EXTENSIONS OF UNIPOTENT CHARACTERS AND THE INDUCTIVE MCKAY CONDITION

GUNTER MALLE

Abstract. We prove that unipotent characters of groups of Lie type extend to their inertia groups in the full automorphism group. We then apply this in order to show that certain groups of Lie type are good for the prime $p = 3$ or $p = 2$. This deals with all the exceptions not covered by our general approach for verifying the McKay conjecture for all groups of Lie type in non-defining characteristic.

1. Introduction

In the 1970's, following an observation of John McKay, Jon Alperin put forward the conjecture that for every finite group G and every prime number p

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathrm{N}_G(P))|,$$

where P is a Sylow p-subgroup of G and, for a finite group H, $\mathrm{Irr}_{p'}(H)$ denotes the number of irreducible complex characters of H of degree prime to p. This conjecture has been proved for various families of finite groups, for example for all p-solvable groups, but a general proof still remains to be found. In [10] we recently showed that the proof for general groups can be reduced to the proof of a stronger statement for simple groups, thus pointing the way to a possible proof of the McKay conjecture via the classification of finite simple groups.

More precisely, it needs to be shown that every finite simple group S is *good for every prime divisor* p of its order (see the definitions below). In [17], the author has shown that all simple groups not of Lie type, and also the groups of Lie type with exceptional Schur multiplier are good for all p. Furthermore, we showed in [16] that for groups of Lie type G and primes p different from the defining characteristic, the proof of being good can be transferred to the investigation of extensions of characters of centralizers of suitable tori in G, at least when $p \geq 5$, and for most types when $p = 2, 3$. Part of the required statements concerning these extensions have meanwhile been proved by Späth [20]. It still remains, though, to check compatibility with outer automorphisms.

The purpose of the present paper is twofold. First, in Section 2 we prove an extendibility result for unipotent characters of groups of Lie type (Theorem 2.4) which may be of independent interest. This is based on results of Lusztig as well as on earlier results on Hecke algebras for disconnected groups. Secondly, we show that all the groups left open in our paper [16] are good. Namely, in Section 3 we

Date: June 12, 2008.

2000 *Mathematics Subject Classification.* Primary 20C33, Secondary 20C15.

The author gratefully acknowledges financial support by the Deutsche Forschungsgemeinschaft.

apply the result of Section 2 in the investigation of the remaining cases from [16] where $p = 3$, different from the defining characteristic, see Theorem 3.2. Also, we show in Proposition 3.17 that the Tits simple group ${}^2F_4(2)'$ is good for all divisors of its order. Finally, in Section 4 we treat the open cases for $p = 2$ by showing that the symplectic groups over fields $\mathbb{F}_q$ where q is an odd power of an odd prime are good for $p = 2$.

Throughout we'll use the notation of the Atlas [3] for the various groups of Lie type, in particular, Artin's single letter convention for the simple groups. Thus the full orthogonal group is denoted $\mathrm{GO}_{2n}^+(q)$, the projective symplectic group $\mathrm{S}_{2n}(q)$, and so on.

2. Maximal extensions of unipotent characters

In this section we prove that the unipotent characters of finite simple groups of Lie type extend to their inertia groups in the full automorphism group.

We consider the following setup. Let $\hat{\mathbf{G}}$ be a simple simply-connected algebraic group over the algebraic closure of a finite field and $F : \hat{\mathbf{G}} \to \hat{\mathbf{G}}$ a Frobenius morphism with finite group of fixed points $\hat{G} := \hat{\mathbf{G}}^F$. Let $S := \hat{G}/Z(\hat{G})$. The simple groups S obtained in this way will be called the simple groups of Lie-type. (Note that by the classification of finite simple groups, apart from the alternating groups, the sporadic groups and the Tits group, all non-abelian finite simple groups can be obtained in this way.)

Let $\mathbf{G} := \hat{\mathbf{G}}/Z(\hat{\mathbf{G}})$ be the adjoint quotient of $\hat{\mathbf{G}}$. Then F induces a Frobenius morphism on $\mathbf{G}$ which we again denote by F, and which commutes with the natural epimorphism $\pi : \hat{\mathbf{G}} \to \mathbf{G}$. Let $G := \mathbf{G}^F$ be the group of fixed points. Then π induces an injection $S \to G$, and G induces the group of diagonal automorphisms on S.

Lusztig has introduced an important subset of $\mathrm{Irr}(G)$, the so-called unipotent characters. These are the irreducible complex characters of G occurring in a generalized Deligne-Lusztig character $R_{\mathbf{T}}^{\mathbf{G}}(1)$, where $\mathbf{T}$ runs over the F-stable maximal tori of $\mathbf{G}$. The unipotent characters are partitioned into Harish-Chandra series, one of which is the principal series, which by definition consists of the constituents of the permutation character of G on B, where B is a Borel subgroup of G. Note that $\mathrm{Ind}_B^G(1)$ is just the Deligne-Lusztig character $R_{\mathbf{T}}^{\mathbf{G}}(1)$ for the maximally split torus $\mathbf{T}$ of $\mathbf{G}$.

By the results of Lusztig [12], the unipotent characters of G remain irreducible upon restriction to S, and these restrictions are just the unipotent characters of S. Hence we have:

Proposition 2.1 (Lusztig)**.** *Let S be a simple group of Lie type. Then any unipotent character ρ of S has an extension $\tilde{\rho}$ to the group G of inner-diagonal automorphism of S such that ρ, $\tilde{\rho}$ have the same inertia group in* $\mathrm{Aut}(S)$.

Thus we are left to investigate extensions from G to $\mathrm{Aut}(S)$.

Proposition 2.2. *Let S be a simple group of Lie type different from $\mathrm{L}_n(q)$ $(n \geq 3)$, $\mathrm{O}_{2n}^+(q)$ $(n \geq 4)$ and $E_6(q)$. Then all unipotent characters of S extend to their inertia group in* $\mathrm{Aut}(S)$.

Proof. The group of outer automorphisms of a simple group of Lie type is generated by the diagonal automorphisms, the graph automorphisms of the underlying Dynkin diagram, and the field automorphisms of the field of definition (see [8, Thm. 2.5.12]). First assume that

$$\mathrm{Aut}(S)/G \cong \mathrm{Out}(S)/\{\text{diagonal automorphisms}\}$$

is cyclic. Then all characters of G extend to their inertia groups in $\mathrm{Aut}(G)$ by [9, Cor. 11.22], since the inertia factor groups are cyclic.

According to [8, Thm. 2.5.12] the only cases where $\mathrm{Aut}(S)/G$ is not cyclic are $S = \mathrm{L}_n(q)$ $(n \geq 3)$, $\mathrm{O}_{2n}^+(q)$ $(n \geq 4)$ and $E_6(q)$. □

Proposition 2.3. *Let S be a simple group of Lie type and χ a cuspidal unipotent character of S. Then χ extends to its inertia group in* $\mathrm{Aut}(S)$.

Proof. By Proposition 2.2 we may assume that $S \in \{\mathrm{L}_n(q), \mathrm{O}_{2n}^+(q), E_6(q)\}$. The groups $\mathrm{L}_n(q)$, $n \geq 3$, do not have cuspidal unipotent characters.

For $S = \mathrm{O}_8^+(q)$ there exists a unique cuspidal unipotent character χ which is hence invariant under all automorphisms. Let $\mathbf{T}$ denote a maximal F-stable torus of $\mathbf{G}$ with $|\mathbf{T}^F| = \Phi_2^2\Phi_6$, where Φ_i denotes the ith cyclotomic polynomial evaluated at q. Let σ be a quasi-central automorphism of $\mathbf{G}$ inducing a graph automorphism of order r, $r \in \{2, 3\}$, and stabilizing $\mathbf{T}$. Digne and Michel [4, Def. 2.2] have defined a generalized Deligne-Lusztig character $R_{\mathbf{T}\sigma}^{\mathbf{G}\sigma}(1)$ for the disconnected group $\mathbf{G}.\langle\sigma\rangle$. The decomposition of $R_{\mathbf{T}\sigma}^{\mathbf{G}\sigma}(1)$ was determined in [15, Thms. 6 and 7]. It contains the r extensions of χ to $G.\langle\sigma\rangle$ with different multiplicities. Now $\mathbf{T}^F$ is stabilized by the field automorphisms of G, so these act on the set of constituents of $R_{\mathbf{T}\sigma}^{\mathbf{G}\sigma}(1)$. It follows that at least one of the extensions of χ is invariant under field automorphisms. Thus χ extends to G extended by all graph and field automorphisms, as claimed.

For $S = E_6(q)$ there are two cuspidal unipotent characters. Write $\tilde{G} := G.\langle\gamma\rangle$ for the extension of G by the graph automorphism γ of order 2. Again the explicit decomposition in [15, Th. 8] shows that the extensions of these two characters to $\tilde{G}$ occur with different multiplicities in some generalized Deligne-Lusztig characters, and we may conclude as above.

