ corresponding to s.

Let us consider the partition λ of n obtained by concatenation of the λ^i's (equivalently, its dual $\lambda^\star$ is the sum of the dual partitions $\lambda^{i\star}$). Then:

(a) The unipotent support of $\rho_{\boldsymbol\lambda}$ is the unipotent class with Jordan normal form $\lambda^\star$. We deduce from [11, 2.4] that the irreducible Brauer character φ_λ appears with multiplicity one in $\rho^0_{\boldsymbol\lambda}$ and that no other irreducible Brauer character φ_μ with $a(\mu)\geq a(\lambda)$ can appear.

(b) The restriction $\rho^0_{\boldsymbol\lambda}$ of $\rho_{\boldsymbol\lambda}$ to the set of ℓ'-elements can be computed in terms of restrictions of unipotent characters by

$$\begin{aligned}\rho^0_{\boldsymbol\lambda} &= \Big(\frac{(-1)^{A(\boldsymbol\lambda)+\ell(w_sw_0)}}{|W(s)|} \sum_{w\in W(s)} \chi_{\boldsymbol\lambda}(ww_s) R^{\mathbf G}_{\mathbf T_{ww_sw_0F}}(1)\Big)^0 \\ &= \frac{(-1)^{A(\boldsymbol\lambda)+\ell(w_sw_0)}}{n_1!\cdots n_m!} \sum_{(w_1,\ldots,w_m)\in\mathfrak S_{n_1}\times\cdots\times\mathfrak S_{n_m}} \chi_{\lambda^1}(w_1)\cdots\chi_{\lambda^m}(w_m)\Big(R^{\mathbf G}_{\mathbf T_{w_1\cdots w_mw_0F}}(1)\Big)^0.\end{aligned}\tag{1}$$

As a consequence of the orthogonality relations of Deligne–Lusztig characters, $\rho^0_{\boldsymbol\lambda}$ will be orthogonal to many virtual projective characters R_w, especially when $\chi_{\boldsymbol\lambda}$ vanishes on many conjugacy classes, which is the case for triangular partitions.

Now assume that all λ^j are triangular partitions. Then up to permutation, the multipartition $\boldsymbol\lambda = (\lambda^1,\ldots,\lambda^m)$ is uniquely determined by λ. For each λ^j one can consider the partition (n_j) if n_j is odd or $(n_j-1,1)$ otherwise, and we shall denote by $\bar{\boldsymbol\lambda}$ their concatenation. It is a partition of n into odd parts, and as such it corresponds to a cuspidal F-conjugacy class of $\mathfrak S_n$ (see §3.2). Let us define

$$\mathcal C_\lambda = \big\{[ww_0]_F \in [\mathfrak S_n]_F \mid w\in\mathfrak S_{n_1}\times\cdots\times\mathfrak S_{n_m} \text{ and } \chi_{\boldsymbol\lambda}(w)\neq 0\big\}.$$

Since a triangular partition is a 2-core, every character χ_{λ^j} is of 2-defect zero, and thus we deduce that $\mathcal C_\lambda$ contains only cuspidal classes. By (1) these are precisely the classes of elements ww_0 such that $(R^{\mathbf G}_{\mathbf T_{ww_0F}})^0$ occurs in $\rho^0_{\boldsymbol\lambda}$. Under rather strong assumptions on λ and $\mathcal C_\lambda$, one can show that $\rho^0_{\boldsymbol\lambda} = \varphi_\lambda$.

Proposition 5.1. *Let λ be a partition of n such that $\lambda^\star$ is a sum of triangular partitions. We assume that:*

(i) *every element of $\mathcal C_\lambda$ is of the form $[\sigma_{\bar{\boldsymbol\mu}}w_0]_F$ for some $\mu\trianglelefteq\lambda$ such that $\mu^\star$ is a sum of triangular partitions,* *and for every such μ we assume that*

(ii) *every element of $\mathcal{C}_\mu$ is of the form $[\sigma_{\bar{\boldsymbol{\nu}}} w_0]_F$ for some $\nu \trianglelefteq \mu$ such that $\nu^\star$ is a sum of triangular partitions,*

(iii) $\mathcal{C}_\mu = \{\mathcal{O}$ *cuspidal* $\mid [\sigma_{\bar{\boldsymbol{\mu}}} w_0]_F \leq \mathcal{O}\}$.

Then $\rho^0_{\boldsymbol{\lambda}} = \varphi_\lambda$.

The proof is based on an argument that we shall use in many other situations.

Lemma 5.2. *Let ρ be a character of $\mathrm{SU}_n(q)$ and $\mathcal{C}$ be a set of F-conjugacy classes such that $\rho^0 = \sum_{[w]_F \in \mathcal{C}} m_w (R^{\mathbf{G}}_{\mathbf{T}_{wF}})^0$ for some rational numbers m_w. Let μ be a partition of n and $x \in W$ such that*

(i) *P_μ occurs in the decomposition of R_x, and*

(ii) *for all $\mathcal{O} \in \mathcal{C}$ we have $\mathcal{O} \not\leq [x]_F$.*

Then $\langle P_\mu, \rho^0 \rangle = 0$. In other words, φ_μ does not occur in the ℓ-restriction of ρ.

Proof. For each $x \in W$ we may and will choose a projective character $\widetilde{R}_x$ whose unipotent part is R_x. Let $\mathcal{C}_{\not\leq}$ be the set of classes $[x]_F$ such that $\mathcal{O} \not\leq [x]_F$ for all $\mathcal{O} \in \mathcal{C}$. It is clearly closed under the partial ordering on F-conjugacy classes. In other words, if $[y]_F \leq [x]_F$ and $[x]_F \in \mathcal{C}_{\not\leq}$ then $[y]_F \in \mathcal{C}_{\not\leq}$. By induction on $[x]_F \in \mathcal{C}_{\not\leq}$, we show that a relation $\langle R_x, \varphi_\mu \rangle \neq 0$ forces $\langle P_\mu, \rho^0 \rangle = 0$. For $[x]_F \in \mathcal{C}_{\not\leq}$, the orthogonality relations of Deligne–Lusztig characters yield

$$0 = \langle (-1)^{\ell(x)} \widetilde{R}_x, \rho^0 \rangle = \sum_{\mu \vdash n} \langle (-1)^{\ell(x)} \widetilde{R}_x, \varphi_\mu \rangle \langle P_\mu, \rho^0 \rangle. \tag{2}$$

Assume by induction that for any $[y]_F < [x]_F$ and any μ such that $\langle R_y, \varphi_\mu \rangle \neq 0$ we have $\langle P_\mu, \rho^0 \rangle = 0$. If $\langle R_x, \varphi_\mu \rangle \neq 0$, then either P_μ already occurred in a Deligne–Lusztig character R_y for $[y]_F < [x]_F$, in which case $\langle P_\mu, \rho^0 \rangle = 0$, or by Proposition 2.1 we have $\langle (-1)^{\ell(x)} \widetilde{R}_x, \varphi_\mu \rangle > 0$. In any case, equation (2) gives a sum of nonnegative numbers which add up to zero, therefore $\langle P_\mu, \rho^0 \rangle = 0$ must be zero whenever $\langle \widetilde{R}_x, \varphi_\mu \rangle$ is not. □

Proof of the proposition. We want to show that $\langle P_\mu, \rho^0_{\boldsymbol{\lambda}} \rangle = \delta_{\lambda,\mu}$. We deduce from [11, 2.4] that φ_λ appears with multiplicity one in $\rho^0_{\boldsymbol{\lambda}}$ and that no other irreducible Brauer character φ_μ with $a(\mu) \geq a(\lambda)$ can appear. In particular it suffices to prove that the coefficient of φ_μ on $\rho^0_{\boldsymbol{\lambda}}$ is zero whenever $\mu \not\trianglelefteq \lambda$. In addition, since $\rho_{\boldsymbol{\lambda}}$ is cuspidal, any irreducible Brauer character occurring in $\rho^0_{\boldsymbol{\lambda}}$ is cuspidal, so we only need to consider partitions μ such that $\mu^\star$ is multiplicity-free, which we will assume now.

Lemma 5.2 applied to $\mathcal{C} = \mathcal{C}_\lambda$, together with the property (iii) shows that $\langle P_\mu, \rho^0_{\boldsymbol{\lambda}} \rangle = 0$ whenever P_μ occurs in R_w with $[w]_F \notin \mathcal{C}_\lambda$. Since every unipotent character of $\mathrm{GU}_n(q)$ is a uniform function, the virtual modules R_w for $[w]_F \notin \mathcal{C}_\lambda$ span a subspace of $K_0(kG$-proj) of codimension $|\mathcal{C}_\lambda|$. Consequently, there are at most $|\mathcal{C}_\lambda|$ projective indecomposable modules P_μ which do not occur as constituents in any R_w for $[w]_F \notin \mathcal{C}_\lambda$.

Since $\rho_{\boldsymbol{\lambda}}$ has unipotent support corresponding to λ, the coefficient of φ_λ on $\rho^0_{\boldsymbol{\lambda}}$ is equal to 1, hence non-zero. According to the previous argument, this proves that P_λ cannot appear as a constituent of a R_w for $[w]_F \notin \mathcal{C}_\lambda$. Now by (i) every element of $\mathcal{C}_\lambda$ is of the form $[\sigma_{\bar{\boldsymbol{\mu}}} w_0]_F$. If we apply the previous argument to μ, we deduce that P_μ cannot appear as a constituent of an R_w for $[w]_F \notin \mathcal{C}_\mu$. It is clear from the property (iii) that $\mathcal{C}_\mu \subset \mathcal{C}_\lambda$, so that the P_μ's for $[\sigma_{\bar{\boldsymbol{\mu}}} w_0]_F \in \mathcal{C}_\lambda$ are exactly the projective modules which do no appear. Consequently, all the projective covers of cuspidal modules P_μ with $\mu \ntrianglelefteq \lambda$ have to occur in an R_w for $[w]_F \notin \mathcal{C}_\lambda$. □

Remark 5.3. The set of partitions λ such that $\lambda^\star$ is multiplicity-free and the set of partitions with odd terms have the same cardinality. It would be interesting to see whether there exists a bijection f with $f(\lambda) = \bar{\boldsymbol{\lambda}}$ when $\lambda^\star$ is a sum of triangular partitions and such that $[\sigma_{f(\lambda)} w_0]_F$ is minimal among the classes $[w]_F$ such that P_λ occurs in R_w.

Proposition 5.4. *The following partitions satisfy the assumptions of Proposition 5.1:*

(1) *The partitions* $\lambda = 2^b 1^{n-2b}$ *with* $n \geq 3b$.
(2) *The partitions* $32^i 1^j$ *with* $1 \leq i \leq 3$ *and* $i \leq j \leq 10$*, or* $i = 4$ *and* $4 \leq j \leq 7$.
(3) *The partitions* $3^2 2^2 1^j$ *with* $2 \leq j \leq 8$.

Proof. First assume that $\lambda = 2^b 1^{n-2b}$ with $n \geq 3b$. Then any partition $\mu \trianglelefteq \lambda$ has the same shape, so we only need to check the properties (i) and (iii) for $\mathcal{C}_\lambda$. The multipartition $\boldsymbol{\lambda}$ is $(21, 21, \ldots, 21, 1, \ldots, 1)$, $\bar{\boldsymbol{\lambda}} = 3^b 1^{n-3b}$. Since the character χ_{21} of $\mathfrak{S}_3$ takes non-zero values exactly on the classes corresponding to 3 and 1^3, we deduce that $\mathcal{C}_\lambda = \{[\sigma_{3^c 1^{n-3c}} w_0]_F \mid c \leq b\}$, which proves (i). Property (iii) follows from Proposition 3.11.

In (2) every partition $\mu \trianglelefteq 32^i 1^j$ such that $\mu^\star$ is a sum of triangular partitions has shape (1) or (2). Here $\boldsymbol{\lambda} = (321, 21, 21, \ldots)$ and $\bar{\boldsymbol{\lambda}} = 53^{i-1} 1^{j-i+1}$. Therefore $[\sigma_\mu w_0]_F \in \mathcal{C}_\lambda$ if and only if μ is the concatenation of a partition with odd terms of 6 and $i-1$ partitions with odd terms of 3 (completed with 1's to get a partition of n). Therefore μ has shape $53^{i'} 1^{j'}$ with $i' \leq i-1$ or $3^{i'} 1^{j'}$ with $i' \leq i+1$. Since $3^{i+1} \trianglelefteq 53^{i-1} 1$, these partitions μ are exactly the partitions with odd terms such that $\mu \trianglelefteq \bar{\boldsymbol{\lambda}}$. Moreover, to any such μ corresponds a unique partition $\nu \trianglelefteq \lambda$ such that $\nu^\star$ is a sum of triangular partitions and $\bar{\boldsymbol{\nu}} = \mu$, with $\boldsymbol{\nu} = (321, 21, 21, \ldots)$ if $\mu = 53^{i'} 1^{j'}$ and $\boldsymbol{\nu} = (21, 21, \ldots)$ if $\mu = 3^{i'} 1^{j'}$. This proves that $\mathcal{C}_\lambda$ satisfies property (i) of Proposition 5.1. Then (iii) follows from Conjecture 3.7 which can be checked with the help of Chevie as long as i and j are small.

The same argument applies to $\lambda = 3^2 2^2 1^j$. In this case $\boldsymbol{\lambda} = (321, 321, 1, 1, \ldots)$, so that $\bar{\boldsymbol{\lambda}} = 5^2 1^j$. The partitions μ with odd terms such that $\mu \trianglelefteq \bar{\boldsymbol{\lambda}}$ are

$$\mu \in \{5^2 1^j, 53^2 1^{j-1}, 531^{j+2}, 51^{j+5}, 3^4 1^{j-2}, 3^3 1^{j+1}, 3^2 1^{j+4}, 31^{j+7}, 1^{j+10}\}.$$

They are in bijection with the set of partitions

$$\nu \in \{3^2 2^2 1^j, 32^3 1^{j+1}, 32^2 1^{j+3}, 321^{j+5}, 2^4 1^{j+2}, 2^3 1^{j+4}, 2^2 1^{j+6}, 21^{j+8}, 1^{j+10}\}$$

via the map $\nu \longmapsto \bar{\boldsymbol{\nu}}$. Therefore $\mathcal{C}_\lambda$ satisfies property (i). Again, (iii) follows from Conjecture 3.7 which holds whenever k is small. □

Remark 5.5. In the proof of Proposition 5.1 we show that P_λ does not occur in R_w for $[w]_F \notin \mathcal{C}_\lambda$. Therefore with $\lambda = 2^b 1^{n-2b}$ and $n \geq 3b$, we obtain from the previous proposition that if $P_{2^b 1^{n-2b}}$ occurs in R_w, then ww_0 is conjugate to a cycle of type $3^c 1^{n-3c}$ with $c \leq b$. From the particular case $b = 0$ we deduce that P_{1^n} occurs only in R_{w_0}.

The previous strategy can be extended to deal with non-cuspidal representations. An example is given in the following lemma.

Lemma 5.6. *Assume that $n \geq 4$. Let $I = \{s_2, \ldots, s_{n-2}\}$ and $\mathbf{L}$ be the corresponding standard Levi subgroup of $\mathbf{G}$. Then the only irreducible Brauer characters which can appear in $R_{\mathbf{L}}^{\mathbf{G}}(\rho_{(1,1,\ldots,1)})^0$ are φ_{1^n}, $\varphi_{21^{n-2}}$ and $\varphi_{31^{n-3}}$.*

Proof. Let $\rho = R_{\mathbf{L}}^{\mathbf{G}}(\rho_{(1,1,\ldots,1)})$. By definition of $\rho_{(1,1,\ldots,1)}$ we have

$$R_{\mathbf{L}}^{\mathbf{G}}(\rho_{(1,1,\ldots,1)})^0 = R_{\mathbf{L}}^{\mathbf{G}}(\rho_{(1,1,\ldots,1)}{}^0) = R_{\mathbf{L}}^{\mathbf{G}}((-1)^{\ell(w_0)-\ell(w_I)} R_{\mathbf{T}_{w_I F}}^{\mathbf{L}}(1)^0) = -(R_{w_I F})^0$$

where w_I is the longest element of W_I. From Lemma 5.2, we deduce that $\langle P_\lambda, \rho^0 \rangle = 0$ whenever P_λ occurs in R_w for some $[w]_F \not\geq [w_I]_F$.

Now, the permutation $w_I w_0$ is just the transposition $(1, n)$, which is a cycle of type 21^{n-2} with maximal length in its conjugacy class (equal to $2n-3$). Let μ be a partition of n and σ_μ be of cycle type μ defined in Section 3. Using the explicit formulae for the length of σ_μ given in [15, Lemma 3.2], one can check that $\ell(\sigma_\mu) < 2n-3$ if and only if $\mu \in \{1^n, 31^{n-3}\}$. In particular, the only F-conjugacy classes that are larger than $[w_I]_F$ correspond to the partitions 1^n and 31^{n-3} (compare with Proposition 3.14).

