THE 2F-MODULES FOR NEARLY SIMPLE GROUPS

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Abstract. We complete the classification of irreducible 2F-modules for groups of Lie type acting in the natural characteristic by dealing with the three open cases from [5]. We also finish the classification of such modules for almost quasi-simple groups, and show that in all cases there is an offender with cubic action.

1. Introduction and statement of results

Let G be a finite group with $F^*(G)$ quasi-simple. For an absolutely irreducible KG-module V over a finite field K of characteristic ℓ, and a subgroup A of G, we let

$$f(A) := f^V(A) := |A|^2 \cdot |C_V(A)|.$$

We say that V is a 2F-module for G if $C_G(V)$ is an ℓ'-subgroup of $Z(F^*(G))$ and if there exists a non-trivial elementary abelian ℓ-subgroup A of G satisfying

$$|V| \leq f(A)\,; \tag{1}$$

the group A is then called an offender.

This situation turns out to be important in several contexts. The results in the first two papers of this series have been used by Aschbacher and Smith [1], [2] in the classification of quasithin simple groups. This condition also comes up in Meierfrankenfeld's program of classifying certain simple groups by alternative means. Note that groups containing transvections and bi-transvections (i.e. unipotent elements trivial on a subspace of codimension 2) on V give examples of 2F-modules over the prime field. Also subgroups of classical groups which contain root subgroups of classical groups give examples of 2F-modules. Thus, our results include some special cases of results of Kantor [8] and the first author and Saxl [6]. We refer the reader to [4] and [5] for further motivation and history for considering this set-up and for other references.

In this paper we first consider the case where G is itself quasi-simple. In [4] and [5] the first and third authors treated this situation, and completely classified all 2F-modules with the exception of a small number of open cases. For groups of Lie type acting in the natural characteristic, the situation covered in [5], the unresolved cases are as follows:

(1) $G = E_7(q)$ and $\dim V = 56$ over $\mathbb{F}_q$;
(2) $G = {}^2E_6(q)$ and $\dim V = 27$ over $\mathbb{F}_{q^2}$;
(3) $G = F_4(q)$, $\ell \geq 3$ and $\dim V = 26 - \delta_{3,\ell}$ over $\mathbb{F}_q$.

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(Note that in [5] it was shown that if $G = F_4(q)$ with q even, then the 26-dimensional irreducible module is a 2F-module.) We shall show that if q is a power of 3 and V is the 25-dimensional module for $F_4(q)$, then V is a 2F-module; to do this we shall produce an explicit offender. We shall also show that the rest of these modules are not 2F-modules: here we use the idea of Mal'cev, who computed the maximal ranks of elementary abelian unipotent subgroups of the finite Chevalley groups, by converting the problem into one about certain sets of roots.

It turns out that a variation of the methods considered in the Lie case can resolve the open cases from [4], of Co_1 and Co_2 acting on their smallest-dimensional modules over $\mathbb{F}_2$; these have been considered by the second author in [10], where it is shown that neither gives rise to a 2F-module. Thus the classification of 2F-modules for quasi-simple groups is complete.

Next we turn to the classification of 2F-modules V for groups G such that $F^*(G)$ is quasi-simple. This more general situation was treated in [4], where complete results were obtained for the cases considered; it was found that all composition factors of the restriction $V|_{F^*(G)}$ are 2F-modules for $F^*(G)$. In [5] partial results were obtained showing that if this conclusion does not hold, then $F^*(G)$ must be an orthogonal group in even dimension. The results here allow us to dispose of this alternative.

Theorem 1. *Let G be a finite group such that $F^*(G)$ is quasi-simple, and V be an absolutely irreducible 2F-module for G; then all composition factors of the restriction $V|_{F^*(G)}$ are 2F-modules for $F^*(G)$.*

Theorem 2. *Let G be a finite group with $F^*(G)$ quasi-simple. If V is an absolutely irreducible 2F-module for G over k with an offender A such that G is the normal closure of A, then V is given in Tables 1-6. Moreover, any V given in the tables is a 2F-module.*

We give in Tables 2 to 6 all instances of such groups G and modules V where $F^*(G)$ is a group of Lie type and the module is in the defining characteristic. For the convenience of the reader we collect together from [4] all other instances of such groups G and modules V in Table 1. Note that in these tables we write $d = \dim V$ for the dimension over the smallest field of definition $\mathbb{F}_{\ell^f}$, and indicate the size of some offender A; in Tables 2 to 5 we denote the type of automorphism applied as inner, field, graph or graph-field.

For the next result we recall that if V is a 2F-module for the group G, an offender A is called cubic if

$$[[[V, A], A], A] = 0.$$

In [4] it was observed that each absolutely irreducible 2F-module in the cases treated there had a cubic offender. We shall complete the proof that this is true for all groups G considered here. We remark that in the tables we always assume that G is the normal closure of an offender A. If G is quasi-simple this is not an issue, but it is easy to see that if G is not perfect, there are many situations where no A not contained in $F^*(G)$ can act cubically. In particular, there may be no cubic offenders whose normal closure generates G. However, if we do not insist on this generation property, we do have:

Table 1. 2F-modules for groups other than of Lie type in defining characteristic

G	d	f	ℓ	conditions	$\log_\ell \lvert A\rvert$
$\mathfrak{A}_n$	$n-2$	1	2 or 3	$\ell \mid n$	1
$\mathfrak{S}_n$	$n-2$	1	2	$2 \mid n$	1
$\mathfrak{A}_n$	$n-1$	1	2 or 3	$\ell \nmid n$	1
$\mathfrak{S}_n$	$n-1$	1	2	$2 \nmid n$	1
$3.\mathfrak{A}_6$	3	2	2		1
$3.\mathfrak{S}_6$	6	1	2		2
$\mathfrak{A}_7$	4	1	2		1
$\mathfrak{S}_7$	8	1	2		3
$\mathfrak{A}_9$	8	1	2		3
$2.\mathfrak{A}_5$	2	2	3		1
$2.\mathfrak{A}_9$	8	1	3		3
$\mathrm{U}_3(3)$	6	1	2		1
$\mathrm{U}_3(3).2$	6	1	2		2
$3_1.\mathrm{U}_4(3)$	6	2	2		3
$3_1.\mathrm{U}_4(3).2_2$	6	2	2		1
$3_1.\mathrm{U}_4(3).(2^2)_{122}$	12	1	2		5
$2.\mathrm{L}_3(4)$	6	1	3		2
$\mathrm{S}_6(2)$	7	1	3		1
$2.\mathrm{S}_6(2)$	8	1	3		3
$2.\mathrm{O}_8^+(2)$	8	1	3		1
M_{12}	10	1	2		3
$M_{12}.2$	10	1	2		4
M_{22}	10	1	2		3
$M_{22}.2$	10	1	2		3
$3.M_{22}$	6	2	2		3
J_2	6	2	2		4
M_{23}	11	1	2		3
M_{24}	11	1	2		3
M_{11}	5	1	3		2
$2.M_{12}$	6	1	3		2

Theorem 3. *Let G be a finite group with $F^*(G)$ quasi-simple; then any absolutely irreducible 2F-module for G has a cubic offender.*

This seems rather curious and suggests that perhaps there is a proof along the lines of a general version of Thompson's replacement lemma which guarantees a quadratic offender for F-modules (see the next paragraph).

The last main result is the classification of F-modules for $F^*(G)$ quasi-simple. Recall that F-modules are defined analogously to 2F-modules, where the function f defined above is replaced by f_1 defined by

$$f_1(A) := f_1^V(A) := |A| \cdot |C_V(A)|.$$

The classification with G quasi-simple was completed many years ago. See [5] for a proof – we reproduce the results in Table 7. We classify those absolutely irreducible F-modules with $F^*(G)$ quasi-simple and an offender A whose conjugates generate $G > F^*(G)$.

More specifically we have:

Theorem 4. *Let G be a finite group with $F^*(G)$ quasi-simple. Let V be an absolutely irreducible F-module for G defined over a finite field of characteristic ℓ such that G is generated by the conjugates of an offender. If $G \neq F^*(G)$ then (G, V) are as given in Table 8. Thus, in particular, if ℓ is odd then $G = \langle F^*(G), \gamma\rangle$ where γ is a field automorphism of order ℓ and V is a natural module for $F^*(G)$.*

The arrangement of the remainder of this paper is as follows. In sections 2-4 we consider the open cases mentioned above for quasi-simple groups G, showing that the 25-dimensional module for $F_4(q)$ in characteristic 3 is a 2F-module but the remainder are not. In sections 5-10 we treat groups G with $F^*(G)$ quasi-simple of Lie type acting on modules in the defining characteristic; we determine all possibilities for 2F-modules in Theorems 6.1, 7.1, 8.3, 9.1, 9.2, 9.3 and 10.1, and obtain Theorems 1 and 2 as consequences. In section 11 we prove Theorem 3 on the existence of cubic offenders. Finally in section 12 we prove Theorem 4 on F-modules.

2. Abelian sets of roots

We begin with the open cases remaining from [5]; as stated above, we shall use a technique of Mal'cev to reduce the situation to a question about sets of roots in a certain root system. We shall deal with the reduction itself in the succeeding sections, as the details will vary a little from case to case, but in this section we explain the basic strategy to be followed in considering the sets of roots which arise.

Let Φ be a root system, with positive system Φ^+; in the cases which arise here, Φ will be of type E_7 or E_8. We write roots as linear combinations of simple roots; thus for example if Φ is of type E_7 we shall write ${\substack{234321\\2}}$ for the highest root. We shall often use dots to denote undetermined coefficients, our convention being that they may be replaced by any integers which yield a root; thus for example if Φ is of type E_8, the statement "${\substack{1\cdot21000\\1}} \in X$" means that the set X contains both roots ${\substack{1121000\\1}}$ and ${\substack{1221000\\1}}$. Moreover, we shall on occasion write equations of the form

$$\substack{0\cdot11000\\ \cdot} + \substack{0\cdot10000\\ \cdot} = \substack{0121000\\1};$$

this means that the roots ${\substack{0\cdot11000\\ \cdot}}$ may be matched with the roots ${\substack{0\cdot10000\\ \cdot}}$ in such a way that the sum in every case is ${\substack{0121000\\1}}$.

Let Ψ be a subsystem of Φ and Ω a subset of $\Phi^+ \setminus \Psi^+$. In the application to the action of the group G on the module V, the sets will be chosen so that the roots in Ψ and Ω will arise from G and V respectively.

Recall that a subset X of Ψ^+ is called abelian if $(X + X) \cap \Psi = \emptyset$, where we write $S_1 + S_2 = \{s_1 + s_2 : s_i \in S_i\}$. Here it will be convenient to consider sets which may contain negative roots; as the natural generalization, we shall call a subset X of Ψ abelian if $(X + X) \cap \Psi_0 = \emptyset$, where we set $\Psi_0 = \Psi \cup \{0\}$.

Using Mal'cev's technique, we shall see that an elementary abelian ℓ-subgroup A of G gives a subset X of Ψ, while its fixed point space $C_V(A)$ gives a subset Y of Ω. The conditions which X and Y must then satisfy are

$$(X+X)\cap\Psi_0=(X+Y)\cap\Omega=\emptyset. \tag{2}$$

Set $x=|X|$ and $y=|Y|$; then A is non-trivial if and only if $x>0$, and the condition for a non-trivial A to be an offender translates into an inequality of the form $2x+y\geq m$, where the value of m is closely related to $\dim V$. We therefore assume that

$$X \text{ and } Y \text{ satisfy condition (2)}, \quad x>0 \quad \text{and} \quad 2x+y\geq m,$$

and work to obtain a contradiction.

We shall say that a root $\alpha\in\Psi$ *excludes* the root $\beta\in\Psi\cup\Omega$ if $\alpha+\beta$ lies in Ψ_0 (for $\beta\in\Psi$) or Ω (for $\beta\in\Omega$). Observe that Y may be assumed to contain all roots of Ω not excluded by those in X; indeed, our strategy will be to begin with $X=\emptyset$, $Y=\Omega$ and build up X successively, at each stage reducing Y by removing the roots in Ω which have been excluded. We shall call a root in Ψ or Ω *available* at a given stage if it is not one of the roots chosen by that stage and has not been excluded either by the roots chosen or by an explicit argument.

There are two principles which will be used several times to restrict the sets which must be considered; both concern a root $\alpha\in\Psi$ which is available at a certain stage. Suppose that α excludes either (i) no available roots from Ψ and at most two from Ω, or (ii) just one available root α' from Ψ and none from Ω. Any set X which may be obtained from this point may or may not contain α; but given any such set which does not, inserting α (and, in (ii), removing α' if it is present) cannot decrease the value of $2x+y$. Thus, among the possible sets X, the maximal value of $2x+y$ occurs at one which contains α; so we may assume $\alpha\in X$ (and thus $\alpha'\notin X$ in (ii)). We shall call these the *first and second insertion principles*.

3. The smallest module for $F_4(q)$

We begin by treating the action of $F_4(q)$ on its module over $\mathbb{F}_q$ of dimension $26-\delta_{3,\ell}$, where ℓ is an odd prime and q is a power of ℓ. We shall view $F_4(q)$ as lying inside the E_6 parabolic subgroup of $E_7(q)$.

Let Φ be a root system of type E_7, with simple roots $\alpha_1,\ldots,\alpha_7$ numbered in the usual fashion. Let $\Psi=\{\sum_{i=1}^{7}n_i\alpha_i\in\Phi: n_7=0\}=\{\substack{\cdot\cdot\cdot\cdot\cdot 0\\ \cdot}\}$, so that Ψ is a root system of type E_6; set $\Omega=\{\substack{\cdot\cdot\cdot\cdot\cdot 1\\ \cdot}\}$, so that we have $\Phi^+=\Psi^+\cup\Omega$. Let τ be the automorphism of Ψ given by the symmetry of the Dynkin diagram, and set $\Psi_l=\{\alpha\in\Psi:\tau(\alpha)=\alpha\}$, $\Psi_s=\Psi\setminus\Psi_l$; then the τ-orbits in Ψ_s are pairs $\{\alpha,\tau(\alpha)\}$. Moreover, the τ-orbits in Ψ may be seen as forming a system Σ of type F_4, with long roots the elements of Ψ_l and short roots the τ-orbits in Ψ_s; we take

$$\beta_1=\substack{000000\\1},\quad \beta_2=\substack{001000\\0},\quad \beta_3=\left\{\substack{010000\\0},\substack{000100\\0}\right\},\quad \beta_4=\left\{\substack{100000\\0},\substack{000010\\0}\right\}$$

as the simple roots of Σ, and write $\sum_{i=1}^{4}n_i\beta_i$ as $n_1n_2n_3n_4$.

We have $F_4(q)<E_6(q)<E_7(q)$, where each short root subgroup of $F_4(q)$ is diagonally embedded in the product of two root subgroups of $E_6(q)$ whose roots are

interchanged by τ. Let Q be the unipotent radical of the E_6 parabolic of $E_7(q)$, so that Q is the product of the root subgroups of $E_7(q)$ whose roots lie in Ω; we may view Q as an irreducible 27-dimensional $E_6(q)$-module. The centralizer C of $F_4(q)$ in Q is 1-dimensional, diagonally embedded in the product of the root subgroups corresponding to the roots

$$\substack{122111\\1}, \quad \substack{112211\\1}, \quad \substack{012221\\1}.$$

If $\ell > 3$ then C is a direct summand of the $F_4(q)$-module Q, whose complement is the irreducible 26-dimensional module $V(\lambda_1)$ (in standard notation). On the other hand if $\ell = 3$ then the $F_4(q)$-module Q is indecomposable and self-dual, with socle C; it contains a 26-dimensional submodule, whose quotient by C is the irreducible module $V(\lambda_1)$.

First assume $\ell = 3$; let V be the 26-dimensional submodule contained in Q. Let A be the subgroup of $F_4(q)$ which is the product of the root subgroups corresponding to the 8 roots

$$2342, \quad 1342, \quad 1242, \quad 1232, \quad 1231, \quad 1222, \quad 1221, \quad 1220,$$

so that $|A| = q^8$; the corresponding roots in E_7 (meaning those lying in the relevant τ-orbits) are

$$\substack{123210\\\cdot}, \quad \substack{\cdot\cdot 2\cdot\cdot 0\\1}.$$

Then $C_V(A)$ contains both C and the 9 root subgroups corresponding to the roots

$$\substack{\cdot\cdot 4321\\2}, \quad \substack{123\cdot\cdot 1\\\cdot},$$

and thus $|C_V(A)| \geq q^{10}$. Therefore $f^V(A) \geq q^{26}$, and so V is a 2F-module; taking the quotient by C shows the following.

Theorem 3.1. *$V(\lambda_1)$ is a 2F-module for $F_4(q)$ in characteristic* 3.

For the remainder of this section we assume $\ell > 3$. Let A be a non-trivial elementary abelian ℓ-subgroup of $F_4(q)$; we may assume that A lies in the product U of the positive root subgroups. We follow the procedure of Mal'cev as expounded in [3, 3.3]: we let h be a linear functional on $\mathbb{Z}\Sigma$ taking positive $\mathbb{Q}$-linearly independent values on the simple roots β_i, and use it to determine an ordering on the roots by setting $\alpha < \beta$ if and only if $h(\alpha) < h(\beta)$. The abelian subgroup A then gives rise to an abelian Lie subalgebra $I(A)$ of the Lie algebra $I(U)$, with corresponding abelian set of roots $S_A \subset \Sigma$, and we have $|A| \leq |I(A)| = q^{|S_A|}$.

Writing W for the Weyl group of $F_4(q)$, we have the following lemma.