Finally, $S = \mathrm{O}_{2n}^+(q)$, $n \geq 5$, has a cuspidal unipotent character ρ if and only if $n = s^2$ is the square of an even number s. In that case ρ is parametrized by the Lusztig symbol

$$\begin{pmatrix} 1 & 2 & \dots & 2s-2 & 2s-1 \\ & & - & & \end{pmatrix}.$$

Let γ denote the graph-automorphism of G of order 2. Let L be a γ- and F-stable Levi subgroup of G of type ${}^2D_{n-(2s-1)}T$ where T is a torus of order $|T| = q^{2s-1} + 1$, and ρ_1 the unipotent character of L parametrized by the symbol

$$\begin{pmatrix} 1 & 2 & \dots & 2s-2 & \\ 0 & & & & \end{pmatrix}.$$

Then by Asai's hook formula $\langle R_L^G(\rho_1), \rho\rangle = \pm 1$. By Proposition 2.2, ρ_1 extends to a character $\tilde{\rho}_1$ of $\tilde{L} := L.\langle\gamma\rangle$. By [4, Cor. 2.4],

$$R_{\tilde{L}}^{\tilde{G}}(\tilde{\rho}_1)|_G = R_L^G(\rho_1),$$

so $R_{\tilde L}^{\tilde G}(\tilde\rho_1)$ contains the two extensions of ρ to $\tilde G$ with different multiplicities. Since both $\tilde L$ and $\tilde\rho_1$ are invariant under field automorphisms, this proves that the two extensions of ρ to $\tilde G$ are invariant under field automorphisms, hence extend to $\mathrm{Aut}(S)$. □

Theorem 2.4. *Let S be a simple group of Lie type. Then all unipotent characters of S extend to their inertia group in* $\mathrm{Aut}(S)$.

Proof. By Proposition 2.2 we may assume that $S \in \{\mathrm{L}_n(q), \mathrm{O}_{2n}^+(q), E_6(q)\}$. In these cases, we first consider the principal series unipotent characters. Let B be a Borel subgroup of G. Let ψ denote a generator of the cyclic group of field automorphisms of G. Then ψ stabilizes B and centralizes the Weyl group W of G. Let $\tilde G := G.\langle\psi\rangle$ denote the extension of G by the group of field automorphisms. By [14, Satz 1.5] the endomorphism algebra of the permutation representation of $\tilde G$ on B is isomorphic to the group algebra of the extension $\tilde W := W.\langle\psi\rangle \cong W \times \langle\psi\rangle$ of the Weyl group of W by ψ, and the corresponding bijection of irreducible characters is compatible with induction from parabolic subgroups. Thus, writing

$$\mathrm{Ind}_B^G(1) = \sum_\chi n_\chi\, \chi$$

with suitable $n_\chi \in \mathbb{N}$, we have

$$\mathrm{Ind}_{\tilde B}^{\tilde G}(1) = \sum_\chi n_\chi\, \tilde\chi$$

with suitable extensions $\tilde\chi$ of χ to $\tilde G$. The graph automorphisms of G commute with ψ and stabilize $\tilde B$. Thus they act on the set of constituents of $\mathrm{Ind}_{\tilde B}^{\tilde G}(1)$. In particular, χ is invariant under some graph automorphism γ if and only if $\tilde\chi$ is γ-invariant. The group of graph automorphisms is cyclic or isomorphic to $\mathfrak{S}_3$. Characters whose inertia group has cyclic Sylow subgroups extend, by [9, Cor. 11.22 and 11.31], so this shows that $\tilde\chi$ extends to the inertia group of χ in the group of graph automorphisms.

In the case of $S = \mathrm{L}_n(q)$, all unipotent characters of $G = \mathrm{PGL}_n(q)$ occur as constituents of the permutation character of G on a Borel subgroup, so the claim follows from our previous consideration.

For $S = \mathrm{O}_8^+(q)$ all unipotent characters are either cuspidal or lie in the principal series. Thus, we are done by Proposition 2.3.

For $S = E_6(q)$ the only unipotent characters which are neither cuspidal nor lie in the principal series are the three characters lying in the Harish-Chandra series of the cuspidal unipotent character of the Levi subgroup L of type D_4. Write $\tilde G := G.\langle\gamma\rangle$ for the extension of G by the graph automorphism γ of order 2. By Proposition 2.3 the cuspidal unipotent character of L extends to the normalizer $\tilde L$ of L in $\tilde G$. The explicit decomposition in [15, Th. 8] shows that the extensions of the three characters to $\tilde G$ occur with different multiplicities in some generalized Deligne-Lusztig characters $R_{\tilde L}^{\tilde G}(\lambda)$, and we may conclude as before.

Finally, for $S = \mathrm{O}_{2n}^+(q)$, $n \geq 5$, let χ be an arbitrary unipotent character of G, lying in the Harish-Chandra series of a cuspidal unipotent character of a split

Levi subgroup L. By Proposition 2.3 we may assume that χ is non-cuspidal, so that $L < G$. The decomposition of R_L^G is determined by the Hecke algebra of a Weyl group of type B_k for some k. In particular, the multiplicities in Harish-Chandra induced unipotent characters is completely determined by corresponding decompositions in $W(B_k)$. Let $M \geq L$ be a split Levi subgroup of type D_{n-1}. Let χ_1 be an irreducible constituent of ${}^*R_M^G(\chi)$. Then $\langle R_M^G(\chi_1), \chi\rangle = 1$, since the induction from $W(B_{k-1})$ to $W(B_k)$ is multiplicity free (see [7, 6.1.9], for example). Now M can be chosen γ-invariant, and then $\langle R_{\tilde{M}}^{\tilde{G}}(\tilde{\chi}_1), \tilde{\chi}\rangle = 1$ by the Mackey-formula, where $\tilde{\chi}_1, \tilde{\chi}$ denote suitable extensions of χ_1, χ to $\tilde{M} := M\langle\gamma\rangle$ respectively $\tilde{G} := G\langle\gamma\rangle$. Since by induction $\tilde{\chi}_1$ is invariant under field automorphisms, we conclude as before that χ extends to $\mathrm{Aut}(S)$. □

We end this section by recalling from [16, Prop. 3.7 and 3.9] the inertia groups of unipotent characters in $\mathrm{Aut}(S)$ according to results of Lusztig:

Theorem 2.5. *Let χ be a unipotent character of a simple group of Lie type S. Then χ is $\mathrm{Aut}(S)$-invariant, except in the following cases:*

(a) *In $S = \mathrm{O}_{2n}^+(q)$ with even $n \geq 4$, the graph automorphism of order 2 interchanges the two unipotent characters in all pairs labelled by the same degenerate symbol of defect 0 and rank n.*

(b) *In $S = \mathrm{O}_4^+(q)$ the graph automorphism of order 3 has two non-trivial orbits with characters labelled by the symbols*

$$\left\{\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}', \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}', \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \end{pmatrix}\right\}.$$

(c) *In $S = \mathrm{Sp}_4(2^{2f+1})$ the graph automorphism of order 2 interchanges the two unipotent principal series characters labelled by the symbols*

$$\left\{\begin{pmatrix} 1 & 2 \\ 0 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & \end{pmatrix}\right\}.$$

(d) *In $S = G_2(3^{2f+1})$ the graph automorphism of order 2 interchanges the two unipotent principal series characters labelled by the characters $\{\phi'_{1,3}, \phi''_{1,3}\}$ of the Weyl group $W(G_2)$.*

(e) *In $S = F_4(2^{2f+1})$ the graph automorphism of order 2 has eight orbits of length 2, consisting of the unipotent characters labelled by*

$$\{\phi'_{8,3}, \phi''_{8,3}\},\ \{\phi'_{8,9}, \phi''_{8,9}\},\ \{\phi'_{2,4}, \phi''_{2,4}\},\ \{\phi'_{2,16}, \phi''_{2,16}\},$$
$$\{\phi'_{9,6}, \phi''_{9,6}\},\ \{\phi'_{1,12}, \phi''_{1,12}\},\ \{\phi'_{4,7}, \phi''_{4,7}\}, \{(B_2, \epsilon'), (B_2, \epsilon'')\}.$$

3. The prime $p = 3$

In [10] we showed that the McKay conjecture holds for a finite group G and a prime p if a certain inductive condition is satisfied for all simple groups involved in G: all finite simple groups X involved in G have to be *good for the prime p*. Here we formulate a stronger condition on a simple group X which implies that it is good for p, and which will be sufficient for the purpose of this paper.

Let X be a non-abelian simple group and p a prime. Let P denote a Sylow p-subgroup of X and let $A = \mathrm{N}_{\mathrm{Aut}(X)}(P)$.

Proposition 3.1. *Assume that the order of the Schur multiplier of X is a power of p and that the following conditions are satisfied:*

(i) *there exists a proper subgroup $\mathrm{N}_X(P) \le M < X$ of X normalized by A,*
(ii) *there exists an A-equivariant bijection*

$$^* : \mathrm{Irr}_{p'}(X) \longrightarrow \mathrm{Irr}_{p'}(M), \quad \theta \mapsto \theta^*,$$

(iii) *all characters in $\mathrm{Irr}_{p'}(X)$ and in $\mathrm{Irr}_{p'}(M)$ extend to their respective inertia groups in AX and in AM.*

Then X is good for the prime p in the sense of [10].

Note that (iii) is not a necessary condition for X to be good, but it suffices.

The aim of this section is to show that certain groups of Lie type are good for the prime $p = 3$. More precisely we deal with the following situations:

Table 1. The groups for $p = 3$

X	conditions
$\mathrm{L}_3(q)$	$q \equiv 4, 7 \pmod 9$
$\mathrm{U}_3(q)$	$q \equiv 2, 5 \pmod 9$, $q \ne 2$
$G_2(q)$	$q \equiv 2, 4, 5, 7 \pmod 9$, $q \ne 2$
$^2F_4(q^2)$	$q^2 \equiv 2, 5 \pmod 9$, $q^2 \ne 2$

Theorem 3.2. *The simple groups X occurring in Table 1 are good for $p = 3$.*

This will be shown in Corollary 3.9, Theorem 3.12 and Propositions 3.14, 3.16 in the next four subsections by verifying the conditions in Proposition 3.1.

