THE NAVARRO–TIEP GALOIS CONJECTURE FOR $p = 2$

GUNTER MALLE

Abstract. We prove a recent conjecture of Navarro and Tiep on the 2-rationality of characters of finite groups in relation with the structure of the commutator factor group of a Sylow 2-subgroup. On the way we complete the characterisation of characters of odd degree in quasi-simple groups of Lie type.

1. Introduction

In this note we prove the case $p = 2$ of a recent conjecture of Navarro and Tiep [7] relating rationality properties of irreducible characters of a finite group to the exponent of the commutator factor group of a Sylow p-subgroup. To phrase it, we introduce the following notation: Fix a prime p. For an integer $e \geq 1$ let $\sigma_e = \sigma_{p,e}$ be the automorphism of the maximal abelian extension $\mathbb{Q}^{\rm ab}$ of $\mathbb{Q}$ that fixes p'-roots of unity and sends any p-power root of unity ζ to ζ^{1+p^e}. As values of irreducible characters of a finite group G lie in $\mathbb{Q}^{\rm ab}$, any $\sigma_{p,e}$ acts on $\mathrm{Irr}(G)$.

The following was conjectured in [7, Conj. A] (in fact for all primes p, but here we only solve the case when $p = 2$), adding to the large body of expected relevance of characters of p'-degree:

Theorem 1. *Let G be a finite group and $P \leq G$ a Sylow 2-subgroup. Let $e \geq 1$. Then the exponent of P/P' is less or equal to 2^e if and only if all $\chi \in \mathrm{Irr}_{2'}(G)$ are $\sigma_{2,e}$-invariant.*

Here, for a finite group G, $\mathrm{Irr}_{p'}(G)$ denotes the set of irreducible characters of G of degree not divisible by p. In fact, Navarro and Tiep [7, Thm. B] already show the "if"-direction of Theorem 1 (again for an arbitrary prime p), and they also give a reduction of the "only if"-direction to a question on quasi-simple groups [7, Conj. 5.4]:

Conjecture 2 (Navarro–Tiep). *Let p be a prime and S be a quasi-simple group with $|Z(S)|$ not divisible by p. Let A be a p-group acting on S centralising $Z(S)$ and let $P \leq S$ be an A-invariant Sylow p-subgroup of S. If every A-invariant linear character of P is $\sigma_{p,e}$-invariant, then every A-invariant $\chi \in \mathrm{Irr}_{p'}(S)$ is $\sigma_{p,e}$-invariant.*

It is this assertion on quasi-simple groups that we verify in this note. More precisely, in Section 2 we show that Conjecture 2 holds for all primes p if S is not of Lie type in cross characteristic, while in Section 3 we discuss the remaining cases when $p = 2$. For this, we extend the characterisation of odd degree characters in groups of Lie type to include the groups of type A, see Theorem 3.3.

Date: April 29, 2019.

2010 *Mathematics Subject Classification.* 20C15, 20C33.

Key words and phrases. McKay conjecture, Galois action, rationality and Sylow subgroups.

The author gratefully acknowledges financial support by SFB TRR 195.

As pointed out in [7], Theorem 1 would follow from Navarro's Galois–McKay conjecture [6], but this seems currently far out of reach; thus, our result should be seen as a further confirmation for the validity of this "stunning" (according to J. Alperin) conjecture.

Acknowledgement: The Chevie system provided by J. Michel [5] proved extremely useful for computations on component groups of centralisers of quasi-isolated elements. I like to thank Mandi Schaeffer Fry for her comments on an earlier version.

2. Non-Lie type groups

In this section we prove Conjecture 2 for all cases apart from groups of Lie type in non-defining characteristic, for all primes. For odd primes, these results were essentially also already obtained in [7, Thm. 5.8]. Our proofs in particular for the defining characteristic case are partly different and also work for $p = 2$.