Finally, the span of $\mathcal{R} = \{R_{\sigma_\mu w_0}\}_{\mu \notin \{1^n, 21^{n-2}, 31^{n-3}\}}$ has codimension 3 in the space of projective unipotent characters, therefore there are at most three partitions λ such that P_λ does not occur in these R_w's. By Remark 5.5, we know already that P_{1^n} (resp. $P_{21^{n-2}}$) occurs only in R_{w_0} (resp. in R_{w_0} and $R_{\sigma_{31^{n-3}} w_0}$). Furthermore, $P_{31^{n-3}}$ cannot occur in any element of $\mathcal{R}$ otherwise by the previous argument $\langle P_{31^{n-3}}, \rho^0 \rangle$ would be zero, which is impossible since the unipotent part of $P_{31^{n-3}}$ equals $R_{\mathbf{L}}^{\mathbf{G}}(\rho_{1^{n-2}})$ by [14, Prop. 4.4 and Lemma 4.6] and

$$\begin{aligned}
\langle R_{\mathbf{L}}^{\mathbf{G}}(\rho_{1^{n-2}}), -R_{w_I} \rangle &= -\langle \rho_{1^n} + \rho_{2^2 1^{n-4}} + \rho_{31^{n-3}}, R_{w_I} \rangle \\
&= -\chi_{1^n}(w_I w_0) - \chi_{2^2 1^{n-4}}(w_I w_0) + \chi_{31^{n-3}}(w_I w_0) \\
&= 1 + \Big(\frac{(n-2)(n-5)}{2} + 1 \Big) + \Big(-\frac{(n-3)(n-4)}{2} + 1 \Big) \\
&= 2
\end{aligned}$$

(see the proof of Proposition 5.16 for the values of characters of $\mathfrak{S}_n$ on the transposition $w_I w_0$). We conclude that for all $\lambda \notin \{1^n, 21^{n-2}, 31^{n-3}\}$ we have $\langle P_\lambda, R_{\mathbf{L}}^{\mathbf{G}}(\rho_{(1,1,\ldots,1)})^0 \rangle = 0$, which means that φ_λ is not a constituent of $R_{\mathbf{L}}^{\mathbf{G}}(\rho_{(1,1,\ldots,1)})^0$. $\square$

Remark 5.7. We have $w_I < s_1 w_I$ and $s_1 w_I w_0 = (1, n, 2)$ has maximal length in its conjugacy class. So by definition of the order on F-conjugacy classes we deduce that $[w_I]_F = \mathcal{O}_{21^{n-2}} < \mathcal{O}_{31^{n-3}} = [s_1 w_I]_F$ although $21^{n-2} \trianglelefteq 31^{n-3}$. However this does not contradict Conjecture 3.7 since $[w_I]_F$ is not cuspidal.

Remark 5.8. Proposition 5.1 and Lemma 5.6 can be used to compute decomposition numbers as follows. Let λ be a partition of n and Ψ_λ be the character of the corresponding PIM. By Brauer reciprocity, the unipotent part of Ψ_λ is $\sum d_{\mu,\lambda} \rho_\mu$. Let ρ be a (not necessarily unipotent) character such that $\rho^0 = \sum x_\mu \rho_\mu^0$. Then if one knows that φ_λ is not a constituent of ρ^0 we obtain the relation $\langle \Psi_\lambda, \rho^0 \rangle = \sum d_{\mu,\lambda} x_\mu = 0$ on the decomposition numbers.

5.2. **Two-column partitions.** We apply the previous results to compute the bottom-right corner of the decomposition matrix of $\mathrm{SU}_n(q)$ (corresponding to two-column partitions) in the case where $\ell \mid (q+1)$.

Theorem 5.9. *Assume $\ell > n$ and $\ell \mid (q+1)$. Let $a := \lfloor n/3 \rfloor + 1$. Then the unipotent parts of the projective indecomposable $\mathrm{SU}_n(q)$-modules indexed by the partitions 2^b1^{n-2b} for $b \leq a$ are given by the columns in the following matrix (where dots "." represent zeroes):*

$$
\begin{array}{c|cccccc}
2^a1^{n-2a} & 1 & . & . & . & . & . \\
2^{a-1}1^{n-2a+2} & n-2a+1 & 1 & . & . & . & . \\
\vdots & \vdots & \ddots & \ddots & . & . & . \\
2^21^{n-4} & \binom{n-a-2}{a-2} & \cdots & n-5 & 1 & . & . \\
21^{n-2} & \binom{n-a-1}{a-1} & \binom{n-a}{a-2} & \cdots & n-3 & 1 & . \\
1^n & \binom{n-a}{a} & \binom{n-a+1}{a-1} & \cdots & \binom{n-2}{2} & n-1 & 1
\end{array}
$$

Proof. For a partition λ of n, the unipotent constituents ρ_μ of the projective indecomposable module P_λ satisfy $\mu \trianglelefteq \lambda$ (see the proof of Proposition 4.3). Therefore the only unipotent constituents of $P_{2^b1^{n-2b}}$ are the unipotent characters associated with partitions 2^c1^{n-2c} with $0 \leq c \leq b$.

If $c \leq n/3$, the dual partition of 2^c1^{n-2c} is a sum of triangular partitions, namely c copies of 21 and $n-3c$ copies of 1. The corresponding character $\rho_{(21,\ldots,21,1,\ldots,1)}$ is cuspidal and by Propositions 5.1 and 5.4 it remains irreducible after ℓ-reduction. More precisely $(\rho_{(21,\ldots,21,1,\ldots,1)})^0 = \varphi_{2^c1^{n-2c}}$. This yields relations on the coefficients of the decomposition matrix, which we shall use in order to prove the assertion.

The character χ_{21} of $\mathfrak{S}_3$ has value 2 on the trivial class, and -1 on the class of 3-cycles. For $0 \leq k \leq c$, let x_k be any product of k disjoint 3-cycles in $\mathfrak{S}_{3c}$. Then the value of the character of $(\mathfrak{S}_3)^c$ corresponding to the multipartition $(21, 21, \ldots, 21)$ on x_k is $(-1)^k2^{c-k}$. Moreover, for a given k, there are $2^k\binom{c}{k}$ elements of $(\mathfrak{S}_3)^c$ in the conjugacy class of x_k in $\mathfrak{S}_{3c}$, so that by (1) above we deduce that $(\rho_{(21,21,\ldots,21,1,\ldots,1)})^0$ is equal to the restriction to the set of ℓ'-elements of the following virtual unipotent character:

$$\rho_c := \frac{(-1)^{n(n-1)/2-c}}{3^c}\sum_{k=0}^{c}(-1)^k\binom{c}{k}R_{\mathbf{T}_{x_kw_0F}}^{\mathbf{G}}(1).$$

From the expression of Deligne–Lusztig characters in terms of almost characters together with Lemma 4.2, we have, for any partition λ of n

$$\begin{aligned}\langle R_{\mathbf{T}_{x_kw_0F}}^{\mathbf{G}}(1), \rho_\lambda\rangle =& (-1)^{a(\lambda)+A(\lambda)}\langle R_{\mathbf{T}_{x_kw_0F}}^{\mathbf{G}}(1), R_{\widetilde{\chi}_\lambda}\rangle \\ =& (-1)^{a(\lambda)+A(\lambda)}\widetilde{\chi}_\lambda(x_kw_0F) = (-1)^{A(\lambda)}\chi_\lambda(x_k).\end{aligned}$$

With the value $A(2^b1^{n-2b}) = n(n-1)/2 - b$ we deduce that the multiplicity of $\rho_{2^b1^{n-2b}}$ in ρ_c is given by

$$\langle \rho_c, \rho_{2^b1^{n-2b}}\rangle = \frac{(-1)^{b+c}}{3^c}\sum_{k=0}^{c}(-1)^k\binom{c}{k}\chi_{2^b1^{n-2b}}(x_k).$$

Let D be the submatrix of the decomposition matrix corresponding to rows and columns labelled by 2^b1^{n-2b} for $a \geq b \geq 0$, and $C = (\langle \rho_i, \rho_{2^j1^{n-2j}}\rangle)_{i,j=a,\ldots,0}$. Since $\rho_i^0 = \varphi_{2^i1^{n-2i}}$ and since the unipotent characters form a basic set of characters, we have $C = D^{-1}$. Therefore Theorem 5.9 is about giving C^{-1} explicitly, which we do in the next three lemmae. $\square$

Lemma 5.10. *Let $0 \leq j \leq \lfloor n/2 \rfloor$ and $0 \leq k \leq \lfloor n/3 \rfloor$. Then the value of $\chi_{2^j 1^{n-2j}}$ on a product of k disjoint 3-cycles is given by*

$$\chi_{2^j 1^{n-2j}}(x_k) = \sum_{r=0}^{k} \binom{k}{r} \left(\binom{n-3k}{j-3r} - \binom{n-3k}{j-3r-1} \right).$$

Proof. For any positive integer n and any integers j, k we denote by $M_{n,j,k}$ the right-hand side of the formula. A straightforward computation using Pascal's rule yields the following relation:

$$M_{n,j,k+1} = M_{n-3,j-3,k} + M_{n-3,j,k}. \tag{3}$$

Under the assumptions on j and k given in the Lemma we show by induction on k that $\chi_{2^j 1^{n-2j}}(x_k) = M_{n,j,k}$. The case $k = 0$ corresponds to $M_{n,j,0} = \binom{n}{j} - \binom{n}{j-1} = \chi_{2^j 1^{n-2j}}(1)$ which follows from the hook-length formula. Assume first that $j \geq 3$ and $n - 2j \geq 3$. Then two 3-hooks (both of height 1) can be removed from the partition $2^j 1^{n-2j}$ and the Murnaghan–Nakayama rule gives $\chi_{2^j 1^{n-2j}}(x_{k+1}) = \chi_{2^{j-3} 1^{n-2j+3}}(x_k) + \chi_{2^j 1^{n-2j-3}}(x_k)$ and we conclude using (3).

Assume $j \in \{0, 1, 2\}$. Then $M_{n,j,k} = \binom{n-3}{j} - \binom{n-3}{j-1}$ and $M_{n,j-3,k} = 0$. If in addition $n - 2j \geq 3$ we have $\chi_{2^j 1^{n-2j}}(x_{k+1}) = \chi_{2^j 1^{n-2j-3}}(x_k) = M_{n-3,j,k} = M_{n-3,j,k} + M_{n-3,j-3,k}$ and again we conclude using formula (3). If $n - 2j \in \{0, 1, 2\}$, then $n \leq 6$ and one checks easily that the nine partitions to consider all satisfy the formula.

We are left with the case $j \geq 3$ and $n - 2j < 3$ which corresponds to the partitions 2^j, $2^j 1$ and $2^j 1^2$. Two 3-hooks (of leg length 1 and 2) can be removed from the partitions 2^j, and the Murnaghan–Nakayama rule yields $\chi_{2^j}(x_{k+1}) = \chi_{2^{j-3} 1^3}(x_k) - \chi_{2^{j-2} 1}(x_k) = M_{n-3,j-3,k} - M_{n-3,j-2,k}$. Since $n = 2j$, we have

$$\begin{aligned} M_{n-3,j-2,k} &= \sum_{r=0}^{k} \binom{k}{r} \left(\binom{2j-3-3k}{j-2-3r} - \binom{2j-3-3k}{j-3-3r} \right) \\ &= \sum_{r=0}^{k} \binom{k}{r} \left(\binom{2j-3-3k}{j-3(k-r)-1} - \binom{2j-3-3k}{j-3(k-r)} \right) \\ &= -M_{n-3,j,k} \end{aligned}$$

and we can invoke formula (3) to conclude. A similar argument shows that $M_{n-3,j-1,k} = -M_{n-3,j,k}$ when $n = 2j+1$ and $M_{n-3,j,k} = 0$ when $n = 2j+2$. We conclude using formula (3) and the Murnaghan–Nakayama rule for $2^j 1$ and $2^j 1^2$. $\square$

Lemma 5.11. *Let $0 \leq i \leq j \leq a$. Then*

$$(-1)^{i+j} \langle \rho_i, \rho_{2^j 1^{n-2j}} \rangle = \binom{n-2i}{j-i} - \binom{n-2i}{j-i-1}.$$

Proof. From the previous Lemma, together with the expression for $\langle \rho_i, \rho_{2^j 1^{n-2j}} \rangle$ given at the end of the proof of Theorem 5.9 we obtain

$$(-1)^{i+j} \langle \rho_i, \rho_{2^j 1^{n-2j}} \rangle = \frac{1}{3^i} \sum_{k=0}^{i} \sum_{r=0}^{k} (-1)^k \binom{i}{k} \binom{k}{r} \left(\binom{n-3k}{j-3r} - \binom{n-3k}{j-3r-1} \right).$$

Let us split this sum into two parts by considering the following integer

$$A_{i,j,n} = \sum_{k=0}^{i}\sum_{r=0}^{k}(-1)^k\binom{i}{k}\binom{k}{r}\binom{n-3k}{j-3r}.$$

We now show by induction on i that $A_{i,j,n} = 3^i\binom{n-2i}{j-i}$ (without any restriction on n or j), which is enough to conclude. For $i = 0$ the relation is straightforward. For $i > 0$, we use Pascal's rule on the binomial coefficients $\binom{i}{k}$ and $\binom{k}{r}$ to derive the relation

$$A_{i,j,n} = A_{i-1,j,n} - A_{i-1,j,n-3} - A_{i-1,j-3,n-3}.$$

Using the induction hypothesis and again Pascal's rule we obtain finally

$$\begin{aligned} A_{i,j,n} &= 3^{i-1}\left(\binom{n-2i+2}{j-i+1} - \binom{n-2i-1}{j-i+1} - \binom{n-2i-1}{j-i-2}\right) \\ &= 3^i\left(\binom{n-2i-1}{j-i} + \binom{n-2i-1}{j-i-1}\right) = 3^i\binom{n-2i}{j-i}, \end{aligned}$$

which is the expected result. □

Lemma 5.12. *Let $C = (c_{i,j})$ be the square matrix of size $a+1$ defined by*

$$c_{i,j} = (-1)^{i+j}\left(\binom{n-2i}{j-i} - \binom{n-2i}{j-i-1}\right)$$

for $i \leq j$ and by $c_{i,j} = 0$ otherwise. Then C is invertible and the coefficients of $C^{-1} = (d_{i,j})$ are given by

$$d_{i,j} = \binom{n-i-j}{j-i}$$

for $i \leq j$ and by $d_{i,j} = 0$ otherwise.

Proof. We start by computing the following sum, for $i \leq j$

$$\begin{aligned} B_{i,j} = \sum_{k=i}^{j}(-1)^k\binom{n-i-k}{k-i}\binom{n-2k}{j-k} &= \sum_{k=i}^{j}(-1)^k\frac{(n-i-k)!}{(k-i)!(j-k)!(n-j-k)!} \\ &= \sum_{k=0}^{j-i}(-1)^k\frac{(n-2i-k)!}{k!(j-i-k)!(n-i-j-k)!} \\ &= \sum_{k=0}^{j-i}(-1)^k\binom{j-i}{k}\binom{n-2i-k}{j-i}. \end{aligned}$$

We claim that this equals $(-1)^{j-i}$. Let $m \geq s \geq 0$ and

$$P_{m,s}(X) = \frac{1}{s!}(m+X)(m+X-1)\cdots(m+X-s+1).$$

This is a polynomial of degree s. If we decompose it in the basis of polynomials $\binom{X}{k}$ for $k = 0,\ldots,s$, the coefficient of $\binom{X}{s}$ is equal to 1. On the other hand, by the discrete Taylor formula, it is also equal to

$$\sum_{k=0}^{s}(-1)^{s-k}\binom{s}{k}P_{m,s}(s-k)=\sum_{k=0}^{s}(-1)^{s-k}\binom{s}{k}\binom{m+s-k}{s}.$$

With $s=j-i$ and $m=n-i-j$ we deduce the previous claim, that is $B_{i,j}=(-1)^{i-j}$.

Let $D=(d_{i,j})$. The coefficient (i,j) of DC is given by $B_{i,i}=1$ if $i=j$, and by $(-1)^{i+j}(B_{i,j}+B_{i,j-1})=0$ if $i<j$. On the other hand since D and C both have triangular shape $(DC)_{i,j}=0$ whenever $j<i$. This proves that $D=C^{-1}$. □

A consequence of Theorem 5.9 is that the decomposition numbers of a large family of characters satisfy the analogue of James's row and column removal rule for $\mathrm{GL}_n(q)$ [22, Rule 5.8] (see also §6.4):

Corollary 5.13. *Assume $\ell>n$ and $\ell|(q+1)$. Let $c\le b\le 1+n/3$. Then the decomposition number $d_{2^c1^{n-2c},2^b1^{n-2b}}=[\rho_{2^c1^{n-2c}}:\varphi_{2^b1^{n-2b}}]$ satisfies James's row removal rule:*

$$d_{2^c1^{n-2c},2^b1^{n-2b}}=d_{1^{n-2c},2^{b-c}1^{n-2b}}.$$

Remark 5.14. This corollary suggests that Theorem 5.9 should hold whenever the partition 2^a1^{n-2a} makes sense, that is when $a\le n/2$. This turns out to be true when $n\le 10$ (see Tables 1–6).

It is generally conjectured that entries of decomposition matrices for a fixed family of groups of Lie type, like $\mathrm{SU}_n(q)$ (n fixed) are bounded independently from q and the prime ℓ. Theorem 5.9 shows that nevertheless such a bound will be rather large:

Corollary 5.15. *The entries of the ℓ-modular decomposition matrices of unipotent blocks of $\mathrm{SU}_n(q)$, $q\equiv -1 \pmod{\ell}$, are not bounded by any polynomial function in n.*

Indeed, one entry has the form $\binom{n-a}{a}$, with $a=\lfloor n/3\rfloor+1$.

5.3. **The partition 321^{n-5}.** As in the previous section, we use the ℓ-reduction of non-unipotent characters to compute certain decomposition numbers in the principal ℓ-block of $\mathrm{SU}_n(q)$ for general n.

Proposition 5.16. *Assume $\ell|(q+1)$ with either $\ell>n\ge 7$ or $n=6$ and $\ell>7$. Then*

$$\Psi_{321^{n-5}}=\rho_{321^{n-5}}+2\rho_{2^31^{n-6}}+(n-4)\rho_{31^{n-3}}+(n-4)\rho_{2^21^{n-4}}+2\rho_{21^{n-2}}+(2n-6)\rho_{1^n}$$

up to adding non-unipotent characters.