3.1. **Special linear groups** $\mathrm{SL}_3(q)$**.** In this section let $S = \mathrm{SL}_3(q)$ be the special linear group over a finite field $\mathbb{F}_q$ with $q \equiv 4, 7 \pmod 9$. We start by recalling some results on the classification of the irreducible complex characters of S (see Simpson and Frame [19]). We denote by Φ_d the dth cyclotomic polynomial, evaluated at q. For the congruences given, the cyclotomic polynomial $\Phi_2 = q + 1$ is not divisible by 3, and $\Phi_1 = q - 1$, $\Phi_3 = q^2 + q + 1$ are divisible by 3 to the first power only.

Let B be the Borel subgroup of S consisting of the upper triangular matrices of determinant 1. Then $|B| = q^3\Phi_1^2$. The permutation character of G on B decomposes as

$$\mathrm{Ind}_B^G(1_B) = \rho_1 + 2\,\rho_2 + \rho_3,$$

where $\rho_1 = 1_S, \rho_2, \rho_3 = \mathrm{St}_S$ are irreducible characters of degrees

$$\rho_1(1) = 1,\ \rho_2(1) = q\Phi_2,\ \rho_3(1) = q^3.$$

These are the unipotent characters of S. Denote by χ the linear character of B of order 3 with

$$\chi(\begin{pmatrix} \zeta_0^m & 0 & 0 \\ 0 & \zeta_0^n & 0 \\ 0 & 0 & \zeta_0^{-m-n} \end{pmatrix}) = \exp(2\pi\mathbf{i}(m-n)/3),$$

where ζ_0 generates $\mathbb{F}_q^\times$. Then $\mathrm{Ind}_B^G(\chi)$ has three irreducible constituents $\varphi_1, \varphi_2, \varphi_3$ of degree $\varphi_i(1) = \frac{1}{3}\Phi_2\Phi_3$. The conjugacy class of regular unipotent elements of $\mathrm{GL}_3(q)$ splits into three conjugacy classes in $\mathrm{SL}_3(q)$, with representatives

$$u_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ u_2 := \begin{pmatrix} 1 & \zeta & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ u_3 := \begin{pmatrix} 1 & \zeta^2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

with $\zeta \in \mathbb{F}_q^\times$ of order 3. These are the only classes on which the three characters $\varphi_1, \varphi_2, \varphi_3$ differ. We choose notation so that

$$\varphi_i(u_i) = (2q+1)/3 \qquad (1 \le i \le 3), \qquad \varphi_i(u_j) = (-q+1)/3 \quad (i \neq j).$$

Lemma 3.3. *Let $q \equiv 4, 7 \pmod 9$ be a prime power. Then $S = \mathrm{SL}_3(q)$ has six irreducible characters of degree prime to 3. These are the three unipotent characters $\rho_1 = 1_S, \rho_2, \rho_3 = \mathrm{St}_S$ and the three semisimple characters $\varphi_1, \varphi_2, \varphi_3$. All six characters have $Z(S)$ in their kernel.*

Proof. Visibly, the ρ_i and φ_i have degree prime to 3. The claim can now be read off from the character table of $\mathrm{SL}_3(q)$ [19]. Alternatively it follows from Lusztig's parametrization of irreducible characters. □

By Theorems 2.4 and 2.5 the unipotent characters ρ_i, $1 \le i \le 3$, extend to $\mathrm{Aut}(S)$. We now investigate the remaining three characters.

Let δ denote the diagonal automorphism of S induced by conjugation with

$$\begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(q).$$

Our congruence conditions ensure that ζ generates the Sylow 3-subgroup of $\mathbb{F}_q^\times$, hence δ generates the full group of diagonal automorphisms. Let γ denote the transpose-inverse automorphism on $\mathrm{GL}_3(q)$ composed with conjugation with the permutation matrix of the transposition $(1,3)$. So γ induces the graph automorphism on S. Write r for the defining characteristic of S, so that $q = r^f$ for some $f \ge 1$, and let ψ be the field automorphism of S of order f which raises each matrix entry to its rth power. We let

$$\psi_0 := \begin{cases} \psi & \text{if } r \equiv 1 \pmod 3, \\ \psi^2 & \text{if } r \equiv 2 \pmod 3. \end{cases}$$

Thus, $\langle \psi_0 \rangle$ is the stabilizer of ζ in the group of field automorphisms. In the sequel we identify δ, γ, ψ with their images in $\mathrm{Out}(S)$.

Lemma 3.4. *We have*

$$\mathrm{Aut}(S) = \langle S, \delta, \gamma, \psi\rangle / Z(S)$$

and $\mathrm{Out}(S) = \langle \delta, \gamma\rangle \times \langle\psi\rangle \cong \mathfrak{S}_3 \times C_f$.

Proof. The outer automorphism group of S is generated by the diagonal, the field and the graph automorphisms, so the first statement holds. The second statement is now clear from the explicit form of δ, γ and ψ. □

Lemma 3.5.
(a) *The inertia group of* φ_1 *in* $\mathrm{Out}(S)$ *equals* $I_{\mathrm{Out}(S)}(\varphi_1) = \langle\psi, \gamma\rangle$.
(b) *The inertia group of* φ_2 *in* $\mathrm{Out}(S)$ *equals*

$$I_{\mathrm{Out}(S)}(\varphi_2) = \begin{cases} \langle\psi, \gamma\delta\rangle & \text{if } r \equiv 1 \pmod 3, \\ \langle\psi\gamma, \gamma\delta\rangle & \text{if } r \equiv 2 \pmod 3. \end{cases}$$

Proof. First note that $\mathrm{Aut}(S)$ must permute the set $\{\varphi_1, \varphi_2, \varphi_3\}$ of characters of degree $\frac{1}{3}\Phi_2\Phi_3$ by Lemma 3.3. Now the diagonal automorphism δ permutes the three regular unipotent elements $u_1, \ldots, u_3$ transitively, so it also fuses the three characters $\varphi_1, \varphi_2, \varphi_3$. Since $\langle\delta, \gamma\rangle \cong \mathfrak{S}_3$, and δ acts non-trivially, γ necessarily interchanges two of the three characters, and leaves one fixed; indeed, it fixes the class of u_1 and hence φ_1. The field automorphisms fix u_1, hence they also fix φ_1. This proves (a).

Clearly, the stabilizer of u_2 in the group of field automorphisms is $\langle\psi_0\rangle$, and if $\psi \neq \psi_0$ then $\psi\gamma$ also stabilizes u_2. Direct calculation show that $\gamma\delta$ fixes u_2. Thus, in all cases we have found a subgroup of index three of $\mathrm{Out}(S)$ stabilizing φ_i. Since $\mathrm{Out}(S)$ permutes the φ_i, $1 \leq i \leq 3$, transitively, this subgroup is the full stabilizer. □

Proposition 3.6. *All characters in* $\mathrm{Irr}_{3'}(S)$ *extend to their inertia groups in* $\mathrm{Aut}(S)$.

Proof. For the unipotent characters this was shown in Theorem 2.4. The semisimple characters φ_i, $1 \leq i \leq 3$, are the three irreducible constituents of $\mathrm{Ind}_B^S(\chi)$ with the linear character χ of B of order 3 defined above.

First assume that $r \equiv 1 \pmod 3$. Then χ is invariant under ψ, so extends to a character $\hat\chi$ of $\hat B := B.\langle\psi\rangle$. Since the φ_i extend to $\hat S := S.\langle\psi\rangle$ by Lemma 3.5, the induced character $\mathrm{Ind}_{\hat B}^{\hat S}(\hat\chi)$ decomposes as

$$\mathrm{Ind}_{\hat B}^{\hat S}(\chi) = \hat\varphi_1 + \hat\varphi_2 + \hat\varphi_3,$$

where $\hat\varphi_i$ denotes an extension of φ_i to $\hat S$. Now $\hat B$ is γ-stable, and since $r \equiv 1 \pmod 3$, so is χ and thus $\mathrm{Ind}_{\hat B}^{\hat S}(\chi)$. As γ interchanges φ_2 with φ_3, it also interchanges $\hat\varphi_2$ with $\hat\varphi_3$, so necessarily fixes $\hat\varphi_1$. Since the inertia factor group of φ_1 is generated by ψ and γ by Lemma 3.5(b), this shows that φ_1 extends to its inertia group.

Furthermore, $\gamma\delta$ interchanges φ_1 with φ_3, so it also interchanges $\hat\varphi_1$ with $\hat\varphi_3$, and thus fixes $\hat\varphi_2$. As above, this shows that φ_2 extends to the inertia factor group $\langle\psi, \gamma\delta\rangle$.

So now assume that $r \equiv 2 \pmod 3$. Then both ψ and γ interchange χ with the complex conjugate $\bar\chi$, so χ is $\psi\gamma$-stable. Consider $\hat B := B.\langle\psi\gamma\rangle$ and $\hat S := S.\langle\psi\gamma\rangle$ and then argue as above. □

Let's now turn to the 3-local situation. The primitive 3-dimensional complex reflection group $H = G_{25}$ (in the notation of Shephard and Todd) is a split extension of an extraspecial group of order 3^{1+2} with the group $\mathrm{SL}_2(3)$ of order 24. Its reflection representation is defined over $R = \mathbb{Z}[\exp(2\pi i/3), 1/3]$. The derived subgroup N of H, of order 216, is a split extension of 3^{1+2} with the quaternion group of order 8. Reduction modulo a prime ideal $\mathfrak{r}$ of R lying above r gives an embedding $H \hookrightarrow \mathrm{GL}_3(q)$ such that N embeds into $\mathrm{SL}_3(q)$ (note that $q \equiv 1 \pmod 3$). Since the Sylow 3-normalizer of $\mathrm{SL}_3(q)$ is isomorphic to N, this gives an explicit realization of this Sylow normalizer. Since H/N does not act by inner automorphisms on N, H is the extension of N with the group of diagonal automorphisms of $\mathrm{SL}_3(q)$.