Proposition 2.1. *The assertion of Conjecture 2 holds for any prime p when S is a covering group of a sporadic simple group or ${}^2F_4(2)'$.*

Proof. It can be checked from the known ordinary character tables that all irreducible p'-degree characters of the groups in question are $\sigma_{p,1}$ fixed for all primes p, except for the characters of ${}^2F_4(2)'$ of degrees 27 and 351, which are not $\sigma_{2,1}$-fixed (but $\sigma_{2,2}$-fixed). All four extend to $\mathrm{Aut}({}^2F_4(2)') = {}^2F_4(2)$. Explicit computation shows that a Sylow 2-subgroup P of ${}^2F_4(2)'$ has $P/P' = C_4 \times C_2$, and this is centralised by the stabiliser in $\mathrm{Aut}({}^2F_4(2)')$ of P. □

Proposition 2.2. *The assertion of Conjecture 2 holds for any prime p when S is an exceptional covering group of a simple group of Lie type, or of $\mathfrak{A}_6$ or $\mathfrak{A}_7$.*

Proof. Again it can be verified from the known ordinary character tables that all irreducible p'-degree characters of the groups in question are $\sigma_{p,1}$ fixed for all primes p. □

Proposition 2.3. *The assertion of Conjecture 2 holds for any prime p when S is a covering group of an alternating group.*

Proof. For $p > 2$ this has been argued in [7, Thm. 5.8], so now assume that $p = 2$. By assumption we only need to consider covering groups with centre of odd order, and so by Proposition 2.2 we may assume that $S = \mathfrak{A}_n$ with $n \geq 5$. As all characters of $\mathfrak{S}_n$ are rational, all of its conjugacy classes are rational. So the irrational conjugacy classes of $\mathfrak{A}_n$ are precisely those classes of $\mathfrak{S}_n$ contained in $\mathfrak{A}_n$ whose centraliser is entirely contained inside $\mathfrak{A}_n$. It is easily seen (and well-known) that such classes contain permutations with all cycle lengths odd and mutually different. Thus the values of any irreducible character of $\mathfrak{A}_n$ on such a class lie in an extension of $\mathbb{Q}$ by $2'$-roots of unity and these are $\sigma_{2,1}$-fixed. The claim follows. □

Proposition 2.4. *Let S be a quasi-simple group of Lie type in characteristic p. Then the assertion of Conjecture 2 holds for S.*

Proof. By Proposition 2.2 we may assume that S is not an exceptional covering group of $S/Z(S)$, and by Proposition 2.1 moreover we have $S \neq {}^2F_4(2)'$. Thus there is a simple algebraic group $\mathbf{G}$ of simply connected type, with a Steinberg map $F : \mathbf{G} \to \mathbf{G}$ such that $S = G/Z$ with $G = \mathbf{G}^F$ and $Z \leq Z(G)$ such that $|Z(S)|$ is prime to p.

Let $\chi \in \mathrm{Irr}_{p'}(G)$. Then, apart from some exceptions which we will deal with below, there is a semisimple element $s \in \mathbf{G}^{*F}$ such that $\chi \in \mathcal{E}(G, s)$ is a semisimple character (see [2, Thm. 6.8]). Let $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding with dual epimorphism $\iota^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$, and denote by F a Steinberg map on $\tilde{\mathbf{G}}$ extending that on $\mathbf{G}$. Let $\tilde{s} \in \tilde{\mathbf{G}}^{*F}$ be a semisimple element with $\iota^*(\tilde{s}) = s$, and let $\tilde{\chi} \in \mathcal{E}(\tilde{\mathbf{G}}^F, \tilde{s})$ be the semisimple character of $\tilde{\mathbf{G}}^F$ lying above χ. Then $\tilde{\chi}$ is σ_e-fixed being the unique semisimple character in the σ_e-stable Lusztig series $\mathcal{E}(\tilde{\mathbf{G}}^F, \tilde{s})$ (see [9, Lemma 3.4]). Now $\tilde{\chi}|_G$ is a multiplicity-free sum of k irreducible characters for some k dividing $|\tilde{\mathbf{G}}^F : \mathbf{G}^F|$, hence prime to p, while the order of σ_e is a power of p. Since conjugation commutes with σ_e, all constituents of $\tilde{\chi}|_G$ are thus also σ_e-fixed, in particular so is χ.