Proof. We write the unipotent part of $\Psi_{321^{n-5}}$ as

$$\rho_{321^{n-5}}+y_1\rho_{2^31^{n-6}}+y_2\rho_{31^{n-3}}+y_3\rho_{2^21^{n-4}}+y_4\rho_{21^{n-2}}+y_5\rho_{1^n}.$$

We first assume that $n\ge 9$, so that we can apply Proposition 5.1 to the partitions 2^31^{n-6}, 2^21^{n-4}, 21^{n-2} and 1^n.

The values of the characters $\chi_{321^{n-6}}$ and $\chi_{31^{n-3}}$ of $\mathfrak{S}_n$ on the classes 3^c1^{n-3c} for $c\le 3$, given in the following table, can be computed using the Murnaghan–Nakayama rule:

λ	3^31^{n-9}	3^21^{n-6}	31^{n-3}	1^n
$\chi_{321^{n-5}}(\sigma_\lambda)$	$\frac{(n-9)(n-11)(n-13)}{3}-3$	$\frac{(n-6)(n-8)(n-10)}{3}-2$	$\frac{(n-3)(n-5)(n-7)}{3}-1$	$\frac{n(n-2)(n-4)}{3}$
$\chi_{31^{n-3}}(\sigma_\lambda)$	$\binom{n-10}{2}$	$\binom{n-7}{2}$	$\binom{n-4}{2}$	$\binom{n-1}{2}$

Note that one can check that the values are correct even when n is smaller than 13. Now, as in the proof of Theorem 5.9, we consider the restriction of the character ρ_c for $c = 0, 1, 2, 3$ to the set of ℓ'-elements and its coefficient on $\varphi_{321^{n-5}}$, which vanish by Propositions 5.1 and 5.4. Equivalently, we have $\langle \rho_c, \Psi_{321^{n-5}} \rangle = 0$. To obtain the necessary relations we need to compute the multiplicity of $\rho_{321^{n-5}}$ and $\rho_{31^{n-3}}$ in each ρ_c using the expression of ρ_c in terms of Deligne–Lusztig characters and the values of the corresponding characters of $\mathfrak{S}_n$. The other scalar products have already been computed in Lemma 5.11. We obtain

c	0	1	2	3
$\langle \rho_{321^{n-5}}, \rho_c \rangle$	$\frac{n(n-2)(n-4)}{3}$	$-(n-3)(n-4)$	$2(n-5)$	-2
$\langle \rho_{2^31^{n-6}}, \rho_c \rangle$	$\binom{n}{2} - \binom{n}{3}$	$\binom{n-2}{2} - n + 2$	$5-n$	1
$\langle \rho_{31^{n-3}}, \rho_c \rangle$	$-\binom{n-1}{2}$	$n-3$	-1	0
$\langle \rho_{2^21^{n-4}}, \rho_c \rangle$	$\binom{n}{2} - n$	$3-n$	1	0
$\langle \rho_{21^{n-2}}, \rho_c \rangle$	$1-n$	1	0	0
$\langle \rho_{1^n}, \rho_c \rangle$	1	0	0	0

and thus the scalar products $\langle \rho_c, \Psi_{321^{n-5}} \rangle = 0$ yield the following relations:

$$
\begin{aligned}
0 &= -2 + y_1, \\
0 &= 2(n-5) + y_1(5-n) - y_2 + y_3, \\
0 &= -(n-3)(n-4) + y_1\left(\binom{n-2}{2} - n + 2\right) + y_2(n-3) + y_3(3-n) + y_4, \\
0 &= \frac{n(n-2)(n-4)}{3} + y_1\left(\binom{n}{2} - \binom{n}{3}\right) - y_2\binom{n-1}{2} + y_3\left(\binom{n}{2} - n\right) + y_4(1-n) + y_5,
\end{aligned}
$$

from which we deduce $y_1 = y_4 = 2$, $y_2 = y_3$ and $y_5 = y_2 + n - 2$.

To obtain the value of y_2 we use a further relation that we deduce from Lemma 5.6. Let $I = \{s_2, \ldots, s_{n-2}\}$ and $w_I = s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1} w_0$ be the longest element of the corresponding parabolic subgroup. Then by Lemma 5.6 we have $\langle \Psi_{321^{n-5}}, R^0_{w_I} \rangle = 0$. Using the following values of characters of $\mathfrak{S}_n$ computed with the help of the Murnaghan–Nakayama rule

λ	321^{n-5}	2^31^{n-6}	31^{n-3}	2^21^{n-4}	21^{n-2}
$-\chi_\lambda(w_I w_0)$	$\frac{(n-2)(n-4)(n-6)}{3}$	$\frac{(n-2)(n-3)(n-7)}{6} + n - 3$	$\binom{n-3}{2} - 1$	$\frac{(n-2)(n-5)}{2} + 1$	$n-3$

we get

$$
\begin{aligned}
0 = \langle \Psi_{321^{n-5}}, R^0_{w_I} \rangle &= \chi_{321^{n-5}}(w_I w_0) - 2\chi_{2^31^{n-6}}(w_I w_0) - y_2 \chi_{31^{n-3}}(w_I w_0) \\
&\quad + y_2 \chi_{2^21^{n-4}}(w_I w_0) - 2\chi_{21^{n-2}}(w_I w_0) + (y_2 + n - 2)\chi_{1^n}(w_I w_0) \\
&= 2n - 8 - 2y_2
\end{aligned}
$$

so that $y_2 = n - 4$.

When $n = 6, 7, 8$ we cannot invoke Proposition 5.1 with the partition $2^3 1^{n-6}$ since it is no longer a concatenation of triangular partitions. Therefore the relation $0 = -2 + y_1$ can no longer be deduced via ℓ-reduction. Let us consider the projective character Ψ_{3215} of $\mathrm{SU}_{10}(q)$. (Here one needs to assume $\ell > 7$ so that one has indeed $\ell > 10$). The multiplicity of $\rho_{2^31^2}$ (resp. ρ_{2^3}) in the Harish-Chandra restriction of Ψ_{321^5} to a split Levi subgroup of semisimple type ${}^2A_7$ (resp. ${}^2A_5$) is 2. Therefore y_1 can be at most 2 when $n = 6, 8$. Now, the restriction $(\chi_{\boldsymbol{\mu}})^0$ for the multipartition $\boldsymbol{\mu} = (2, 2, 2, 1, 1)$ (resp. $\boldsymbol{\mu} = (2, 2, 2)$) forces y_1 to be at least 2, therefore it must be equal to 2. The same argument works when $n = 7$, starting with the projective character Ψ_{321^4} of $\mathrm{SU}_9(q)$. □

Remark 5.17. Proposition 5.16 is another example where the analogue of James's column removal rule holds, since in this case $d_{2^31^{n-6},321^{n-5}} = d_{1^3,21} = 2$.

6. Tables for small rank

In this section, we give the decomposition matrices (respectively approximations with few unknown entries) of unipotent blocks of $\mathrm{SU}_n(q)$, $n \leq 10$, for unitary primes $\ell > n$ (see the introduction to Section 4 for why we exclude smaller primes). In addition, we determine the ℓ-modular Harish-Chandra series for the unipotent Brauer characters in all considered cases. They are known when $\ell|(q+1)$ by [14, Thm. 4.12] (again when $\ell > n$), but not for the other unitary primes. Thus our results may give hints on properties of this distribution into Harish-Chandra series.

The Brauer trees in the cases when the Sylow ℓ-subgroups are cyclic were determined by Fong and Srinivasan [8]. In the cases considered here, we recover their results by our methods.

Recall from §2 that for $w \in W$ we denote by R_w (resp. $R_w[\lambda]$) the virtual character afforded by the cohomology of the Deligne–Lusztig variety $\mathrm{X}(w)$ (resp. the generalized λ-eigenspace of F^δ on that cohomology). By [24, Cor. 3.9], the eigenvalues of F^δ on a unipotent character ρ in the cohomology of $\mathrm{X}(w)$ are of the form $\lambda_\rho q^{\delta m/2}$ where λ_ρ is a root of unity which depends only on ρ and m is a nonnegative integer. We fix an indeterminate v and we shall denote by $v^{\delta m}\rho$ the class in the Grothendieck group (with scalars extended to the polynomial ring $\mathbb{Z}[v]$) of $G \times \langle F^\delta \rangle$-modules of such a representation.

Let $\mathcal{H}_v(W)$ be the Iwahori–Hecke algebra of W with equal parameters v. By convention, the standard basis $(t_w)_{w\in W}$ of $\mathcal{H}_v(W)$ will satisfy the relation $(t_s + v)(t_s - v^{-1}) = 0$ for all $s \in S$. For $\chi \in (\mathrm{Irr} W)^F$ we denote by $\widetilde{\chi}_v$ the character of $\mathcal{H}_v(W) \rtimes \langle F \rangle$ which specializes to $\widetilde{\chi}$ at $v = 1$. The virtual $G \times \langle F^\delta \rangle$-modules afforded by the cohomology and the intersection cohomology of Deligne–Lusztig varieties can be computed by means of these bases. For the following result, see [3, §III, Prop. 1.2, Thm. 1.3 and 2.3]:

Theorem 6.1 (Digne–Michel, Lusztig)**.** *Let $w \in W$. The class in the Grothendieck group of $G \times \langle F^\delta \rangle$-modules of the cohomology of* $\mathrm{X}(w)$ *is given by*

$$R_w := \sum_{i\in\mathbb{Z}} (-1)^i \left[H^i(\mathrm{X}(w))\right] = v^{\ell(w)} \sum_{\chi\in(\mathrm{Irr}\, W)^F} \widetilde{\chi}_v(t_w F) R_{\widetilde{\chi}}.$$

This formula can be evaluated by the Chevie-programs written by Jean Michel [29]. Up to adding non-unipotent characters, R_w is a virtual projective character and Proposition 2.1 gives some control on how it can be decomposed on the basis of projective indecomposable modules.

Convention. To simplify the proofs, we shall write equalities between characters whenever they hold up to adding/removing non-unipotent characters. Since the unipotent characters form a basic set, this will be sufficient for our purposes.

Table 1. Unipotent blocks of unitary groups $\mathrm{SU}_n(q)$ for $\ell|(q+1)$, $\ell > n$

$\mathrm{SU}_2(q)$

2	1	1	
1^2	q	1	1
		ps	c

$\mathrm{SU}_3(q)$

3	1	1		
21	$q\Phi_1$	.	1	
1^3	q^3	1	2	1
		ps	c	c

$\mathrm{SU}_4(q)$

4	1	1				
31	$q\Phi_6$	1	1			
2^2	$q^2\Phi_4$	1	1	1		
21^2	$q^3\Phi_6$	.	.	1	1	
1^4	q^6	.	1	1	3	1
		ps	1^2	ps	c	c

$\mathrm{SU}_5(q)$

5	1	1						
41	$q\Phi_1\Phi_4$	.	1					
32	$q^2\Phi_{10}$	.	.	1				
31^2	$q^3\Phi_4\Phi_6$	1	2	1	1			
2^21	$q^4\Phi_{10}$	1	2	.	1	1		
21^3	$q^6\Phi_1\Phi_4$	.	1	.	.	2	1	
1^5	q^{10}	.	2	1	1	3	4	1
		ps	21	ps	1^3	c	c	c

$\mathrm{SU}_6(q)$

6	1	1										
51	$q\Phi_{10}$	1	1									
42	$q^2\Phi_3\Phi_6^2$	1	1	1								
41^2	$q^3\Phi_4\Phi_{10}$	.	.	1	1							
3^2	$q^3\Phi_3\Phi_{10}$	1	1	1	.	1						
321	$q^4\Phi_1^3\Phi_3\Phi_4$	.	.	.	.	.	1					
2^3	$q^6\Phi_3\Phi_{10}$	.	.	1	1	1	2	1				
31^3	$q^6\Phi_4\Phi_{10}$	.	1	1	3	1	2	.	1			
2^21^2	$q^7\Phi_3\Phi_6^2$	.	1	1	3	1	2	1	1	1		
21^4	$q^{10}\Phi_{10}$	.	.	.	1	.	2	1	.	3	1	
1^6	q^{15}	.	.	.	3	1	6	1	1	6	5	1
		ps	1^2	ps	21^2	1^2	c	ps	1^4	c	c	c

6.1. **The case $\ell|(q+1)$.**

Theorem 6.2. *Let ℓ be a prime. Then all unipotent characters of* $\mathrm{SU}_n(q)$*, where* $q \equiv -1 \pmod{\ell}$*, lie in the principal block. Its ℓ-modular decomposition matrix for*

- $2 \le n \le 5$ *and* $\ell > n$,
- $n = 6, 7$ *and* $\ell > 7$,
- $n = 8$ *and* $\ell > 11$, *and*
- $n = 9$ *and* $\ell > 13$

is as given in Tables 1–4.

The corresponding modular Harish-Chandra series are given in the bottom rows of the tables. For $n \le 7$ we also print the degrees of the unipotent characters.

As customary, the principal series is indicated by 'ps', cuspidal characters are denoted by 'c', and for all other series the Harish-Chandra source is given.

Proof. It is well-known that all unipotent characters of $\mathrm{SU}_n(q)$ lie in the principal ℓ-block, see e.g. [2]. The decomposition matrix for $\mathrm{SU}_2(q) \cong \mathrm{SL}_2(q)$ is well-known, and the last unknown entry in the one for $\mathrm{SU}_3(q)$ was determined by Okuyama and Waki

TABLE 2. $\mathrm{SU}_7(q)$, $\ell|(q+1)$, $\ell > 7$

7	1	1														
61	$q\Phi_1\Phi_3\Phi_6$	.	1													
52	$q^2\Phi_4\Phi_{14}$	.	.	1												
51^2	$q^3\Phi_3\Phi_6\Phi_{10}$	1	2	1	1											
43	$q^3\Phi_1\Phi_3\Phi_{14}$	.	.	.	.	1										
421	$q^4\Phi_{10}\Phi_{14}$	1	2	.	1	.	1									
3^21	$q^5\Phi_3\Phi_6\Phi_{14}$	1	2	1	1	2	1	1								
41^3	$q^6\Phi_1\Phi_3\Phi_4\Phi_{10}$	.	1	.	.	1	2	.	1							
32^2	$q^6\Phi_3\Phi_6\Phi_{14}$	.	.	1	.	2	.	1	.	1						
321^2	$q^7\Phi_{10}\Phi_{14}$	.	.	.	.	.	.	.	.	1	1					
2^31	$q^9\Phi_1\Phi_3\Phi_{14}$	.	1	.	.	.	2	.	1	.	2	1				
31^4	$q^{10}\Phi_3\Phi_6\Phi_{10}$	.	2	1	1	2	3	1	4	1	3	.	1			
2^21^3	$q^{11}\Phi_4\Phi_{14}$	.	2	1	1	2	4	1	4	.	3	2	1	1		
21^5	$q^{15}\Phi_1\Phi_3\Phi_6$	.	.	.	.	1	2	.	1	.	2	3	.	4	1	
1^7	q^{21}	.	.	.	.	2	3	1	4	1	8	4	1	10	6	1
		ps	21	ps	1^3	21	2^21	1^3	21^3	ps	c	c	1^5	c	c	c

[30]. It is also an immediate consequence of Theorem 5.9. It was shown by Geck [10] that the decomposition matrix of the unipotent block of $\mathrm{SU}_n(q)$ is unitriangular and that the unipotent characters form a basic set. The distribution of modular characters into Harish-Chandra series was proved in [14, Thm. 4.12].

Harish-Chandra induction of PIMs from proper Levi subgroups yields most of the columns in the tables. Next, Theorem 5.9 and Proposition 5.16 yield the PIMs in the lower right-hand corner of the decomposition matrices. For $\mathrm{SU}_4(q)$ and for $\mathrm{SU}_5(q)$ all other listed projectives are indecomposable, since any direct summand would have to lie in the same or a smaller Harish-Chandra series, and clearly there is no such decomposition possible. This completes the proof for $\mathrm{SU}_4(q)$ and for $\mathrm{SU}_5(q)$.

For $\mathrm{SU}_6(q)$, after computing all Harish-Chandra induced projectives, we are left to show that the fourth column is indecomposable. Using the ℓ-reduction of the non-unipotent irreducible character associated with the multipartition $\boldsymbol{\mu} = (2,2,2)$ one finds that the $(7,4)$-entry of the decomposition matrix is at least 1 (so that the fourth column is indecomposable).

For $\mathrm{SU}_7(q)$ a combination of Theorem 5.9 and Proposition 5.16 gives the PIMs corresponding to cuspidal modules.