Lemma 3.2. *Any abelian set of short roots in Σ is the image under an element of W of one of the following:*

$$\emptyset, \quad \{1232\}, \quad \{123\cdot\}, \quad \{123\cdot, 1221\}.$$

Proof. Let S be an abelian set of short roots in Σ. If $S \neq \emptyset$, the transitivity of W on short roots means that we may assume $1232 \in S$; this excludes all negative short roots, together with $\cdot\cdot 10$. Similarly, if $|S| > 1$, the transitivity of $\mathrm{stab}_W(1232)$ on the roots $\cdot\cdot\cdot 1$ means that we may assume $1231 \in S$; this excludes 0001, $\cdot\cdot 11$. Finally, if $|S| > 2$, the transitivity of $\mathrm{stab}_W(1232, 1231)$ on the roots $\cdot\cdot 21$ means that we may assume $1221 \in S$; this excludes $\cdot 121$, and there are no further roots which could lie in S. $\square$

Observe that the stabilizers in W of these sets are

$$W, \quad \langle w_{\beta_1}, w_{\beta_2}, w_{\beta_3} \rangle, \quad \langle w_{\beta_1}, w_{\beta_2}, w_{\beta_4} \rangle, \quad \langle w_{\beta_1}, w_{\beta_3}, w_{\beta_4} \rangle$$

respectively.

We now view $F_4(q)$ as a subgroup of $E_6(q)$ and therefore of $E_7(q)$; we let $B = C_Q(A)$, so that AB is an elementary abelian ℓ-subgroup of $E_7(q)$. We use the linear functional h to obtain one on $\mathbb{Z}\Phi$ as follows. Write $a_i = h(\beta_i)$ for $i = 1, 2, 3, 4$; choose ϵ, ϵ' small and positive and $\omega > h(2342)$ such that $a_1, a_2, a_3, a_4, \epsilon, \epsilon', \omega$ are $\mathbb{Q}$-linearly independent, and set

$$\begin{aligned} &h'(\alpha_2) = a_1, \quad h'(\alpha_3) = a_3 - \epsilon, \quad h'(\alpha_1) = a_4 - \epsilon', \\ &h'(\alpha_4) = a_2, \quad h'(\alpha_5) = a_3 + \epsilon, \quad h'(\alpha_6) = a_4 + \epsilon', \quad h'(\alpha_7) = \omega. \end{aligned}$$

Again determine an ordering on Φ by setting $\alpha < \beta$ if and only if $h'(\alpha) < h'(\beta)$. The choice of ω means that all roots in Ψ are smaller than those in Ω; by taking ϵ and ϵ' sufficiently small we may ensure that the ordering on Ψ is compatible with that on Σ (and that roots interchanged by τ are adjacent). Note that since $\epsilon, \epsilon' > 0$ we have $\alpha < \tau(\alpha)$ for $\alpha \in \{ \begin{smallmatrix} 1221\cdot 0 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 112100 \\ 1 \end{smallmatrix} \}$; these three τ-orbits $\{\alpha, \tau(\alpha)\}$ are the short roots arising in Lemma 3.2.

Now the elementary abelian subgroup AB gives rise to an abelian set S_{AB} of roots in Φ; write $X = S_{AB} \cap \Psi$ and $Y = S_{AB} \cap \Omega$, and subdivide X into $X_l = X \cap \Psi_l$ and $X_s = X \cap \Psi_s$. The choice of h' means that the roots in X_l are just the long roots in S_A, while the roots in X_s all lie in different τ-orbits, and these τ-orbits are just the short roots in S_A. Set $x_l = |X_l|$, $x_s = |X_s|$, $x = x_l + x_s = |X|$ and $y = |Y|$.

By the above and Lemma 3.2, if we now allow the possibility that X contains negative roots we may assume that X_s is one of

$$\emptyset, \quad \{ \begin{smallmatrix} 122110 \\ 1 \end{smallmatrix} \}, \quad \{ \begin{smallmatrix} 1221\cdot 0 \\ 1 \end{smallmatrix} \}, \quad \{ \begin{smallmatrix} 1221\cdot 0 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 112100 \\ 1 \end{smallmatrix} \}.$$

Moreover, for each of these possibilities we may apply elements of the stabilizer in W of the corresponding set of short roots; these stabilizers are

$$\langle w_2, w_4, w_3 w_5, w_1 w_6 \rangle, \quad \langle w_2, w_4, w_3 w_5 \rangle, \quad \langle w_2, w_4, w_1 w_6 \rangle, \quad \langle w_2, w_3 w_5, w_1 w_6 \rangle$$

respectively, where we write w_i for w_{α_i}. We shall prove the following.

Theorem 3.3. *If X and Y are as above and $x > 0$, then $2x + y \leq 26$.*

We have $|A| \leq q^x$, $|B| \leq q^y$; thus Theorem 3.3 will give $|A|^2 |B| \leq q^{26}$, showing that Q is not a 2F-module. As $Q = V(\lambda_1) \oplus C_Q(F_4(q))$, an immediate consequence will be the following.

Theorem 3.4. *$V(\lambda_1)$ is not a 2F-module for $F_4(q)$ in characteristic greater than 3.*

As explained in the previous section, we shall prove Theorem 3.3 by contradiction; thus for the remainder of this section we shall consider the condition

(3) $\qquad X$ and Y are as above, $\quad x > 0 \quad$ and $\quad 2x + y \geq 27$.

Note that if we write z for the number of roots of Ω excluded by X, we have $z = 27 - y$; the inequality $2(x_l + x_s) + y \geq 27$ thus becomes $x_l \geq \frac{z}{2} - x_s$. Note moreover that $x_l \leq 6$ since the root system Ψ_l is of type D_4. Write $\rho = \begin{smallmatrix}234321\\2\end{smallmatrix}$.

We now work through the four possibilities for X_s.

Lemma 3.5. *If X and Y satisfy condition 3, then $X_s \neq \{\begin{smallmatrix}1221\cdot0\\1\end{smallmatrix}, \begin{smallmatrix}112100\\1\end{smallmatrix}\}$.*

Proof. Assume the converse. The roots in X_s exclude the 12 roots $\begin{smallmatrix}0\cdot\cdot\cdot\cdot1\\0\end{smallmatrix}$, $\begin{smallmatrix}0\cdot1111\\1\end{smallmatrix}$, $\begin{smallmatrix}\cdot12211\\1\end{smallmatrix}$, $\begin{smallmatrix}\cdot\cdot2221\\1\end{smallmatrix}$ from Ω, giving $x_l \geq \frac{12}{2} - 3$; since $x_l \leq 6$ we must have $z \leq 18$. The $\langle w_2, w_3w_5, w_1w_6\rangle$-orbits on the available roots in Ψ_l are

$$\{\begin{smallmatrix}123210\\\cdot\end{smallmatrix}\}, \quad \{\begin{smallmatrix}122210\\1\end{smallmatrix}, \begin{smallmatrix}112110\\1\end{smallmatrix}, \begin{smallmatrix}012100\\1\end{smallmatrix}\}, \quad \{\begin{smallmatrix}111110\\\cdot\end{smallmatrix}, \begin{smallmatrix}011100\\\cdot\end{smallmatrix}, \begin{smallmatrix}001000\\\cdot\end{smallmatrix}\} \quad \text{and} \quad \{\begin{smallmatrix}000000\\\cdot\end{smallmatrix}\}.$$

Suppose if possible that X_l were contained in the first two orbits; thus $x_l \leq 5$ and so $z \leq 16$. Since $x_l \geq 3$ we must have a root from the second orbit present, which we may take to be $\begin{smallmatrix}122210\\1\end{smallmatrix}$, which excludes $\begin{smallmatrix}\cdot12111\\1\end{smallmatrix}$ from Ω, giving $z \geq 14$ and so $x_l \geq 4$. Thus we must have another root from the second orbit present, which we may take to be $\begin{smallmatrix}112110\\1\end{smallmatrix}$, which excludes $\begin{smallmatrix}122211\\1\end{smallmatrix}$ from Ω, giving $z \geq 15$ and so $x_l \geq 5$. We therefore in fact must have $\begin{smallmatrix}123210\\\cdot\end{smallmatrix}, \begin{smallmatrix}012100\\1\end{smallmatrix} \in X_l$ as well; but these exclude $\begin{smallmatrix}111111\\\cdot\end{smallmatrix}$ from Ω so that $z = 17$, contrary to assumption.

Now suppose X_l meets the fourth orbit; we may assume $\begin{smallmatrix}000000\\1\end{smallmatrix} \in X_l$, which excludes $\begin{smallmatrix}001000\\0\end{smallmatrix}$, $\begin{smallmatrix}011100\\0\end{smallmatrix}$, $\begin{smallmatrix}111110\\0\end{smallmatrix}$, $\begin{smallmatrix}123210\\1\end{smallmatrix}$ from Ψ_l and $\begin{smallmatrix}111111\\0\end{smallmatrix}$, $\begin{smallmatrix}123\cdot\cdot1\\1\end{smallmatrix}$ from Ω, so $z \geq 16$ and $x_l \geq 5$. The remaining available roots in Ψ_l are $\begin{smallmatrix}001000\\1\end{smallmatrix}$, $\begin{smallmatrix}01\cdot100\\1\end{smallmatrix}$, $\begin{smallmatrix}11\cdot110\\1\end{smallmatrix}$, $\begin{smallmatrix}122210\\1\end{smallmatrix}$, $\begin{smallmatrix}123210\\2\end{smallmatrix}$; of these only $\begin{smallmatrix}123210\\2\end{smallmatrix}$ excludes no further roots from Ω, so that the value of z will increase and we must have $x_l = 6$, but of these seven roots there are three pairs summing to $\begin{smallmatrix}123210\\2\end{smallmatrix}$ so that X_l can contain at most four of them. Thus $\begin{smallmatrix}000000\\\cdot\end{smallmatrix} \notin X_l$.

It follows that X_l must meet the third orbit; we may assume $\begin{smallmatrix}111110\\1\end{smallmatrix} \in X_l$, which excludes $\begin{smallmatrix}001000\\0\end{smallmatrix}$, $\begin{smallmatrix}011100\\0\end{smallmatrix}$, $\begin{smallmatrix}012100\\1\end{smallmatrix}$ from Ψ_l and $\begin{smallmatrix}012111\\1\end{smallmatrix}$, $\begin{smallmatrix}123211\\1\end{smallmatrix}$ from Ω, giving $z \geq 14$ and $x_l \geq 4$. For each of the 7 available roots in Ψ_l there is a different root in Ω which it excludes: for each $\alpha \in \{\begin{smallmatrix}12\cdot210\\\cdot\end{smallmatrix}, \begin{smallmatrix}112110\\1\end{smallmatrix}, \begin{smallmatrix}111110\\0\end{smallmatrix}\}$ we may take $\rho - \alpha$, while $\begin{smallmatrix}011100\\1\end{smallmatrix}$ and $\begin{smallmatrix}001000\\1\end{smallmatrix}$ exclude $\begin{smallmatrix}123221\\1\end{smallmatrix}$ and $\begin{smallmatrix}123321\\1\end{smallmatrix}$ respectively. Since $z \leq 18$ we can exclude at most 4 more roots from Ω, and so may choose at most 4 more roots in X_l, giving $x_l \leq 5$; but we must choose at least 3 more, giving $z \geq 17$, forcing $x_l \geq 6$, contrary to assumption. $\square$

Lemma 3.6. *If X and Y satisfy condition 3, then $X_s \neq \{\begin{smallmatrix}1221\cdot0\\1\end{smallmatrix}\}$.*

Proof. Assume the converse. The roots in X_s exclude the 9 roots $\begin{smallmatrix}00\cdot\cdot\cdot1\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}\cdot122\cdot1\\1\end{smallmatrix}$ from Ω, giving $x_l \geq \frac{9}{2} - 2$, i.e., $x_l \geq 3$; since $x_l \leq 6$ we must have $z \leq 16$. The $\langle w_2, w_4, w_1w_6\rangle$-orbits on the available roots in Ψ_l are $\{\begin{smallmatrix}12\cdot210\\\cdot\end{smallmatrix}\}$, $\{\begin{smallmatrix}11\cdot110\\\cdot\end{smallmatrix}, \begin{smallmatrix}01\cdot100\\\cdot\end{smallmatrix}\}$ and $\{\begin{smallmatrix}00\cdot000\\\cdot\end{smallmatrix}\}$.

Suppose if possible that X_l meets the third orbit; then we may assume $\begin{smallmatrix}001000\\1\end{smallmatrix} \in X_l$, which excludes $\begin{smallmatrix}122210\\1\end{smallmatrix}$, $\begin{smallmatrix}111110\\0\end{smallmatrix}$, $\begin{smallmatrix}011100\\0\end{smallmatrix}$ together with the remaining negative roots from Ψ_l and $\begin{smallmatrix}\cdot11111\\0\end{smallmatrix}$, $\begin{smallmatrix}1222\cdot1\\1\end{smallmatrix}$, $\begin{smallmatrix}123321\\1\end{smallmatrix}$ from Ω, giving $z \geq 14$ and so $x_l \geq 5$. The remaining available roots in Ψ_l are $\begin{smallmatrix}000000\\1\end{smallmatrix}$, $\begin{smallmatrix}001000\\0\end{smallmatrix}$, $\begin{smallmatrix}01\cdot100\\1\end{smallmatrix}$, $\begin{smallmatrix}11\cdot110\\1\end{smallmatrix}$, $\begin{smallmatrix}123210\\\cdot\end{smallmatrix}$; of these only $\begin{smallmatrix}123210\\2\end{smallmatrix}$ excludes no further roots from Ω, so that the value of z will increase and we must have $x_l = 6$. However, we cannot choose $\begin{smallmatrix}001000\\0\end{smallmatrix}$ as it excludes the 3 roots $\begin{smallmatrix}\cdot11111\\1\end{smallmatrix}$, $\begin{smallmatrix}123321\\2\end{smallmatrix}$ from Ω, whereas z can increase by at most 2; and of the remaining seven roots there are three pairs summing to $\begin{smallmatrix}123210\\2\end{smallmatrix}$ so that X_l can contain at most four of them. Thus $\begin{smallmatrix}00\cdot000\\\cdot\end{smallmatrix} \notin X_l$.

As each of the available roots in Ψ_l excludes at least 2 further roots from Ω, the value of z will increase by at least 2 and so $x_l \geq 4$; so we must have some root from the second orbit, which we may take to be $\substack{112110\\1}$, which excludes $\substack{011100\\\cdot}$ from Ψ_l and $\substack{011111\\\cdot}$, $\substack{122211\\1}$ from Ω, giving $z \geq 12$ and so $x_l \geq 4$. For each of the 6 available roots in Ψ_l there is a different root in Ω which it excludes: for each $\alpha \in \{\substack{12\cdot210\\\cdot}, \substack{111110\\\cdot}\}$ we may take $\rho - \alpha$, while $\substack{012100\\1}$ excludes $\substack{122221\\1}$. Since $z \leq 16$ we can exclude at most 4 more roots from Ω, and so may choose at most 4 more roots in X_l, giving $x_l \leq 5$; but we must choose at least 3 more, giving $z \geq 15$, forcing $x_l \geq 6$, contrary to assumption. $\square$

Lemma 3.7. *If X and Y satisfy condition 3, then $X_s \neq \{\substack{122110\\1}\}$.*

Proof. Assume the converse. The root in X_s excludes the 6 roots $\substack{000001\\0}$, $\substack{00\cdot111\\\cdot}$, $\substack{\cdot12211\\1}$ from Ω, giving $x_l \geq \frac{6}{2} - 1 = 2$; since $x_l \leq 6$ we must have $z \leq 14$. The $\langle w_2, w_4, w_3w_5 \rangle$-orbits on the available roots in Ψ_l are $\{\substack{12\cdot210\\\cdot}, \substack{11\cdot110\\\cdot}\}$ and $\{\pm\substack{01\cdot100\\\cdot}, \substack{00\cdot000\\\cdot}\}$. As each of these roots excludes at least 3 roots from Ω, we must have $x_l \geq 4$.

Suppose if possible that X_l were contained in the first orbit; using w_3w_5 we may assume that X_l contains at least two of $\substack{12\cdot210\\\cdot}$, and using w_2 and w_4 we may assume $\substack{123210\\\cdot} \in X_l$, which excludes $\substack{000011\\0}$, $\substack{\cdot11111\\\cdot}$ from Ω, so $z \geq 11$ and $x_l \geq 5$. We may therefore further assume $\substack{122210\\1}$, $\substack{11\cdot110\\1} \in X_l$, which excludes $\substack{\cdot12111\\1}$, $\substack{12\cdot211\\1}$ from Ω, giving $z \geq 15$, contrary to assumption.

It follows that X_l must meet the second orbit; we may assume $\substack{012100\\1} \in X_l$, which excludes $-\substack{011100\\\cdot}, -\substack{001000\\\cdot}, \substack{111110\\\cdot}$ from Ψ_l and $\substack{000011\\0}$, $\substack{111111\\\cdot}$, $\substack{1\cdot2221\\1}$ from Ω, giving $z \geq 11$ and so $x_l \geq 5$. The root $\substack{000000\\1}$ excludes the 4 roots $\substack{011111\\0}$, $\substack{123\cdot\cdot1\\1}$ from Ω, and likewise $-\substack{000000\\1}$ excludes the 4 roots $\substack{011111\\1}$, $\substack{123\cdot\cdot1\\2}$ from Ω; since z can increase by at most 3, neither of these can be chosen. For each of the other 8 available roots in Ψ_l there is a different root in Ω which it excludes: $\substack{001000\\\cdot}$ excludes $\substack{123321\\\cdot}$; $\substack{011100\\\cdot}$ excludes $\substack{123221\\\cdot}$; $\substack{112110\\1}$ excludes $\substack{122211\\1}$; $\substack{12\cdot210\\\cdot}$ excludes $\substack{01\cdot111\\\cdot}$. Since $z \leq 14$ we can exclude at most 3 more roots from Ω, and so may choose at most 3 more roots in X_l, giving $x_l \leq 4$, contrary to assumption. $\square$

Lemma 3.8. *If X and Y satisfy condition 3, then $X_s \neq \emptyset$.*

Proof. Assume the converse, so that $X = X_l$ and $x_l \geq \frac{z}{2}$; since $x_l \leq 6$ we must have $z \leq 12$. Using W we may assume that $\substack{123210\\2} \in X_l$, which excludes the 6 roots $\substack{\cdot\cdot\cdot\cdot\cdot1\\0}$ from Ω, giving $x_l \geq \frac{6}{2} = 3$, as well as the negative roots $-\substack{123210\\2}, -\substack{12\cdot210\\1}, -\substack{11\cdot110\\1}, -\substack{01\cdot100\\1}, -\substack{00\cdot000\\1}$ from Ψ_l.