The exceptions mentioned above occur for

$$S/Z(S) \in \{{}^2B_2(2),\ {}^2G_2(3),\ G_2(2),\ G_2(3),\ {}^2F_4(2),\ F_4(2)\}$$

and for $\mathrm{Sp}_{2n}(2)$, $n \geq 2$. The group ${}^2B_2(2)$ is solvable of order 20, and ${}^2F_4(2)$ was excluded. The character tables of the remaining four groups in the list are known and the claim can be checked directly. Note that ${}^2G_2(3)' \cong \mathrm{L}_2(8)$ has $P/P' \cong C_9$ for $p = 3$ and there are three characters of degree 7 that are fixed by $\sigma_{3,2}$ but not by $\sigma_{3,1}$. Finally, to deal with $\mathrm{Sp}_{2n}(2)$, we use that the unipotent characters of groups of classical type are determined by their multiplicities in Deligne–Lusztig characters, which are p-rational (again by [9, Lemma 3.4]). Since Sp_{2n} has connected centre in characteristic 2, all centralisers of semisimple elements in the dual group SO_{2n+1} are connected, and clearly of classical type. So by Jordan decomposition, all irreducible characters of $\mathrm{Sp}_{2n}(2)$ are determined by their multiplicities in the $\sigma_{2,1}$-invariant Deligne–Lusztig characters, and thus are $\sigma_{2,1}$-fixed. □

3. Groups of Lie type in odd characteristic

Here we consider Conjecture 2 for groups of Lie type in odd characteristic for the prime $p = 2$. Ultimately, for these groups, the validity of the conjecture hinges on the fact that the commutator factor group of a Sylow 2-subgroup has the same exponent as the centre of a Sylow 2-subgroup of the dual group (see Proposition 3.2).

We introduce the following setup. Let $\mathbf{G}$ be a simple algebraic group of simply connected type over an algebraically closed field of characteristic $r > 0$ and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg map such that $G = \mathbf{G}^F$ is quasi-simple. Let $\mathbf{G}^*$ be dual to $\mathbf{G}$ with corresponding Steinberg map also denoted F and $G^* = \mathbf{G}^{*F}$. For the first result, the characteristic r of $\mathbf{G}$ can be arbitrary:

Proposition 3.1. *Let $\mathbf{G}$ be as above and let $p \neq r$ be a prime. Let $s \in \mathbf{G}^{*F}$ be semisimple with connected centraliser and assume that $\mathcal{E}(G, s)$ is $\sigma_{p,e}$-stable.*

(a) *If $\mathbf{G}^F$ is of classical type, then any $\chi \in \mathcal{E}(G, s)$ is $\sigma_{p,e}$-invariant.*
(b) *If $\chi \in \mathcal{E}(G, s) \cap \mathrm{Irr}_{p'}(G)$ lies in the principal series of G then χ is $\sigma_{p,e}$-invariant.*

Proof. First assume that $\mathbf{G}^F$ is of classical type. Then $C_{\mathbf{G}^*}(s)$ is also of classical type and connected by assumption. Now the unipotent characters of classical type groups are uniquely determined by their multiplicities in the Deligne–Lusztig characters, hence by Lusztig's Jordan decomposition the same is true for the characters in $\mathcal{E}(G, s)$. As $\mathcal{E}(G, s)$ is $\sigma_{p,e}$-stable, so is the $\mathbf{G}^{*F}$-class of s and hence are the corresponding Deligne–Lusztig characters. Hence χ as in (a) is also $\sigma_{p,e}$-invariant.