For $\mathrm{SU}_8(q)$ the previous arguments leave to determine the missing entries — denoted by $c_1, \ldots, c_7$ — for the column corresponding to the cuspidal Brauer character 32^21. Relations on the c_i's are first obtained using Propositions 5.1 and 5.4: the restriction to ℓ'-elements of the non-unipotent characters corresponding to the multipartitions $(321,1,1)$, $(21,21,1,1)$, $(21,1,1,1,1,1)$ and $(1,1,1,1,1,1,1,1)$ are the irreducible Brauer characters φ_{3213}, $\varphi_{2^21^4}$, φ_{21^6} and φ_{1^8} of $\mathrm{SU}_8(q)$. From the fact that the coefficient of each of these on φ_{32^21} is zero, one gets four relations on the c_i's, from which we deduce $c_1 = 2$, $c_5 = -2c_2 + 3c_3 + c_4 - 3$,

Table 3. $\mathrm{SU}_8(q)$, $\ell|(q+1)$, $\ell > 11$

	ps	1^2	ps	21^2	1^2	ps	321	21^2	1^4	ps	2^21^2	1^2	1^4	21^4	c	c	ps	c	1^6	c	c	c
8	1																					
71	1	1																				
62	1	1	1																			
61^2	.	.	1	1																		
53	1	1	1	.	1																	
4^2	1	1	1	.	1	1																
521	.	.	.	.	.	.	1															
431	.	.	.	.	.	1	.	1														
51^3	.	1	1	3	1	.	2	.	1													
42^2	.	.	1	1	1	1	2	1	.	1												
421^2	.	1	1	3	1	.	2	.	1	1	1											
3^22	.	.	1	1	1	.	2	.	.	1	.	1										
3^21^2	.	1	1	3	2	1	4	3	1	1	1	1	1									
41^4	.	.	.	1	.	.	2	1	.	1	3	.	.	1								
32^21	.	.	.	.	1	1	2	3	.	.	.	.	1	.	1							
321^3	.	.	.	.	.	.	1	.	.	.	.	.	.	.	2	1						
2^4	.	.	.	.	1	1	4	4	.	1	1	1	1	.	3	.	1					
2^31^2	.	.	.	1	.	.	4	1	.	1	3	1	.	1	3	2	1	1				
31^5	.	.	.	3	1	.	8	3	1	1	6	1	1	5	3	4	.	.	1			
2^21^4	.	.	.	3	1	.	8	3	1	1	7	1	1	5	3	4	1	3	1	1		
21^6	.	.	.	.	.	.	2	1	.	.	3	.	.	1	3	2	1	6	.	5	1	
1^8	.	.	.	.	.	.	6	3	.	.	6	1	1	5	8	10	1	10	1	15	7	1

$c_6 = -5c_2 + 6c_3$ and $c_7 = 2 - 9c_2 + 10c_3 + c_4$. Using Lemma 5.6 we obtain another relation given by $10 - 16c_1 - 4c_2 + 10c_3 + 9c_4 - 10c_5 + 5c_6 - c_7 = 0$, which together with the previous relations yields $c_4 = 3$. For the lower bounds, one uses the multipartitions $(2,2,2,2)$ and $(21,21,2)$ to get respectively $c_2 \geq 3$ and $c_3 \geq c_2$. To deduce the missing entries we compute the column of $\mathrm{SU}_{12}(q)$ corresponding to the partition 32^21^5 and then look at its Harish-Chandra restriction, assuming that $\ell > 12$ (or equivalently $\ell > 11$). Apart from 31^9, all the partitions of 12 that are smaller than 32^21^5 satisfy the assumptions of Proposition 5.1, and therefore we obtain almost as many relations as unknown entries for the corresponding column. The missing relation is obtained from Lemma 5.6. This yields

$$\Psi_{32^21^5} = \rho_{32^21^5} + 6\rho_{321^7} + 21\rho_{31^9} + 3\rho_{2^41^4} + 12\rho_{2^31^6} + 24\rho_{2^21^8} + 21\rho_{21^{10}} + 84\rho_{1^{12}}.$$

Since $c_1 = 2$, we deduce that the restriction of this character to $\mathrm{SU}_8(q)$ has necessarily $\Psi_{32^21} + 4\Psi_{321^3}$ as a direct summand, giving upper bounds on the multiplicity of some unipotent constituents in Ψ_{32^21}, namely $c_2 \leq 3$ and $c_3 \leq 4$. Together with the previous lower bounds, we deduce that $c_2 = 3$ and $c_4 \in \{3,4\}$. The relations on the c_i's become $c_5 = 3c_3 - 6$, $c_6 = 6c_3 - 15$ and $c_7 = 10c_3 - 22$. Finally, to determine c_3 we decompose

TABLE 4. $\mathrm{SU}_9(q)$, $\ell|(q+1)$, $\ell > 13$

9	1																													
81	.	1																												
72	.	.	1																											
71^2	1	2	1	1																										
63	.	.	.	.	1																									
621	1	2	.	1	.	1																								
54	.	.	.	.	.	.	1																							
531	1	2	1	1	2	1	1	1																						
61^3	.	1	.	.	1	2	.	.	1																					
52^2	.	.	1	.	2	.	1	1	.	1																				
4^21	1	2	1	1	2	1	.	1	.	.	1																			
521^2	.	.	.	.	.	.	.	.	.	1	.	1																		
432	.	.	.	.	.	.	.	.	.	.	.	.	1																	
431^2	.	.	.	.	1	.	.	.	.	.	2	.	1	1																
42^21	.	1	.	.	1	2	.	.	1	.	2	2	.	1	1															
3^3	.	.	1	.	2	.	.	1	.	1	1	.	2	.	.	1														
51^4	.	2	1	1	2	3	.	1	4	1	.	3	.	.	.	.	1													
3^221	.	1	.	.	.	2	.	.	1	.	.	2	.	.	1	.	.	1												
421^3	.	2	1	1	2	4	.	1	4	.	1	3	.	.	2	.	1	.	1											
32^3	.	.	.	.	.	.	.	.	.	1	.	1	2	.	.	1	.	2	.	1										
3^21^3	.	2	1	1	4	4	1	2	4	1	4	6	2	4	2	1	1	2	1	.	1									
41^5	.	.	.	.	1	2	.	.	1	.	2	2	1	1	3	.	.	.	4	.	.	1								
32^21^2	.	.	.	.	2	.	1	1	.	1	3	3	2	4	.	1	.	2	.	1	1	.	1							
321^4	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	2	.	1	.	.	3	1						
2^41	.	.	.	.	2	1	1	1	.	.	3	3	.	4	2	.	.	2	1	.	1	.	3	.	1					
2^31^3	.	.	.	.	1	2	.	.	1	.	2	4	.	1	4	.	.	4	4	.	.	1	6	2	2	1				
31^6	.	.	.	.	2	3	.	1	4	1	3	11	2	4	4	1	1	6	10	1	1	6	6	5	.	.	1			
2^21^5	.	.	.	.	2	3	.	1	4	1	4	11	2	4	6	1	1	6	11	.	1	6	9	5	3	4	1	1		
21^7	.	.	.	.	.	.	.	.	.	.	2	2	1	1	3	.	.	3	4	.	.	1	12	2	4	10	.	6	1	
1^9	.	.	.	.	.	.	.	.	.	.	3	8	2	4	4	1	.	12	10	1	1	6	30	12	5	20	1	21	8	1
	p	21	p	1^3	21	2^21	p	1^3	21^3	p	2^21	321^2	21	21^3	2^31	1^3	1^5	c	2^21^3	p	1^5	21^5	c	c	c	c	1^7	c	c	c

the Deligne–Lusztig character R_w for $w = s_1s_2s_3s_4$ a Coxeter element. We find

$$R_w = \Psi_8 - \Psi_{71} - \Psi_{62} + \Psi_{61^2} + \Psi_{53} - \Psi_{521} + \Psi_{42^2} + \Psi_{32^21} - \Psi_{2^4} + (3 - c_3)\Psi_{2^31^2}$$

so that by Proposition 2.1 we must have $c_3 \leq 3$, and therefore $c_3 = 3$.

For $\mathrm{SU}_9(q)$, let us denote by $d_1, \ldots, d_{17}$ the remaining missing entries for the two columns corresponding to the cuspidal Brauer characters 3^221 and 32^21^2 of $\mathrm{SU}_9(q)$. (Note that the multiplicity of ρ_{421^3} and ρ_{41^5} in $\Psi_{32^21^2}$ is zero.) As for $\mathrm{SU}_8(q)$, the ℓ-reduction of suitably chosen non-unipotent characters gives relations on the d_i's, from which we deduce $d_3 = d_1 + d_2 - 2$, $d_4 = d_1$, $d_6 = 4 - 2d_2 + 2d_5$, $d_7 = d_1 + d_2 + 2$, $d_8 = 4 - 2d_2 + 3d_5$, $d_9 = 3 - 4d_2 + 4d_5$, $d_{10} = 8 + d_1 - 4d_2 + 5d_5$, $d_{11} = 3$, $d_{13} = 2d_{12}$, $d_{14} = 6$, $d_{15} = 3d_{12}$, $d_{16} = 4d_{12}$ and $d_{17} = 5d_{12} + 15$. The lower bounds are obtained by looking at the

ℓ-reduction of non-unipotent characters $\rho_{\boldsymbol{\mu}}$ for the following multipartitions:

multipartition $\boldsymbol{\mu}$	relations
$(2,2,1,1,1,1,1)$	$d_2 \geq 2$
$(2,2,2,2,1)$	$d_{12} \geq 3$
$(3,2,2,2)$	$d_1 \geq 2$
$(21,2,2,2)$	$d_5 \geq d_2$

To get the value of d_{12} one computes the column of the decomposition matrix of $\mathrm{SU}_{13}(q)$ corresponding to the partition 32^21^6 (assuming that $\ell > 13$). It is given by

$$\Psi_{32^21^6} = \rho_{32^21^6} + 7\rho_{321^8} + 28\rho_{31^{10}} + 3\rho_{2^41^5} + 14\rho_{2^31^7} + 31\rho_{2^21^9} + 24\rho_{21^{11}} + 108\rho_{1^{13}},$$

and its Harish-Chandra restriction to $\mathrm{SU}_9(q)$ involves $\Psi_{32^21^2} + 4\Psi_{321^4}$. This forces $d_{12} \leq 3$ and therefore $d_{12} = 3$ by the previous inequality. From the previous relations we deduce $d_{13} = 6$, $d_{15} = 9$, $d_{16} = 12$ and $d_{17} = 30$.

To compute d_2 we compute the coefficient of $P_{3^21^3}$ in the Deligne–Lusztig character R_w for $w = s_1s_2s_3s_5s_4s_5$. It is given by $\langle \widetilde{R}_w, \varphi_{3^21^3}\rangle = 4 - 2d_2$. Now one checks that for all $v < w$, the PIM $P_{3^21^3}$ does not occur in R_v, so that by Proposition 2.1 we must have $d_2 \leq 2$, and therefore $d_2 = 2$ by the previous inequalities. The other relations on the d_i's become $d_3 = d_1$, $d_6 = 2d_5$, $d_7 = d_1 + 4$, $d_8 = 3d_5$, $d_9 = 4d_5 - 5$ and $d_{10} = d_1 + 5d_5$.

Finally, to compute d_1 and d_5 we consider the PIM of $\mathrm{SU}_{15}(q)$ corresponding to the partition 3^221^7. It has at most 12 unipotent constituents, including $\rho_{3^21^9}$, $\rho_{32^31^6}$ and $\rho_{2^41^7}$. As usual, we obtain relations on the multiplicities of these characters using Propositions 5.1 and 5.4 assuming that $\ell > 15$ (or equivalently $\ell > 13$). With the restriction to ℓ'-elements of the non-unipotent characters associated with the multipartitions

$$(321,21,21,1,1,1),\ (321,21,1,1,1,1,1,1),\ (321,1,1,1,1,1,1,1,1,1),$$
$$(21,21,21,21,21),\ (21,21,21,21,1,1,1),$$

we can deduce that $\langle \Psi_{3^221^7}, \rho_{32^31^6}\rangle = 2$ and $\langle \Psi_{3^221^7}, \rho_{3^21^9}\rangle = \langle \Psi_{3^221^7}, \rho_{2^41^7}\rangle =: d$. Let Ψ be the restriction of $\Psi_{3^221^7}$ to a standard Levi subgroup of semisimple type ${}^2A_8$. We have $\langle \Psi, \rho_{32^3}\rangle = 2$ and $\langle \Psi, \rho_{3^21^3}\rangle = \langle \Psi, \rho_{2^41}\rangle = d$. Moreover, $\lambda = 3^221$ is the highest partition such that ρ_λ is a constituent of Ψ. From the shape of the decomposition matrix and the value of d_2 we deduce that $\Psi_{3^221} + (d-2)\Psi_{3^21^3}$ is a direct summand of Ψ. This forces d_1 (resp. d_5) to be bounded by 2 (resp. by $d - (d-2) = 2$). With the previous inequalities this forces $d_1 = d_5 = 2$ which finishes the determination of all the decomposition numbers of the principal block of $\mathrm{SU}_9(q)$. □

The situation is more complicated for $\mathrm{SU}_{10}(q)$ but we can use our methods to compute all columns of the decomposition matrix but the one for the ordinary cuspidal character. For this specific column we are left with 4 possibilities.

Theorem 6.3. *Assume $\ell > 17$ and $\ell|(q+1)$. Then the ℓ-modular decomposition matrix of the principal ℓ-block of* $\mathrm{SU}_{10}(q)$ *is given in Tables 5 and 6, where $\alpha, \beta \in \{0,1\}$.*

In the tables we have also given the labelling of unipotent characters by bipartitions of 5.

In order to prove this theorem, and especially in order to determine the column corresponding to the partition 4321, we will need to restrict a projective indecomposable module from $\mathrm{SU}_{18}(q)$ to $\mathrm{SU}_{10}(q)$. Although we will not need to determine explicitly

Table 5. Decomposition matrix for $\mathrm{SU}_{10}(q)$, $\ell|(q+1)$, $\ell > 17$

10	5.	1																					
91	.5	1	1																				
82	4.1	1	1	1																			
81^2	41.	.	.	1	1																		
73	1.4	1	1	1	.	1																	
721	${}^2A_5 : 2.$	.	.	.	.	.	1																
71^3	.41	.	1	1	3	1	2	1															
64	3.2	1	1	1	.	1	.	.	1														
631	32.	.	.	.	.	.	.	.	1	1													
62^2	31.1	.	.	1	1	1	2	.	1	1	1												
621^2	3.1^2	.	1	1	3	1	2	1	.	.	1	1											
61^4	$31^2.$	.	.	.	1	.	2	.	.	1	1	3	1										
5^2	2.3	1	1	1	.	1	.	.	1	.	.	.	.	1									
541	${}^2A_5 : 1^2.$	.	.	.	.	.	.	.	.	.	.	.	.	.	1								
532	$1^2.3$	.	.	1	1	1	2	.	.	.	1	.	.	.	.	1							
531^2	1.31	.	1	1	3	2	4	1	1	3	1	1	.	1	2	1	1						
52^21	.32	.	.	.	.	1	2	.	1	3	.	.	.	1	2	.	1	1					
521^3	${}^2A_5 : 1.1$	.	.	.	.	.	1	.	.	.	.	.	.	.	1	.	.	2	1				
51^5	$.31^2$	.	.	.	3	1	8	1	.	3	1	6	5	.	2	1	1	3	4	1			
4^22	21.2	.	.	1	1	1	2	.	1	1	1	.	.	1	2	1	.	.	.	.	1		
4^21^2	2.21	.	1	1	3	2	4	1	1	3	1	1	.	1	2	1	1	.	.	.	1	1	
43^2	$2^2.1$	.	.	.	.	.	.	.	1	1	.	.	.	1	2	.	.	.	.	.	1	.	1
4321	${}^2A_9$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
431^3	$2^21.$	.	.	.	.	.	.	.	.	1	.	.	.	.	2	.	.	.	.	.	1	3	1
42^3	21.1^2	.	.	.	.	1	4	.	1	4	1	1	.	1	4	1	1	3	.	.	1	1	1
42^21^2	$21^2.1$	.	.	.	1	.	4	.	.	2	1	3	1	.	4	1	.	3	2	.	1	3	1
421^4	2.1^3	.	.	.	3	1	8	1	.	3	1	7	5	.	2	1	1	3	4	1	.	1	.
41^6	$21^3.$	.	.	.	.	.	2	.	.	1	.	3	1	.	2	.	.	3	2	.	.	3	1
3^31	1.2^2	.	.	.	.	1	2	.	1	3	.	.	.	1	4	1	1	1	.	.	1	1	3
3^22^2	$1^2.21$	.	.	.	.	1	4	.	1	4	1	1	.	1	6	2	1	3	.	.	1	1	3
3^221^2	$1^3.2$	.	.	.	1	.	4	.	.	1	1	3	1	.	2	1	.	3	2	.	.	.	.
3^21^4	1.21^2	.	.	.	3	1	10	1	.	6	1	7	5	1	10	2	2	6	8	1	1	7	3
32^31	${}^2A_5 : .2$	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	2	1	.	.	.	.
32^21^3	$.2^21$	.	.	.	.	.	2	.	.	3	.	.	.	1	8	1	1	4	4	.	1	6	3
321^5	${}^2A_5 : .1^2$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	2	1	.	.	.	.
31^7	$.21^3$	.	.	.	.	.	6	.	.	3	.	6	5	.	8	1	1	11	14	1	.	6	3
2^5	$1^3.1^2$	.	.	.	.	.	2	.	.	1	.	.	.	.	6	1	.	6	2	.	1	4	4
2^41^2	$1^2.1^3$	.	.	.	.	.	2	.	.	3	.	1	.	1	10	1	1	6	4	.	1	7	4
2^31^4	$1^4.1$	.	.	.	.	.	2	.	.	1	.	3	1	.	4	.	.	6	4	.	.	3	1
2^21^6	1.1^4	.	.	.	.	.	6	.	.	3	.	6	5	.	8	1	1	11	14	1	.	7	3
21^8	$1^5.$	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	3	2	.	.	3	1
1^{10}	$.1^5$	.	.	.	.	.	.	.	.	.	.	.	.	.	6	.	.	8	10	.	.	6	3
		ps	1^2	ps	21^2	1^2	321	1^4	ps	21^2	ps	2^21^2	21^4	1^2	321	1^2	1^4	32^21	321^3	1^6	ps	2^21^2	21^2

the character of this module, we will require relations on the decomposition numbers of $\mathrm{SU}_{18}(q)$ which are given in the following Lemma.

TABLE 6. Decomposition matrix for $\mathrm{SU}_{10}(q)$, $\ell|(q+1)$, $\ell > 17$, cntd.