Extending H by the transpose-inverse automorphism we obtain a group $\hat{H}$ of order 1296 with a 6-dimensional faithful representation over R. Again reduction modulo $\mathfrak{r}$ shows that $\hat{N}$ is the extension of the Sylow 3-normalizer of $\mathrm{SL}_3(q)$ by the diagonal and graph automorphisms. The character tables of $N, H, \hat{H}$ can be determined explicitly using GAP [6], and one obtains the $3'$-character degrees given in Table 2 below. Here, the symbol n^k signifies that the group in question has k characters of degree n.

Table 2. Character degrees of Sylow 3-normalizers

group	isom. type	rational $\mathrm{Irr}_{3'}$	non-rational $\mathrm{Irr}_{3'}$
N	$3^{1+2}.Q_8$	$1^4, 2, 8$	–
$N.\langle\gamma\rangle$	$3^{1+2}.Q_8.2$	$1^4, 2, 8^2$	2^2
$N.\langle\delta\rangle = H$	$3^{1+2}.\mathrm{SL}_2(3)$	$1, 2, 8$	$1^2, 2^2, 8^2$
$N.\langle\gamma,\delta\rangle = \hat{H}$	$3^{1+2}.\mathrm{SL}_2(3).2$	$1^2, 2, 4, 8^2, 16$	2^2

Let's denote the principal character, the character of degree 2 and the character of degree 8 of N by ρ'_1, ρ'_2, ρ'_3 respectively. Further, denote the three non-principal linear characters of N by φ'_i, $1 \le i \le 3$. The transpose-inverse automorphism γ acts non-trivially on H (see Table 2). Choose φ'_1 to be the non-trivial linear character of N invariant under γ.

Lemma 3.7. *Let $A := \mathrm{N}_{\mathrm{Aut}(S)}(N)/N$. Then $A \cong \mathrm{Out}(S)$, and we have:*

(a) *The characters ρ'_i are A-invariant.*
(b) *The inertia group of φ'_1 in A equals $I_A(\varphi'_1) = \langle\psi, \gamma\rangle$.*
(c) *The inertia group of φ'_2 in A equals*

$$I_A(\varphi'_2) = \begin{cases} \langle\psi, \gamma\delta\rangle & \text{if } r \equiv 1 \pmod 3, \\ \langle\psi\gamma, \gamma\delta\rangle & \text{if } r \equiv 2 \pmod 3. \end{cases}$$

All six characters extend to their inertia groups in $\mathrm{N}_{\mathrm{Aut}(S)}(N)$.

Proof. Assertion (a) is immediate since ρ_1 is the principal character and ρ_2, ρ_3 are the unique characters of N of their respective degrees. By Table 2, the diagonal automorphism of order 3 permutes the φ'_i transitively.

Again, first assume that $r \equiv 1 \pmod 3$. Then $\mathrm{SL}_3(r)$ contains a subgroup of order 3 with normalizer isomorphic to N, so up to conjugation, ψ acts trivially on

N, H and $\hat{H}$. Hence $\mathrm{N}_{\mathrm{Aut}(S)}(N) \cong \hat{H} \times \langle\psi\rangle$. By Table 2, the ρ_i' extend to $\hat{H}$, hence to $\mathrm{N}_{\mathrm{Aut}(S)}(N)$. Also, by definition φ_1' is invariant by γ, and also by ψ, so we obtain (b) in this case. By the table, φ_1' extends to $N.\langle\gamma\rangle$, and hence to its inertia group $N.\langle\gamma\rangle \times \langle\psi\rangle$. The character φ_2' is invariant under $\gamma\delta$, showing (c), and extends by the table.

If $r \equiv 2 \pmod 3$ then $\mathrm{SL}_3(r)$ does not contain a conjugate of N, so ψ acts non-trivially on N and H. The outer automorphism group of $H = G_{25}$ has order 2, the only non-trivial outer automorphism being induced by the embedding of G_{25} into the complex reflection group G_{26}. So ψ and γ have the same image in $\mathrm{Aut}(H)$, and $\psi\gamma$ acts as an inner automorphism on H. We may now argue as before, with ψ replaced by $\psi\gamma$. □

The previous result together with Lemma 3.5 and Proposition 3.6 now shows:

Proposition 3.8. *The map*

$$\Psi : \mathrm{Irr}_{3'}(\mathrm{SL}_3(q)) \longrightarrow \mathrm{Irr}_{3'}(N), \qquad \rho_i \mapsto \rho_i', \quad \varphi_i \mapsto \varphi_i',$$

is $\mathrm{N}_{\mathrm{Aut}(S)}(N)$*-equivariant.*

Corollary 3.9. *The groups* $\mathrm{L}_3(q)$*,* $q \equiv 4, 7 \pmod 9$*, are good for the prime* $p = 3$.

Proof. We apply the criterion in Proposition 3.1 to the simple group $X = \mathrm{L}_3(q)$. We may suppose that $q \neq 4$ since the group $\mathrm{L}_3(4)$ has been shown to be good for all primes in [17, Th. 4.1]. Then the order of the Schur multiplier of X divides 3 as required. All the characters of $\mathrm{SL}_3(q)$ and of N considered above, as well as their extensions, have $Z(\mathrm{SL}_3(q))$ in their kernel, so by factoring out $Z(\mathrm{SL}_3(q))$ the statements above can be transferred to statements about X. In condition (i) we take $M = N$. The equivariant bijection in (ii) is provided by Ψ in Proposition 3.8. Finally, all characters extend to their respective inertia groups by Proposition 3.6 and Lemma 3.7, which proves (iii). □

3.2. **Special unitary groups** $\mathrm{SU}_3(q)$**.** The situation for unitary groups is slightly easier, since the outer automorphism group has a simpler structure and the local situation is the same as for the special linear groups, so has been treated already.

Let $\mathbf{G}$ be the algebraic group SL_3 defined over the prime field of order r, with corresponding Frobenius endomorphism $\psi : \mathbf{G} \to \mathbf{G}$. Let $f \geq 1$ and $F := \psi^f$, and let γ denote the transpose-inverse automorphism on $\mathbf{G}$. Then the group of fixed points $\mathbf{G}^{F\gamma} =: S = \mathrm{SU}_3(q)$ of $F\gamma$ is the special unitary group with $q = r^f$. In this section we assume that $q \equiv 2, 5 \pmod 9$.

Lemma 3.10. *Let* $q \equiv 2, 5 \pmod 9$ *be a prime power. Then* $S = \mathrm{SU}_3(q)$ *has six irreducible characters of degree prime to 3. These are the three unipotent characters* $\rho_1 = 1_S, \rho_2, \rho_3 = \mathrm{St}_S$ *of degrees* $1, q\Phi_1, q^3$*, and three further semisimple characters* $\varphi_1, \varphi_2, \varphi_3$ *of degree* $\frac{1}{3}\Phi_1\Phi_6$*. All six characters have* $Z(S)$ *in their kernel.*

Proof. This can be read off from the character table of $\mathrm{SU}_3(q)$ [19]. □

The conjugacy class of regular unipotent elements of $\mathrm{GL}_3(q^2)$ splits into three conjugacy classes in $\mathrm{SU}_3(q)$, with representatives

$$u_1 := \begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix},\ u_2 := \begin{pmatrix}1&\zeta&0\\0&1&1\\0&0&1\end{pmatrix},\ u_3 := \begin{pmatrix}1&\zeta^2&0\\0&1&1\\0&0&1\end{pmatrix},$$

where $\zeta \in \mathbb{F}_{q^2}^\times$ has order 3. These are the only classes on which the three characters $\varphi_1, \varphi_2, \varphi_3$ differ. We choose notation so that

$$\varphi_i(u_i) = (2q-1)/3 \qquad (1 \leq i \leq 3), \qquad \varphi_i(u_j) = -(q+1)/3 \quad (i \neq j).$$

Let δ denote the diagonal automorphism of S induced by conjugation with

$$\begin{pmatrix}\zeta&0&0\\0&1&0\\0&0&1\end{pmatrix} \in \mathrm{GL}_3(q^2).$$

Then δ generates the full group of diagonal automorphisms of S. By construction ψ^f acts like γ on S, hence ψ^{2f} centralizes S. It is well-known that $\mathrm{Aut}(S) = \langle S, \delta, \psi\rangle/Z(S)$. Again by our congruence condition on q, f is necessarily odd, so that

$$\mathrm{Out}(S) = \langle \delta, \gamma\rangle \times \langle \psi^2\rangle \cong \mathfrak{S}_3 \times C_f.$$

Lemma 3.11.
(a) *The inertia group of* φ_1 *in* $\mathrm{Out}(S)$ *equals* $I_{\mathrm{Out}(S)}(\varphi_1) = \langle\psi\rangle$.
(b) *The inertia group of* φ_2 *in* $\mathrm{Out}(S)$ *equals* $I_{\mathrm{Out}(S)}(\varphi_2) = \langle\psi^2\gamma\delta\rangle$.
All characters in $\mathrm{Irr}_{3'}(S)$ *extend to their inertia groups in* $\mathrm{Aut}(S)$.

Proof. The diagonal automorphism δ permutes the three regular unipotent elements $u_1, \ldots, u_3$ transitively, so it also fuses the three characters $\varphi_1, \varphi_2, \varphi_3$. Since $\langle\delta,\gamma\rangle \cong \mathfrak{S}_3$, and δ acts non-trivially, γ necessarily interchanges two of the three characters, and leaves one fixed; indeed, it fixes the class of u_1 and hence φ_1. The field automorphisms fix u_1, hence they also fix φ_1.