In (b), χ lies in the principal series of G. That is, if $\mathbf{T}$ denotes a maximally split maximal torus of $\mathbf{G}$, then there exists $\lambda \in \mathrm{Irr}(\mathbf{T}^F)$ such that χ lies in the Harish-Chandra series of $(\mathbf{T}, \lambda)$. Let $(\mathbf{T}^*, s')$ be dual to $(\mathbf{T}, \lambda)$, so $\lambda \in \mathcal{E}(\mathbf{T}^F, s')$. As Deligne–Lusztig induction preserves labels of Lusztig series we have that s' and s are conjugate. Thus by [8, Lemma 4.5(1)] the relative Weyl group $W_G(\mathbf{T}, \lambda)$ identifies to $W(s)$, the F-fixed points of the Weyl group of $C_{\mathbf{G}^*}(s)$.

The action of Galois automorphisms on Harish-Chandra series has been determined by Schaeffer Fry [8, Thm. 3.8]: $\sigma_{p,e}$ acts by permuting the Harish-Chandra sources λ, and if it fixes λ (up to G-conjugation) then it acts by $\sigma_{p,e}$ on the irreducible characters of the Hecke algebra $\mathcal{H}_G(\mathbf{T}, \lambda)$. Note that the characters r_σ and $\delta'_{\lambda,\sigma}$ occurring in [8] are both trivial as $C_{\mathbf{G}^*}(s)$ is connected and thus $W_G(\mathbf{T}, \lambda) = W^\circ(s) = R(\lambda)$ by [8, Lemma 4.5(2)]. Now the characters of all Hecke algebras of Weyl groups having no factor of type E_7 or E_8 are rational. Furthermore, the non-rational characters in types E_7 or E_8 only involve $\sqrt{q}$, where q is a power of the underlying characteristic. Thus they are $\sigma_{p,1}$-invariant unless $p = 2$. But for $p = 2$ they do label characters of even degree. As χ was supposed to be of p'-degree the action of $\sigma_{p,e}$ on χ is trivial. $\square$

From now on we assume that *the characteristic r of $\mathbf{G}$ is odd.*

Proposition 3.2. *Let G, G^* be as above. Let $P \in \mathrm{Syl}_2(G)$ be a Sylow 2-subgroup of G and $P^* \in \mathrm{Syl}_2(G^*)$ a Sylow 2-subgroup of the dual group G^*. Then the exponent of $Z(P^*)$ is bounded above by the exponent of P/P'.*

Proof. We prove this case-by-case. First assume that G is of exceptional type. Then $Z(P^*)$ is elementary abelian, for $P^* \in \mathrm{Syl}_2(\mathbf{G}^{*F})$, unless $\mathbf{G}^F = E_6(\epsilon q)$ with $4|(q - \epsilon)$, see Table 1. (The structure of $Z(P^*)$ can easily be deduced from the shape of a Sylow 2-subgroup of the centraliser of a suitable 2-central element $t \in \mathbf{G}^{*F}$ as given in the table.)

TABLE 1. Centres of Sylow 2-subgroups $P^* \in \mathrm{Syl}_2(G^*)$ in exceptional types

G	$C_G(t)$	$Z(P^*)$	G	$C_G(t)$	$Z(P^*)$
${}^2G_2(q^2)$	$A_1(q^2) \times 2$	C_2^3	$E_6(q)$	$D_5(q).T_1$	$C_{\lvert q-1\rvert_2}$
$G_2(q)$	$A_1(q)^2$	C_2	${}^2E_6(q)$	${}^2D_5(q).T_1$	$C_{\lvert q+1\rvert_2}$
${}^3D_4(q)$	$A_1(q^3)A_1(q)$	C_2	$E_7(q)$	$D_6(q)A_1(q)$	C_2
$F_4(q)$	$B_4(q)$	C_2	$E_8(q)$	$D_8(q)$	C_2

Assume that $\mathbf{G}^F = E_6(\epsilon q)$ with $q \equiv \epsilon \pmod 4$. Let $t \in \mathbf{G}^F$ be an involution with centraliser of type D_5. Then the quotient of $C_{\mathbf{G}}(t)$ by the D_5-factor is a 1-dimensional torus, with F-fixed points a cyclic group of order $C_{q-\epsilon}$, whence P/P' has exponent at least $|q - \epsilon|_2$, the exponent of $Z(P^*)$.