4321	${}^2A_9$	1																			
431^3	$2^21.$	2	1																		
42^3	21.1^2	2	.	1																	
42^21^2	$21^2.1$	2	1	1	1																
421^4	2.1^3	2	.	1	3	1															
41^6	$21^3.$	6	1	1	6	5	1														
3^31	1.2^2	2	.	.	.	.	.	1													
3^22^2	$1^2.21$	2	.	1	.	.	.	1	1												
3^221^2	$1^3.2$	2	.	1	1	.	.	.	1	1											
3^21^4	1.21^2	8	5	1	3	1	.	1	1	3	1										
32^31	${}^2A_5 : .2$	$3+\alpha$	.	.	.	.	.	.	.	2	.	1									
32^21^3	$.2^21$	$6+2\alpha$	5	.	.	.	.	1	.	3	1	2	1								
321^5	${}^2A_5 : .1^2$	$1+3\alpha$	.	.	.	.	.	.	.	2	.	3	4	1							
31^7	$.21^3$	$14+4\alpha$	5	1	10	15	7	1	1	8	1	4	10	6	1						
2^5	$1^3.1^2$	$10+4\alpha$	1	1	1	.	.	1	1	3	.	4	.	.	.	1					
2^41^2	$1^2.1^3$	$10+4\alpha-\beta$	5	1	3	1	.	1	1	3	1	4	3	.	.	1	1				
2^31^4	$1^4.1$	$10+4\alpha-3\beta$	1	1	7	5	1	.	1	4	.	4	8	2	.	1	3	1			
2^21^6	1.1^4	$20+4\alpha-6\beta$	5	1	13	16	7	1	1	8	1	4	13	6	1	1	6	5	1		
21^8	$1^5.$	$14+4\alpha-10\beta$	1	.	6	5	1	.	.	3	.	4	15	2	.	1	10	15	7	1	
1^{10}	$.1^5$	$26+10\alpha-15\beta$	5	.	10	15	7	1	1	15	1	10	45	14	1	1	15	35	28	9	1
		c	21^4	ps	2^31^2	2^21^4	21^6	1^4	1^2	c	1^6	c	c	c	1^8	ps	c	c	c	c	c

Here, $\alpha, \beta \in \{0, 1\}$.

Lemma 6.4. *Assume $\ell > 17$ and $\ell|(q+1)$. Let us write the unipotent part of the projective character associated with the partition 4321^9 as*

$$\begin{aligned}\Psi_{4321^9} = \rho_{4321^9} &+ a_1\rho_{431^{11}} + a_2\rho_{42^31^8} + a_3\rho_{42^21^{10}} + a_4\rho_{421^{12}} + a_5\rho_{41^{14}} + a_6\rho_{3^31^9} + a_7\rho_{3^22^21^8} \\ &+ a_8\rho_{3^221^{10}} + a_9\rho_{3^31^9} + a_{10}\rho_{32^41^7} + a_{11}\rho_{32^31^9} + a_{12}\rho_{32^21^{11}} + a_{13}\rho_{321^{13}} + a_{14}\rho_{31^{15}} \\ &+ a_{15}\rho_{2^61^6} + a_{16}\rho_{2^51^8} + a_{17}\rho_{2^41^{10}} + a_{18}\rho_{2^31^{12}} + a_{19}\rho_{2^21^{14}} + a_{20}\rho_{21^{16}} + a_{21}\rho_{1^{18}}.\end{aligned}$$

Then the a_i's are subject to the following relations: $a_2 = 2$, $a_3 = a_1$, $a_7 = a_6$, $a_{11} = 2a_8$ and $4 + a_1 + a_6 + 3a_8 - a_{15} - a_{16} = 0$.

Proof. We proceed as follows to obtain relations on the a_i's: we consider several characters ρ such that ρ^0 can be expressed in terms of "a few" ℓ-restrictions of Deligne–Lusztig characters $R_{\mathbf{T}_{wF}}^{\mathbf{G}}(1)$ and then apply Lemma 5.2 to get $\langle \Psi_{4321^9}, \rho^0 \rangle = 0$. We start with the non-unipotent characters ρ associated with the multipartitions

$$\begin{gathered}(321, 321, 1, 1, 1, 1, 1, 1), (321, 21, 21, 21, 1, 1, 1), (321, 21, 21, 1, 1, 1, 1, 1, 1), \\ (21, 21, 21, 21, 21, 21), (21, 21, 21, 21, 21, 1, 1, 1) \\ (21, 21, 21, 21, 1, 1, 1, 1, 1, 1), (21, 21, 21, 1, 1, 1, 1, 1, 1, 1, 1, 1).\end{gathered}$$

By Propositions 5.1 and 5.4 their restrictions to the set of ℓ'-elements are irreducible Brauer characters different from φ_{4321^9}, so that we obtain seven relations $\langle \Psi_{4321^9}, \rho^0 \rangle = 0$, from which we deduce

$$\begin{aligned} a_7 &= -2 + a_2 + a_6, & a_{16} &= 3a_1 - 2a_3 + 3a_8, \\ a_{10} &= a_6, & a_{17} &= -2 + a_2 + 3a_4 - 2a_9 + 3a_{12}, \\ a_{11} &= 2a_1 - 2a_3 + 2a_8, & a_{18} &= -a_1 + a_3 - a_5 + 2a_{13}, \\ a_{15} &= 2 + a_2 + a_6. \end{aligned}$$

Further relations could be easily obtained but we will not need them for our purpose.

We now want to consider the non-unipotent characters $R_{\mathbf{L}}^{\mathbf{G}}(\eta)$ where $\mathbf{L}$ is a Levi subgroup of semisimple type ${}^2A_{15}$ and η is a lift of the Brauer character $\varphi_{2^k1^{16-2k}}$. It is not clear whether φ_{4321^9} occurs in $R_{\mathbf{L}}^{\mathbf{G}}(\eta)^0$ or not, unless $k=0$. However we can adapt the proof of Lemma 5.6 to these characters. With $I = \{s_2, \ldots, s_{16}\}$, the product $w_I w_0$ is just the transposition $(1,18)$, of cycle type 21^{16}. Then from the definition of η, we deduce that $R_{\mathbf{L}}^{\mathbf{G}}(\eta)^0$ can be written as a linear combination of Deligne–Lusztig characters $R_{\mathbf{T}_{\sigma_\mu w_0 F}}^{\mathbf{G}}(1)^0$ for $\mu \in \{21^{16}, 321^{13}, \ldots, 3^k21^{16-3k}\}$. We write $\mathcal{C}$ for the corresponding set of F-conjugacy classes, and $\mathcal{C}_{\not\leq}$ for the set of classes $[w]_F$ satisfying $\mathcal{O} \not\leq [w]_F$ for all $\mathcal{O} \in \mathcal{C}$. By Proposition 3.14 and Remark 3.15, we have $[w]_F \in \mathcal{C}_{\not\leq}$ if and only if ww_0 is not of cycle type μ with

$$\mu \in \{1^{18}, 31^{15}, \ldots, 3^{k+1}1^{15-3k}\} \cup \{51^{13}, \ldots, 53^{k-1}1^{16-3k}\} \cup \{21^{16}, 321^{13}, \ldots, 3^k21^{16-3k}\}.$$

We now want to determine PIMs P_λ which occur in some Deligne–Lusztig character R_w for $[w]_F \in \mathcal{C}_{\not\leq}$, since by Lemma 5.2 this will force $\langle \Psi_\lambda, R_{\mathbf{L}}^{\mathbf{G}}(\eta)^\circ \rangle = 0$. By Remark 5.5 we know that for $\lambda \in \{1^{18}, 21^{16}, \ldots, 2^{k+1}1^{16-2k}\} \cup \{321^{13}, \ldots, 32^k1^{15-2k}\}$ and $k \leq 4$, the corresponding PIM is the projective cover of a cuspidal module, and it can occur only in a Deligne–Lusztig character $R_{\sigma_\mu w_0}$ with $\mu \in \{1^{18}, 31^{15}, \ldots, 3^{k+1}1^{15-3k}\} \cup \{51^{13}, \ldots, 53^{k-1}1^{16-3k}\}$, therefore it can not appear in R_w for $[w]_F \in \mathcal{C}_{\not\leq}$. By induction on k, we can assume that for $\lambda \in \{31^{15}, 41^{14}, 421^{12}, \ldots, 42^{k-2}1^{18-2k}\}$, P_λ does not occur in any R_w with $[w]_F \in \mathcal{C}_{\not\leq} \cup \{3^{k+1}1^{15-3k}, 53^{k-1}1^{16-3k}, 3^k21^{16-3k}\}$ so in particular not in any R_w with $[w]_F \in \mathcal{C}_{\not\leq}$. We claim that $P_{42^{i-1}1^{16-2i}}$ does not occur in R_w with $[w]_F \in \mathcal{C}_{\not\leq}$ either. Otherwise by Lemma 5.2 we would have $\langle P_{42^{k-1}1^{16-2k}}, R_{\mathbf{L}}^{\mathbf{G}}(\eta)^0 \rangle = 0$. But for $k > 0$, the projective indecomposable module $P_{42^{k-1}1^{16-2k}}$ is just the Harish-Chandra induction of $P_{2^k1^{16-2k}}$ since it is indecomposable by [14, Prop. 4.4 and Lemma 4.6]. The latter is known explicitly by Theorem 5.9, and we can check that $\langle \Psi_{42^{k-1}1^{16-2k}}, R_{\mathbf{L}}^{\mathbf{G}}(\eta)^0 \rangle = \langle R_{\mathbf{L}}^{\mathbf{G}}(\Psi_{2^k1^{16-2k}}), R_{\mathbf{L}}^{\mathbf{G}}(\eta) \rangle = 2$.

Thus, we have found $3k+3$ partitions λ such that P_λ cannot occur in R_w for $[w]_F \in \mathcal{C}_{\not\leq}$. Since $3k+3$ is exactly the number of F-conjugacy classes that are not in $\mathcal{C}_{\not\leq}$, we deduce from the usual dimension argument that for every λ outside the set

$$\{1^{18}, 21^{16}, \ldots, 2^{k+1}1^{16-2k}\} \cup \{321^{13}, \ldots, 32^k1^{15-2k}\} \cup \{31^{15}, 41^{14}, 421^{12}, \ldots, 42^{k-1}1^{16-2k}\}$$

the projective module P_λ occurs in some Deligne–Lusztig character R_w with $w \in \mathcal{C}_{\not\leq}$. By Lemma 5.2 this forces in particular $\langle \Psi_{4321^9}, R_{\mathbf{L}}^{\mathbf{G}}(\eta) \rangle = 0$.

This scalar product with $k=3$ and $k=4$ yields two relations on the a_i's, which together with the previous give $a_2 = 2$ and $a_1 = a_3$. $\square$

Proof of Theorem 6.3. As before, Harish-Chandra induction of PIMs from proper Levi subgroups yields most of the columns in the tables. In addition, the lower-right corner is obtained by Theorem 5.9 and Proposition 5.16 so that the only missing columns correspond to the projective characters Ψ_{4321}, $\Psi_{3^2 21^2}$, $\Psi_{32^3 1}$ and $\Psi_{32^2 1^3}$. We will denote by $e_1, \ldots, e_{47}$ the missing entries below the diagonal in these columns.

Recall that if ρ_μ is a unipotent constituent of Ψ_λ then $\mu \trianglelefteq \lambda$ (see the proof of Proposition 4.3), so that $e_{42} = 0$. For the other relations, we use Propositions 5.1 and 5.4 with the partitions $32^2 1^3$, 321^5, $2^3 1^4$, $2^2 1^6$, 21^8 and 1^{10}. This yields[1] $e_{22} = e_{20} + 2e_{21} - 4$, $e_{23} = 3e_{21} - 4$, $e_{27} = -3e_{20} - 2e_{25} + 3e_{26} + 10$, $e_{28} = 23 - 6e_{20} - 4e_{21} + e_{24} - 5e_{25} + 6e_{26}$, $e_{29} = 30 - 10e_{20} - 9e_{25} + 10e_{26}$, $e_{30} = 45 - 15e_{20} + 2e_{21} + e_{24} - 14e_{25} + 15e_{26}$, $e_{31} = 2$, $e_{32} = 3$, $e_{36} = -2e_{34} + 3e_{35}$, $e_{37} = e_{33} - 5e_{34} + 6e_{35} - 4$, $e_{38} = -9e_{34} + 10e_{35}$, $e_{39} = e_{33} - 14e_{34} + 15e_{35} + 2$, $e_{40} = 4$, $e_{44} = 3e_{43} - 1$, $e_{45} = e_{41} + 6e_{43} - 15$, $e_{46} = 10e_{43} - 15$ and $e_{47} = e_{41} + 15e_{43} - 10$. Further relations are obtained from Lemma 5.6. Together with the previous ones, they yield $e_{24} = e_{20} + 4e_{21} - 3$, $e_{33} = 4$ and $e_{41} = 10$.

In addition, we look at the restriction to ℓ'-elements of other non-unipotent characters. They no longer give irreducible Brauer characters, but they still yield the following inequalities:

multipartition $\boldsymbol{\mu}$	relations
$(2,2,1,1,1,1,1,1)$	$e_{20} \geq 3$
$(2,2,2,2,1,1)$	$e_{43} \geq 3$
$(2,2,2,2,2)$	$e_{34} \geq 4$
$(3,2,2,2,1)$	$e_{21} \geq 2$
$(21,2,2,2,1)$	$e_{26} \geq e_{25} + e_{20} - e_{21} - 1$
$(2^3,2,2)$	$e_{25} \geq 2e_{21} - 1$
$(2^3 1, 21)$	$e_{35} \geq e_{34}$

Next, as in the case of $\mathrm{SU}_8(q)$, we compute the character of well-chosen PIMs for unitary groups of larger rank and then deduce upper bounds for the e_i's by Harish-Chandra restriction. We start with the restriction of the projective character $\Psi_{32^2 1^5}$ of $\mathrm{SU}_{12}(q)$, which we have computed previously. It contains $\Psi_{32^2 1^3}$, and since the coefficient of $\rho_{2^4 1^2}$ in this restriction is 3, we deduce that $e_{43} \leq 3$, hence $e_{43} = 3$ with the previous inequalities. This yields $e_{44} = 8$, $e_{45} = 13$, $e_{46} = 15$ and $e_{47} = 45$.

The partitions $32^3 1^7$, $32^2 1^9$ and 321^{11} of 16 satisfy the assumptions of Proposition 5.1 (see Proposition 5.4.(2)), so that we can compute the projective characters $\Psi_{32^3 1^7}$ and $\Psi_{3^2 21^8}$ of $\mathrm{SU}_{16}(q)$ (using Lemma 5.6 in addition). They are given by:

$$\begin{aligned}
\Psi_{32^3 1^7} &= \rho_{32^3 1^7} + 8\rho_{32^2 1^9} + 36\rho_{321^{11}} + 120\rho_{31^{13}} + 4\rho_{2^5 1^6} + 24\rho_{2^4 1^8} + 76\rho_{2^3 1^{10}} \\
&\quad + 156\rho_{2^2 1^{12}} + 180\rho_{21^{14}} + 660\rho_{1^{16}}, \\
\Psi_{3^2 21^8} &= \rho_{3^2 21^8} + x\rho_{3^2 1^{10}} + 2\rho_{32^3 1^7} + x\rho_{32^2 1^9} + 2\rho_{321^{11}} + (x+11)\rho_{31^{13}} \\
&\quad + 3\rho_{2^5 1^6} + x\rho_{2^4 1^8} + 4\rho_{2^3 1^{10}} + (x+11)\rho_{2^2 1^{12}} + 3\rho_{21^{14}} + (x+24)\rho_{1^{16}},
\end{aligned}$$

for some undetermined x. Using again Harish-Chandra restriction to $\mathrm{SU}_{10}(q)$ one finds $e_{21} \leq 2$, $e_{25} \leq 3$ and $e_{34} \leq 4$. The previous inequalities force equalities to hold and we

[1] Relations on $e_1, \ldots, e_{19}$ are also obtained, but they are not sufficient to determine entries in the column of Ψ_{4321} and we shall rather deal with them at the end of the proof.

deduce the following part of the decomposition matrix:

$$\begin{array}{r|cccccccc}
3^221^2 & 1 \\
3^21^4 & e_{20} & 1 \\
32^31 & 2 & . & 1 \\
32^21^3 & e_{22} & 1 & 2 & 1 \\
321^5 & e_{23} & . & 3 & 4 & 1 \\
31^7 & e_{24} & 1 & 4 & 10 & 6 & 1 \\
2^5 & 3 & . & 4 & . & . & . & 1 \\
2^41^2 & e_{26} & 1 & e_{35} & 3 & . & . & 1 & 1
\end{array}$$

Let us have a closer look at the restriction of $\Psi_{3^221^8}$: the coefficients of $\rho_{3^21^4}$, $\rho_{32^21^3}$ and $\rho_{2^41^2}$ in this restriction are equal. No matter how many copies of $\Psi_{3^21^4}$, Ψ_{32^31}, $\Psi_{32^21^3}$ we remove from this restriction, we observe from the previous matrix than the multiplicity of $\rho_{3^21^4}$ will be larger that the multiplicity of ρ_{32^31}, which in turn will be larger than the multiplicity of $\rho_{32^21^3}$, so that $e_{20} \geq e_{22} \geq e_{26}$. On the other hand, $e_{26} \leq e_{25} + e_{20} - e_{21} - 1$, that is $e_{26} \leq e_{20}$ since $e_{21} = 2$ and $e_{25} = 3$. We deduce that $e_{26} = e_{20}$, and the relations on the e_i's become $e_{22} = e_{20}$, $e_{23} = 2$, $e_{27} = 4$, $e_{24} = e_{20} + 5$, $e_{28} = e_{20} + 5$, $e_{29} = 3$, $e_{30} = e_{20} + 12$, $e_{36} = 3e_{35} - 8$, $e_{37} = 6e_{35} - 20$, $e_{38} = 10e_{35} - 36$ and $e_{39} = 15e_{35} - 50$.