Now suppose if possible that X_l contains no pair of orthogonal roots. Using $\langle w_4, w_3w_5, w_1w_6 \rangle$ we may assume $\substack{123210\\1} \in X_l$, which excludes the 3 roots $\substack{\cdot\cdot1111\\1}$ from Ω, giving $x_l \geq \frac{9}{2}$; by assumption the only available roots in Ψ_l are then $\substack{122210\\1}$, $\substack{112110\\1}$, $\substack{012100\\1}$, so all three must lie in X_l and we must have $x_l = 5$. However, between them these exclude the 6 roots $\substack{\cdot12111\\1}$, $\substack{012211\\1}$, $\substack{122211\\1}$, $\substack{1\cdot2221\\1}$ from Ω, giving $z \geq 15$, contrary to assumption.

Thus X_l must contain a pair of orthogonal roots; using $\langle w_4, w_3w_5, w_1w_6 \rangle$ we may therefore assume $\substack{111110\\0} \in X_l$, which excludes the 5 roots $\substack{0\cdot\cdot\cdot11\\1}$, $\substack{123211\\2}$ from Ω, giving $x_l \geq \frac{11}{2}$, as well as the roots $-\substack{111110\\0}$, $\substack{00\cdot000\\1}$, $\substack{01\cdot100\\1}$ from Ψ_l. Since $z \leq 12$, at most one further root may be excluded from Ω. The $\langle w_4, w_3w_5 \rangle$-orbits on the remaining available roots in Ψ_l are $\{\substack{12\cdot210\\1}, \substack{11\cdot110\\1}\}$ and $\{\pm\substack{011100\\0}, \pm\substack{001000\\0}\}$. However, each root in the second orbit excludes a further 4 roots from Ω, and so cannot be included in X_l; each root in the first orbit excludes just one further root from Ω, but the roots excluded are all different. $\square$

By combining Lemmas 3.5, 3.6, 3.7 and 3.8, we have therefore proved Theorem 3.3 and so also Theorem 3.4.

4. The 56-dimensional module for $E_7(q)$

We next consider the action of $E_7(q)$ on its 56-dimensional module over $\mathbb{F}_q$; here we shall view $E_7(q)$ as lying inside the E_7 parabolic subgroup of $E_8(q)$.

Let Φ be a root system of type E_8, with simple roots $\alpha_1, \ldots, \alpha_8$ numbered in the usual fashion; we shall write the root $\sum_{i=1}^{8} n_i\alpha_i$ as $\substack{n_1n_3n_4n_5n_6n_7n_8\\n_2}$. Let

$$\Psi = \{\sum_{i=1}^{8} n_i\alpha_i \in \Phi : n_8 = 0\} = \{\substack{\cdots\cdots0\\\cdot}\},$$

so that Ψ is a root system of type E_7; set $\Omega = \{\substack{\cdots\cdots1\\\cdot}\}$, and denote the highest root $\substack{2465432\\3}$ of Φ by δ, so that we have $\Phi^+ = \Psi^+ \cup \Omega \cup \{\delta\}$.

Let Q be the unipotent radical of the E_7 parabolic subgroup of $E_8(q)$, so that Q is the product of the root subgroups of $E_8(q)$ whose roots lie in $\Omega \cup \{\delta\}$. Let Q' be the root subgroup with root δ, and set $V = Q/Q'$; then V is an irreducible $E_7(q)$-module of dimension 56.

Let A be a non-trivial unipotent elementary abelian subgroup of $E_7(q)$; we may assume that A lies in the product U of the positive root subgroups of $E_8(q)$. We then obtain an abelian Lie subalgebra $I(A)$ of the Lie algebra $I(U)$, and hence an abelian set of roots X. Let $B = C_A(Q)$; let $I(B)$ be its Lie algebra and the corresponding set of roots (excluding δ) be Y. Then $I(AB) = I(A) + I(B)$ is a Lie subalgebra of $I(U)$ that is abelian modulo $I(Q')$, so as in Section 2 we have $(X + X) \cap \Psi_0 = (X + Y) \cap \Omega = \emptyset$, where we allow X to contain negative roots so as to permit the application of the Weyl group W of $E_7(q)$. With $x = |X|$ and $y = |Y|$, we shall prove the following.

Theorem 4.1. *If X and Y are as above and $x > 0$, then $2x + y \leq 55$.*

We have $|A| \leq |I(A)| = q^x$ and $|B| \leq |I(B)| = q^{y+1}$, so that $f(A) = |A|^2|B/Q'| \leq q^{2x+y} \leq q^{55} < |V|$; so we shall have shown the following.

Theorem 4.2. *$V(\lambda_1)$ is not a 2F-module for $E_7(q)$.*

As an immediate consequence we obtain the following.

Corollary 4.3. *The 54-dimensional module for ${}^2E_6(q)$ (defined over $\mathbb{F}_q$) is not a 2F-module.*

Proof. Let V be the module concerned; then $V^+ = V \oplus U$, where V^+ is the 56-dimensional module for $E_7(q)$ and U consists of the fixed points of G (cf. [7]). Thus, if A is an elementary abelian unipotent subgroup of ${}^2E_6(q)$, we have $f^V(A)q^2 = f^{V^+}(A) < |V^+|$, so $f^V(A) < |V|$. $\square$

As in the previous section, we shall prove Theorem 4.1 by contradiction; thus for the remainder of this section we shall consider the condition

$$\text{(4)} \qquad X \text{ and } Y \text{ are as above}, \quad x > 0 \quad \text{and} \quad 2x + y \geq 56.$$

Note that if we write z for the number of roots of Ω excluded by X, we have $z = 56 - y$; the inequality $2x + y \geq 56$ thus becomes $x \geq \frac{z}{2}$.

Since $\begin{smallmatrix}2343210\\2\end{smallmatrix}$ may be added to each of the 12 roots $\begin{smallmatrix}0\cdot\cdot\cdot\cdot\cdot1\\\cdot\end{smallmatrix}$ but to no others in Ω, the transitivity of W on Ψ shows that any root in X excludes 12 roots of Ω. We shall write $X = \{\xi_1, \xi_2, \dots\}$. As explained in Section 2, our strategy will be to begin with $X = \emptyset$, $Y = \Omega$ and build up X successively, at each stage reducing Y by removing the roots in Ω which have been excluded; at various stages we shall employ the first and second insertion principles.

We begin with a straightforward result.

Lemma 4.4. *If X and Y satisfy condition 4, then X contains a pair of orthogonal roots.*

Proof. Assume that X does not contain two orthogonal roots. As $X \neq \emptyset$, using W we may assume $\xi_1 = \begin{smallmatrix}2343210\\2\end{smallmatrix}$; thus $z \geq 12$ and so $|X| \geq 6$. The root ξ_1 excludes the negative roots $-\begin{smallmatrix}2343210\\2\end{smallmatrix}$ and $-\begin{smallmatrix}1\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot\end{smallmatrix}$, and by assumption no root $\begin{smallmatrix}0\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot\end{smallmatrix}$ lies in X, so the remaining roots in X must be of the form $\begin{smallmatrix}1\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot\end{smallmatrix}$; as the stabilizer in W of ξ_1 is transitive on these, we may assume $\xi_2 = \begin{smallmatrix}1343210\\2\end{smallmatrix}$. This excludes the 7 roots $\begin{smallmatrix}11\cdot\cdot\cdot\cdot1\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}0011111\\1\end{smallmatrix}$ from Ω, giving $z \geq 19$ and so $|X| \geq 10$; it also excludes from Ψ the root $\begin{smallmatrix}1000000\\0\end{smallmatrix}$, and by assumption no root $\begin{smallmatrix}11\cdot\cdot\cdot\cdot0\\\cdot\end{smallmatrix}$ can now lie in X, so we must have $X \subseteq \{\xi_1, \xi_2, \begin{smallmatrix}12\cdot\cdot\cdot\cdot0\\\cdot\end{smallmatrix}\}$. Again using transitivity we may assume $\xi_3 = \begin{smallmatrix}1243210\\2\end{smallmatrix}$, and arguing similarly we have $X \subseteq \{\xi_1, \xi_2, \xi_3, \begin{smallmatrix}123\cdot\cdot\cdot0\\\cdot\end{smallmatrix}\}$; then we may assume $\xi_4 = \begin{smallmatrix}1233210\\2\end{smallmatrix}$, and we obtain $X \subseteq \{\xi_1, \xi_2, \xi_3, \xi_4, \begin{smallmatrix}1232\cdot\cdot0\\2\end{smallmatrix}, \begin{smallmatrix}1233210\\1\end{smallmatrix}\}$. However, this contradicts $|X| \geq 10$; so X must in fact contain a pair of orthogonal roots. $\square$

By Lemma 4.4 and the transitivity of $W(D_6)$ on the roots $\begin{smallmatrix}0\cdot\cdot\cdot\cdot\cdot0\\\cdot\end{smallmatrix}$, we may henceforth assume

$$\xi_1 = \begin{smallmatrix}2343210\\2\end{smallmatrix}, \qquad \xi_2 = \begin{smallmatrix}0122210\\1\end{smallmatrix};$$

between them these exclude the 22 roots $\begin{smallmatrix}0\cdot\cdot\cdot\cdot\cdot1\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}1\cdot\cdot\cdot111\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}23432\cdot1\\2\end{smallmatrix}$ from Ω, so that $|X| \geq 11$. The stabilizer in W of $\{\xi_1, \xi_2\}$ has four orbits on the remaining available roots in Ψ:

$$O_1 = \{\begin{smallmatrix}1\cdot\cdot\cdot210\\\cdot\end{smallmatrix}\}, \quad O_2 = \{\begin{smallmatrix}\cdot\cdot\cdot\cdot1\cdot0\\\cdot\end{smallmatrix}\}, \quad O_3 = \{\begin{smallmatrix}0\cdot\cdot\cdot000\\\cdot\end{smallmatrix}\}, \quad O_4 = \{\begin{smallmatrix}00000\cdot0\\0\end{smallmatrix}\},$$

of sizes 8, 32, 24 and 2 respectively. The roots in O_1 are orthogonal to neither ξ_1 nor ξ_2, and each excludes 2 available roots from Ω; those in O_2 are orthogonal to exactly one of ξ_1 and ξ_2, and each excludes 5 available roots from Ω; those in the remaining two orbits are orthogonal to both ξ_1 and ξ_2, but those in O_3 form a D_4 subsystem with each excluding 8 available roots from Ω, while those in O_4 form an A_1 subsystem orthogonal to the D_4 with each excluding 9 available roots from Ω.

Our proof will consist of three propositions, which successively limit the possibilities for the orbits O_i in which the remaining roots in X may lie.

Proposition 4.5. *If X and Y satisfy condition 4, then X does not contain three mutually orthogonal roots which are orthogonal to a D_4 subsystem in Ψ.*

Proof. Assume the result false; then we may take $\xi_3 = \substack{0000010\\0}$, which excludes the 9 roots $\substack{1\cdot\cdot\cdot211\\\cdot}, \substack{2465421\\3}$ from Ω, giving $z \geq 31$ and so $|X| \geq 16$, as well as the 17 roots $\substack{\cdot\cdot\cdot\cdot100\\\cdot}, -\substack{0000010\\0}$ from Ψ. Of the 48 available roots in Ψ, the 24 roots $\substack{1\cdot\cdot\cdot210\\\cdot}, \substack{\cdot\cdot\cdot\cdot110\\\cdot}$ each exclude just one available root from Ω (different in each case) and no two may be added, while the 24 roots $\substack{0\cdot\cdot\cdot000\\\cdot}$ each exclude 6 available roots from Ω. Thus if X contained no root $\substack{0\cdot\cdot\cdot000\\\cdot}$, the value of $2x+y$ would be maximized by including each of the 24 roots $\substack{1\cdot\cdot\cdot210\\\cdot}, \substack{\cdot\cdot\cdot\cdot110\\\cdot}$, giving $X = \{\substack{\cdot\cdot\cdot\cdot\cdot10\\\cdot}\}$, $Y = \{\substack{2465431\\3}\}$; but then $2x + y = 2.27 + 1 = 55$, contrary to assumption. So X must contain some root $\substack{0\cdot\cdot\cdot000\\\cdot}$; using $W(D_4)$ we may assume $\xi_4 = \substack{0121000\\1}$, which excludes the 6 roots $\substack{1\cdot22221\\1}, \substack{1233321\\\cdot}, \substack{234\cdot321\\2}$ from Ω, giving $z \geq 37$ and so $|X| \geq 19$, as well as the 15 roots $-\substack{0121000\\1}, -\substack{0\cdot1\cdot000\\\cdot}, \substack{000\cdot110\\0}, \substack{1111110\\\cdot}, \substack{1\cdot22210\\1}$ from Ψ. By the second insertion principle we may now assume

$$\xi_5 = \substack{0121110\\1},\ \xi_6 = \substack{0122110\\1},\ \xi_7 = \substack{1232110\\1},\ \xi_8 = \substack{1232110\\2},\ \xi_9 = \substack{1243210\\2},\ \xi_{10} = \substack{1343210\\2},$$

so that $\pm\substack{0000000\\1}, \pm\substack{0100000\\0}, \pm\substack{0001000\\0} \notin X$.

Now of the 20 available roots in Ψ, the 12 roots $\substack{0\cdot11110\\\cdot}, \substack{1\cdot2\cdot110\\1}, \substack{123\cdot210\\\cdot}$ again each exclude just one available root from Ω (different in each case) and no two may be added, while the 8 roots $\substack{0\cdot1\cdot000\\\cdot}$ each exclude 3 available roots from Ω. As before, if X contained no root $\substack{0\cdot1\cdot000\\\cdot}$, the value of $2x+y$ would be maximized by including each of the former, giving $X = \{\substack{0000010\\0}, \substack{0\cdot11110\\\cdot}, \substack{0121000\\1}, \substack{012\cdot\cdot10\\1}, \substack{1\cdot2\cdot110\\1}, \substack{123\cdot\cdot10\\1}, \substack{\cdot\cdot\cdot\cdot\cdot10\\2}\}$, $Y = \{\substack{1\cdot4\cdot321\\2}, \substack{1354321\\\cdot}, \substack{246\cdot321\\3}, \substack{2465431\\3}\}$; but then $2x+y = 2.22+7 = 51$, contrary to assumption. So X must contain some root $\substack{0\cdot1\cdot000\\\cdot}$; using $\langle w_2, w_3, w_5\rangle$ we may assume $\xi_{11} = \substack{0111000\\1}$, which excludes the 3 roots $\substack{1232221\\1}, \substack{1243321\\2}, \substack{2354321\\2}$ from Ω, giving $z \geq 40$ and so $|X| \geq 20$, as well as the 4 roots $\substack{0010000\\0}, \substack{0011110\\0}, \substack{1121110\\1}, \substack{1232210\\1}$ from Ψ. By the second insertion principle we may now assume

$$\xi_{12} = \substack{0111110\\1},\ \xi_{13} = \substack{1222110\\1},\ \xi_{14} = \substack{1233210\\2},$$

so that $\substack{0011000\\0}, \substack{0010000\\1}, \substack{0110000\\0} \notin X$.

This leaves just 9 available roots in Ψ, each excluding either 1 or 2 available roots from Ω; since we have still roots in X to select, the value of z will increase, and so $|X| \geq 21$, meaning that at least a further 7 roots of the 9 must be chosen. However, among the 9 we have three disjoint pairs which may be added:

$$\substack{0111000\\0} + \substack{1232210\\2} = \substack{1343210\\2},\quad \substack{0110000\\1} + \substack{1122110\\1} = \substack{1232110\\2},\quad \substack{0011000\\1} + \substack{0111110\\0} = \substack{0122110\\1}.$$

Thus at most 6 further roots may be chosen, a contradiction. $\square$

Thus all remaining roots in X must be chosen from O_1, O_2 and O_3.

Proposition 4.6. *If X and Y satisfy condition 4, then X does not contain three mutually orthogonal roots.*

Proof. Assume the result false; then by Proposition 4.5 we may take $\xi_3 = \substack{0121000\\1}$, which excludes the 8 roots $\substack{1\cdot222\cdot1\\1}, \substack{1233321\\\cdot}, \substack{234\cdot321\\2}$ from Ω, giving $z \geq 30$ and so $|X| \geq 15$, as well as the 19 roots $-\substack{0121000\\1}, -\substack{0\cdot1\cdot000\\\cdot}, \substack{000\cdot1\cdot0\\0}, \substack{11111\cdot0\\\cdot}, \substack{1\cdot22210\\1}$ from Ψ. By Proposition 4.5 again we have $\pm\substack{0001000\\0} \notin X$ (since these roots form the A_1 subsystem of the D_4A_1 orthogonal to ξ_1 and ξ_3), and $\pm\substack{0000000\\1} \notin X$ (similarly with the D_4A_1 orthogonal to ξ_2

and ξ_3). By the second insertion principle we may assume $\xi_4 = \genfrac{}{}{0pt}{}{1343210}{2}$, $\xi_5 = \genfrac{}{}{0pt}{}{1243210}{2}$, so that $\pm\genfrac{}{}{0pt}{}{0100000}{0} \notin X$.