Clearly, the stabilizer of u_2 in the group of field automorphisms is $\langle\psi^2\rangle$. Direct calculation show that $\gamma\delta$ fixes u_2. Thus, $\langle\psi^2, \gamma\delta\rangle$ stabilizes φ_2. Since $\mathrm{Out}(S)$ permutes the φ_i, $1 \leq i \leq 3$, transitively, this subgroup is the full stabilizer. Note that ψ^2 has odd order, while $\gamma\delta$ is an involution, so the inertia group is cyclic. This completes the proof of (a) and (b).

By Theorems 2.4 and 2.5 the unipotent characters ρ_i, $1 \leq i \leq 3$, extend to $\mathrm{Aut}(S)$. The inertia groups of φ_i in $\mathrm{Aut}(S)$ are cyclic by the first part of the proof, so these characters also extend. □

Theorem 3.12. *The groups* $\mathrm{U}_3(q)$, $q \equiv 2, 5 \pmod 9$, $q \neq 2$, *are good for the prime* $p = 3$.

Proof. The order of the Schur multiplier of $\mathrm{U}_3(q)$ divides 3 as required in Proposition 3.1. The characters of $\mathrm{U}_3(q)$ are the characters of $S = \mathrm{SU}_3(q)$ having $Z(S)$ in their kernel. This allows to make use of the results proved above.

Let N denote the normalizer of a Sylow 3-subgroup P of S and $A := \mathrm{N}_{\mathrm{Aut}(S)}(N)$. The special unitary group $\mathrm{SU}_3(q)$ embeds in a natural way into $\mathrm{SL}_3(q^2)$, such that all automorphisms of $\mathrm{SU}_3(q)$ extend to $\mathrm{SL}_3(q^2)$ in an obvious fashion. If $q \equiv 2, 5 \pmod 9$, then $q^2 \equiv 4, 7 \pmod 9$. Thus, comparison of Lemma 3.11 with Lemma 3.5 shows the existence of an A-equivariant bijection

$$\Xi : \mathrm{Irr}_{3'}(\mathrm{SU}_3(q)) \to \mathrm{Irr}_{3'}(\mathrm{SL}_3(q^2)).$$

Under the above embedding, P is a Sylow 3-subgroup of $\mathrm{SL}_3(q^2)$, as follows from the order formulas, and N is the full normalizer of P in $\mathrm{SL}_3(q^2)$. Composing Ξ above with the bijection Ψ in Proposition 3.6 we thus obtain an A-equivariant bijection

$$\Psi \circ \Xi : \mathrm{Irr}_{3'}(\mathrm{SU}_3(q)) \to \mathrm{Irr}_{3'}(N).$$

Proposition 3.1 together with Lemma 3.7 and Lemma 3.11 now shows that $\mathrm{U}_3(q)$ is good for $p = 3$. □

3.3. **The groups $G_2(q)$.** In this section we consider the simple groups $X = G_2(q)$, $q \equiv 2, 4, 5, 7 \pmod 9$, for the prime $p = 3$. Let z be a 3-central 3-element of X. When $q \equiv 4, 7 \pmod 9$, the normalizer M of $\langle z \rangle$ is a maximal subgroup $\mathrm{SL}_3(q).2$ (extension with the graph automorphism), when $q \equiv 2, 5 \pmod 9$, it is a maximal subgroup $\mathrm{SU}_3(q).2$ (extension with the graph-field automorphism). By Kleidman [11, Prop. 2.2], M contains the normalizer of a Sylow 3-subgroup centralizing z. The outer automorphism group of X is cyclic, generated by a field automorphism ψ of order f, when $q = r^f$. By choosing z in the centralizer $G_2(r)$ of ψ, we obtain M normalized by ψ.

Lemma 3.13. *The group $G_2(q)$, $q \equiv 2, 4, 5, 7 \pmod 9$, has precisely 9 irreducible characters of degree prime to 3. These are six unipotent characters and three further characters of degrees $q^3+\epsilon, q(q+\epsilon)(q^3+\epsilon), q^3(q^3+\epsilon)$ belonging to 3-central 3-elements in the dual group in Lusztig's parametrization, where $q \equiv \epsilon \pmod 3$.*

Proof. This can be read off from the known character tables (see [2, 5]) or using Lusztig's Jordan decomposition of characters. □

Proposition 3.14. *The groups $G_2(q)$, $q \equiv 2, 4, 5, 7 \pmod 9$, $q \neq 2$, are good for the prime 3.*

Proof. The group $G_2(4)$ with an exceptional Schur multiplier has been shown to be good for 3 in [17, Th. 4.1], so we may assume that $q \neq 4$. Then $X = G_2(q)$ has trivial Schur multiplier. By Theorems 2.4 and 2.5 the unipotent characters of X extend to $\mathrm{Aut}(X)$. The three other characters in $\mathrm{Irr}_{3'}(X)$ are distinguished by their degrees, hence they are stabilized by $\mathrm{Aut}(X)$, and they extend since $\mathrm{Aut}(X)/X$ is cyclic.

The characters in $\mathrm{Irr}_{3'}(M)$ have been determined in Sections 3.1 and 3.2; these are the six extensions to M of the unipotent characters ρ_i, $1 \leq i \leq 3$, of M', of degrees $1, q(q+\epsilon), q^3$, the two extensions of φ_1, and the induced of φ_2. Let $\hat{M} := M.\langle\psi\rangle$, the normalizer of M in $\mathrm{Aut}(X)$. According to Theorem 2.4, the unipotent characters ρ_i extend from M' to $\hat{M}$. Also, the induced of φ_2 is the only character of its degree, so it also extends to $\hat{M}$. Finally, it was shown in Proposition 3.6 that φ_1 extends to $\hat{M}$. Thus any bijection $\mathrm{Irr}_{3'}(X) \to \mathrm{Irr}_{3'}(M)$ is $\mathrm{Aut}(X)$-equivariant, and on both sides, all characters extend maximally. Since M is a proper subgroup of X containing the normalizer of a Sylow 3-subgroup of X, it satisfies the conditions in Proposition 3.1 needed to show that X is good for 3. □

3.4. **The groups $^2F_4(q^2)$.** In this section we deal with the simple groups $X = {}^2F_4(q^2)$, $2 < q^2 \equiv 2, 5 \pmod 9$, for the prime $p = 3$, and we also handle $^2F_4(2)'$ for all primes. For $q^2 > 2$, the situation is quite similar to the one for $G_2(q)$. Let z be a 3-central 3-element of X. By [13, Prop. 1.2] the normalizer $M := \mathrm{N}_X(\langle z\rangle)$ of $\langle z\rangle$ is a maximal subgroup $\mathrm{SU}_3(q^2).2$ (extension with the graph-field automorphism) containing the normalizer of a Sylow 3-subgroup centralizing z. The outer automorphism group of X is cyclic, generated by a field automorphism ψ of order $2f+1$, when $q^2 = 2^{2f+1}$. By chosing z in the centralizer $^2F_4(2)$ of ψ, we obtain M normalized by ψ.

Lemma 3.15. *The group $^2F_4(q^2)$, $q^2 \equiv 2, 5 \pmod 9$, has precisely 9 irreducible characters of degree prime to 3. These are six unipotent characters, and three further characters of degrees*

$$(q^2-1)(q^4+1)(q^{12}+1),\ q^2(q^8-1)(q^{12}+1),\ q^6(q^2-1)(q^4+1)(q^{12}+1)$$

belonging to 3-central 3-elements in the dual group.

Proof. This can be read off from the known character table [13]. □

Proposition 3.16. *The groups $^2F_4(q^2)$, $q^2 \equiv 2, 5 \pmod 9$, $q^2 \neq 2$, are good for the prime 3.*

Proof. By Theorems 2.4 and 2.5 the unipotent characters of $X = {}^2F_4(q^2)$ extend to $\mathrm{Aut}(X)$. The three other characters in $\mathrm{Irr}_{3'}(X)$ are distinguished by their degrees, hence they are stabilized by $\mathrm{Aut}(X)$, and they extend since $\mathrm{Aut}(X)/X$ is cyclic.

It has already been argued in the proof of Proposition 3.14 that all characters of M extend to $\mathrm{N}_{\mathrm{Aut}(X)}(M)$. Since M contains the normalizer of a Sylow 3-subgroup of X, it satisfies the conditions in Proposition 3.1 needed to show that X is good for 3. □

The group $^2F_4(2)$ is not simple; its derived subgroup $^2F_4(2)'$ is a non-abelian simple group not occurring elsewhere in the classification, hence should be considered to be a sporadic simple group. We prove the following:

Proposition 3.17. *The Tits simple group $^2F_4(2)'$ is good for all prime divisors of its order.*

Proof. Since $X = {}^2F_4(2)'$ has trivial Schur multiplier, according to [17, Cor. 2.2] it suffices to show that the ordinary McKay conjecture holds for X and for $G := \mathrm{Aut}(X) = {}^2F_4(2)$ for all prime divisors p of $|X| = 2^{11}.3^3.5^2.13$.

The Sylow 13-subgroups of G are cyclic, so X is good for $p = 13$ by [17, Thm. 2.3]. For $p = 5$, the Sylow 5-normalizer in X respectively G is isomorphic to $5^2.4\mathfrak{A}_4$ respectively $5^2.4\mathfrak{S}_4$ by [3], and these subgroups of G can easily be constructed explicitly in the 26-dimensional representation of G using GAP [6]. It is now straightforward to check the ordinary McKay conjecture for X and G with $p = 5$.