To obtain the values of $e_1, \ldots, e_{20}$ and e_{35} we decompose Deligne–Lusztig characters R_w for various w. Starting with a Coxeter element $w = s_1s_2s_3s_4s_5$ we find

$$\begin{aligned} R_w = {} & \Psi_{10} - \Psi_{91} - \Psi_{82} + \Psi_{81^2} + \Psi_{73} - \Psi_{721} + \Psi_{62^2} + \Psi_{52^21} - \Psi_{42^3} - \Psi_{32^31} \\ & + \Psi_{2^5} + (e_{35} - 4)\Psi_{2^41^2}. \end{aligned}$$

From Proposition 2.1 we deduce that $e_{35} \leq 4$, and therefore $e_{35} = 4$ by the previous inequalities.

Now let $w' = s_1s_2s_3s_4s_6s_5s_6$. We can first check that Ψ_{4321} occurs with multiplicity -1 in $R_{w'}$. Then for $v < w'$, we decompose R_v to find that the only PIMs which can occur in R_v correspond to partitions lying in the set $\{\mu \ntrianglelefteq 4321\} \cup \{42^3, 42^21^2, 3^22^2, 32^31, 2^5\}$. We deduce from Proposition 2.1 that for any partition λ outside of this set, the multiplicity of Ψ_λ in $R_{w'}$ is non-positive. This yields 14 inequalities giving upper bounds for 13 variables among $e_1, \ldots, e_{19}$. As usual, lower bounds are obtained by ℓ-restrictions of non-unipotent characters corresponding to well-chosen partitions. We give, in the following table, the partition to consider for each variable.

e_1	$(4,3,1,1,1)$	e_{12}	$(321,1,1,1,1)$
e_4	$(4,2,1,1,1,1)$	e_{13}	$(3,1,1,1,1,1,1,1)$
e_5	$(4,1,1,1,1,1,1)$	e_{15}	$(2^31,21)$
e_6	$(3,3,3,1)$	e_{16}	$(21,21,21,1)$
e_8	$(3,3,2,1,1)$	e_{17}	$(21,21,1,1,1,1)$
e_9	$(3,3,1,1,1,1)$	e_{18}	$(21,1,1,1,1,1,1,1)$
e_{11}	$(321,21,1)$	e_{19}	$(1,1,1,1,1,1,1,1,1,1)$

Except for e_{15}, the lower and upper bounds all match and we obtain $e_1 = e_6 = 2$, $e_4 = -2e_2 + 3e_3$, $e_5 = 4 - 5e_2 + 6e_3$, $e_8 = -e_2 + e_3 + e_7$, $e_9 = 6 - 3e_2 + 3e_3 + e_7$, $e_{11} = 2e_{10}$, $e_{12} = 3e_{10} - 8$, $e_{13} = 10(e_3 - e_2) + e_7 + 4e_{10}$, $e_{16} = -2e_{14} + 3e_{15}$, $e_{17} = 10 - 5e_{14} + 6e_{15}$, $e_{18} = 6 + 15(e_2 - e_3) - e_7 - 9e_{14} + 10e_{15}$ and $e_{19} = 20e_2 - 21e_3 + 6e_{10} - 14e_{14} + 15e_{15}$. For e_{15} we can only deduce that $-1 \leq e_{15} - e_{14} \leq 2(e_3 - e_2)$.

As before, to get further upper bounds we consider the Harish-Chandra restriction of Ψ_{4321^9} using its expression given in Lemma 6.4. In the following table we give the multiplicity of Ψ_λ in this restriction for various partitions λ:

λ	coefficient
42^3	$2 - e_2$
42^21^2	$e_2 - e_3$
3^22^2	$e_2 - e_7$
32^31	$4 - e_{10}$
2^5	$-4 - e_2 + e_3 + e_7 + 4e_{10} - e_{14}$.

Since these coefficients must be nonnegative, we deduce upper bounds for e_2, e_3, e_7, e_{10} and e_{14}. Lower bounds are obtained by ℓ-reduction of non-unipotent characters corresponding to the following multipartitions:

$\boldsymbol{\mu}$	relation
$(4,2,2,2)$	$e_2 \geq 2$
$(2,21,21)$	$e_3 \geq 2$
$(32,32)$	$e_7 \geq e_2$
$(321,1^4)$	$e_{10} \geq 3$
$(2,2,2,2,2)$	$e_{14} \geq 4 - 5e_2 + e_3 + e_7 + 4e_{10}$.

This yields successively $2 = e_2 = e_3 = e_7$, $e_{10} \in \{3,4\}$ and $e_{14} = 4e_{10} - 2$. If we set $\alpha = e_{10} - 4$ and $\beta = e_{14} - e_{15}$, we deduce that $\alpha, \beta \in \{0,1\}$ and any e_i with $i = 1,\ldots,19$ can be expressed in terms of α and β. The explicit relations are written in Table 6

Finally we consider the Deligne–Lusztig character for $w'' = s_1s_2s_3s_6s_5s_4s_6s_5s_6$. One can check that the PIM $\Psi_{3^21^4}$ does not occur in R_v for any $v < w''$, therefore by Proposition 2.1 it must occur in $R_{w''}$ with a non-positive coefficient. This coefficient is actually given by $6e_{20} - 18$ so that $e_{20} \leq 3$. With the previous inequalities this proves that $e_{20} = 3$. □

Remark 6.5. One striking observation to make on the decomposition matrices is the repetition of large portions of certain rows. We point out a few examples in the largest case, that is, for $\mathrm{SU}_{10}(q)$ when $q \equiv -1 \pmod{\ell}$. If λ is one of the partitions $71^3, 64, 531^2, 31^7$, the first part of the corresponding row of the decomposition matrix repeats itself for the partition μ obtained by moving one box from the first row of λ to the second row. The same happens for other partitions λ as well. This process does not preserve Harish-Chandra series; we do not have any explanation for this phenomenon.

6.2. **The case $\ell | (q^2 - q + 1)$.**

Theorem 6.6. *Let $\ell > n$ be a prime. Then the ℓ-modular decomposition matrices for the unipotent ℓ-blocks of* $\mathrm{SU}_n(q)$, $3 \leq n \leq 10$, $\ell | (q^2 - q + 1)$, *are as given in Tables 7–13.*

The modular Harish-Chandra series are as indicated in the bottom rows of the tables.

Proof. As before, we label the characters of PIMs by the partition labelling the corresponding Brauer character. For $n \leq 5$, the Sylow ℓ-subgroups of $\mathrm{SU}_n(q)$ are cyclic. The Brauer trees in these cases were determined by Fong and Srinivasan [8], but can also easily be found by Harish-Chandra induction. Now let $n = 6$. Harish-Chandra induction of PIMs from proper Levi subgroups gives all the columns in Table 8 except for the 6th

Table 7. $\mathrm{SU}_n(q)$, $3 \leq n \leq 5$, $\ell|(q^2-q+1)$, $\ell > n$

$$\mathrm{SU}_3(q): \quad 3 \underset{ps}{\text{——}} 1^3 \underset{c}{\text{——}} \bigcirc \underset{c}{\text{——}} 21$$

$$\mathrm{SU}_4(q): \quad 4 \underset{ps}{\text{——}} 2^2 \underset{ps}{\text{——}} 1^4 \underset{c}{\text{——}} \bigcirc$$

$$\mathrm{SU}_5(q): \quad \begin{array}{c} 5 \text{ —— } 2^21 \text{ —— } \bigcirc \text{ —— } 21^3 \\ 32 \text{ —— } 1^5 \text{ —— } \bigcirc \text{ —— } 41 \end{array}$$

$$\mathrm{SU}_9(q): \quad \begin{array}{c} 72 \text{ —— } 421^3 \text{ —— } \bigcirc \text{ —— } 42^21 \\ 521^2 \underset{ps}{\text{——}} 2^21^5 \underset{1^3}{\text{——}} \bigcirc \underset{21}{\text{——}} 431^2 \end{array}$$

Table 8. $\mathrm{SU}_6(q)$, $\ell|(q^2-q+1)$, $\ell > 6$

6	1	1										
51	$q\Phi_{10}$	.	1									
42	$q^2\Phi_3\Phi_6^2$	.	.	1								
41^2	$q^3\Phi_4\Phi_{10}$	1	.	.	1							
3^2	$q^3\Phi_3\Phi_{10}$	1	1	.	.	1						
321	$q^4\Phi_1^3\Phi_3\Phi_4$	.	.	.	.	.	1					
2^3	$q^6\Phi_4\Phi_{10}$	1	.	.	1	1	.	1				
31^3	$q^6\Phi_3\Phi_{10}$	.	1	.	.	1	.	.	1			
2^21^2	$q^7\Phi_3\Phi_6^2$	.	.	.	.	.	.	.	.	1		
21^4	$q^{10}\Phi_{10}$	.	.	.	1	.	.	1	.	.	1	
1^6	q^{15}	.	.	.	.	1	2	1	1	.	.	1
		ps	ps	ps	ps	ps	c	B	1^4	ps	c	c

Here, B stands for the (cuspidal) Steinberg PIM of $\mathrm{SL}_3(q^2)$.

and the last two, corresponding to the characters Ψ_{321}, Ψ_{21^4} and Ψ_{1^6}. By triangularity, we just have to determine the five entries below the diagonal for the 6th projective, which we denote by $a_1, \dots, a_5$, and the one missing entry in the 10th projective. As observed by Gerhard Hiss, the (projective) tensor product of Ψ_6 with ρ_{321} has the following unipotent constituents: $(q+1)\rho_{321} + q^2(q-1)\rho_{1^6}$. This implies that $a_1 = \cdots = a_4 = 0$. Furthermore,

TABLE 9. $\mathrm{SU}_7(q)$, $\ell|(q^2-q+1)$, $\ell>7$

7	1	1														
61	$q\Phi_1\Phi_3\Phi_6$	.	1													
52	$q^2\Phi_4\Phi_{14}$	1	.	1												
51^2	$q^3\Phi_3\Phi_6\Phi_{10}$	.	.	.	1											
43	$q^3\Phi_1\Phi_3\Phi_{14}$	.	.	.	.	1										
421	$q^4\Phi_{10}\Phi_{14}$	1	.	.	.	.	1									
3^21	$q^5\Phi_3\Phi_6\Phi_{14}$	.	.	.	1	.	.	1								
41^3	$q^6\Phi_1\Phi_3\Phi_4\Phi_{10}$	.	.	.	.	1	.	.	1							
32^2	$q^6\Phi_3\Phi_6\Phi_{14}$	.	.	.	.	.	.	.	.	1						
321^2	$q^7\Phi_{10}\Phi_{14}$	.	.	1	.	.	.	.	.	.	1					
2^31	$q^9\Phi_1\Phi_3\Phi_{14}$	.	.	.	.	.	.	.	1	.	.	1				
31^4	$q^{10}\Phi_3\Phi_6\Phi_{10}$	.	.	.	.	.	.	.	.	1	.	.	1			
2^21^3	$q^{11}\Phi_4\Phi_{14}$	1	.	1	.	.	1	.	.	.	.	.	.	1		
21^5	$q^{15}\Phi_1\Phi_3\Phi_6$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	
1^7	q^{21}	.	.	1	.	.	.	.	.	.	1	2	.	1	.	1
		ps	21	ps	ps	21	1^3	1^3	21	ps	B	c	1^3	1^3	21	c

the (projective) tensor product of Ψ_{42} with ρ_{21^4} has unipotent constituents

$$\rho_{41^2}+2\rho_{2^3}+\rho_{2^21^2}+(q^2+2)\rho_{21^4}+\rho_{1^6}$$

hence decomposes as $\Psi_{41^2}+\Psi_{2^3}+\Psi_{2^21^2}+q^2\Psi_{21^4}$, and necessarily the last entry in the 10th column vanishes as well.

From the cohomology of the Deligne–Lusztig variety for $w=s_1s_2s_3$ we get the virtual projective character

$$\begin{aligned} R_w =&\rho_6-\rho_{42}-\rho_{321}+\rho_{2^21^2}-\rho_{1^6}\\ =&\Psi_6-\Psi_{42}-\Psi_{41^2}-\Psi_{3^2}-\Psi_{321}+\Psi_{2^3}+\Psi_{31^3}+\Psi_{2^21^2}-(2-a_5)\Psi_{1^6}\end{aligned}$$

where $\Psi_{321}=\rho_{321}+a_5\rho_{1^6}$ so that $a_5\leq 2$. If $\ell>6$ then $a_5\geq 2$, which force $a_5=2$.

For $n=7$, there are only two entries to be determined, belonging to the projective character Ψ_{2^31} whose unipotent part equals $\rho_{2^31}+b_1\rho_{2^21^3}+b_2\rho_{1^7}$. The existence of an ℓ-character in general position in T_w for $w=s_2s_3s_4s_3s_2s_5s_6$ forces the relation $b_2\geq b_1+2$ to hold. On the other hand the character $R_{s_1s_2s_3}[1]$ coming from the cohomology of the Deligne–Lusztig variety $\mathrm{X}(s_1s_2s_3)$ is given by

$$\begin{aligned} R_{s_1s_2s_3}[1] =&\rho_7+\rho_{41^3}-\rho_{1^7}\\ =&\Psi_7-\Psi_{52}-\Psi_{421}+\Psi_{41^3}+\Psi_{321^2}-\Psi_{2^31}+(1+b_1)\Psi_{2^21^3}+(b_2-b_1-2)\Psi_{1^7}.\end{aligned}$$

From Proposition 2.1 we deduce the relation $b_2-b_1-2\leq 0$ so that $b_2=b_1+2$.

We claim that $b_1=0$. For this, we view $\mathrm{GU}_7(q)$ as a Levi subgroup of $\mathrm{SU}_9(q)$. The projective character $\Psi_{2^31^3}$, cut by the principal block of $\mathrm{SU}_9(q)$, involves only the unipotent characters $\rho_{2^31^3}$ (with multiplicity 1), ρ_{21^7} and ρ_{1^9}. Now, the unipotent part of the

Table 10. $\mathrm{SU}_8(q)$, $\ell|(q^2-q+1)$, $\ell>8$

8	1	1																					
71	$q\Phi_{14}$	.	1																				
62	$q^2\Phi_4\Phi_8\Phi_{10}$	.	.	1																			
61^2	$q^3\Phi_3\Phi_6\Phi_{14}$	.	.	.	1																		
53	$q^3\Phi_4\Phi_8\Phi_{14}$	1	.	.	.	1																	
4^2	$q^4\Phi_3\Phi_8\Phi_{14}$	.	1	1	.	.	1																
521	$q^4\Phi_1^3\Phi_3\Phi_4^2\Phi_8$	.	.	.	.	.	.	1															
431	$q^5\Phi_8\Phi_{10}\Phi_{14}$	1	.	.	.	.	.	.	1														
51^3	$q^6\Phi_3\Phi_{10}\Phi_{14}$	.	.	.	.	1	.	.	.	1													
42^2	$q^6\Phi_4^2\Phi_8\Phi_{14}$	.	.	1	.	.	1	.	.	.	1												
421^2	$q^7\Phi_3\Phi_6^2\Phi_8\Phi_{10}$	.	.	.	.	.	.	.	.	.	.	1											
3^22	$q^7\Phi_3\Phi_6\Phi_8\Phi_{14}$	.	.	.	1	.	.	.	.	.	.	.	1										
3^21^2	$q^8\Phi_4^2\Phi_8\Phi_{14}$	1	.	.	.	1	.	.	.	1	.	.	.	1									
41^4	$q^{10}\Phi_3\Phi_{10}\Phi_{14}$	.	.	.	.	.	.	.	.	.	1	.	.	.	1								
32^21	$q^9\Phi_8\Phi_{10}\Phi_{14}$	.	1	.	.	.	1	.	.	.	.	.	.	.	.	1							
321^3	$q^{11}\Phi_1^3\Phi_3\Phi_4^2\Phi_8$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1						
2^4	$q^{12}\Phi_3\Phi_8\Phi_{14}$	1	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.	1					
2^31^2	$q^{13}\Phi_4\Phi_8\Phi_{14}$	.	.	.	.	.	1	.	.	.	1	.	.	.	1	.	.	.	1				
31^5	$q^{15}\Phi_3\Phi_6\Phi_{14}$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1			
2^21^4	$q^{16}\Phi_4\Phi_8\Phi_{10}$	.	.	.	.	.	.	2	.	1	.	.	.	1	.	.	.	1	.	.	1		
21^6	$q^{21}\Phi_{14}$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	.	.	.	1	
1^8	q^{28}	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	2	.	1	.	.	.	1
		ps	*ps*	*ps*	*ps*	*ps*	*ps*	321	*ps*	1^4	*ps*	*ps*	*ps*	*ps*	21^4	1^4	321	B	B	1^4	1^6	21^4	1^6

Harish-Chandra restriction of $\Psi_{2^31^3}$ to $\mathrm{GU}_7(q)$ has only ρ_{2^31}, ρ_{21^5} and ρ_{1^7} as unipotent constituents, and not $\rho_{2^21^3}$, which forces $b_1=0$ and therefore $b_2=2$.

In the case $n=8$, it turns out that no simple unipotent kG-module is cuspidal, so that the decomposition matrix can be determined using representations of various Hecke algebras and the decomposition matrices of proper Levi subgroups like $\mathrm{SU}_6(q)$.