The stabilizer in W of $\{\xi_1, \ldots, \xi_5\}$ has two orbits on the 36 available roots in Ψ:

$$\left\{\genfrac{}{}{0pt}{}{012\cdot1\cdot0}{1}, \genfrac{}{}{0pt}{}{123\cdot\cdot\cdot0}{\cdot}\right\}, \quad \left\{\genfrac{}{}{0pt}{}{0\cdot1\cdot\cdot\cdot0}{\cdot}, \genfrac{}{}{0pt}{}{1\cdot2\cdot1\cdot0}{1}\right\},$$

of sizes 12 and 24 respectively. Suppose if possible that the remaining roots in X all lie in the first orbit; then $|X| \le 5 + 12 = 17$. However, the inclusion of any of the 12 roots excludes 2 from Ω, giving $z \ge 32$ and so $|X| \ge 16$, so that in fact at least 11 of the 12 must be included; but then between them the 11 exclude all 8 roots $\genfrac{}{}{0pt}{}{123\cdot2\cdot1}{\cdot}$ from Ω, giving $z \ge 38$ and hence $|X| \ge 19$, a contradiction. Thus X contains some root from the second orbit, so we may assume $\xi_6 = \genfrac{}{}{0pt}{}{0111110}{0}$; this excludes the 4 roots $\genfrac{}{}{0pt}{}{12\cdot\cdot211}{2}$, $\genfrac{}{}{0pt}{}{2354321}{3}$ from Ω, giving $z \ge 34$ and so $|X| \ge 17$, as well as the 6 roots $\genfrac{}{}{0pt}{}{001\cdot\cdot00}{1}$, $\genfrac{}{}{0pt}{}{112\cdot100}{1}$, $\genfrac{}{}{0pt}{}{1232100}{2}$ from Ψ. Moreover by Proposition 4.5 X also does not contain $\genfrac{}{}{0pt}{}{0011110}{1}$ or $\genfrac{}{}{0pt}{}{0011100}{0}$ (as may be seen by considering the D_4A_1 subsystems orthogonal to ξ_6 and ξ_1, and ξ_6 and ξ_3 respectively). By the second insertion principle we may now assume $\xi_7 = \genfrac{}{}{0pt}{}{1232110}{1}$, so that $\genfrac{}{}{0pt}{}{0111100}{1} \notin X$.

We now consider the set $S = \left\{\genfrac{}{}{0pt}{}{123\cdot211}{1}, \genfrac{}{}{0pt}{}{123\cdot221}{2}\right\}$ of 4 roots among those available in Ω; the stabilizer in W of $\{\xi_1, \ldots, \xi_7\}$ is transitive on S. Since we have so far chosen just 7 roots in X, and each available root in Ψ excludes some available root from Ω, the value of z must increase, as must therefore the lower bound on $|X|$; so we must choose at least 11 further roots in X. Assume if possible that after X has been completed at most 2 of the 4 roots in S have been excluded; by transitivity we may assume $\genfrac{}{}{0pt}{}{1232211}{1}$ and at least one other root γ from S fail to be excluded. We cannot have $\gamma = \genfrac{}{}{0pt}{}{1232221}{2}$ or $\genfrac{}{}{0pt}{}{1233221}{2}$ because there are just 11 available roots in Ψ which exclude neither $\genfrac{}{}{0pt}{}{1232211}{1}$ nor γ, two of which are $\genfrac{}{}{0pt}{}{0011110}{0}$ and $\genfrac{}{}{0pt}{}{0110000}{1}$, which may be added; so we must instead have $\gamma = \genfrac{}{}{0pt}{}{1233211}{1}$. Since there are just 13 available roots in Ψ which exclude neither $\genfrac{}{}{0pt}{}{1232211}{1}$ nor $\genfrac{}{}{0pt}{}{1233211}{1}$, and they include two pairs $\genfrac{}{}{0pt}{}{001\cdot000}{0} + \genfrac{}{}{0pt}{}{122\cdot100}{1} = \genfrac{}{}{0pt}{}{1232100}{1}$, all the other 9 roots must be included in X; but one of the 9 is $\genfrac{}{}{0pt}{}{0111100}{0}$, which excludes the 3 available roots $\genfrac{}{}{0pt}{}{12\cdot\cdot221}{2}$ from Ω, giving $z \ge 37$ and so $|X| \ge 19$, which is not possible. Thus at least 3 of the 4 roots in S must be excluded; again using transitivity we may assume $\genfrac{}{}{0pt}{}{123\cdot211}{1}$, $\genfrac{}{}{0pt}{}{1232221}{2}$ are excluded. Since each of $\genfrac{}{}{0pt}{}{012\cdot110}{1}$, $\genfrac{}{}{0pt}{}{1233210}{1}$ excludes one of the three (different in each case) and no other available roots from Ω, and just one available root from Ψ, by the second insertion principle we may assume $\xi_8 = \genfrac{}{}{0pt}{}{0121110}{1}$, $\xi_9 = \genfrac{}{}{0pt}{}{0122110}{1}$, $\xi_{10} = \genfrac{}{}{0pt}{}{1233210}{1}$, so that $\genfrac{}{}{0pt}{}{122\cdot100}{1}$, $\genfrac{}{}{0pt}{}{0110000}{1} \notin X$.

By this point we have $z \ge 37$, so that $|X| \ge 19$; by the second insertion principle again we may now assume $\xi_{11} = \genfrac{}{}{0pt}{}{1232110}{2}$, so that $\genfrac{}{}{0pt}{}{0111100}{0} \notin X$. We have 17 available roots in Ψ, and must choose at least 8 of them to be included in X; as each excludes at least 1 available root in Ω, and no available root in Ω is excluded by more than 5 available roots in Ψ, the value of z will need to increase by at least 2, giving $|X| \ge 20$, so that in fact at least 9 of the available roots in Ψ must be chosen. However, we have eight disjoint pairs which may be added:

$$\genfrac{}{}{0pt}{}{001\cdot000}{0} + \genfrac{}{}{0pt}{}{122\cdot110}{1} = \genfrac{}{}{0pt}{}{1232110}{1}, \quad \genfrac{}{}{0pt}{}{012\cdot100}{1} + \genfrac{}{}{0pt}{}{112\cdot110}{1} = \genfrac{}{}{0pt}{}{1243210}{2},$$

$$\genfrac{}{}{0pt}{}{011\cdot000}{0} + \genfrac{}{}{0pt}{}{123\cdot210}{2} = \genfrac{}{}{0pt}{}{0111000}{1} + \genfrac{}{}{0pt}{}{1232210}{1} = \genfrac{}{}{0pt}{}{0111110}{1} + \genfrac{}{}{0pt}{}{1232100}{1} = \genfrac{}{}{0pt}{}{1343210}{2}.$$

Thus ξ_{12} must be the remaining root $\genfrac{}{}{0pt}{}{0011110}{0}$, and we must have one from each of the eight pairs; since $\genfrac{}{}{0pt}{}{0011110}{0}$ excludes $\genfrac{}{}{0pt}{}{0111000}{1}$ we must in fact have $\xi_{13} = \genfrac{}{}{0pt}{}{1232210}{1}$. However, between them ξ_{12} and ξ_{13} exclude the 3 available roots $\genfrac{}{}{0pt}{}{1233221}{2}$, $\genfrac{}{}{0pt}{}{1343211}{2}$, $\genfrac{}{}{0pt}{}{2454321}{3}$ from Ω, and ξ_{14} is either $\genfrac{}{}{0pt}{}{0110000}{0}$ or $\genfrac{}{}{0pt}{}{1233210}{2}$ and thus excludes either $\genfrac{}{}{0pt}{}{1244321}{2}$ or $\genfrac{}{}{0pt}{}{1232221}{1}$ from Ω, so that $z \geq 41$, giving $|X| \geq 21$, which is impossible. $\square$

Thus all remaining roots in X must be chosen from O_1 and O_2.

Proposition 4.7. *If X and Y satisfy condition 4, then X does not contain three roots of which one is orthogonal to the other two.*

Proof. Assume the result false; then by Proposition 4.6 we may take $\xi_3 = \genfrac{}{}{0pt}{}{1232110}{2}$, which excludes the 5 roots $\genfrac{}{}{0pt}{}{1\cdot\cdot\cdot211}{1}$, $\genfrac{}{}{0pt}{}{1233321}{1}$ from Ω, as well as the 5 roots $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot100}{0}$ from Ψ. Moreover, the 7 available roots $\genfrac{}{}{0pt}{}{1111110}{0}$, $\genfrac{}{}{0pt}{}{1\cdot\cdot\cdot100}{1}$ are orthogonal to both ξ_2 and ξ_3, so by Proposition 4.6 again are excluded. By the first insertion principle we may assume $\{\xi_4, \ldots, \xi_{15}\} = \{\genfrac{}{}{0pt}{}{0\cdot\cdot\cdot110}{1}, \genfrac{}{}{0pt}{}{1\cdot\cdot\cdot210}{\cdot}\}$; these roots exclude the 16 roots $\genfrac{}{}{0pt}{}{1\cdot\cdot\cdot221}{1}$, $\genfrac{}{}{0pt}{}{1\cdot\cdot\cdot2\cdot1}{2}$, $\genfrac{}{}{0pt}{}{2\cdot\cdot\cdot321}{2}$ from Ω. By Proposition 4.6 we then cannot have any root $\genfrac{}{}{0pt}{}{0\cdot\cdot\cdot100}{1}$, $\genfrac{}{}{0pt}{}{0\cdot\cdot\cdot110}{0}$ in X, since each of these is orthogonal to ξ_3 and ξ_i for some $4 \leq i \leq 15$. By the first insertion principle again we may now assume $\{\xi_{16}, \ldots, \xi_{22}\} = \{\genfrac{}{}{0pt}{}{1232100}{2}, \genfrac{}{}{0pt}{}{1\cdot\cdot\cdot110}{1}\}$. However, then $Y = \{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot\cdot1}{3}\}$, and so $2x + y = 2.22 + 7 = 51$, a contradiction. $\square$

Thus all remaining roots in X must be chosen from O_1; but this is impossible, as we have $|X| \geq 11$. This completes the proof of Theorem 4.1.

5. Auxiliary results

We now turn to the consideration of groups G with $F^*(G)$ quasi-simple. The following easy observation allows us to study 2F-modules for cyclic extensions.

Proposition 5.1. *Let N be normal in G of prime index ℓ. Let V be a 2F-module for G of characteristic ℓ with offender A. Let $x \in A$ with $G = \langle N, x\rangle$ and assume that $W := C_V(x)$ is an absolutely irreducible $C_N(x)$-module, with $\dim V = \ell \dim W$. Then either*

(a) *W is an F-module for $C_N(x)$ with offender $B := A \cap N$, or*
(b) *$A = \langle x\rangle$, $\ell \leq 3$, $\dim W \leq 4 - \ell$.*

Proof. By assumption we have

$$f^V(A) = |A|^2|C_V(A)| = \ell^2|B|^2|C_W(B)| \geq |V|.$$

But then

$$\begin{aligned}\left(f_1^W(B)\right)^2 = |B|^2|C_W(B)|^2 \quad &= \ell^2|B|^2|C_W(B)|\frac{|C_W(B)|}{\ell^2} \\ &\geq \frac{|C_W(B)|}{\ell^2}|V| \geq \frac{|C_W(B)|}{\ell^2}|W|^\ell \geq \frac{1}{\ell}|W|^2.\end{aligned}$$

Thus $f_1^W(B) \geq |W|/\sqrt{\ell}$, but since both are integral powers of ℓ, this forces $f_1^W(B) \geq |W|$. Hence W is an F-module for $C_N(x)$ if $B \neq 1$, that is, if $A \neq \langle x\rangle$. Otherwise $\ell^2|W| \geq |V| \geq |W|^\ell$, so $\ell^2 \geq |W|^{\ell-1}$. $\square$

We'll apply this result to the situation where G is an extension of the quasi-simple group N of Lie-type in characteristic ℓ by a field automorphism x of order ℓ and V is the induced to G of an absolutely irreducible N-module. Then $C_N(x)$ is a group of Lie-type of the same type as N (but over a smaller field), and W is automatically irreducible for $C_N(x)$.

The following result is very useful for inductive arguments along parabolic subgroups (see [4, Proposition 2.4]):

Proposition 5.2. *Let V be a 2F-module for G. Let $H \leq G$ such that H contains an offender. Then either there exists an offender $A \leq U := O_\ell(H)$, or $C_V(U)$ is a 2F-module for H/U.*

Throughout, we will freely make use of the results expounded in [3] on automorphisms of order p of groups of Lie type in characteristic p. More precisely, the classes of field and graph-field automorphisms and their centralizers are described in [3, Proposition 4.9.1], while the classes of graph automorphisms and their centralizers are investigated in [3, Proposition 4.9.2].

6. Linear groups

In this section we consider the case $S = \mathrm{L}_n(q)$, $q = \ell^a$. Let Y_n denote the natural module for $\mathrm{SL}_n(q)$ of dimension n, with right $\mathrm{SL}_n(q)$-action.

Theorem 6.1. *The absolutely irreducible 2F-modules for groups G with $F^*(G) = \mathrm{SL}_n(q)$, $q = \ell^a$, $n \geq 3$, with an offender whose conjugates generate G are as given in Table 2.*

Proof. By [5, Proposition 4.7] we may assume that G involves proper outer automorphisms. Moreover, if V is a 2F-module for G then $V|_{F^*(G)}$ has a composition factor W which is a 2F-module for $F^*(G)$ by [5, Proposition 4.8]. Thus, up to twists and taking duals, W is either the natural module Y_n, the exterior square $\wedge^2(Y_n)$ for $n \geq 4$, the symmetric square $\Sigma^2(Y_n)$ for $\ell > 2$, the third exterior power $\wedge^3(Y_6)$ for $n = 6$, or a tensor product $Y_n \otimes Y_n^{(a/2)}$ of the natural module with a Frobenius twist thereof.

1. First assume that $G = \mathrm{SL}_n(\ell^a).\ell$ involves a field automorphism of order ℓ (so in particular $\ell|a$). We go through the cases for W. If V is induced then application of Proposition 5.1 together with the list of F-modules in [5, Table 2] for the centralizer $\mathrm{SL}_n(\ell^{a/\ell})$ shows that either $W = Y_n$ or $W = \wedge^2(Y_n)$. If $W = Y_n$ is the natural module, the bounds on ranks of ℓ-subgroups with given centralizer in [5, Proposition 3.10] lead to the condition

$$n(\ell - 1) + c \leq 2(c(n - c) + \ell/a)$$

for an offender A on V with $A \cap F^*(G)$ of centralizer dimension $n - c \geq n/2$ on Y_n. This is only satisfied for $\ell \leq n/2 + 1$. When $\ell \leq (n + 1)/2$ this is Example 6.6. If $\ell = n/2 + 1$ then moreover $\ell \leq 3$, so in fact $n = 4$, $\ell = 3$, or $n = 2$, $\ell = 2$. An application of [5, Proposition 3.10] now shows that $a = 3$ in the first case, $a \leq 4$ in the second, which is again Example 6.6.

For $W = \wedge^2(Y_n)$, we arrive at

$$\binom{n}{2}(\ell-1)+c(n-c-1) \leq 2(c(n-c)+\ell/a)$$

by the bounds on centralizers on exterior squares in [5, Corollary 2.11] applied to inner elements with commutator space of dimension c on Y_n. This forces $\ell = 2$, $n \leq 5$. If $n = 5$ then necessarily $c = 2$, and A contains elements with distinct centralizers. This can be ruled out using [5, Lemma 2.10]. The case $n = 4$ concerns the natural module for $\mathrm{SO}_6^+(q)$, and the better bounds for orthogonal groups in [5, Proposition 3.14] allow us to exclude this case.

The only 2F-module invariant under field automorphisms is $W = Y_n \otimes Y_n^{(a/2)}$ with $\ell = 2$ and $2|a$. Let $A \leq G$ be an offender such that $A' := A \cap F^*(G)$ has centralizer dimension at least $n-c \geq n/2$ on Y_n. Any outer element $g \in A \setminus A'$ acts on $C_{Y_n}(A')$, thus by [5, Lemma 2.12] its commutator on $C_V(A')$ is of dimension at least $(n-c)(n-c-1)/2$. By [5, Corollary 2.11] this gives the condition $2c(n-c) + (n-c)(n-c-1)/2 \leq 2c(n-c)+4/a$, hence $n/2 \leq n-c \leq 2$. Thus we have $n \leq 4$, $a \leq 4$. When $n = 4$ then A has to have rank $4a$, hence will contain elements with distinct centralizers on Y_n. Application of [5, Lemma 2.10] rules out this case. For $n = 3$ the rank equals $2a$, but then the centralizer on V is at most 4-dimensional. This forces $a = 2$, and we arrive at Example 6.7.

2. Next consider the case that $\ell = 2$ and $G = \mathrm{SL}_n(2^a).2$ is obtained by adjoining the graph automorphism γ. The centralizer in $\mathrm{SL}_n(2^a)$ of a graph automorphism of order 2 is contained in a symplectic group $\mathrm{Sp}_{\lfloor \frac{n}{2} \rfloor}(2^a)$. Thus by [5, Proposition 3.12] its 2-rank equals $an(n+2)/8$ for n even, $a(n^2-1)/8$ for n odd. If V is induced from a tensor product $W = V_1 \otimes \ldots \otimes V_r$ for $\mathrm{SL}_n(q)$ with $d_i = \dim(V_i)$, then

$$d_1 \cdots d_r \frac{a}{r} \leq 2\left(a\lfloor\frac{n(n+2)}{8}\rfloor+1\right).$$

This is only satisfied when $r = 1$ and either $d = n$, or $n = 4$, $d = 6$. But in the latter case, W is invariant, hence does not occur in this case. Now assume that $d = n$, so V is induced from a Frobenius twist of the natural module. If $n = 4$ or $n \geq 6$, this is Example 6.9. Clearly an offender A has to contain inner elements. Denote by c the codimension of the centralizer of $A \cap F^*(G)$ on W. Since V is induced we reach the condition $3 + c \leq 2(1 + 1/a)$ for $n = 3$, hence $c = a = 1$. Similarly, for $n = 5$ we find $5 + c \leq 2(r + 1/a)$, where $|A| = 2^{ar+1}$. Hence $r \geq 2$, but then $c \geq 2$, which forces $a \leq 2$ (since $r \leq 3$). This is again Example 6.9.