For $p = 3$, the Sylow 3-normalizer in X respectively G is isomorphic to $3^{1+2}.8$ respectively $3^{1+2}.8.2$ by [3, corrections]. Again, both groups are easily constructed using GAP, and the assertion can be seen to hold.

The Sylow 2-subgroups of X and of G are self-normalizing. The commutator factor group of the Sylow 2-subgroup has order 8 respectively 16, and the assertion easily follows from the character degrees printed in [3]. □

4. The prime $p = 2$

In this section we show that the symplectic groups $X = \mathrm{S}_{2n}(q)$, $q \equiv 3, 5 \pmod 8$, $n \geq 2$, are good for the prime $p = 2$. For this we need to parametrize the irreducible characters of $G = \mathrm{Sp}_{2n}(q)$ of odd degree. These are contained in Lusztig series $\mathcal{E}(G, s)$ of $\mathrm{Irr}(G)$ indexed by semisimple elements s in the dual groups $\mathrm{SO}_{2n+1}(q)$ centralizing a Sylow 2-subgroup. Furthermore, in order to deal with diagonal automorphisms we need to know the 2-central elements of $\mathrm{Spin}_{2n+1}(q)$. We therefor first recall the classification of such elements.

4.1. **2-central elements in odd-dimensional orthogonal groups.** Let q be an odd prime power. Let V be a $2n + 1$ dimensional vector space over the algebraic closure $\bar{\mathbb{F}}_q$ with a non-degenerate quadratic form defined over $\mathbb{F}_q$, and denote by $\mathbf{G}^* = \mathrm{SO}(V)$ its isometry group. Let $F : \mathbf{G}^* \to \mathbf{G}^*$ denote the standard Frobenius endomorphism on $\mathrm{GL}(V)$. Then the group of fixed points $G^* := (\mathbf{G}^*)^F$ is the special orthogonal group $\mathrm{SO}_{2n+1}(q)$.

Lemma 4.1. *Let $s \in G^*$ be semisimple centralizing a Sylow 2-subgroup of G^*. Then s is an involution.*

Proof. Let $s \in G^*$ be semisimple. Then s lies in an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}^*$. The longest element w_0 in the Weyl group W of $\mathbf{G}^*$ with respect to $\mathbf{T}$ inverts all elements of $\mathbf{T}$, hence normalizes the centralizer $\mathrm{C}_{G^*}(s)$. Since $\mathbf{T}$ is F-stable so is w_0. If $\mathrm{C}_{G^*}(s)$ contains a Sylow 2-subgroup of G^*, the Sylow 2-subgroups of $\mathrm{N}_{G^*}(\langle s \rangle)$ must lie in $\mathrm{C}_{G^*}(s)$, so w_0 has to centralize s. Hence s is equal to its inverse. □

Geometrically, the involutions in $\mathrm{SO}_{2n+1}(q)$ can be described as follows. Let t be an involution in $\mathbf{G}^*$. Denote its eigenspaces for the eigenvalues 1 and -1 by V^+, V^- respectively. Both eigenspaces are non-degenerate with respect to the orthogonal form on V, so $V = V^+ \perp V^-$. Clearly, V^- is of even dimension $\dim(V^-) = 2i > 0$, say, so V^+ is odd-dimensional. The centralizer $\mathrm{C}_{\mathbf{G}^*}(t)$ of t is then isomorphic to $\mathrm{SO}(V^+) \times \mathrm{GO}(V^-)$. Here, the first factor is connected while the second factor $\mathrm{GO}(V^-)$ is disconnected, with connected component of the identity $\mathrm{SO}(V^-)$ of index 2. Clearly, the class of t is F-stable, so the F-fixed points split into two G^*-conjugacy classes. Write t_i for a representative of the class with untwisted centralizer $\mathrm{SO}_{2(n-i)+1}(q) \times \mathrm{GO}_{2i}^+(q)$, and t_i' for a representative of the other class. This leads to the following statement:

Proposition 4.2. *The conjugacy classes of involutions in $\mathrm{SO}_{2n+1}(q)$, q odd, are as indicated in Table 3. Let $J \subset \mathbb{N}$ be such that $n = \sum_{j \in J} 2^j$ is the 2-adic decomposition of n. Then t_i resp. t_i' is 2-central if and only if there exists a subset $\emptyset \neq I \subseteq J$ with $i = \sum_{j \in I} 2^j$ and furthermore*

(a) $q \equiv 1 \pmod 4$ *or* $0 \notin I$ *in the case* t_i *(i.e.,* $q^i \equiv 1 \pmod 4$*),*
(b) $q \equiv 3 \pmod 4$ *and* $0 \in I$ *in the case* t_i' *(i.e.,* $q^i \equiv 3 \pmod 4$*).*

In particular all 2-central involutions of $\mathrm{SO}_{2n+1}(q)$ *are contained in its derived group* $\mathrm{O}_{2n+1}(q)$.

Table 3. Involutions in $\mathrm{SO}_{2n+1}(q)$, q odd

type	centralizer	2-central for
t_i $(1 \le i \le n)$	$\mathrm{SO}_{2(n-i)+1}(q) \times \mathrm{GO}_{2i}^+(q)$	see text
t_i' $(1 \le i \le n)$	$\mathrm{SO}_{2(n-i)+1}(q) \times \mathrm{GO}_{2i}^-(q)$	see text

Proof. The classification of involutions and their centralizers as described above is proved in [8, 4.6]. From this it is an easy exercise using the order formulas and Lemma 4.3 below to work out the conditions for being 2-central. It follows from [8, Table 4.5.1] that all such involutions induce the trivial diagonal automorphism on the derived subgroup $\mathrm{O}_{2n+1}(q)$. □

Lemma 4.3. *Let* $k, n \in \mathbb{N}$ *with* $k \le n$. *The parabolic subgroup* $W(B_k) \times W(B_{n-k})$ *of the Weyl group* $W(B_n)$ *has odd index if and only if* $k = \sum_{j \in I} 2^j$ *for some subset* $I \subseteq J$ *where* $n = \sum_{j \in J} 2^j$ *is the 2-adic decomposition of* n.

Proof. We have $|W(B_k)| = 2^k\, k!$, so the relevant index equals $\binom{n}{k}$. The claim is now easily established. □

Next we consider $\mathrm{Spin}_{2n+1}(q)$, the odd-dimensional spin group over $\mathbb{F}_q$, that is, the $\mathbb{F}_q$-points of a simple simply-connected group $\tilde{\mathbf{G}}^*$ of type B_n over $\bar{\mathbb{F}}_q$.

Lemma 4.4. *Let* $s \in \mathrm{Spin}_{2n+1}(q)$ *be semisimple centralizing a Sylow 2-subgroup. Then* s *lies in* $Z(\mathrm{Spin}_{2n+1}(q))$.

Proof. Let $s \in \tilde{\mathbf{G}}^* = \mathrm{Spin}_{2n+1}(\bar{\mathbb{F}}_q)$ be semisimple. Since $\tilde{\mathbf{G}}^*$ is simply-connected, the centralizer $\mathbf{C} = \mathrm{C}_{\tilde{\mathbf{G}}^*}(s)$ of s is a connected reductive group (see [21, 3.9]). The possible types of centralizers in $\tilde{\mathbf{G}}^*$ can be determined by the algorithm of Borel–de Siebenthal from the extended Dynkin diagram of type B_n. It follows that the semisimple part of $\mathbf{C}$ has a Weyl group contained in a reflection subgroup of type $B_k \times D_{n-k}$ for some $0 \le k \le n$. This has index 2 in the reflection subgroup of type $B_k \times B_{n-k}$, and hence even index in $W(B_n)$, unless $k = n$. Thus, the group of $\mathbb{F}_q$-points of the connected centralizer $\mathbf{C}$ has even index in $\mathrm{Spin}_{2n+1}(q)$ unless $\mathbf{C} = \tilde{\mathbf{G}}^*$, i.e., $s \in Z(\mathrm{Spin}_{2n+1}(q))$. □

4.2. **Characters of odd degree.** Now let $\mathbf{G}$ be the $2n$-dimensional symplectic group over $\bar{\mathbb{F}}_q$, defined over $\mathbb{F}_q$ with corresponding Frobenius $F : \mathbf{G} \to \mathbf{G}$. The group of fixed points $G := \mathbf{G}^F$ is the symplectic group $\mathrm{Sp}_{2n}(q)$. Let $\mathbf{G}^*$ be a group dual to $\mathbf{G}$ with corresponding Frobenius also denoted by $F : \mathbf{G}^* \to \mathbf{G}^*$. Then $G^* \cong \mathrm{SO}_{2n+1}(q)$.

We introduce the following notation. For a semisimple element $s \in G^*$ let

$$\mathcal{E}_{2'}(G, s) := \mathrm{Irr}_{2'}(G) \cap \mathcal{E}(G, s) = \{\chi \in \mathcal{E}(G, s) \mid \chi(1) \equiv 1 \pmod 2\}$$

denote the irreducible characters of odd degree in the Lusztig series $\mathcal{E}(G, s)$.

Proposition 4.5. *The irreducible characters of G of odd degree are given as*

$$\mathrm{Irr}_{2'}(G) = \mathcal{E}_{2'}(G, 1) \quad \amalg \coprod_{s \in G^* \text{ 2-central}} \mathcal{E}_{2'}(G, s)$$

where the disjoint union runs over 2-central involutions $s \in G^*$ modulo conjugation.