For $n=9$, let $w=s_1s_2s_1s_3s_2s_4s_3s_8s_7s_6s_5s_4$. Then $\mathbf{T}^{wF}$ contains a Sylow ℓ-subgroup of G. If $\ell>9$, there exist semisimple ℓ-elements $t,t',t''\in\mathbf{T}^{wF}$ such that $(C_{\mathbf{G}}(t),wF)$ (resp. $(C_{\mathbf{G}}(t'),wF)$, resp. $(C_{\mathbf{G}}(t''),wF)$) has semisimple type $A_0(q)$, (resp. ${}^2A_2(q)$, resp. ${}^2A_2(q^3)$). Through the Jordan decomposition of characters, we get three non-unipotent cuspidal characters ρ, ρ' and ρ''. The restriction to the set of ℓ'-elements of the last two

TABLE 11. $\mathrm{SU}_9(q)$, $\ell|(q^2-q+1)$, $\ell>9$

9	1	1																					
81	$q\Phi_1\Phi_4\Phi_8$	.	1																				
71^2	$q^3\Phi_4\Phi_8\Phi_{14}$	1	.	1																			
63	$q^3\Phi_1\Phi_4^2\Phi_8\Phi_{18}$	.	1	.	1																		
621	$q^4\Phi_{10}\Phi_{14}\Phi_{18}$	1	.	.	.	1																	
54	$q^4\Phi_8\Phi_{14}\Phi_{18}$	1	.	.	.	.	1																
61^3	$q^6\Phi_1\Phi_3\Phi_4\Phi_8\Phi_{14}$	.	1	.	1	.	.	1															
52^2	$q^6\Phi_4^2\Phi_8\Phi_{10}\Phi_{18}$	1	.	1	.	.	1	.	1														
4^21	$q^6\Phi_4\Phi_8\Phi_{14}\Phi_{18}$	1	.	1	.	1	.	.	.	1													
432	$q^7\Phi_1\Phi_4\Phi_8\Phi_{14}\Phi_{18}$	.	.	.	1	.	.	.	.	.	1												
3^3	$q^9\Phi_3\Phi_8\Phi_{14}\Phi_{18}$	.	.	1	.	.	.	.	1	1	.	1											
51^4	$q^{10}\Phi_3\Phi_8\Phi_{10}\Phi_{14}$	.	.	1	.	.	.	.	1	.	.	.	1										
3^221	$q^{10}\Phi_1\Phi_4\Phi_8\Phi_{14}\Phi_{18}$	.	.	.	.	.	.	1	.	.	.	.	.	1									
32^3	$q^{12}\Phi_4\Phi_8\Phi_{14}\Phi_{18}$	.	.	.	.	.	1	.	1	.	.	1	.	.	1								
3^21^3	$q^{12}\Phi_4^2\Phi_8\Phi_{10}\Phi_{18}$	1	.	1	.	1	1	.	1	1	.	1	1	.	.	1							
41^5	$q^{15}\Phi_1\Phi_3\Phi_4\Phi_8\Phi_{14}$	.	.	.	1	.	.	1	.	.	1	.	.	.	.	.	1						
321^4	$q^{16}\Phi_{10}\Phi_{14}\Phi_{18}$	.	.	.	.	.	1	.	.	.	.	.	.	.	1	.	.	1					
2^41	$q^{16}\Phi_8\Phi_{14}\Phi_{18}$	1	.	.	.	1	1	.	.	.	.	.	.	.	.	1	.	.	1				
2^31^3	$q^{18}\Phi_1\Phi_4^2\Phi_8\Phi_{18}$	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	1	.	.	1			
31^6	$q^{21}\Phi_4\Phi_8\Phi_{14}$	.	.	.	.	.	1	.	1	.	.	1	1	2	1	1	.	.	.	.	1		
21^7	$q^{28}\Phi_1\Phi_4\Phi_8$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	2	1	.	1	
1^9	q^{36}	.	.	.	.	.	1	.	.	.	.	1	.	2	1	1	.	3	1	2	1	.	1
		ps	21	ps	21	1^3	ps	21	ps	1^3	$21.B$	$1^3.B$	1^3	2^31	B	1^3	21	c	c	c	1^7	c	c

is expressed in terms of the basic set of unipotent characters as follows:

$$(\rho')^0 = \big(-\rho_{81}+\rho_{63}-2\rho_{54}+2\rho_{52^2}-\rho_{432}-2\rho_{3^3}-2\rho_{51^4}+\rho_{3^221}+2\rho_{3^21^3}-2\rho_{2^41}$$
$$-\rho_{2^31^3}+\rho_{21^7}\big)^0,$$
$$(\rho'')^0 = \big(-\rho_{63}+\rho_{621}+\rho_{54}+\rho_{61^3}-\rho_{52^2}-\rho_{4^21}+\rho_{432}+2\rho_{3^3}+\rho_{51^4}-\rho_{3^221}-\rho_{32^3}$$
$$\rho_{3^21^3}-\rho_{41^5}+\rho_{321^4}+\rho_{2^41}+\rho_{2^31^3}\big)^0.$$

Since the restriction of ρ' (resp. ρ'') has a positive coefficient on the Brauer character φ_{21^7} (resp. the characters φ_{321^4}, φ_{2^41} and $\varphi_{2^31^3}$), we deduce that φ_{321^4}, φ_{2^41}, $\varphi_{2^31^3}$ and φ_{21^7} are cuspidal. Let us denote the entries below the diagonal for these columns by $c_1,\ldots,c_{13}$.

In order to obtain information on the c_i we start with Harish-Chandra restriction of projective characters of $\mathrm{SU}_{11}(q)$. The partitions that are smaller than 321^6 and that have the same 3-core (that is 1^2) are 321^6, 2^31^5 and 1^{11}. Therefore the corresponding unipotent characters are the only unipotent constituents of Ψ_{321^6}. Since ρ_{31^6} does not occur in the restriction of these characters to $\mathrm{SU}_9(q)$, we deduce that $c_3=0$. The only unipotent constituent of Ψ_{21^9} is ρ_{21^9}, whose restriction is ρ_{21^7}, which proves that ρ_{21^7} is the only unipotent constituent of Ψ_{21^7} and forces $c_{13}=0$. For c_2 and c_4 we need

Table 12. $\mathrm{SU}_{10}(q)$, $\ell|(q^2-q+1)$, $\ell>10$, principal block

10	$5.$	1																					
82	4.1	1	1																				
73	1.4	1	.	1																			
721	${}^2A_5:2.$	.	.	.	1																		
71^3	$.41$	.	.	1	.	1																	
621^2	3.1^2	.	1	.	.	.	1																
5^2	2.3	1	1	1	.	.	.	1															
52^21	$.32$	.	.	1	.	1	.	1	1														
521^3	${}^2A_5:1.1$	.	.	.	1	.	.	.	.	1													
4^21^2	2.21	1	1	1	.	1	1	1	.	.	1												
43^2	$2^2.1$	1	1	.	.	.	.	1	.	.	.	1											
4321	${}^2A_9$	.	.	.	.	.	.	.	.	.	.	.	1										
431^3	$2^21.$	1	.	.	.	.	.	.	.	.	.	1	.	1									
42^3	21.1^2	1	1	.	.	.	1	1	.	.	1	1	.	.	1								
421^4	2.1^3	.	.	.	2	1	1	.	.	.	1	.	.	.	1	1							
41^6	$21^3.$	.	.	.	.	.	.	.	.	.	.	1	d_4	1	1	.	1						
3^31	1.2^2	1	.	1	.	1	.	1	1	.	1	.	d_5	.	.	.	.	1					
321^5	${}^2A_5:.1^2$	.	.	.	.	.	.	.	.	1	.	.	d_6	.	.	.	.	.	1				
2^5	$1^3.1^2$	1	.	.	.	.	.	1	.	.	1	1	d_7	1	1	.	.	1	.	1			
2^31^4	$1^4.1$	.	.	.	.	.	.	1	.	.	.	1	d_8	1	1	.	1	.	.	1	1		
2^21^6	1.1^4	.	.	.	2	1	.	1	1	2	1	.	d_9	.	1	1	.	1	.	1	.	1	
1^{10}	$.1^5$	.	.	.	.	.	.	1	1	2	.	.	d_{10}	.	.	.	.	1	2	1	3	1	1
		ps	ps	ps	321	1^4	ps	ps	1^4	321	ps	ps	c	21^4	1^3	1^6	21^4	$1^4.1^3$	c	1^3	c	1^6	c

Here $d_{10} = 6 - 3d_4 + 3d_5 + 2d_6 - 3d_7 + 3d_8 + d_9$

Table 13. $\mathrm{SU}_{10}(q)$, $\ell|(q^2-q+1)$, blocks 31 and 21^2, $\ell>10$

91	1								
64	1	1							
62^2	.	1	1						
61^4	.	.	1	1					
4^22	1	1	1	.	1				
3^221^2	.	.	1	1	1	1			
32^31	.	.	.	.	.	.	1		
32^21^3	1	.	.	.	1	.	.	1	
31^7	.	.	.	.	1	1	2	1	1
	ps	ps	ps	21^4	ps	1^3	321	1^4	1^6

81^2	1								
631	1	1							
541	.	.	1						
532	1	.	.	1					
51^5	.	.	.	1	1				
3^22^2	1	1	.	1	.	1			
3^21^4	.	.	2	1	1	1	1		
2^41^2	.	1	2	.	.	1	1	1	
21^8	.	1	.	.	.	.	.	1	1
	ps	ps	321	ps	1^4	ps	1^6	1^3	21^4

to go up to $\mathrm{SU}_{13}(q)$ and consider the projective character $\Psi_{32^31^4}$ whose unipotent part is $\rho_{32^31^4} + x\rho_{32^21^6} + y\rho_{31^{10}}$ for some $x, y \geq 0$ (since 32^31^4, 32^21^6 and 31^{10} are the only partitions which are smaller than 32^31^4 and have 31 as a 3-core). The Harish-Chandra restriction of this character to $\mathrm{SU}_9(q)$, cut by the block, is $\rho_{32^3} + 2\rho_{321^4} + (2x+y)\rho_{31^6} + (x+2y)\rho_{1^9}$.

From the decomposition matrix of $\mathrm{SU}_9(q)$ we deduce that one copy of Ψ_{321^4} occurs in this restriction, which forces $c_2 = c_4 = 0$.

Further relations are provided by the virtual characters $R_w[\lambda]$. Starting with $w = s_1s_2s_3s_4$ we have

$$\begin{aligned} R_w[1] = & -\rho_{71^2} - \rho_{41^5} + \rho_{1^9} \\ = & -\Psi_{71^2} + \Psi_{52^2} + \Psi_{4^21} - \Psi_{432} - \Psi_{41^5} + \Psi_{2^31^3} + (1-c_{11})\Psi_{21^7} + (2-c_{12})\Psi_{1^9} \end{aligned}$$

so that by Proposition 2.1 we obtain $c_{11} \leq 1$ and $c_{12} \leq 2$. On the other hand, the expression of $(\rho')^0$ in terms of irreducible Brauer characters forces $c_{11} \geq 1$ and therefore $c_{11} = 1$. Also, the ℓ-reduction of ρ gives $c_{11} + c_{12} \geq 3$. Together with the previous inequality, this forces $c_{12} = 2$.

With a class of eigenvalues congruent to q^4 modulo ℓ we have

$$\begin{aligned} R_w[q^4] = & -\rho_{81} + \rho_{51^4} + \rho_{21^7} \\ = & -\Psi_{81} + \Psi_{63} - \Psi_{432} + \Psi_{51^4} - \Psi_{3^21^3} + \Psi_{2^41} - c_6\Psi_{2^31^3} \\ & + (2 - c_8 + c_6)\Psi_{21^7} + (1 - c_9 - 2c_6)\Psi_{1^9}. \end{aligned}$$

From Proposition 2.1 we deduce $c_6 = 0$, $c_8 \leq 2$ and $c_9 \leq 1$. Again, we use ℓ-reduction of non-unipotent characters for finding lower bounds: ρ yields the relation $c_8 - c_6 \geq 2$ so that $c_8 = 2$, and ρ'' the relation $-3c_6 + c_8 + c_9 \geq 3$ which forces $c_9 = 1$.

The last eigenspace of F^2 corresponds to the eigenvalues congruent to q^2 modulo ℓ and is given by

$$\begin{aligned} R_w[q^2] &= \rho_9 + \rho_{61^3} - \rho_{31^6} \\ &= \Psi_9 - \Psi_{71^2} - \Psi_{621} - \Psi_{54} + \Psi_{61^3} + \Psi_{52^2} + \Psi_{4^21} - \Psi_{3^3} - \Psi_{3^221} + \Psi_{32^3} \\ &\quad + \Psi_{3^21^3} - \Psi_{41^5} + \Psi_{2^31^3}. \end{aligned}$$

In particular, none of Ψ_{321^4}, Ψ_{21^7} and Ψ_{1^9} appears in R_w for $w = s_1s_2s_3s_4$.

Let us now consider the virtual characters coming from the cohomology of the Deligne–Lusztig variety associated with $w = s_1s_2s_3s_4s_5s_4$. Its contribution to the principal block b is given by

$$\begin{aligned} bR_w &= \rho_9 + \rho_{81} - \rho_{63} + \rho_{52^2} + \rho_{3^21^3} + \rho_{2^31^3} - \rho_{21^7} + \rho_{1^9} \\ &= \Psi_9 + \Psi_{81} - \Psi_{71^2} - 2\Psi_{63} - \Psi_{621} - \Psi_{54} + \Psi_{61^3} + 2\Psi_{52^2} + \Psi_{4^21} + 2\Psi_{432} - 2\Psi_{3^3} \\ &\quad - \Psi_{51^4} - \Psi_{3^221} + \Psi_{32^3} + 3\Psi_{3^21^3} - \Psi_{41^5} - 2\Psi_{2^41} + 2\Psi_{2^31^3} \end{aligned}$$

and we observe that no new projective indecomposable module appears.

With $w = s_1s_2s_3s_4s_5s_6s_5s_4$, the decomposition of R_w, cut by the block is given by

$$\begin{aligned} bR_w &= \rho_9 + \rho_{63} - \rho_{61^3} - \rho_{4^21} + \rho_{3^3} - \rho_{32^3} + \rho_{41^5} - \rho_{2^31^3} + \rho_{1^9} \\ &= \Psi_9 - \Psi_{71^2} + \Psi_{63} - \Psi_{621} - \Psi_{54} - 2\Psi_{61^3} + \Psi_{52^2} - \Psi_{432} + \Psi_{3^3} + \Psi_{3^221} - 2\Psi_{32^3} \\ &\quad + 3\Psi_{41^5} + 3\Psi_{321^4} + \Psi_{2^41} - 4\Psi_{2^31^3} - 3\Psi_{31^6} + 3\Psi_{21^7} + 3(3 - c_5)\Psi_{1^9}. \end{aligned}$$

Since Ψ_{1^9} does not occur in any R_v for $v < w$ (such v are either non-cuspidal or conjugate to one of the above w's), we deduce from Proposition 2.1 that $3 - c_5 \geq 0$. From the ℓ-reduction of ρ'' we get $-3 + c_5 \geq 0$ and therefore $c_5 = 3$.

Finally, for $n = 10$ the non-principal blocks in Table 13 have no cuspidal Brauer characters and their decomposition matrices are easily determined. For the principal block b we denote by d_i the entries in the columns corresponding to the projective characters

Ψ_{4321}, Ψ_{321^5} and $\Psi_{2^31^4}$. Note that we must have $d_{11} = 0$ since $2^5 \not\trianglelefteq 321^5$. Other easy relations are obtained by Harish-Chandra restriction, as in the case $n = 9$. The restriction of Ψ_{4321^3} (resp. Ψ_{432^31}, resp. Ψ_{321^7}, resp. $\Psi_{2^31^6}$) yields $d_1 = 0$ (resp. $d_2 = d_3 = 0$, resp. $d_{13} = 0$, resp. $d_{15} = 0$). The other relations are obtained by decomposing virtual projective modules R_w. As usual, we start with a Coxeter element $w = s_1s_2s_3s_4s_5$. We find

$$\begin{aligned} bR_w &= \rho_{10} - \rho_{82} - \rho_{721} + \rho_{621^2} + \rho_{521^3} - \rho_{421^4} - \rho_{321^5} + \rho_{2^21^6} - \rho_{1^{10}} \\ &= \Psi_{10} - 2\Psi_{82} - \Psi_{73} - \Psi_{721} + \Psi_{71^3} + 3\Psi_{621^2} + 2\Psi_{5^2} - 2\Psi_{52^21} + 2\Psi_{521^3} \\ &\quad - 4\Psi_{4^21^2} - \Psi_{43^2} + \Psi_{42^3} + 3\Psi_{3^31} - 3\Psi_{321^5} - 2\Psi_{2^5} + 3d_{12}\Psi_{2^31^4} \\ &\quad + 3(d_{14} - d_{12}d_{16} - 2)\Psi_{1^{10}}. \end{aligned}$$

Since $\Psi_{2^31^4}$ and $\Psi_{1^{10}}$ are the characters of projective covers of cuspidal modules, by Proposition 2.1 we must have $d_{12} \leq 0$ and $d_{14} - d_{12}d_{16} - 2 \leq 0$. This forces $d_{12} = 0$ and $d_{14} \leq 2$. The converse relation is obtained by considering as usual the ℓ-reduction of the Deligne–Lusztig induction ρ of an ℓ-character in general position, which exists as soon as $\ell > 10$. We obtain $d_{14} = 2$. As a byproduct, neither $\Psi_{2^31^4}$ nor $\Psi_{1^{10}}$ occurs in R_w.