Thus we may assume that $V = W$ restricts irreducibly to $F^*(G)$, so has highest weight invariant under the graph automorphism. That is, W is either $\wedge^2(Y_4)$ or $\wedge^3(Y_6)$. The first is Example 6.8. For $\wedge^3(Y_6)$ we use the fact that the centralizer of a graph automorphism in $F^*(G)$ is $H := \mathrm{Sp}_6(2^a)$. The H-composition factors of $\wedge^3(Y_6)$ are two natural modules and the 8-dimensional spin module Z_3. The graph automorphism acts by interchanging the two natural modules. Using the bounds in [5, Proposition 3.12] for sizes of 2-subgroups of H and the fact that any non-trivial element has at most 6-dimensional centralizer on Z_3 we deduce that no example occurs.

3. Next consider the case that $\ell = 2$ and $G = \mathrm{SL}_n(2^a).2$ is obtained by adjoining a graph-field automorphism (so a is even). Here, the centralizer of γ of order 2 in $\mathrm{SL}_n(2^a)$ is $\mathrm{SU}_n(2^{a/2})$. So by [5, Proposition 3.11] it has 2-rank $\binom{n}{2}^2 a/2$ if n is even, respectively $(\binom{n-1}{2}^2+1)a/2$ if n is odd. None of the 2F-modules W for $F^*(G)$ extend in this case, so V is induced. If V is an induced basic representation, the fundamental inequality leads to the cases $d = n$ or $n = 4$, $V = \wedge^2(V_0)$. In the second case, if $A' = A \cap \mathrm{SL}_n(q)$ has centralizer of codimension c on V_0, then $6+c \le 2g+4/a$, where ag denotes the rank of A'. So clearly $g \ge 3$, but then necessarily $c \ge 3$. This implies $g \ge 4$, whence $c \ge 5$, and thus a contradiction. When $d = n$, so V is induced from a Frobenius twist of the natural module, clearly A contains inner elements. Let c denote the codimension of the centralizer of $A' = A \cap F^*(G)$ on Y_n. We find the condition $n + c \le 2(r + 1/a)$, where $|A'| = 2^{ar}$. Using the fact that $a \ge 2$ and $c > 1$ when $r > 1$ we conclude that $n \ge 6$. When $n = 7$, the precise bounds given in [5, Proposition 3.11] show that necessarily $a = 2$. Together with the cases $n = 6$ and $n \ge 8$ this is Example 6.10.

4. Finally assume that $\ell = 2$, $2|a$, and $G = \mathrm{SL}_n(2^a).2^2$ contains both the graph and the field automorphism of order 2. Here again none of the 2F-modules for $F^*(G)$ are invariant. For representations of G whose restriction to $F^*(G)$ contains a basic representation we get the dimension bound $d \le n(n+2)/4 + 8/a$ and we conclude that either $W = Y_n$, or $n = 4$, $W = \wedge^2(Y_4)$ and $a \in \{2, 4\}$. The latter case is the natural module for $\mathrm{SO}_6^+(2^a)$; we obtain the condition $6 + c \le 2r + 8/a$ for the rank ar of the subgroup A' with centralizer of codimension c on $\wedge^2(Y_4)$. By [5, Proposition 3.14] we have $r \le \binom{c}{2}$, so the inequality only leaves $a = 2$, which is Example 6.11. For the case of $W = Y_n$ the natural module we may argue as before.

The standard inequality shows that the case of tensor products does not lead to examples in this situation. □

We now list the examples for 2F-modules for nearly simple groups G with $F^*(G) = \mathrm{SL}_n(q)$ in defining characteristic.

Case 6.1: On its natural module Y_n, $G = \mathrm{SL}_n(q)$ acts as a transvection group, so we get an example with $|A| = q$.

Case 6.2: When $n \ge 4$ the alternating square $\wedge^2(Y_n)$ of the natural module is irreducible for $\mathrm{SL}_n(q)$. The subgroup

$$A := \left\langle \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ a_1 & \dots & a_{n-1} & 1 \end{pmatrix} \middle| \; a_i \in \mathbb{F}_q \right\rangle \tag{5}$$

of order q^{n-1} visibly has centralizer of dimension at least $\binom{n-1}{2}$ on $\wedge^2(Y_n)$, so this yields an example.

Case 6.3: For $\ell > 2$ the symmetric square $\Sigma^2(Y_n)$ is irreducible. Again the subgroup A in (5) gives an example: its centralizer has dimension at least $\binom{n}{2}$.

Case 6.4: Let $q = \ell^{2a}$ and $Y_n^{(a)}$ the a-th Frobenius twist of the natural module. Then $Y_n \otimes Y_n^{(a)}$ is irreducible for $\mathrm{SL}_n(q)$ and defined over the subfield $\mathbb{F}_{\ell^a}$ of $\mathbb{F}_q$ of

TABLE 2. 2F-modules for linear groups in defining characteristic

G	d	V	f	conditions	$\log_\ell \lvert A\rvert$	type
$\mathrm{SL}_n(\ell^a)$	n	Y_n	a		a	i
$\mathrm{SL}_n(\ell^a)$	$\binom{n}{2}$	$\wedge^2(Y_n)$	a	$n \geq 4$	$(n-1)a$	i
$\mathrm{SL}_n(\ell^a)$	$\binom{n+1}{2}$	$\Sigma^2(Y_n)$	a	$\ell \geq 3$	$(n-1)a$	i
$\mathrm{SL}_n(\ell^{2a})$	n^2	$Y_n \otimes Y_n^{(a)}$	a		$\lfloor n^2/4\rfloor a$	i
$\mathrm{SL}_6(\ell^a)$	20	$\wedge^3(Y_6)$	a		$5a$	i
$\mathrm{SL}_n(\ell^{\ell a}).\ell$	ℓn	$Y_n \uparrow^G$	a	$\ell \leq (n+1)/2$		f
				or $n = \ell = 2, a \leq 2$		f
				or $(n, \ell, a) = (4, 3, 1)$	5	f
$\mathrm{SL}_n(2^{2a}).2$	n^2	$Y_n \otimes Y_n^{(a)}$	a	$n = 2,\ a \leq 2$	$2a-1$	f
				or $n = 3, a = 1$	2	f
$\mathrm{SL}_n(2^a).2$	$2n$	$Y_n \uparrow^G$	a	$n \geq 4,\ n \neq 5$		g
				or $n = 3, a = 1$	2	g
				or $n = 5, a \leq 2$	$3a+1$	g
$\mathrm{SL}_4(2^a).2$	6	$\wedge^2(Y_4)$	a		$3a+1$	g
$\mathrm{SL}_n(2^{2a}).2$	$2n$	$Y_n \uparrow^G$	$2a$	$n \geq 6,\ n \neq 7$		gf
				or $n = 7,\ a = 1$	10	gf
$\mathrm{SL}_n(2^{2a}).2^2$	$4n$	$Y_n \uparrow^G$	a	$n \geq 12,\ n \neq 13$		f, g
				or $n = 13,\ a = 1$	23	f, g
$\mathrm{SL}_4(4).2^2$	12	$\wedge^2(Y_4) \oplus \wedge^2(Y_4^{(1)})$	1		5	f, g

index 2. Condition (1) now reads

$$n^2a - za \leq 2 \cdot 2a \left\lceil \frac{n^2-1}{4} \right\rceil$$

where $z \geq 1$ denotes the dimension of the centralizer on $Y_n \otimes Y_n^{(a)}$ of a maximal elementary abelian ℓ-subgroup. Thus this gives an example.

Case 6.5: For $n = 6$ the alternating cube $\wedge^3(Y_6)$ gives an example with the subgroup A from (5): its rank equals $5a$, and it centralizes at least the alternating cube of a 5-dimensional subspace, of dimension 10.

Case 6.6: Consider the extension G of $\mathrm{SL}_n(\ell^{\ell a})$ by the field automorphism σ of order ℓ. This acts irreducibly on the $\mathbb{F}_{\ell^a}\mathrm{SL}_n(\ell^{\ell a})$-module

$$Y_n \oplus Y_n^{(a)} \oplus \ldots \oplus Y_n^{((\ell-1)a)} = Y_n \uparrow^G_{\mathrm{SL}_n(\ell^{\ell a})},$$

and outer ℓ-elements have centralizer of dimension n. Let A be a maximal elementary abelian ℓ-subgroup in the centralizer $\mathrm{SL}_n(\ell^a)$ of σ with $\lfloor (n+1)/2\rfloor$-dimensional centralizer on Y_n, extended by σ. Then (1) becomes

$$a\left(n(\ell-1) + \left\lfloor \frac{n+1}{2} \right\rfloor\right) \leq 2\left(a\left\lfloor \frac{n^2}{4} \right\rfloor + 1\right),$$

and we get an example when $\ell \leq (n+1)/2$. Furthermore, when $n = \ell = 2$ we get an example for $\mathrm{L}_2(2^{2a}).2$, $a \in \{1, 2\}$, the extension of $\mathrm{O}_4^-(2^a)$ by the graph automorphism. A further solution of the above equation is given by $n = 4$, $\ell = 3$, $a = 1$; in this case $|A| = 3^5$.

Case 6.7: With G as in the previous example, where $\ell = 2$, the tensor product $Y_n \otimes Y_n^{(a)}$ is defined over $\mathbb{F}_{2^a}$. For $n = 2$, the case $a = 1$ is the deleted permutation module for $\mathfrak{S}_5$, hence occurs. The case $a = 2$, that is, the 4-dimensional representations of $\mathrm{L}_2(16).2$ defined over $\mathbb{F}_4$, yield examples since the 2-rank of the centralizer of the outer involution is 3. When $n = 3$, $a = 1$, we get an example by choosing $A' := A \cap \mathrm{SL}_n(2^{2a})$ as in (5).

Case 6.8: Consider $\mathrm{SL}_n(q)$, $q = 2^a$, $n \geq 3$, extended by the graph automorphism γ. Then $Y_n \oplus Y_n^*$ is an irreducible $\mathbb{F}_q\mathrm{SL}_n(q).2$-module. When $n = 3$, $a = 1$, the Sylow 2-subgroup of the centralizer of γ, of order 2^2, gives an example with centralizer of dimension 2. When $n \geq 4$, a maximal elementary abelian 2-subgroup in the centralizer of γ, extended by γ, gives an example, unless $n = 5$, $a \geq 3$, with centralizer of dimension $\lfloor (n+1)/2 \rfloor$.

Case 6.9: The group $G = \mathrm{SL}_4(2^a).2$ (extension by the graph automorphism) on $\wedge^2(Y_4)$ is just the orthogonal group $\mathrm{GO}_6^+(2^a)$ on its natural module. A suitable conjugate γ of the graph automorphism centralizes an elementary abelian ℓ-subgroup

$$A' := \left\{ \left(\begin{array}{cccc} 1 & & & \\ 0 & 1 & & \\ a_1 & a_2 & 1 & \\ a_3 & a_1 & 0 & 1 \end{array} \right) \,\middle|\, a_i \in \mathbb{F}_{2^a} \right\}$$

of order 2^{3a}. Thus $A = \langle A', \gamma \rangle$ gives an example. There also exists an offender of order 2^{2a+1} centralizing a 2-dimensional subspace. This example actually fits in the infinite series of examples for orthogonal groups in Case 9.9.

Case 6.10: Consider $\mathrm{SL}_n(q)$, $q = 2^{2a}$, $n \geq 3$, extended by the graph-field automorphism γ. Then $Y_n \oplus Y_n^{*(a)}$ is an irreducible $\mathbb{F}_q\mathrm{SL}_n(q)$-module. Take a maximal elementary abelian 2-subgroup in the centralizer of γ, extended by γ. For $n \geq 6$, this gives an example with centralizer of dimension $\lfloor (n+1)/2 \rfloor$, unless $n = 7$, $a \geq 2$.

Case 6.11: Let G be the extension of $\mathrm{SL}_n(q)$, $q = 2^{2a}$, $n \geq 3$, by the graph and the field automorphism, both of order 2. Then a maximal elementary abelian 2-subgroup of the centralizer in $\mathrm{SL}_n(q)$ of the two outer automorphisms, extended by both outer automorphisms, gives an example on the module induced from the natural module when $n \geq 12$, unless $n = 13$, $a \geq 2$.

Case 6.12: The extension G of the general orthogonal group $\mathrm{GO}_6^+(4)$ by the field automorphism gives an example on the natural module $\wedge^2(Y_4)$ of $\mathrm{GO}_6^+(4)$ induced to G with $|A| = 32$ and centralizer of dimension 2. This example actually fits in the infinite series of examples for orthogonal groups in Case 9.13.

7. Unitary groups

Here we consider the case $S = \mathrm{U}_n(q)$, $q = \ell^a$. Let Y_n denote the natural module for $\mathrm{SU}_n(q)$ of dimension n.

Theorem 7.1. *The absolutely irreducible 2F-modules for groups G with $F^*(G) = \mathrm{SU}_n(q)$, $q = \ell^a$, $n \geq 3$, with an offender whose conjugates generate G are as given in Table 3.*

Proof. By [5, Proposition 4.11] we may assume that G and A involve proper outer automorphisms. Moreover either the restriction of a 2F-module for G to $\mathrm{SU}_n(\ell^a)$ contains the natural module Y_n, or $n = 4$ and it contains $\wedge^2(Y_4)$.

First assume that ℓ is odd and $G = \mathrm{SU}_n(\ell^a).\ell$ involves a field automorphism of order ℓ. For the module $Y_n \uparrow^G$ induced from the natural module the bounds on the size of centralizer spaces in [5, Proposition 3.11] force

$$\ell \leq \begin{cases} \frac{n+2}{4} & n \text{ even} \\ \frac{n-1}{4} & n \text{ odd.} \end{cases}$$

For $n = 4$, $\wedge^2(Y_4)$ is the natural module for $\mathrm{SO}_6^-(\ell^a)$, and using the bounds for orthogonal groups in [5, Proposition 3.14] we see that necessarily $\ell \geq 3$ cannot occur.

It remains to consider the case that $\ell = 2$ and G is obtained from $\mathrm{SU}_n(2^a)$ by adjoining the graph-field automorphism of order 2. For the natural module the same arguments as above show that $n \geq 4$ for n even, and $n \geq 7$ or $(n, a) = (5, 1)$ for n odd.

The group $\mathrm{SU}_4(2^a).2$ on $\wedge^2(Y_4)$ is the reflection group $\mathrm{GO}_6^-(2^a)$ on its natural module and gives an example. $\square$

We now list the examples for 2F-modules for nearly simple groups G with $F^*(G) = \mathrm{SU}_n(q)$ in defining characteristic.

TABLE 3. 2F-modules for unitary groups in defining characteristic

G	d	V	f	conditions	$\log_\ell \lvert A\rvert$	type
$\mathrm{SU}_n(\ell^a)$	n	Y_n	$2a$		a	i
$\mathrm{SU}_4(\ell^a)$	6	$\wedge^2(Y_4)$	a		a	i
$\mathrm{SU}_n(\ell^{\ell a}).\ell$	ℓn	$Y_n \uparrow^G$	$2a$	n even, $2 \neq \ell \leq (n+2)/4$	$an^2/4$	f
				or n odd, $2 \neq \ell \leq (n-1)/4$	$a(n-1)^2/4$	f
$\mathrm{SU}_n(2^a).2$	$2n$	$Y_n \uparrow^G$	a	$n \geq 4,\ n \neq 5$		gf
				or $n = 5, a \leq 2$	$3a+1$	gf
$\mathrm{SU}_4(2^a).2$	6	$\wedge^2(Y_4)$	a		$3a+1$	gf

Case 7.1: The special unitary group $\mathrm{SU}_n(q)$ acts as a transvection group on its natural module Y_n over $\mathbb{F}_{q^2}$, thus we get an example with $|A| = q$.

Case 7.2: The 4-dimensional unitary group $\mathrm{SU}_4(q)$ is isomorphic to the 6-dimensional orthogonal group of minus-type. The latter is a bi-transvection group on its natural module, so we get an example with $|A| = q$.

Case 7.3: The extension G of $\mathrm{SU}_n(\ell^{\ell a})$ by the field automorphism of order ℓ acts irreducibly on the $\mathbb{F}_{\ell^{2a}}G$-module $Y_n \oplus Y_n^{(a)} \oplus \ldots \oplus Y_n^{((\ell-1)a)}$. We obtain an example by taking $A' = A \cap \mathrm{SU}_n(\ell^{\ell a})$ maximal elementary abelian of order $\ell^{an^2/4}$ for n even, with centralizer of dimension $n/2$ on the natural module, respectively elementary abelian of order $\ell^{a(n-1)^2/4}$ with centralizer of dimension $(n+1)/2$ on Y_n when n is odd.

Case 7.4: The extension G of $\mathrm{SU}_n(2^a)$ by the graph-field automorphism σ of order 2 acts irreducibly on the $\mathbb{F}_{2^a}G$-module $V = Y_n \oplus Y_n^*$. The centralizer C of σ is $\mathrm{SO}_n(2^a)$ for n odd, and $\mathrm{Sp}_n(2^a)$ for n even. Let A be the extension of a maximal elementary abelian 2-subgroup of C with σ. Then this gives an offender on V for $n \geq 4$ even, as well as for $n \geq 7$ odd and for $n = 5$, $a \leq 2$.