Proof. By [16, Prop. 7.2] a character $\chi \in \mathrm{Irr}(G)$ has odd degree if and only if $\chi \in \mathcal{E}_{2'}(G, s)$ for some semisimple element $s \in G^*$ centralizing a Sylow 2-subgroup of G^*. Now $s \in G^* = \mathrm{SO}_{2n+1}(q)$ centralizes a Sylow 2-subgroup if and only if s is a 2-central involution by Lemma 4.1. □

The previous statement can be made more precise as follows:

Proposition 4.6. $\mathrm{Irr}_{2'}(G)$ *is in bijection with the set of pairs (s, ϕ) where*

(i) *$s \in G^*$ is either 1 or a 2-central involution, modulo G^*-conjugation, and*
(ii) *$\phi \in \mathrm{Irr}_{2'}(W_M)$, where $M = \mathrm{C}_{G^*}(s)$.*

Proof. Let $\chi \in \mathrm{Irr}_{2'}(G)$. By Proposition 4.5 there exists a uniquely determined $s \in G^*$ up to conjugation with $\chi \in \mathcal{E}_{2'}(G, s)$. Hence, under the Jordan decomposition $\psi_s : \mathcal{E}(G, s) \to \mathcal{E}(\mathrm{C}_{G^*}(s), 1)$, χ corresponds to a character $\psi_s(\chi)$ of odd degree. Let $\mathbf{M} := \mathrm{C}_{\mathbf{G}^*}(s)$,

$$e := \begin{cases} 1 & \text{if } q \equiv 1 \pmod 4, \\ 2 & \text{if } q \equiv 3 \pmod 4, \end{cases}$$

and $\mathbf{S}^*$ a Sylow e-torus of $\mathbf{M}$. By [16, Th. 5.9] this is also a Sylow e-torus of $\mathbf{G}^*$. Then $|(\mathbf{S}^*)^F| = \Phi_e(q)^n$, so $\mathbf{S}^*$ is a maximal torus of $\mathbf{G}^*$. In particular it is self-centralizing, whence $\mathrm{C}_M(\mathbf{S}^*) = \mathbf{S}^*$. By [16, Cor. 6.6] a character $\chi \in \mathcal{E}(\mathrm{C}_{G^*}(s), 1)$ has odd degree if and only if it lies in the e-Harish-Chandra series of (L, λ), with $L = \mathrm{C}_M(\mathbf{S}^*) = \mathbf{S}^*$ and $\lambda \in \mathcal{E}(L, 1)$ e-cuspidal, indexed by a character $\phi \in \mathrm{Irr}_{2'}(W_M(L, \lambda))$. But the unique unipotent character λ of $L = \mathbf{S}^*$ is the trivial character, and the relative Weyl group $W_M(\mathbf{S}^*)$ is just the Weyl group of M. This completes the proof. □

The 2-central involutions of G^* and their centralizers have been determined in Proposition 4.2. The characters of $W(B_n)$ of odd degree are easily described. Recall that the irreducible characters of $\mathfrak{S}_n$, $W(B_n)$ respectively, are naturally parametrized by partitions λ, respectively pairs of partitions (λ, μ), of n.

Lemma 4.7. *Let $n = 2^i \geq 2$.*

(a) *The irreducible character $\chi_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ has odd degree if and only if λ is a hook partition $(n-k, 1^k)$ for some k.*
(b) *The irreducible character $\chi_{\lambda,\mu} \in \mathrm{Irr}(W(B_n))$ has odd degree if and only if $(\lambda, \mu) = ((n-k, 1^k), -)$ or $(\lambda, \mu) = (-, (n-k, 1^k))$ for some k.*

In particular $|\mathrm{Irr}_{2'}(\mathfrak{S}_n)| = n$, $|\mathrm{Irr}_{2'}(W(B_n))| = 2n$.

Proof. The character χ_λ of $\mathfrak{S}_n$ parametrized by the hook partition $(n-k, 1^k)$ has degree $\binom{n-1}{k}$, which is odd for all k. There are n such hook partitions. On the other hand $|\mathrm{Irr}_{2'}(\mathfrak{S}_n)| = n$ by Olsson [18, Cor. 3.6 and 3.7].

The irreducible character $\chi_{\lambda,\mu}$ of $W(B_n)$ is obtained by inducing the exterior tensor product $\chi_\lambda \otimes (\chi_\mu \otimes \epsilon)$ from $W_{\lambda,\mu} := W(B_k) \times W(B_{n-k})$ to $W(B_n)$, where $\lambda \vdash k$, $\mu \vdash n-k$, and ϵ denotes the non-trivial linear character of $W(B_k)$ which trivially restricts to the complement $\mathfrak{S}_k$. Thus $\chi_{\lambda,\mu}$ has odd degree if and only if the Young subgroup $W_{\lambda,\mu}$ has odd index and both χ_λ, χ_μ have odd degree. The first condition forces $k \in \{0, n\}$ by Lemma 4.3, and from the second it follows by the first part of the proof that (λ, μ) is as claimed. □

Finally, we need to determine the irreducible characters of odd degree of the adjoint type group $\mathrm{PCSp}_{2n}(q)$. Since G is a two-fold central extension of $X := \mathrm{S}_{2n}(q)$, all $\chi \in \mathrm{Irr}_{2'}(G)$ have $Z(G)$ in their kernel, so can be considered as characters of X, and all irreducible characters of X of odd degree arise this way. We will therefor identify $\chi \in \mathrm{Irr}_{2'}(G)$ with $\chi \in \mathrm{Irr}_{2'}(X)$.

Proposition 4.8. *The irreducible characters of $\hat{X} := \mathrm{PCSp}_{2n}(q)$ of odd degree are the extensions to $\hat{X}$ of the unipotent characters of $X = \mathrm{S}_{2n}(q)$ of odd degree. In particular, the non-unipotent characters of X of odd degree do not extend to $\hat{X}$.*

Proof. The group $\hat{X}$ is the group of fixed points of a simple algebraic group of adjoint type C_n. Thus, again [16, Prop. 7.2] applies: a character $\chi \in \mathrm{Irr}(\hat{X})$ has odd degree if and only if $\chi \in \mathcal{E}_{2'}(\hat{X}, s)$ for some semisimple element s in the dual group $\hat{X}^* \cong \mathrm{Spin}_{2n+1}(q)$, centralizing a Sylow 2-subgroup of $\hat{X}^*$. By Lemma 4.4 these are just the two elements in $Z(\hat{X}^*)$. The Lusztig series $\mathcal{E}(\hat{X}, 1)$ are just the unipotent characters of $\hat{X}$, which by [12] are extensions of unipotent characters of X, while the other Lusztig series consists of the products of unipotent characters with the non-trivial linear character of $\hat{X}$. In particular, both series only contain extensions of unipotent characters of X to $\hat{X}$. □

4.3. **Action of automorphisms.** We next determine the inertia groups of odd degree characters of $X = \mathrm{S}_{2n}(q)$ in $\mathrm{Aut}(X)$. The outer automorphism group of X is generated by the diagonal automorphism δ of order 2 and the field automorphism ψ of order f, where $q = p^f$ with p prime. Furthermore, δ and ψ commute, so $\mathrm{Out}(X) = \langle\delta\rangle \times \langle\psi\rangle$.

Proposition 4.9. *Assume that q is not a square. Let $\chi \in \mathrm{Irr}_{2'}(X)$ have odd degree.*

(a) *If χ is unipotent, then $I_{\mathrm{Out}(X)}(\chi) = \mathrm{Out}(X)$.*

(b) *If $\chi \in \mathcal{E}_{2'}(G, s)$ for some 2-central involution s then $I_{\mathrm{Out}(X)}(\chi) = \langle\psi\rangle$.*

In particular, all $\chi \in \mathrm{Irr}_{2'}(\bar{G})$ extend to their inertia groups in $\mathrm{Aut}(\bar{G})$.

Proof. For the unipotent characters this is Proposition 2.2. If χ is not unipotent, $I_{\mathrm{Out}(X)}(\chi)$ does not contain δ, by Proposition 4.8. Let $\hat{\chi} \in \mathrm{Irr}(\mathrm{PCSp}_{2n}(q))$ be the induced of χ. Since PCSp_{2n} is of adjoint type, so has connected centre, $\hat{\chi}$ is uniquely determined by its multiplicities in the various Deligne-Lusztig characters, by [12, proof of 8.1]. Since all conjugacy classes of involutions and all conjugacy classes of maximal tori of $\hat{X} = \mathrm{PCSp}_{2n}(q)$ are stable under field automorphisms, this implies that $\hat{\chi}$ is stable under field automorphisms. Since q is not a square, f and hence the order of ψ is odd and $\mathrm{Out}(X)$ is cyclic. This proves that $I_{\mathrm{Out}(X)}(\chi) = \langle\psi\rangle$. Extendibility is clear since the inertia group is cyclic. □

4.4. $\mathrm{S}_{2n}(q)$ **is good for** $p = 2$.

Theorem 4.10. *Let q be an odd power of an odd prime and $n = 2^i \geq 1$. Then the symplectic group $\mathrm{S}_{2n}(q)$ is good for the prime 2.*

Proof. We proceed by induction, the induction base $n = 1$ being given by [10, Th. 15.3] since $\mathrm{Sp}_2(q) = \mathrm{SL}_2(q)$. So now assume that $n = 2^i > 1$. The Schur multiplier of $X := \mathrm{S}_{2n}(q)$ is cyclic of order 2, and the full covering group is isomorphic to $G := \mathrm{Sp}_{2n}(q)$. Thus all faithful characters of G have even degree, and $\mathrm{Irr}_{2'}(\mathrm{S}_{2n}(q)) = \mathrm{Irr}_{2'}(G)$. Let $m := n/2$ and denote by M the normalizer in G of the natural subgroup $\mathrm{Sp}_{2m}(q) \times \mathrm{Sp}_{2m}(q)$ consisting of 2-by-2 block matrices, a wreath product $\mathrm{Sp}_{2m}(q) \wr 2$. Then M is a proper subgroup containing the normalizer of a Sylow 2-subgroup of G by Carter and Fong [1, Th. 4].