Next assume that $\mathbf{G}$ is of classical type B_n, C_n or D_n. Then the centre of a Sylow 2-subgroup of $\mathbf{G}^{*F}$ is again elementary abelian, as follows from the description given in [1, Thm. 4.10.6], so the claim follows.

Finally, we consider $\mathbf{G}^F = \mathrm{SL}_n(\epsilon q)$ with $\epsilon \in \{\pm 1\}$. First assume that $4|(q - \epsilon)$ and that n is a 2-power. As a Sylow 2-subgroup of $\mathrm{GL}_n(\epsilon q)$ is contained in the normaliser of a maximal torus of order $(q - \epsilon)^n$, it is isomorphic to the wreath product of $C_{|q-\epsilon|_2}$

with a Sylow 2-subgroup of $\mathfrak{S}_n$. Its centre consists of the diagonally embedded subgroup $C_{|q-\epsilon|_2}$ of the base group. Thus the centre of a Sylow 2-subgroup of $\mathrm{PGL}_n(\epsilon q)$ has order 2, consisting of (images of) elements of the form $(1, -1, 1, -1, \ldots)$ in the base group, and the same holds for $\mathrm{L}_n(\epsilon q)$.

Now assume that n is not a 2-power, with 2-adic decomposition $n = \sum 2^{a_i}$. Then we use that a Sylow 2-subgroup of $\mathrm{GL}_n(\epsilon q)$ is contained in a subgroup $\prod_i \mathrm{GL}_{2^{a_i}}(\epsilon q)$, and our previous considerations show that the centre of a Sylow 2-subgroup of $\mathrm{GL}_n(\epsilon q)$, $\mathrm{PGL}_n(\epsilon q)$ and $\mathrm{L}_n(\epsilon q)$ has exponent $|q - \epsilon|_2$. An entirely similar calculation shows that a Sylow 2-subgroup P of $\mathrm{SL}_n(\epsilon q)$ or $\mathrm{L}_n(\epsilon q)$ has P/P' of exponent at least $|q - \epsilon|_2$ if n is not a 2-power.

On the other hand, if $4|(q + \epsilon)$ then a Sylow 2-subgroup of $\mathrm{GL}_n(\epsilon q)$ is contained in the normaliser of a maximal torus T of order $(q^2 - 1)^r(q - \epsilon)^m$ with $n = 2r + m$, $m \in \{0, 1\}$. This is an extension of T with the Weyl group of type B_r. Thus the centre of a Sylow 2-subgroup of $\mathrm{SL}_n(\epsilon q)$, $\mathrm{PGL}_n(\epsilon q)$ or $\mathrm{L}_n(\epsilon q)$ is elementary abelian, and we are done. □

We next complete our earlier characterisation of odd degree characters from [4] to include groups of type A:

Theorem 3.3. *Let $\mathbf{G}$ be simple of simply connected type with a Steinberg endomorphism $F : \mathbf{G} \to \mathbf{G}$. Then any odd degree character χ of $G = \mathbf{G}^F$ lies in the principal series of G, unless $G = \mathrm{Sp}_{2n}(q)$ with $n \geq 1$ odd and $q \equiv 3 \pmod 4$, $\chi \in \mathcal{E}(G, s)$ with $C_{G^*}(s) = B_{2k}(q) \cdot {}^2D_{n-2k}(q).2$ where $0 \leq k \leq (n-1)/2$, and χ lies in the Harish-Chandra series of a cuspidal character of degree $\frac{1}{2}(q - 1)$ of a Levi subgroup $\mathrm{Sp}_2(q) \times C_{q-1}^{n-1}$.*