Case 7.5: The graph automorphism of the special orthogonal group $\mathrm{SO}_6^-(2^a)$ centralizes a subgroup $H = \mathrm{SO}_5(2^a)$. Let A be a maximal elementary abelian 2-subgroup of H, extended by this graph automorphism. Then A has rank $3a+1$ (see [5, Proposition 3.14]) and at least a 1-dimensional centralizer on the natural module. Thus this gives an example for the natural module of $\mathrm{SO}_6^-(2^a).2$, so for $\mathrm{SU}_4(2^a).2$ on $\wedge^2(Y_4)$. This example actually fits in the infinite series of examples for orthogonal groups in Case 9.9.

8. Symplectic groups

Here we consider the case $S = \mathrm{Sp}_{2n}(q)$, $q = \ell^a$. Let $\lambda_1, \ldots, \lambda_n$ denote the fundamental weights for Sp_{2n}, ordered along the Dynkin diagram of type C_n, with the double bond between the first two nodes. Let Y_{2n} denote the natural $2n$-dimensional symplectic module for $\mathrm{Sp}_{2n}(q)$; if $\ell = 2$ let Z_n denote the 2^n-dimensional spin module.

To deal with the spin module for $\mathrm{Sp}_6(2^a)$ and 8-dimensional orthogonal groups we need the following lemma:

Lemma 8.1. *Let A be an elementary abelian unipotent subgroup of $\mathrm{Sp}_6(q)$ with $|A| > q^5$ and q even. Then the fixed space of A on the natural module Y_6 is a maximal totally isotropic subspace. In particular, A is contained in the unipotent radical of the maximal parabolic subgroup that is the stabilizer of this 3-dimensional totally singular subspace.*

Proof. Let P be the stabilizer of a totally singular 3-space with radical U and Levi subgroup $L \cong \mathrm{GL}_3(q)$. Note that U is elementary abelian of order $|U| = q^6$. We may assume that $A \leq P$ and we will show that $A \leq U$, whence the result.

Set $B = A \cap U$. Suppose that $B \neq A$. Set $X = A/B \leq P/U \cong L$. Note that we can enlarge A by considering $AC_U(X)$ which is still elementary abelian. Thus, we may assume that $B = C_U(X)$. In particular, $|B| = q^b$ for some b. Moreover, $b \leq 4$ since a transvection in L centralizes only a 4-dimensional subspace of U.

Thus, $|X| > q$. Note that X is contained in the unipotent radical Y of a maximal parabolic of L. We claim that $|C_U(X)| \leq q^3$. Note that U has two composition factors as an L-module: the dual of the natural module and a Frobenius twist of the natural module. On either of these modules, X and Y have the same fixed points and Y has a 2-dimensional fixed space on one of these modules and a 1-dimensional fixed space on the other. Let M be the direct sum of these two modules. Thus,

$$|C_U(X)| \leq |C_M(X)| = |C_M(Y)| = q^3$$

and so $B = C_U(X)$ has order at most q^3. Hence $|A| = |X||B| \le q^5$, a contradiction. $\square$

Lemma 8.2. *Let $V = Z_3$ be the 8-dimensional spin module for $\mathrm{Sp}_6(q)$, $q = 2^a > 2$. Then $4|A|^2\,|C_V(A)| < |V|^2$ for every elementary abelian subgroup $A \ne 1$.*

Proof. Let $A \ne 1$ be a counterexample. By [5, Lemma 5.1] we have $|C_V(A)| \le q^6$, whence $|A| > q^4$. We may consider $G := \mathrm{Sp}_6(q)$ as a subgroup of $H := \mathrm{SO}_8^+(q)$ on the spin module V, which under triality becomes the natural module of H. The bounds in [5, Proposition 3.14] then show that $|C_V(A)| \le q^4$, whence $|A| \ge q^6/2 > q^5$ since $q \ge 4$. The previous lemma now shows that A is contained in the unipotent radical U of a maximal parabolic subgroup of G of type GL_3. By a closure argument we may assume that $A = U$, which leads to a contradiction, since U has only a 1-dimensional centralizer on V by [5, Theorem 3.1]. $\square$

Theorem 8.3. *The absolutely irreducible 2F-modules for groups G with $F^*(G) = \mathrm{Sp}_{2n}(q)$, $q = \ell^a$, $n \ge 2$, with an offender whose conjugates generate G are as given in Table 4.*

Proof. The 2F-modules for $G = \mathrm{Sp}_{2n}(q)$ were determined in [5]. So first assume that $G = \mathrm{Sp}_{2n}(\ell^a).\ell$ involves a field automorphism of order ℓ. By [5, Proposition 6.2 and 6.12] the restriction of a 2F-module V for G to $\mathrm{Sp}_{2n}(\ell^a)$ contains a 2F-module W of G, so either the natural module Y_{2n}, or the spin module Z_n when $n \in \{3, 4, 5\}$ and $\ell = 2$, or the heart of the exterior square $\tilde{\Lambda}^2(Y_{2n})$ when $n \le 4$, or the tensor product module $Y_4 \otimes Y_4^{(a)}$ when $n = 2$. The standard inequality shows that the last case does not arise.

In the remaining cases V is induced, so Proposition 5.1 together with [5, Table 2] shows that either W is the natural module Y_{2n}, or $n = 3$, $\ell = 2$ and $W = Z_3$ is the spin module. If V is induced from the natural module, then the bounds in [5, Proposition 3.12] on the size of centralizer spaces on Y_{2n} force $\ell \le (n + 2)/2$.

Finally, assume that $\ell = 2$, $n = 3$, and V is induced from the spin module Z_3 of dimension 8. If $q = 4$ this is Example 8.6. The case $q > 4$ is ruled out by Lemma 8.2.

This completes the proof for $n \ge 3$, since there the only outer automorphisms are field automorphisms. So finally assume that G is an extension of $\mathrm{Sp}_4(2^a)$, a odd, with the exceptional graph automorphism, where $a \ge 2$ (since $\mathrm{Sp}_4(2) \cong \mathfrak{S}_6$ was treated in [4]). Our standard inequalities allow us to rule out this case immediately. $\square$

We now list the examples for 2F-modules for nearly simple groups G with $F^*(G) = \mathrm{Sp}_{2n}(q)$ in defining characteristic.

Case 8.1: The symplectic group $\mathrm{Sp}_{2n}(q)$ acts as a transvection group on its natural module Y_{2n} over $\mathbb{F}_q$, thus we get an example with $|A| = q$.

Case 8.2: The irreducible representation with highest weight λ_{n-1} of $\mathrm{Sp}_{2n}(\ell^a)$ is the heart of the exterior square of the natural representation. Consider the maximal parabolic subgroup of type GL_n. The centralizer of its unipotent radical U has weight λ_{n-2}, hence dimension $\binom{n}{2}$. The unipotent radical U is elementary abelian of ℓ-rank $\binom{n+1}{2}$. Thus we find an example when $n \le 3$. In the case $n = 2$ we have an alternative interpretation: The 4-dimensional symplectic group is isomorphic to the 5-dimensional orthogonal group. The latter is a bi-transvection group on its natural

TABLE 4. 2F-modules for symplectic groups in defining characteristic.

G	d	V	f	conditions	$\log_\ell \lvert A\rvert$	type
$\mathrm{Sp}_{2n}(\ell^a)$	$2n$	Y_{2n}	a		a	i
$\mathrm{Sp}_{2n}(\ell^a)$	$\binom{2n}{2}-1-\delta_{\ell,n}$	$\tilde{\wedge}^2(Y_{2n})$	a	$n=2,3$	$\binom{n+1}{2}a$	i
				or $(n,\ell)=(4,2)$	$10a$	i
$\mathrm{Sp}_{2n}(2^a)$	2^n	Z_n	a	$n=3,4,5$	$(2n-1)a$	i
$\mathrm{Sp}_4(\ell^{2a})$	16	$Y_{2n}\otimes Y_{2n}^{(a)}$	a		$3a$	i
$\mathrm{Sp}_{2n}(\ell^{\ell a}).\ell$	$2n\ell$	$Y_{2n}\uparrow^G$	a	$\ell\leq(n+2)/2$		f
$\mathrm{Sp}_6(4).2$	16	$Z_3\uparrow^G$	1		6	f

module, so we get an example for $\mathrm{Sp}_4(q)$ in dimension 5, when $\ell\neq 2$. When $\ell=2$ this representation is reducible, and its irreducible constituent of dimension 4 yields an example.

Case 8.3: The spin module Z_n for $\mathrm{Sp}_{2n}(2^a)$ has highest weight λ_1. Consider the maximal parabolic subgroup of type Sp_{2n-2}. The centralizer of its unipotent radical U is again the spin module for $\mathrm{Sp}_{2n-2}(\ell^a)$, hence has dimension 2^{n-1}. But U is elementary abelian of rank $(2n-1)a$. Thus U is an offender on Z_n if $2^n-2^{n-1}\leq 2(2n-1)$, so for $n\leq 5$.

Case 8.4: The tensor product of the natural module Y_4 for $\mathrm{Sp}_4(\ell^{2a})$ with its twist $Y_4^{(a)}$ is defined over $\mathbb{F}_{\ell^a}$. A maximal elementary abelian ℓ-subgroup A of order ℓ^{3a} has a 2-dimensional centralizer on Y_4, hence at least a 4-dimensional centralizer on $Y_4\otimes Y_4^{(a)}$. Thus A is an offender.

Case 8.5: The extension G of $\mathrm{Sp}_{2n}(\ell^{\ell a})$ by the field automorphism g of order ℓ acts irreducibly on the induced natural module $V:=Y_{2n}\oplus Y_{2n}^{(a)}\oplus\ldots\oplus Y_{2n}^{((\ell-1)a)}$, which is defined over $\mathbb{F}_{\ell^a}$. Let A be a maximal elementary abelian subgroup of the centralizer $\mathrm{Sp}_{2n}(\ell^a)$ of G, of rank $a\binom{n+1}{2}$, with centralizer of dimension n on the natural module, extended by g. Then A has centralizer of dimension n on V, so gives an example when $\ell\leq(n+2)/2$.

Case 8.6: The abelian unipotent radical of the Sp_4-parabolic of $\mathrm{Sp}_6(4)$, of order 2^{10}, is normalized by the field automorphism σ of order 2, with centralizer A' of order 2^5. It centralizes a 4-dimensional spin module for the Levi factor. Thus $A:=\langle A',\sigma\rangle$ of order 2^6 centralizes a 4-dimensional subspace of Z_3 induced to $\langle\mathrm{Sp}_6(4),\sigma\rangle$.

9. ORTHOGONAL GROUPS IN DIMENSION AT LEAST SEVEN

In this section we consider the case $S=\mathrm{O}_m(q)$, $q=\ell^a$. We consider separately the subcases where m is odd and m is even.

9.1. **The odd-dimensional orthogonal groups.** Let Y_{2n+1} denote the natural $(2n+1)$-dimensional module for $\mathrm{Spin}_{2n+1}(q)$; let Z_n denote the 2^n-dimensional spin module.

Theorem 9.1. *The absolutely irreducible 2F-modules for groups G with $F^*(G)=\mathrm{Spin}_{2n+1}(q)$, $q=\ell^a$, $n\geq 3$, $\ell\neq 2$, with an offender whose conjugates generate G are as given in Table 5.*

Proof. Only field automorphisms arise in this situation, since diagonal automorphisms have order at most 2, so $G = \mathrm{Spin}_{2n+1}(\ell^a).\ell$ with $\ell|a$. By [5, Proposition 7.3] either the restriction $V|_{F^*(G)}$ of a 2F-module V for G contains the natural representation Y_{2n+1}, or $n \leq 5$ and it contains the spin representation Z_n. Neither of them is invariant under field automorphisms, so V is induced from $\mathrm{Spin}_{2n+1}(\ell^a)$. The case of spin representations is ruled out by the standard inequality. For the natural representation the bounds for elementary abelian ℓ-subgroups with given centralizer dimension in [5, Proposition 3.14] show that $\ell \leq (n-1)/2$ or $(n, \ell, a) = (6, 3, 3)$. □

9.2. **The even-dimensional orthogonal groups.** Let $\lambda_1, \ldots, \lambda_n$ denote the fundamental weights for Spin_{2n}, ordered along the Dynkin diagram of type D_n, with the first two nodes the ones interchanged by the graph automorphism of order 2. Let Y_{2n} denote the natural $2n$-dimensional module for $\mathrm{Spin}_{2n}^{(\pm)}(q)$; let Z_{n-1} denote either one of the two 2^{n-1}-dimensional half-spin modules.

Theorem 9.2. *The absolutely irreducible 2F-modules for groups G with $F^*(G) = \mathrm{Spin}_{2n}^+(q)$, $q = \ell^a$, $n \geq 4$, with an offender whose conjugates generate G are as given in Table 5.*

Proof. 1. First let G be the extension of $\mathrm{Spin}_{2n}^+(q)$ by a field automorphism of order ℓ, so $\ell|a$.

1A. If there exists an offender contained in the unipotent radical of a proper parabolic subgroup, then V is already an example for $\mathrm{Spin}_{2n}^+(q)$, hence induced from Y_{2n} or from the half-spin modules Z_{n-1} when $n \leq 6$ by [5, Theorem 7.5]. If V is induced from Y_{2n}, then our usual inequality gives $\ell \leq n/2$, which is Example 9.6. For Z_{n-1} Proposition 5.1 gives $n \leq 5$, and our inequality forces $\ell = 2$. The case $n = 4$ is Example 9.8. For $n = 5$ the outer involution centralizes just half the space, so our offender A has to have order at least q^5. But then [5, Lemma 5.2] shows that $A \cap \mathrm{Spin}_{10}^+(q)$ centralizes at most a 10-dimensional subspace of the spin module, whence A also has centralizer of dimension at most 10. Thus $|A| \geq q^{10}$. Then [5, Proposition 5.3] gives the final contradiction to the 2F-condition.

1B. Otherwise, by Proposition 5.2, $C_V(U)$ is an example for all proper Levi subgroups L of parabolics $P = U.L$, extended by the field automorphism. But the only example for $n = 3$, that is for $\mathrm{SL}_4(q)$, is the half-spin module, by Theorem 6.1. Applying this for all three end-node parabolics we conclude that for $n = 4$ we only find Y_8 and Z_3. Induction now shows that there are no new examples for $n \geq 5$.

2. Consider G the extension of $\mathrm{SO}_{2n}^+(2^a)$ by the graph automorphism γ of order 2, with centralizer of 2-rank $\binom{n-1}{2}a + 1$. The usual estimates yield the following possibilities for 2F-modules V of G: V is the extension to G of the natural module Y_{2n}, or the tensor product of the natural module with its $a/2$th Frobenius twist with a even, or we have $n = 4$ and either V is induced from the spin module Z_3 or V is the extension of the basic module with highest weight $\lambda_1 + \lambda_2$.

The natural module is Example 9.9; the spin module for $a = 1$ is Example 9.10. The case $a \geq 2$ is ruled out by Lemma 8.2. On the other hand the centralizer dimensions on the tensor product $Y_{2n} \otimes Y_{2n}^{(a/2)}$ can be estimated by [5, Corollary 2.11], showing that this does not give an example. The module with highest weight $\lambda_1 + \lambda_2$

is not an example by [5, Proposition 2.4], since the centralizer of the unipotent radical of the $\mathrm{SO}_6^+(2^a)$-parabolic is the adjoint module for $\mathrm{GL}_4(2^a)$ by [5, Theorem 3.1], which is not a 2F-module by Proposition 3.1.

3. Consider G the extension of $\mathrm{Spin}_{2n}^+(2^a)$ by the graph-field automorphism of order 2, so $2|a$. Here, we have to deal with the module induced from the natural module, and with the tensor product $Y_n \otimes Y_n^{(a/2)}$. The natural module is Example 9.11. The tensor product can be ruled out by the usual calculation of centralizer dimensions.

4. Consider $G = \mathrm{SO}_{2n}^+(2^a).2^2$, the extension by the graph and the field automorphism of order 2, so $2|a$. The standard estimates only allow for modules lying over the natural module and for the tensor product $Y_n \otimes Y_n^{(a/2)}$. The natural module is Example 9.13. The tensor product can be eliminated by using the more precise information on centralizers of groups with given size on the natural module from [5, Corolllary 2.11 and Proposition 3.14].

5. Consider G the extension of $\mathrm{Spin}_8^+(3^a)$ by the graph automorphism of order 3, with centralizer of 3-rank $4a+1$. The standard inequalities show that only the exterior square $\wedge^2(Y_8)$ of the natural module Y_8 might give an example. But computation of possible centralizer spaces using [5, Corollary 2.11 and Proposition 3.14] shows that this does not occur. The cases where G contains a graph-field automorphism of $\mathrm{Spin}_8^+(3^a)$ are immediately ruled out by the standard inequalities. □

Theorem 9.3. *The absolutely irreducible 2F-modules for groups G with $F^*(G) = \mathrm{Spin}_{2n}^-(q)$, $q = \ell^a$, $n \geq 4$, with an offender whose conjugates generate G are as given in Table 5.*

Proof. By [5, Theorem 7.6] we may assume that G involves some proper outer automorphisms.

1. First let $G = \mathrm{Spin}_{2n}^-(\ell^a).\ell$ be an extension of $F^*(G)$ by a field automorphism of odd order ℓ. Let V be a 2F-module for G and assume that V is induced from a module W for $F^*(G)$. By Proposition 5.1 this implies that W is an F-module for $\mathrm{Spin}_{2n}^-(\ell^{a/\ell})$, hence by [5, Table 2] up to twists W is the natural module. Here the usual argument shows that necessarily $\ell \leq n/2$, giving Example 9.7.