Other relations are obtained by considering the Deligne–Lusztig character associated with $w' = s_1s_2s_3s_4s_6s_5s_6$ for different eigenvalues of F^2. The virtual projective module corresponding to the generalized q^4-eigenspace decomposes as

$$\begin{aligned} bR_w[q^4] &= \rho_{10} + \rho_{4^21^2} + \rho_{321^5} \\ &= \Psi_{10} - \Psi_{82} - \Psi_{73} + \Psi_{71^3} + \Psi_{621^2} + \Psi_{5^2} - \Psi_{52^21} - \Psi_{4^21^2} - \Psi_{43^2} - \Psi_{421^4} + \Psi_{41^6} \\ &\quad + \Psi_{321^5} - \Psi_{2^31^4} + \Psi_{2^21^6} + (d_{16} - 3)\Psi_{1^{10}}, \end{aligned}$$

which, by Proposition 2.1, forces $d_{16} - 3 \leq 0$. One the other hand, we have $d_{16} - 3 \geq 0$ by looking at the coefficient of $\varphi_{2^31^4}$ in ρ^0. Therefore $d_{16} = 3$. Finally, the generalized q^2-eigenspace yields

$$\begin{aligned} bR_w[q^2] = &\; - \rho_{73} - \rho_{4321} + \rho_{2^31^4} \\ = &\; - \Psi_{73} + \Psi_{71^3} + \Psi_{5^2} - \Psi_{52^21} - \Psi_{4^21^2} - \Psi_{43^2} - \Psi_{4321} + \Psi_{431^3} + \Psi_{42^3} \\ &\; - \Psi_{421^4} - (1 - d_4)\Psi_{41^6} + (1 + d_5)\Psi_{3^31} + d_6\Psi_{321^5} - (d_5 - d_7 + 2)\Psi_{2^5} \\ &\; + x\Psi_{2^31^4} + (1 - d_7 + d_9)\Psi_{2^21^6} - (2d_6 + d_9 - d_{10} + 3x)\Psi_{1^{10}} \end{aligned}$$

with $x = -d_4 + d_5 - d_7 + d_8 + 2$. From Proposition 2.1 we obtain $x \leq 0$ and $2d_6 + d_9 - d_{10} + 3x \geq 0$. The latter is exactly the coefficient of φ_{4321} in $-\rho^0$, and therefore it must be zero, which yields the relation $-3d_4 + 3d_5 + 2d_6 - 3d_7 + 3d_8 + d_9 - d_{10} + 6 = 0$. □

Remark 6.7. Conjecture 1.2 in [7] (and more precisely the last example in [7, §3.2]) predicts that the unknown entries $d_4, \ldots, d_{10}$ in Table 12 are bounded above by the corresponding entries in the decomposition matrix at $\ell|(q + 1)$ in Table 5 for $\alpha = 1$, $\beta = 0$, that is

$$d_4 \leq 6,\ d_5 \leq 2,\ d_6 \leq 4,\ d_7 \leq 14,\ d_8 \leq 14,\ d_9 \leq 24,\ d_{10} \leq 36.$$

6.3. **The case $\ell|(q^4 - q^3 + q^2 - q + 1)$.** For convenience, we recall the Brauer trees for unipotent ℓ-blocks of cyclic defect from [8]. Note that the result there was only shown for odd prime powers q, but it is easily seen to hold for even q as well.

Theorem 6.8. *Let $\ell > n$ be a prime. Then the ℓ-modular decomposition matrices for the unipotent ℓ-blocks of* $\mathrm{SU}_n(q)$, $5 \le n \le 10$, $\ell|(q^4 - q^3 + q^2 - q + 1)$, *are as given in Tables 14–15.*

Table 14. $\mathrm{SU}_n(q)$, $5 \le n \le 10$, $\ell|(q^4 - q^3 + q^2 - q + 1)$, $\ell > n$

$\mathrm{SU}_5(q)$:	5	—	31^2	—	1^5	—	○	—	21^3	—	41
$\mathrm{SU}_7(q)$:	7	—	3^21	—	2^21^3	—	○	—	21^5	—	43
	52	—	32^2	—	1^7	—	○	—	2^31	—	61
$\mathrm{SU}_9(q)$:	9	—	4^21	—	421^3	—	○	—	41^5	—	431^2
	72	—	3^3	—	2^21^5	—	○	—	2^31^3	—	63
	71^2	—	531	—	2^41	—	○	—	21^7	—	432
	54	—	32^21^2	—	31^6	—	○	—	3^221	—	81
	521^2	—	32^3	—	1^9	—	○	—	42^21	—	61^3
		ps		ps		1^5		21^3		21	

$\mathrm{SU}_6(q)$:	6	—	42	—	2^21^2	—	1^6	—	○	—	321
$\mathrm{SU}_8(q)$:	8	—	4^2	—	3^21^2	—	31^5	—	○	—	321^3
	61^2	—	42^2	—	2^4	—	1^8	—	○	—	521
$\mathrm{SU}_{10}(q)$:	82	—	64	—	3^31	—	32^21^3	—	○	—	321^5
	631	—	43^2	—	2^41^2	—	2^21^6	—	○	—	721
		ps		ps		ps		1^6		321	

$\mathrm{SU}_8(q)$:	71	—	53	—	3^22	—	2^31^2	—	21^6	—	○
		ps		ps		ps		ps		c	

Proof. It remains to consider $\mathrm{SU}_{10}(q)$. We denote by a_i (resp. b_i) the entries in the column corresponding to the projective characters Ψ_{4321} (resp. Ψ_{32^31}). The projective character $\Psi_{21^{10}}$ of $\mathrm{SU}_{12}(q)$ has only $\rho_{21^{10}}$ as unipotent constituent, so that by Harish-Chandra restriction Ψ_{21^8} involves only ρ_{21^8}, which gives the second to last column of the

Table 15. $\mathrm{SU}_{10}(q)$, $\ell|(q^4-q^3+q^2-q+1)$, $\ell>10$

10	$5.$	1																			
91	$.5$	.	1																		
81^2	$41.$	1	.	1																	
71^3	$.41$	.	1	.	1																
61^4	$31^2.$	.	.	1	.	1															
5^2	2.3	1	1	.	.	.	1														
541	${}^2A_5:1^2.$	.	.	.	.	.	.	1													
531^2	1.31	.	1	.	1	.	1	.	1												
521^3	${}^2A_5:1.1$	.	.	.	.	.	.	1	.	1											
51^5	$.31^2$	.	.	.	1	.	.	.	1	.	1										
4^22	21.2	1	.	1	.	.	1	.	.	.	.	1									
4321	${}^2A_9$	.	.	.	.	.	.	.	.	.	.	.	1								
42^21^2	$21^2.1$	.	.	1	.	1	.	.	.	.	.	1	.	1							
41^6	$21^3.$	.	.	.	.	1	.	.	.	.	.	.	.	1	1						
3^22^2	$1^2.21$	.	.	.	.	.	1	.	1	.	.	1	.	.	.	1					
32^31	${}^2A_5:.2$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1				
31^7	$.21^3$	.	.	.	.	.	.	.	1	.	1	.	.	.	.	1	.	1			
2^5	$1^3.1^2$	.	.	.	.	.	.	.	.	.	.	1	a	1	.	1	.	.	1		
21^8	$1^5.$	.	.	.	.	.	.	.	.	.	.	.	$a+2$	1	1	.	.	.	1	1	
1^{10}	$.1^5$	.	.	.	.	.	.	.	.	.	.	.	a	.	.	1	2	1	1	.	1
		ps	ps	ps	ps	ps	ps	321	ps	321	1^6	ps	c	ps	21^6	ps	c	1^6	1^5	c	c

decomposition matrix. For the same reason, the Harish-Chandra restriction of Ψ_{43215} (resp. Ψ_{432^31}, resp. $\Psi_{32^31^5}$, resp. Ψ_{32^51}) yields $a_1=a_2=a_4=0$ (resp. $a_3=a_5=0$, resp. $b_1=0$, resp. $b_2=b_3=0$). We use the Deligne–Lusztig characters R_w for suitable $w\in W$ to obtain relations among the remaining indeterminates a_6, a_7, a_8 and b_4.

As usual, we start by decomposing the virtual projective module R_w corresponding to a Coxeter element $w=s_1s_2s_3s_4s_5$. We find the following unipotent constituents from the principal block b:

$$\begin{aligned} bR_w &= \rho_{10}+\rho_{521^3}-\rho_{1^{10}} \\ &= \Psi_{10}-\Psi_{81^2}+\Psi_{61^4}-\Psi_{5^2}+\Psi_{531^2}+\Psi_{521^3}-\Psi_{51^5}+\Psi_{4^22}-\Psi_{42^21^2} \\ &\quad -\Psi_{3^22^2}-\Psi_{32^31}+\Psi_{31^7}+\Psi_{2^5}-(2-b_4)\Psi_{1^{10}}. \end{aligned}$$

We deduce from Proposition 2.1 that $2-b_4\geq 0$. Assume $\ell>10$. Then there exists an ℓ-character in general position, and we denote by ρ the corresponding non-unipotent character obtained via Deligne–Lusztig induction. The coefficient of φ_{32^31} on ρ^0 gives $-2+b_4\geq 0$, which forces $b_4=2$. In particular, Ψ_{4321}, Ψ_{21^8} and $\Psi_{1^{10}}$ do not occur in bR_w.

Finally we consider the generalized q^4-eigenspace with the element $w' = s_4s_5s_4s_6s_7s_8s_9$, which decomposes as follows:

$$\begin{aligned} bR_{w'}[q^4] &= \rho_{91} - \rho_{4321} - \rho_{21^8} \\ &= \Psi_{91} - \Psi_{71^3} - \Psi_{5^2} + \Psi_{531^2} + \Psi_{4^22} - \Psi_{4321} - \Psi_{42^21^2} + \Psi_{41^6} - \Psi_{3^22^2} \\ &\quad + (a_6+1)\Psi_{2^5} - (2+a_6-a_7)\Psi_{21^8} - (a_6-a_8)\Psi_{1^{10}}. \end{aligned}$$

By Proposition 2.1 we must have $2 + a_6 - a_7 \geq 0$ and $a_6 - a_8 \geq 0$. On the other hand, the ℓ-reduction of ρ yields the relation $-a_6 + a_8 \geq 2 + a_6 - a_7$, and therefore we deduce that $2+a_6-a_7 = -a_6+a_8 = 0$. This gives $a_7 = a_6+2$ and $a_8 = a_6$. In Table 15 we have written a in place of the only remaining unknown a_6. □

Remark 6.9. Again by Conjecture 1.2 in [7] the unknown entry a in Table 15 should be bounded above by $a \leq 14$, as in Remark 6.7.

6.4. **James's row and column removal rule for** $\mathrm{SU}_n(q)$ **.** Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$ and $\mu = (\mu_1 \geq \cdots \geq \mu_s > 0)$ be two partitions of n. Following [22, Thm 6.18], we would like to relate the decomposition number $d_{\lambda,\mu} = \langle \rho_\lambda, \Psi_\mu \rangle$ of $\mathrm{SU}_n(q)$ to a decomposition number of a smaller unitary group. Two cases can be considered:

- (Row removal) If $\lambda_1 = \mu_1$, we set $\bar\lambda = (\lambda_2 \geq \cdots \geq \lambda_r > 0)$ and $\bar\mu = (\mu_2 \geq \cdots \geq \mu_s > 0)$. They are partitions of $\bar n = n - \lambda_1$.
- (Column removal) If $r = s$, we set $\underline{\lambda} = (\lambda_1 - 1 \geq \lambda_2 - 1 \geq \cdots \geq \lambda_r - 1)$ and $\underline{\mu} = (\mu_1 - 1 \geq \mu_2 - 1 \geq \cdots \geq \mu_s - 1)$. They are both partitions of $\underline{n} = n - r$.

We say that λ and μ satisfy *James's row* (resp. *column*) *removal rule* if we are in the first (resp. second) case and $d_{\lambda,\mu} = d_{\bar\lambda,\bar\mu}$ (resp. $d_{\lambda,\mu} = d_{\underline{\lambda},\underline{\mu}}$). James showed in [22, Rule 5.8] that this rule holds for ℓ-decomposition matrices of $\mathrm{GL}_n(q)$. From the matrices that we have determined in this section we observe that this should also be the case for $\mathrm{SU}_n(q)$.

Proposition 6.10. *James's row and column removal rule holds for the decomposition matrices of* $\mathrm{SU}_n(q)$ *with* $n \leq 10$ *given in Tables 1–15.*

Remark 6.11. If James's rule holds in general, it would give some of the entries that we could not determine, namely $d_4 = d_{41^6,4321} = d_{1^6,321} = 2$ and $d_5 = d_{3^31,4321} = d_{2^3,321} = 0$ in Table 12.

Remark 6.12. Other rules used by James in [22] to determine the decomposition matrices of unipotent blocks of $\mathrm{SL}_n(q)$ for $n \leq 10$ will no longer hold for $\mathrm{SU}_n(q)$. For example, by Proposition 4.3, $d_{\lambda,\mu} = 0$ whenever λ is a triangular partition and $\mu \neq \lambda$ but $d_{\mu^\star,\lambda^\star}$ can be non-zero, as shown in Tables 1–15. This proves that the analogue of [22, Rule 5.7] does not hold for $\mathrm{SU}_n(q)$.

References

[1] C. Bonnafé, R. Rouquier, Catégories dérivées et variétés de Deligne–Lusztig. Publ. Math. Inst. Hautes Études Sci. **97** (2003), 1–59.

[2] M. Broué, G. Malle, J. Michel, Generic blocks of finite reductive groups. Représentations unipotentes génériques et blocs des groupes réductifs finis. Astérisque **212** (1993), 7–92.

[3] F. Digne, J. Michel, *Fonctions L des variétés de Deligne–Lusztig et descente de Shintani.* Mém. Soc. Math. France (N.S.) No. 20 (1985).

[4] F. Digne, J. Michel, *Representations of finite groups of Lie type*. London Math. Soc. Student Texts, 21. Cambridge University Press, 1991.

[5] F. Digne, J. Michel, R. Rouquier, Cohomologie des variétés de Deligne–Lusztig. Adv. Math. **209** (2007), 749–822.

[6] O. Dudas, A note on decomposition numbers for groups of Lie type of small rank. J. Algebra **388** (2013), 364–373.

[7] O. Dudas, G. Malle, Projective modules in the intersection cohomology of Deligne–Lusztig varieties. C. R. Acad. Sci. Paris Sér. I Math. **352** (2014), 467–471.

[8] P. Fong, B. Srinivasan, Brauer trees in classical groups. J. Algebra **131** (1990), 179–225.

[9] M. Geck, *Verallgemeinerte Gelfand–Graev Charaktere und Zerlegungszahlen endlicher Gruppen vom Lie-Typ*. Dissertation, RWTH Aachen, 1990.

[10] M. Geck, On the decomposition numbers of the finite unitary groups in nondefining characteristic. Math. Z. **207** (1991), 83–89.

[11] M. Geck, Basic sets of Brauer characters of finite groups of Lie type II. J. London Math. Soc. **47** (1993), 255–268.

[12] M. Geck, G. Hiss, Basic sets of Brauer characters of finite groups of Lie type. J. Reine Angew. Math. **418** (1991), 173–188.

[13] M. Geck, G. Hiss, G. Malle, Cuspidal unipotent Brauer characters. J. Algebra **168** (1994), 182–220.

[14] M. Geck, G. Hiss, G. Malle, Towards a classification of the irreducible representations in non-describing characteristic of a finite group of Lie type. Math. Z. **221** (1996), 353–386.

[15] M. Geck, S. Kim, G. Pfeiffer, Minimal length elements in twisted conjugacy classes of finite Coxeter groups. J. Algebra **229** (2000), 570–600.

[16] M. Geck, G. Pfeiffer, On the irreducible characters of Hecke algebras. Adv. Math. **102** (1993), 79–94.

[17] J. Gruber, G. Hiss, Decomposition numbers of finite classical groups for linear primes. J. Reine Angew. Math. **485** (1997), 55–91.

[18] X. He, Minimal length elements in some double cosets of Coxeter groups. Adv. Math. **215** (2007), 469–503.

[19] X. He, S. Nie, Minimal length elements of finite Coxeter groups. Duke Math. J. **161** (2012), 2945–2967.

[20] F. Himstedt, F. Noeske, Decomposition numbers of $\mathrm{SO}_7(q)$ and $\mathrm{Sp}_6(q)$. J. Algebra **413** (2014), 15–40.

[21] J. E. Humphreys, *Reflection groups and Coxeter groups*. Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990.

[22] G. James, The decomposition matrices of $\mathrm{GL}_n(q)$ for $n \leq 10$. Proc. London Math. Soc. (3) **60** (1990), 225–265.

[23] S. Kim, Mots de longueur maximale dans une classe de conjugaison d'un groupe symétrique, vu comme groupe de Coxeter. C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), 617–622.

[24] G. Lusztig, *Representations of finite Chevalley groups*. CBMS Regional Conference Series in Mathematics, 39. American Mathematical Society, Providence, R.I., 1978.

[25] G. Lusztig, *Characters of reductive groups over a finite field*. Annals of Mathematics Studies, 107. Princeton University Press, Princeton, NJ, 1984.

[26] G. Lusztig, Characters sheaves IV. Adv. Math. **59** (1986), 1–63.

[27] G. Lusztig, From conjugacy classes in the Weyl group to unipotent classes. Represent. Theory **15** (2011), 494–530.

[28] G. Lusztig, From conjugacy classes in the Weyl group to unipotent classes III. Represent. Theory **16** (2012), 450–488.

[29] J. Michel, The GAP-part of the Chevie system. GAP 3-package available for download from http://people.math.jussieu.fr/~jmichel/chevie/chevie.html

[30] T. Okuyama, K. Waki, Decomposition numbers of $\mathrm{SU}(3, q^2)$. J. Algebra **255** (2002), 258–270.

Université Paris Diderot, UFR de Mathématiques, Bâtiment Sophie Germain, 5 rue Thomas Mann, 75205 Paris CEDEX 13, France.
E-mail address: `dudas@math.jussieu.fr`

FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany.
E-mail address: `malle@mathematik.uni-kl.de`