We investigate the characters of odd degree of G and of M in turn. By Proposition 4.6 the dual group $\mathrm{SO}_{2n+1}(q)$ has a unique class $[s]$ of 2-central involutions, with centralizer $\mathrm{GO}_{2n}^{\epsilon}(q)$, where $q^n \equiv \epsilon \pmod 4$. Thus by Proposition 4.6 there is a natural bijection

$$\mathrm{Irr}_{2'}(G) \xleftrightarrow{1-1} \mathrm{Irr}_{2'}(W(B_n)) \amalg \mathrm{Irr}_{2'}(W(B_n))$$

where the second factor on the right comes from the Weyl group of the disconnected group $\mathrm{GO}_{2n}^{\epsilon}(q)$. Thus, by Lemma 4.7 the group X has $2n$ unipotent and $2n$ non-unipotent characters of odd degree.

Exactly the same arguments apply to $\mathrm{Irr}_{2'}(\mathrm{Sp}_{2m}(q))$. The characters of the direct product $\mathrm{Sp}_{2m}(q) \times \mathrm{Sp}_{2m}(q)$ are the exterior tensor products $\mu_1 \otimes \mu_2$ of characters $\mu_i \in \mathrm{Irr}(\mathrm{Sp}_{2m}(q))$. Thus $\mathrm{Irr}_{2'}(M)$ consists of the extensions to M of pairs (μ, μ) of characters $\mu \in \mathrm{Irr}_{2'}(\mathrm{Sp}_{2m}(q))$, whence M has $2n$ extensions of unipotent characters and $2n$ extensions of non-unipotent characters of odd degree. In particular, $|\mathrm{Irr}_{2'}(G)| = |\mathrm{Irr}_{2'}(M)|$.

Both factors $\mathrm{Sp}_{2m}(q)$ are stabilized by diagonal automorphisms induced by diagonal matrices, and by the field automorphism induced by raising matrix entries to their pth power, thus M is also stabilized by these. They induce the corresponding diagonal respectively field automorphisms on the factors $\mathrm{Sp}_{2m}(q)$. By Proposition 4.9 the unipotent characters of $\mathrm{Sp}_{2m}(q)$ extend to $\mathrm{Aut}(\mathrm{Sp}_{2m}(q))$, hence to the wreath product $\mathrm{Aut}(\mathrm{Sp}_{2m}(q)) \wr 2$, hence to the normalizer in $\mathrm{Aut}(X)$ of M. Thus, the $2n$ unipotent characters of X and the $2n$ unipotent characters of M have the same inertia group and extend maximally.

On the other hand, the $2n$ non-unipotent characters of $\mathrm{Sp}_{2m}(q)$ are only stabilized by the field automorphisms by Proposition 4.9, thus the same is true for the extensions to M. So, the $2n$ non-unipotent characters of X and the $2n$ non-unipotent characters of M all have the same cyclic inertia group in $\mathrm{Out}(X)$, to which they extend.

If we choose any map $^* : \mathrm{Irr}_{2'}(X) \to \mathrm{Irr}_{2'}(M)$ sending unipotent characters to unipotent characters, then the conditions of Proposition 3.1 are satisfied. Hence X is good for 2. □

Theorem 4.11. *Let q be an odd power of an odd prime and $n \geq 3$ not a power of 2. Then the symplectic group* $\mathrm{S}_{2n}(q)$ *is good for the prime 2.*

Proof. Let $n = \sum_{j \in J} 2^j$ denote the 2-adic decomposition of n. Then by assumption $|J| \geq 2$. Let $j := \max J$, $m := 2^j$, and M the natural subgroup $\mathrm{Sp}_{2m}(q) \times \mathrm{Sp}_{2(n-m)}(q)$ consisting of block matrices. Since $|J| \geq 2$ we have $m \neq n$, so M is a proper subgroup, which contains the normalizer of a Sylow 2-subgroup of G by [1, Th. 4].

According to Proposition 4.5 the irreducible characters of X of odd degree lie in Lusztig series parametrized by the identity and or the class of a 2-central involution in $G^* = \mathrm{SO}_{2n+1}(q)$. By Proposition 4.2 these are in bijection with

$$\mathcal{J} := \{I \subseteq J \mid 0 \in I \Leftrightarrow q \equiv 3 \pmod 4\},$$

where $I = \emptyset$ corresponds to the identity element. Again by Proposition 4.5 the irreducible characters of $\mathrm{Sp}_{2(n-m)}(q)$ lie in Lusztig series parametrized by

$$\mathcal{J}_1 := \{I \in \mathcal{J} \mid m \notin I\} = \{I \subseteq J \setminus \{m\} \mid 0 \in I \Leftrightarrow q \equiv 3 \pmod 4\},$$

while those of $\mathrm{Sp}_{2m}(q)$ lie in series parametrized by $\mathcal{J}_2 := \{\emptyset, \{m\}\}$. Now the natural bijection

$$\mathcal{J}_1 \times \mathcal{J}_2 \longrightarrow \mathcal{J}, \qquad (I_1, I_2) \mapsto I_1 \cup I_2,$$

shows that the Lusztig series containing odd degree characters of M and of X are parametrized by the same sets. Moreover, by Proposition 4.6 and Lemma 4.7, the bijection preserves the number of odd degree characters in corresponding series.

As in the previous proof, it follows from Proposition 4.9 that the bijection also preserves inertia groups. Since $\mathrm{Out}(X)$ is cyclic, maximal extensibility is automatically implied on both sides. Hence X is good for the prime 2 by Proposition 3.1. □

Since squares are not $\equiv 3, 5 \pmod 8$, the previous two results imply:

Corollary 4.12. *Let $q \equiv 3, 5 \pmod 8$ and $n \geq 2$. Then* $\mathrm{S}_{2n}(q)$ *is good for $p = 2$.*

Taken together with Theorem 3.2 and [10, Th. 17.1] for the Ree groups of type ${}^2G_2$ this proves:

Corollary 4.13. *Let X be a simple group of Lie type and p a prime, such that (X, p) was excluded in Theorem 7.8 or 8.5 of* [16]*. Then X is good for the prime p.*

References

[1] R. Carter, P. Fong, The Sylow 2-subgroups of the finite classical groups. J. Algebra **1** 1964, 139–151.

[2] B. Chang, R. Ree, The characters of $G_2(q)$. Pp. 395–413 in: *Symposia Mathematica*, Vol. XIII, Academic Press, London, 1974.

[3] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, *Atlas of Finite Groups*. Clarendon Press, Oxford, 1985.

[4] F. Digne, J. Michel, Groupes réductifs non connexes. Ann. Sci. École Norm. Sup. **27** (1994), 345–406.

[5] H. Enomoto, H. Yamada, The characters of $G_2(2^n)$. Japan. J. Math. **12** (1986), 325–377.

[6] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*; 2004, `http://www.gap-system.org`.

[7] M. Geck, G. Pfeiffer, *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*. Oxford University Press, Oxford, 2000.

[8] D. Gorenstein, R. Lyons, R. Solomon, *The Classification of the Finite Simple Groups, Number 3.* Mathematical Surveys and Monographs, Amer. Math. Soc., Providence, 1994.
[9] M. Isaacs, *Character Theory of Finite Groups.* Dover, New York, 1994.
[10] M. Isaacs, G. Malle, G. Navarro, A reduction theorem for the McKay conjecture. Invent. Math. **170** (2007), 33–101.
[11] P. Kleidman, The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$, and their automorphism groups. J. Algebra **117** (1988), 30–71.
[12] G. Lusztig, On the representations of reductive groups with disconnected centre. Pp. 157–166 in: *Orbites Unipotentes et Représentations, I.* Astérisque **168** (1988).
[13] G. Malle, Die unipotenten Charaktere von ${}^2F_4(q^2)$. Comm. Algebra **18** (1990), 2361-2381.
[14] ———, Darstellungstheoretische Methoden bei der Realisierung einfacher Gruppen vom Lie Typ als Galoisgruppen. Pp. 443–459 in: *Representation Theory of Finite Groups and Finite-Dimensional Algebras*, Progr. Math., 95, Birkhäuser, Basel, 1991.
[15] ———, Generalized Deligne-Lusztig characters. J. Algebra **159** (1993), 64–97.
[16] ———, Height 0 characters of finite groups of Lie type. Represent. Theory **11** (2007), 192–220.
[17] ———, The inductive McKay condition for simple groups not of Lie type. Comm. Algebra **36** (2008), 455–463.
[18] J.B. Olsson, McKay numbers and heights of characters. Math. Scand. **38** (1976), 25–42.
[19] W.A. Simpson, J.S. Frame, The character tables for $\mathrm{SL}(3,q)$, $\mathrm{SU}(3,q^2)$, $\mathrm{PSL}(3,q)$, $\mathrm{PSU}(3,q^2)$. Canad. J. Math. **25** (1973), 486–494.
[20] B. Späth, Die McKay-Vermutung für quasi-einfache Gruppen vom Lie-Typ. Dissertation, TU Kaiserslautern, 2007.
[21] T.A. Springer, R. Steinberg, Conjugacy classes. Pp. 167–266 in: *Seminar on Algebraic Groups and Related Finite Groups*, Springer, Berlin, 1970.

FB Mathematik, Universität Kaiserslautern, Postfach 3049, D–67653 Kaiserslautern, Germany.
E-mail address: malle@mathematik.uni-kl.de