Now assume that V is not induced, hence restricts irreducibly to $F^*(G)$. Let $P = U.L$ be a proper parabolic subgroup of $F^*(G)$. If the unipotent radical U contains an offender, then V is a 2F-module for $F^*(G)$. But by [5, Theorem 7.6] none of those is invariant under field automorphisms. Thus, $C_V(U)$ is a 2F-module for the extension of the Levi subgroup L by the field automorphism by [5, Proposition 2.4]. First let $n = 4$. Then we may choose L of type $\mathrm{SU}_4(q)$, and by Theorem 7.1 the only 2F-module invariant under field automorphisms is the trivial one. The same conclusion holds for L of type $\mathrm{GL}_3(q)$ by Theorem 6.1. Thus, by the Theorem of Smith and Timmesfeld [5, Theorem 3.1] we have that V is the trivial module. For $n \geq 5$ we use induction, arguing with the Levi subgroups of type $\mathrm{SO}_{2n-2}^-(q)$ and $\mathrm{GL}_{n-1}(q)$, to show that no examples arise.

2. Next consider the case that $G = \mathrm{SO}_{2n}^-(2^a).2$ is the extension by the graph-field automorphism of order 2. By virtually the same estimates as in the case of the graph automorphism of $\mathrm{SO}_{2n}^+(2^a)$ in Theorem 9.2 we arrive at the same four cases as

there. The module with highest weight $\lambda_1+\lambda_2$ and the tensor product of the natural module are excluded by the same arguments. The natural module is Example 9.9, while the spin module for $a=1$ is Example 9.12. The case $a>1$ is ruled out by Lemma 8.2. □

Table 5. 2F-modules for orthogonal groups in defining characteristic

G	d	V	f	conditions	$\log_\ell \lvert A\rvert$	type
$\mathrm{Spin}_n^{(\pm)}(\ell^a)$	n	Y_n	a	$n\ge 7$	a	i
$\mathrm{Spin}_{2n+1}(\ell^a)$	2^n	Z_n	a	$n=3,4,5$	$(2n-1)a$	i
$\mathrm{Spin}_{2n}^+(\ell^a)$	2^{n-1}	Z_{n-1}	a	$n=4,5,6$	$(2n-2)a$	i
$\mathrm{Spin}_{2n}^-(\ell^a)$	2^{n-1}	Z_{n-1}	$2a$	$n=4,5$	$(2n-2)a$	i
$\mathrm{Spin}_{2n+1}(\ell^{\ell a}).\ell$	$(2n+1)\ell$	$Y_{2n+1}\uparrow^G$	a	$\ell\le (n-1)/2$ or		f
				$(n,\ell,a)=(6,3,1)$		f
$\mathrm{Spin}_{2n}^+(\ell^{\ell a}).\ell$	$2n\ell$	$Y_{2n}\uparrow^G$	a	$\ell\le n/2$		f
$\mathrm{Spin}_{2n}^-(\ell^{\ell a}).\ell$	$2n\ell$	$Y_{2n}\uparrow^G$	a	$2\ne\ell\le n/2$		f
$\mathrm{Spin}_8^+(2^{2a}).2$	16	$Z_3\uparrow^G$	a			f
$\mathrm{Spin}_{2n}^\pm(2^a).2$	$2n$	Y_{2n}	a		$3a+1$	g
$\mathrm{Spin}_8^+(2).2$	16	$Z_3\uparrow^G$	1		6	g
$\mathrm{Spin}_{2n}^+(2^{2a}).2$	$4n$	$Y_{2n}\uparrow^G$	a			gf
$\mathrm{Spin}_8^-(2).2$	16	$Z_3\uparrow^G$	1		6	gf
$\mathrm{Spin}_{2n}^+(2^{2a}).2^2$	$4n$	$Y_{2n}\oplus Y_{2n}^{(a)}$	a			f, g

9.3. The examples for $\mathrm{Spin}_n^{(\pm)}(q)$, $n\ge 7$.

Case 9.1: The orthogonal group $\mathrm{SO}_n^{(\pm)}(q)$ acts as a bi-transvection group on its natural module, hence we obtain an example with offender of order q.

Case 9.2: The spin module Z_n for $\mathrm{Spin}_{2n+1}(\ell^a)$ has highest weight λ_1. Consider the maximal parabolic subgroup of type Spin_{2n-1}. The centralizer of its unipotent radical U is again the spin module for $\mathrm{Spin}_{2n-1}(\ell^a)$, hence has dimension 2^{n-1}. But U is elementary abelian of rank $(2n-1)a$. Thus U is an offender on Z_n if $2^n-2^{n-1}\le 2(2n-1)$, so for $n\le 5$.

Case 9.3: For the half-spin modules $V=Z_{n-1}$ of dimension 2^{n-1} of $G=\mathrm{SO}_{2n}^+(q)$ let A be the unipotent radical of the maximal parabolic subgroup with Levi complement of type $\mathrm{SO}_{2n-2}^+(q)$. Then the centralizer $C_V(A)$ is a half-spin module for $\mathrm{SO}_{2n-2}^+(q)$, of dimension 2^{n-2}, while A is elementary abelian of order q^{2n-2}, so we find an example whenever $n\le 6$.

Case 9.4: The half-spin modules $V=Z_{n-1}$ of dimension 2^{n-1} of $G=\mathrm{SO}_n^-(q)$ are defined over $\mathbb{F}_{q^2}$. Let A be the unipotent radical of the maximal parabolic subgroup with Levi complement of type $\mathrm{SO}_{2n-2}^-(q)$. Then the centralizer $C_V(A)$ is a spin module for $\mathrm{SO}_{2n-2}^-(q)$, of dimension 2^{n-2}, while A is elementary abelian of order q^{2n-2}, so we find an example whenever $n\le 5$.

Case 9.5: Let $G=\mathrm{SO}_{2n+1}(\ell^{\ell a}).\ell$, the extension by the field automorphism σ of order ℓ. Let A' be a maximal elementary abelian ℓ-subgroup in the centralizer $\mathrm{SO}_{2n+1}(\ell^a)$ of σ, of order $\ell^{a\binom{n}{2}}$ and with centralizer of dimension n on the natural module Y_{2n+1}. Then $A := \langle A', \sigma\rangle$ is an offender on the induced module for $\ell \leq (n-1)/2$ or $(n, \ell, a) = (6, 3, 1)$.

Case 9.6: Let $G = \mathrm{SO}^+_{2n}(\ell^{\ell a}).\ell$, the extension by the field automorphism σ of order ℓ. Let A' be a maximal elementary abelian ℓ-subgroup in the centralizer $\mathrm{SO}^+_{2n}(\ell^a)$ of σ, of order $\ell^{a\binom{n}{2}}$ and with centralizer of dimension n on the natural module Y_{2n}. Then $A := \langle A', \sigma\rangle$ is an offender on the induced module $Y_{2n} \uparrow^G_{\mathrm{SO}^+_{2n}(\ell^{\ell a})}$ for $\ell \leq n/2$.

Case 9.7: Let $G = \mathrm{SO}^-_{2n}(\ell^{\ell a}).\ell$, $\ell > 2$, the extension by the field automorphism σ of order ℓ. Let A' be a maximal elementary abelian ℓ-subgroup in the centralizer $\mathrm{SO}^-_{2n}(\ell^a)$ of σ, of order $\ell^{a\binom{n}{2}}$ and with centralizer of dimension n on the natural module Y_{2n}. Then $A := \langle A', \sigma\rangle$ is an offender on the induced module for $\ell \leq n/2$.

Case 9.8: The half-spin modules for $\mathrm{SO}^+_8(\ell^{\ell a})$ are images under the triality automorphism of the natural module. Since this gives Example 9.6 for the extension with a field automorphism for $\ell = 2$, the same is true for the half-spin modules.

Case 9.9: The graph automorphism of $\mathrm{SO}^\pm_{2n}(2^a)$ of order 2 leaves the natural module Y_{2n} invariant. Let A be a maximal elementary abelian ℓ-subgroup of $\mathrm{GO}^\pm_6(2^a) \leq \mathrm{GO}^\pm_{2n}(2^a)$. Then A has order 2^{3a+1} and $\dim C_{Y_{2n}}(A) \geq 2n-5$, so this gives an example.

Case 9.10: Let G be the extension of $\mathrm{SO}^+_8(2)$ by the graph automorphism γ of order 2. The graph automorphism interchanges the two half-spin modules of $\mathrm{SO}^+_8(2)$. Denote by V the induced module for G. Let A be generated by γ together with the unipotent radical U of order 2^5 of the maximal parabolic subgroup of $\mathrm{Sp}_6(2) = C_{\mathrm{SO}^+_8(2)}(\gamma)$ of type $\mathrm{SO}_5(2)$. Then U centralizes a 4-dimensional spin module for $\mathrm{SO}_5(2)$, so we get an example.

Case 9.11: Let G be the extension of $\mathrm{SO}^+_{2n}(2^{2a})$ by the graph-field automorphism σ of order 2. Let A' be a maximal elementary abelian ℓ-subgroup in the centralizer $\mathrm{SO}_{2n-1}(2^a)$ of σ, of order $\ell^{a\binom{n-1}{2}}$ and with centralizer of dimension $n-1$ on the natural module Y_{2n}. Then $A := \langle A', \sigma\rangle$ is an offender on the induced module.

Case 9.12: This is obtained from Example 9.10 by exchanging the orthogonal group of plus type with the one of minus type.

Case 9.13: Let G be the extension of $\mathrm{SO}^+_{2n}(2^{2a})$ by the group of graph-field automorphisms of order 4. We get an example with $A \cap \mathrm{SO}^+_{2n}(2^{2a})$ a maximal elementary abelian 2-subgroup in the centralizer $\mathrm{SO}_{2n-1}(2^a)$ of the group of automorphisms, on the module induced from the natural module.

10. Exceptional groups

Theorem 10.1. *Let V be an absolutely irreducible 2F-module for a group G with $F^*(G)$ an exceptional group of Lie type. Then $G = F^*(G)$ and V is as given in Table 6.*

Proof. Let V be a 2F-module for G, where $F^*(G)$ is an exceptional group of Lie type. Let V_0 denote a composition factor of $V|_{F^*(G)}$. By [5, Theorem 8.1] we know that V_0 is a 2F-module for $F^*(G)$, hence contained in [5, Table 1]. More precisely, $F^*(G)$

is one of ${}^2B_2(2^{2a+1})$, $G_2(\ell^a)$, $F_4(\ell^a)$, $E_6(\ell^a)$, ${}^2E_6(\ell^a)$, $E_7(\ell^a)$, and V_0 is a non-trivial irreducible representation of $F^*(G)$ of minimal possible dimension.

1. Since Suzuki groups are defined over fields of even order but have no outer automorphisms of even order, this case does not arise.

2. Now let $F^*(G) = G_2(\ell^a)$, $\ell^a \neq 2$ (the group $G_2(2) = \mathrm{U}_3(3)$ was treated in [4, Proposition 4.3]). Only field automorphisms (of order ℓ) matter, so $G = G_2(\ell^a).\ell$ with $\ell|a$. Then V is the induction to G of the $d := 7-\delta_{\ell,2}$-dimensional module V_0 for $F^*(G)$. On this, the field automorphism has centralizer dimension d. The ℓ-rank of $G_2(\ell^a)$ is at most $4a$ by [5, Table 13]. Thus we find $\ell = 2$, hence $d = 6$. But in this case the ℓ-rank is only $3a$. Moreover inner ℓ-elements have at most 4-dimensional centralizer on V_0. This forces $a = 2$, so $F^*(G) = G_2(4)$, and $A \cap F^*(G)$ has order 8. But subgroups of order 8 have at most a 3-dimensional centralizer on V_0, hence no example arises.

3. For $F^*(G) = F_4(\ell^a)$, V_0 is a 25- or 26-dimensional module, invariant under neither the field automorphisms nor the graph automorphism in characteristic 2. The standard estimate leads to a contradiction here.

4. Now let $G = E_6(\ell^a)_{sc}.\ell$ involve a field automorphism of order ℓ. For the non-invariant 27-dimensional module V_0 of $F^*(G)$ the standard estimate forces $\ell = 2$. The commutator space of non-trivial ℓ-elements on V_0 is at least 6-dimensional, and at least 10-dimensional for all but long root elements by [9, Table 5]. On the other hand, the maximal order of an ℓ-subgroup consisting only of long root elements is ℓ^{5a} by [5, Table 5]. This gives a contradiction. The case of graph and graph-field automorphisms was already ruled out in the proof of [5, Lemma 8.10].

5. For $F^*(G) = {}^2E_6(q)_{sc}$ and $F^*(G) = E_7(q)_{sc}$ only field automorphisms arise. The modules of smallest possible dimension 27 respectively 56 are not invariant under field automorphisms, and the standard estimate allows us to conclude that no 2F-modules arise. $\square$

Table 6. 2F-modules for exceptional groups in defining characteristic

G	d	V	f	conditions	$\log_\ell \lvert A\rvert$
${}^2B_2(2^{2a+1})$	4	V_0	$2a+1$	$a>1$	$2a+1$
$G_2(\ell^a)$	$7-\delta_{\ell,2}$	V_0	a		$3a$
$F_4(2^a)$	26	V_0	a		$10a$
$F_4(3^a)$	25	V_0	a		$8a$
$E_6(\ell^a)$	27	V_0	a		$16a$

11. Cubic action

In this section we determine which of the examples (G,V), where G is a group of Lie type and V is a 2F-module for G in its defining characteristic, have cubic action, that is, there exists an elementary abelian ℓ-subgroup $1 \neq A \leq G$ such that

$$[[[V,A],A],A] = 0 \qquad \text{and} \qquad |V/C_V(A)| \leq |A|^2. \tag{6}$$

We call these the *2F-modules with cubic offender*. In the case where V is a 2F-module for G not a group of Lie type in its defining characteristic, this classification was already achieved in [4, §6].

Theorem 11.1. *Let V be an absolutely irreducible 2F-module for a finite group G such that $F^*(G)$ is quasi-simple. Then there exists a non-trivial elementary abelian offender A acting cubically on V.*

Proof. If $F^*(G)$ is not of Lie type in the same characteristic as V, then this was already proved in [4, Theorem 6.2, 6.4, 6.6].

For groups of Lie type in their defining characteristic we use the classification of 2F-modules in [5, Table 1] and results of the previous sections. We start with some general observations. If V is in fact an F-module for G, then by Thompson's replacement lemma there exists an offender which actually acts quadratically. Thus the result is proved for such modules. Secondly, if A acts quadratically on V and on W, then it acts cubically on $V \otimes W$, hence also on $\wedge^2(V)$ and on $\Sigma^2(V)$.

We look at the remaining cases in detail. We start with the quasi-simple case, i.e., $G = F^*(G)$. For $G = \mathrm{SL}_n(q)$ the only remaining example is Case 6.5. Here clearly the offending subgroup A actually acts quadratically on $\wedge^3(Y_6)$. All examples for the unitary groups are already F-modules except for the natural module of $\mathrm{SU}_3(q)$. But in dimension 3 all A act cubically. For symplectic and orthogonal groups, only the spin modules remain. Here the offender A given in the examples is the unipotent radical of a parabolic subgroup P of orthogonal type. The spin module Z_n splits into a sum of two spin modules Z_{n-1} for the Levi subgroup, one of which is the centralizer $C_{Z_n}(A)$ of A, and A acts trivially on $Z_n/C_{Z_n}(A)$. Thus we actually have quadratic action.

The 2F-modules for exceptional G are listed in Table 6. The offending subgroup for the Suzuki group ${}^2B_2(2^{2a+1})$ has 2-dimensional centralizer on the natural 4-dimensional module, hence acts cubically. For $G_2(\ell^a)$, $\ell \neq 3$, consider the centralizer C of type A_2 of an element of order 3. The $(7 - \delta_{\ell,2})$-dimensional module restricts as $Y_3 \oplus Y_3^* \oplus 1$ to C when $\ell > 2$, respectively as $Y_3 \oplus Y_3^*$ when $\ell = 2$ (the constituents are the eigenspaces for the element of order 3). The unipotent radical of a maximal parabolic subgroup of C has order q^2 and thus clearly acts cubically and has centralizer dimension at least $3 - \delta_{\ell,2}$, so is an offender. In characteristic 3 $G_2(3^a)$ contains bitransvections on the module of dimension 7, see for example [9, Table 1], which gives a quadratic offender.

The 26-dimensional module for $F_4(2^a)$ is the heart of the exterior square of the natural module for $\mathrm{Sp}_8(2^a)$, on which there exists a cubic offender by what we said above. For the 25-dimensional module $V(\lambda_1)$ for $F_4(3^a)$, the offender A exhibited above in proving Theorem 3.1 is cubic: for the elements of A are products of root elements for E_7-roots with α_4-coefficient at least 2, while the elements of the 26-dimensional module having $V(\lambda_1)$ as quotient are products of root elements with α_4-coefficient ranging from 0 to 4. Finally, each of the 27-dimensional modules for $E_6(\ell^a)$ may be treated by a similar easy argument: let V be the unipotent radical of the E_6-parabolic of $E_7(q)$, and let A be the product of the root subgroups of $E_6(q)$

with α_1-coefficient equal to 1; since the elements of V are products of root elements with α_1-coefficient ranging from 0 to 2, it follows as before that A is a cubic offender.

It remains to consider 2F-modules V for groups $G \neq F^*(G)$ with $F^*(G)$ of Lie type, in the same characteristic as V. By our classification, the restriction $V|_{F^*(G)}$ again only contains 2F-modules for $F^*(G)$, so we are done by the previous part if $V|_{F^*(G)}$ is irreducible. Furthermore, if $(G : F^*(G)) = \ell$ and $V = \mathrm{Ind}_{F^*(G)}^G W$ is induced but defined over a proper subfield (of index ℓ), then a cubic offender on W inside $F^*(G)$ trivially is a cubic offender on V. Hence finally assume that V is induced and defined over the same field. For Examples 6.8, 6.10 and 6.12 a maximal elementary abelian subgroup does the job, as well as for Example 7.4. No cases for symplectic groups remain. For Examples 9.10 and 9.12 the unipotent radical of an end-node parabolic of orthogonal type is a quadratic offender, and for Example 9.13 we may again choose a maximal elementary abelian subgroup. This completes the proof of the theorem. □

Note that, in general, we cannot expect to find an offender A acting cubically such that $G = \langle A^G \rangle$. Counter-examples are given by induced natural modules for groups of Lie type extended by field automorphisms. Here no nontrivial outer ℓ-element acts cubically, for ℓ large enough.

12. F-modules

In this section we determine the F-modules for groups G such that $F^*(G)$ is quasi-simple and thus prove Theorem 4. The result for F-modules for quasi-simple groups is in [5, Theorem B]. We reproduce this in Table 7.

Table 7. F-modules for quasi-simple groups

G	d	V	ℓ	f	conditions
$\mathrm{SL}_n(\ell^a)$	n	Y_n	ℓ	a	
$\mathrm{SL}_n(\ell^a)$	$\binom{n}{2}$	$\wedge^2(Y_n)$	ℓ	a	$n \geq 4$
$\mathrm{SU}_n(\ell^a)$	n	Y_n	ℓ	$2a$	$n \geq 4$
$\mathrm{Sp}_{2n}(\ell^a)$	$2n$	Y_{2n}	ℓ	a	
$\mathrm{Spin}_n^{(\pm)}(\ell^a)$	n	Y_n	ℓ	a	$n \geq 6$
$\mathrm{Spin}_7(\ell^a)$	8	Z_3	ℓ	a	
$\mathrm{Spin}_{2n}^+(\ell^a)$	2^{n-1}	Z_{n-1}	ℓ	a	$n = 4, 5$
$G_2(2^a)$	6	$M(\lambda_2)$	2	a	
$\mathfrak{A}_{2n}$	$2n-2$		2	1	
$3.\mathfrak{A}_6$	3		2	2	
$\mathfrak{A}_7$	4		2	1	

We now consider the cases where $N := F^*(G)$ is quasi-simple and V is an F-module (absolutely irreducible) with an offender A whose conjugates generate G and that properly contains N.

We note that in particular, V is necessarily a 2F-module. So we only need consider those G that are not themselves quasi-simple and we go through them one at a time. If the characteristic ℓ is odd, it follows immediately from this that $F^*(G)$ is a classical group and V is the natural module for G. It also follows that if $F^*(G)$ is a Chevalley group in defining characteristic, then it must be a classical group.

Let A be an offender whose conjugates generate G. Let $B := A \cap N$.

12.1. **Alternating, sporadic and cross characteristic cases.** In this subsection, we consider modules in characteristic $\ell = 2$ for sporadic and alternating groups and for Chevalley groups in odd characteristic.

Case 1. $G = \mathfrak{S}_n$ and V the heart of the permutation module. Since the module is defined over the prime field and G contains transvections, this is an example.

Case 2. $G = 3.\mathfrak{S}_6$ and $d = 6$. Any outer involution has a 3-dimensional centralizer (since the module is induced). In particular, $B \neq 1$. It follows that $C_V(A)$ has dimension at most 2. Since $|A| \leq 8$, this is not an example.

Case 3. $G = \mathfrak{S}_7$, $d = 8$. Any outer involution has a 4-dimensional centralizer and $C_V(A)$ has dimension at most 3. However, $|A| \leq 16$, so this is not an example.

Case 4. $U_3(2).2 = G_2(2)$, $d = 6$. This is an example (as for $G_2(2^a)$ in general – see Table 7).

Case 5. $G = 3_1.U_4(3).2_2$ with $d = 6$ over $\mathbb{F}_4$. The 2-rank of G equals 5. Explicit computation shows that elementary abelian subgroups A of order 16 containing outer involutions centralize at most a 3-dimensional subspace, and subgroups of order 4 centralize at most a 4-dimensional subspace. Hence this is not an F-module.

Case 6. $G = 3_1.U_4(3).(2^2)_{122}$ with $d = 12$. Then the centralizer of some outer involution in A has a 6-dimensional centralizer. Thus, $|A| = 2^6$ and $C_V(A) = C_V(x)$, which is easily seen not to be the case.

Case 7. $G = M_{12} : 2$, $d = 10$. There is only one class of outer involutions and its fixed space is 5-dimensional. The 2-rank of the centralizer of an outer involution is 4. So this is not an example.

Case 8. $G = M_{22} : 2$, $d = 10$. There are two classes of outer involutions. If x is an involution of class 2B, then the fixed space has dimension 7. One computes directly that any elementary abelian subgroup of order 4 containing x has fixed space of dimension at most 5 and any elementary abelian subgroup of rank 3 containing x has a fixed space of dimension at most 4 and so there is no example (as the 2-rank of G is 5).

If x is in class 2C, then the fixed space of x is 5-dimensional. Any rank 2 subgroup containing x has fixed space of dimension at most 4 and so there is no example (as the 2-rank of G is 5).

12.2. **Classical groups.** In this subsection, we consider modules in defining characteristic ℓ for classical groups.

Linear groups:

1. Natural module (field, graph, graph-field, graph and field automorphisms)

(a) $G = \mathrm{SL}_n(\ell^{\ell a}).\ell$, $n \geq 2$, the extension by a field automorphism of order ℓ, on the module $V = Y_n^G$ induced from the natural module for $F^*(G)$ over the field $\mathbb{F}_q$

with $q := \ell^a$. Then $C_N(x) = \mathrm{SL}_n(q)$ and for n even, $|B| \le q^{n^2/4}$ and for n odd, $|B| \le q^{(n^2-1)/4}$ (see [5, Proposition 3.10]).

Suppose that n is even. So a necessary condition for V to be an F-module is that $an^2/4 + an/2 + 1 \ge a\ell n$, that is, $\ell \le (n+2)/4 + 1/an$. This is satisfied if and only if $\ell \le (n+2)/4$. Conversely, if $\ell \le (n+2)/4$, we choose B to be of maximal order (i.e. of rank $an^2/4$) and then $C_V(A)$ has dimension $an/2$ and so $an^2/4 + an/2 + 1 \ge a\ell n$.

Similarly, for n odd, we see that this is an example for $\ell \ge (n+2)/4 + 1/an + 1/4n$. This is satisfied when $\ell \le (n+2)/4$, as well as when $n = 5$, $\ell = 2$, $a = 1$.

(b) $G = \mathrm{SL}_n(2^a).2$, $n \ge 3$, the extension with the graph automorphism γ, on the module $V = Y_n^G$ induced from the natural module. First suppose that $n = 2m$ is even. Then we can choose γ so that $C_N(\gamma) = \mathrm{Sp}_{2m}(2^a)$. Let B be the subgroup of $C_N(\gamma)$ acting trivially on a maximal totally isotropic subspace. So the rank of B is $am(m+1)/2$ (see [5, Proposition 3.12]). Thus, we obtain the condition $am(m+1)/2 + am + 1 \ge 4am$, that is, $m \ge 5$, hence $n \ge 10$.

Now suppose that $n = 2m+1$ is odd. Here $C_N(\gamma)$ is an orthogonal group $\mathrm{SO}_{2m+1}(2^a)$. Let B be the subgroup of $C_N(\gamma)$ trivial on a totally singular subspace of dimension $m+1$ in the natural module (and also the dual). Then B contains an elementary abelian subgroup of rank $m(m+1)a/2$. Set $A = \langle B, \gamma\rangle$. Then $C_V(A)$ has dimension $m+1$. So A is an offender if $n \ge 11$.

(c) $G = \mathrm{SL}_n(2^{2a}).2$, $n \ge 3$, the extension by a graph-field automorphism γ on the module $V = Y_n^G$ induced from the natural module. Set $q := 2^a$, so V has dimension $2n$ over $\mathbb{F}_q$. The centralizer $C_N(\gamma)$ is a unitary group $\mathrm{SU}_n(q)$. If n is even, this has a maximal elementary abelian ℓ-subgroup B of order $q^{n^2/4}$ with centralizer dimension $n/2$ (see [5, Proposition 3.11]). We arrive at $n \ge 14$ for this to yield an offender.

If n is odd, there exists such B of order $q^{(n-1)^2/4}$. This leads to the condition $n \ge 17$.

(d) $G = \mathrm{SL}_n(2^a).2^2$, $n \ge 3$, on $V = Y_n^G$ with Y_n the natural module. Here, the centralizer of a field and a graph automorphism is at most a symplectic group $\mathrm{Sp}_{\lfloor n/2\rfloor}(q)$ with $q = 2^a$. Using the bounds for ℓ-ranks and centralizer dimensions in [5, Proposition 3.12] we arrive at $n \ge 26$ when n is even, respectively $n \ge 29$ when n is odd.

2. $G = \mathrm{SL}_n(2^{2a}).2$, the extension by the field automorphism, on $V := Y_n \otimes Y_n^{(a)}$, for $n = 2$, $a \le 2$, or $n = 3$, $a = 1$.

(a) $G = \mathrm{SL}_2(4).2$, so $n = 2$, $a = 1$. This is $\mathfrak{S}_5$ on the heart of the permutation module, so an example.

(b) $G = \mathrm{SL}_2(16).2$, so $n = 2$, $a = 2$. The centralizer of an outer involution in $F^*(G)$ is $\mathrm{SL}_2(4) = \mathfrak{A}_5$, whence $|A| \le 8$. It follows that this is not an F-module.

(c) $G = \mathrm{SL}_3(4).2$, so $n = 3$, $a = 1$. Here, V is defined over $\mathbb{F}_2$ and has dimension 9. If γ is an outer involution, then its centralizer in G is $\mathrm{SL}_3(2) \times \langle\gamma\rangle$. Moreover, $C_V(\gamma)$ has dimension 6 and has two non-trivial composition factors for $\mathrm{SL}_3(2)$, whence this is not an F-module.

3. $G = \mathrm{SL}_4(2^a).2$ the extension with a graph automorphism acting on $\wedge^2(Y_4)$, the outer square of the natural module. This is $\mathrm{SO}_6^+(q)$ in its natural representation,

where $q = 2^a$. The centralizer of a graph automorphism is a 5-dimensional orthogonal group. If $q = 2$, this is also $\mathfrak{S}_8$ on the heart of the permutation module, hence an example with A of order 2 generated by a transvection. If $q > 2$, the bounds for the 2-rank of subgroups with given centralizer in [5, Proposition 3.14] show that it is not an F-module for G.

4. $G = \mathrm{SL}_4(4).2^2$ with $V = \wedge^2(Y_4) \oplus \wedge^2(Y_4^{(1)})$. Here A must contain a field automorphism γ of order 2. Then $C_G(\gamma) = \mathfrak{S}_8 \times \langle\gamma\rangle$ and $C_V(\gamma)$ has dimension 6. Since the 2 rank of $\mathfrak{S}_8$ is 4, this is not an F-module.

Unitary groups:

1. $G = \mathrm{SU}_n(\ell^{\ell a}).\ell$, extension by a field automorphism γ, acting on the module Y_n^G induced from the natural module over $\mathbb{F}_{q^2}$, $q := \ell^a$, with ℓ odd. Then $C_N(\gamma) = \mathrm{SU}_n(q)$ and for n even, $|B| \leq q^{n^2/4}$ while for n odd, $|B| \leq q^{(n-1)^2/4+1}$ (see [5, Proposition 3.10]).

Suppose that n is even. So a necessary condition for V to be an F-module is that $an^2/4 + an/2 + 1 \geq 2a\ell n$ or $\ell \leq (n+2)/8 + 1/2an$. This is satisfied if and only if $\ell \geq (n+2)/8$.

Similarly, for n odd, we see that this is an example for $\ell \leq n/8 + 3/8n + 5/8an$. This is satisfied when $\ell \leq (n-1)/8$.

2. $G = \mathrm{SU}_n(2^a).2$, extension by a graph-field automorphism, acting on the module Y_n^G induced from the natural module over $\mathbb{F}_{2^a}$. Here the centralizer of an outer automorphism is contained in a symplectic group. The bounds for ℓ-ranks in [5, Proposition 3.12] lead to the condition $n \geq 10$ for n even, respectively $n \geq 13$ or $(n, a) = (11, 1)$ for n odd.

3. $G = \mathrm{SU}_4(2^a).2$ acting as $\mathrm{SO}_6^-(2^a)$ on its natural module. As in the SO_6^+ case, this is an example when $a = 1$ and not an example for $a > 1$, by application of [5, Proposition 3.14].

Symplectic groups:

1. $G = \mathrm{Sp}_{2n}(\ell^{\ell a}).\ell$, extension by a field automorphism, acting on the module Y_{2n}^G induced from the natural module over $\mathbb{F}_{\ell^a}$. Here the maximal centralizer of an outer ℓ-element is $\mathrm{Sp}_{2n}(\ell^a)$. The bounds in [5, Proposition 3.12] show that we get an example if and only if $\ell \geq (n+3)/4$.

2. $G = \mathrm{Sp}_6(4).2$ with V being induced from the 8-dimensional spin module. Let $\gamma \in A$ be an outer involution. Then $C_V(\gamma)$ has dimension 8 and $C_G(\gamma) = \mathrm{Sp}_6(2) \times \langle\gamma\rangle$. So $|A| > 2$ and $C_V(A)$ has dimension at most 6. Since $|A| \leq 16$, this is not an example.

Orthogonal groups in dimension at least 7:

1. Natural module with field, graph, field-graph or field and graph automorphism.

(a) $G = \mathrm{SO}_{2n+1}(\ell^{\ell a}).\ell$, extension by the field automorphism, on the module induced from the natural module. The standard estimate using [5, Proposition 3.14] gives the condition $\ell \leq n/4$.

(b) $G = \mathrm{SO}_{2n}^{\pm}(\ell^{\ell a}).\ell$, extension by the field automorphism, on the module induced from the natural module. The standard estimate gives the condition $\ell \leq (n+1)/4$.

(c) $G = \mathrm{SO}_{2n}^{\pm}(2^a).2$, extension by the graph automorphism of order 2, on the natural module. The standard estimate gives the condition that either $a = 1$ or $n \geq 5$.

(d) $G = \mathrm{SO}_{2n}^+(2^{2a}).2$, extension by the graph-field automorphism of order 2, on the module induced from the natural module. Here we arrive at $n \geq 7$.

(e) $G = \mathrm{SO}_{2n}^+(2^{2a}).2^2$, extension by the graph and field automorphisms, on $V = Y_{2n} \oplus Y_{2n}^{(a)}$. Here we get $n \geq 9$.

2. $G = \mathrm{Spin}_8^+(2^{2a}).2$, extension by a field automorphism and V the module induced from the spin module of dimension 8, defined over the field of size 2^{2a}. This is the image under triality of case 1(b) above, hence not an example for $\ell = 2$.

3. $G = \mathrm{Spin}_8^\pm(2).2$ with V the module induced from the spin module. It follows that $|A| \leq 2^7$ and $C_V(A)$ has dimension less than 8, so this is not an example.

This concludes the proof of Theorem 4.

TABLE 8. F-modules for groups G with $F^*(G) \neq G$ quasi-simple

G	d	V	f	conditions	type
$\mathfrak{S}_n$	$n-1$	*nat.*	1	$5 \leq n \equiv 1 \pmod 2$	
$\mathfrak{S}_n$	$n-2$	*nat.*	1	$6 \leq n \equiv 0 \pmod 2$	
$\mathrm{U}_3(3).2$	6	1			
$\mathrm{SL}_n(\ell^{\ell a}).\ell$	ℓn	$Y_n \uparrow^G$	a	$\ell \leq (n+2)/4$	f
				or $(n, \ell, a) = (5, 2, 1)$	f
$\mathrm{SL}_n(2^a).2$	$2n$	$Y_n \uparrow^G$	a	$n \geq 10$, $n \neq 11$	g
				or $(n, a) = (11, 1)$	g
$\mathrm{SL}_4(2).2$	6	$\wedge^2(Y_4)$	1		g
$\mathrm{SL}_n(2^{2a}).2$	$2n$	$Y_n \uparrow^G$	$2a$	$n \geq 14$, $n \neq 15$	gf
$\mathrm{SL}_n(2^{2a}).2^2$	$4n$	$Y_n \uparrow^G$	a	$n \geq 26$, $n \neq 27$	f, g
$\mathrm{SU}_n(\ell^{\ell a}).\ell$	ℓn	$Y_n \uparrow^G$	$2a$	n even, $2 \neq \ell \leq (n+2)/8$	f
				or n odd, $2 \neq \ell \leq (n-1)/8$	f
$\mathrm{SU}_n(2^a).2$	$2n$	$Y_n \uparrow^G$	a	$n \geq 10$, $n \neq 11$	gf
				or $(n, a) = (11, 1)$	gf
$\mathrm{SU}_4(2).2$	6	$\wedge^2(Y_4)$	1		gf
$\mathrm{Sp}_{2n}(\ell^{\ell a}).\ell$	$2n\ell$	$Y_{2n} \uparrow^G$	a	$\ell \leq (n+3)/4$	f
$\mathrm{Spin}_{2n+1}(\ell^{\ell a}).\ell$	$(2n+1)\ell$	$Y_{2n+1} \uparrow^G$	a	$\ell \leq n/4$	f
$\mathrm{Spin}_{2n}^+(\ell^{\ell a}).\ell$	$2n\ell$	$Y_{2n} \uparrow^G$	a	$\ell \leq (n+1)/4$	f
$\mathrm{Spin}_{2n}^-(\ell^{\ell a}).\ell$	$2n\ell$	$Y_{2n} \uparrow^G$	a	$2 \neq \ell \leq (n+1)/4$	f
$\mathrm{Spin}_{2n}^\pm(2^a).2$	$2n$	Y_{2n}	a	$n \geq 5$ or $(n, a) = (4, 1)$	g
$\mathrm{Spin}_{2n}^+(2^{2a}).2$	$4n$	$Y_{2n} \uparrow^G$	a	$n \geq 7$	gf
$\mathrm{Spin}_{2n}^+(2^{2a}).2^2$	$4n$	$Y_{2n} \oplus Y_{2n}^{(a)}$	a	$n \geq 9$	f, g

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