Proof. For all types different from type A, this was shown in [4, Thm. 7.7]. So now assume that $\mathbf{G} = \mathrm{SL}_n$ with a Frobenius map F such that $G = \mathbf{G}^F = \mathrm{SL}_n(\epsilon q)$ with $n \geq 3$, where $\epsilon \in \{\pm 1\}$. If $q \equiv 1 \pmod 4$ then the argument in the proof of loc. cit. applies. So now let $q \equiv 3 \pmod 4$. Assume that $\chi \in \mathrm{Irr}_{2'}(G)$ lies in the Harish-Chandra series of the cuspidal character λ. As the degree of λ divides the degree of any member of its Harish-Chandra series, λ has to have odd degree as well. Thus we are done if we can show that all non-linear cuspidal characters of all Levi subgroups of G have even degree.

For $\mathbf{L} \leq \mathbf{G}$ an F-stable split Levi subgroup, $L = \mathbf{L}^F$ is isomorphic to $\mathrm{SL}_n(q) \cap \prod_{i \geq 1} \mathrm{GL}_{n_i}(q)$ for suitable n_i with $\sum n_i = n$ when $\epsilon = 1$, and to $\mathrm{SU}_n(q) \cap \mathrm{GU}_{n_0}(q) \times \prod_{i \geq 1} \mathrm{GL}_{n_i}(q^2)$ for suitable n_i with $n_0 + 2\sum n_i = n$ when $\epsilon = -1$. For convenience we set $n_0 = 0$ when $\epsilon = 1$.

Let λ be a cuspidal character of L of odd degree. Then the degree polynomial f_λ of λ is divisible by $(X - 1)^r$, where $r = \lfloor n_0/2 \rfloor + \sum_{i \geq 1}(n_i - 1)$ is the semisimple $\mathbb{F}_q$-rank of $\mathbf{L}$ (see [4, Lemma 7.1]). Furthermore, there is some divisor m of $|Z(\mathbf{G})| = n$ such that $mf_\lambda \in \mathbb{Z}[X]$ is a product of cyclotomic polynomials. First assume that $\mathbf{L} = \mathbf{G}$. Then by the above, $\lambda(1)$ is divisible by $2^{n-1}/n_2$ when $\epsilon = 1$, respectively $4^{\lfloor n/2 \rfloor}/n_2$ when $\epsilon = -1$. (Recall that $q \equiv 3 \pmod 4$.) In particular, $\lambda(1)$ is even unless $n = 2$, which was excluded. So G has no non-linear cuspidal characters of odd degree.

So now let $\mathbf{L}$ be proper and set $L' = [\mathbf{L}, \mathbf{L}]^F$. Then there exists a surjection

$$\hat{L} := \prod_{i \geq 1} \mathrm{SL}_{n_i}(q) \to L', \quad \text{respectively} \quad \hat{L} := \mathrm{SU}_{n_0}(q) \times \prod_{i \geq 1} \mathrm{SL}_{n_i}(q^2) \to L',$$

with central kernel. Let λ' be an irreducible character of L' lying below λ and denote the inflation of λ' to $\hat{L}$ again by λ'. Then λ' is also cuspidal and of odd degree. Now we have $\lambda' = \bigotimes_i \lambda_i$ with cuspidal characters λ_i of the direct factors of $\hat{L}$, all necessarily of odd degree. But by what we proved before, cuspidal characters of $\mathrm{SL}_{n_i}(\pm q)$ of odd degree are linear unless $n_i = 2$. So L' is a central product of factors $\mathrm{SL}_2(q)$ and the λ_i are of degree $(q-1)/2$. But the diagonal automorphism induced by L on L' does not fix the λ_i, so that λ has to be of even degree. This achieves the proof. □

We are now ready to prove the main result of this section:

Theorem 3.4. *Let S be a quasi-simple group of Lie type in odd characteristic. Then the assertion of Conjecture 2 holds for S at $p = 2$.*

Proof. Let S be a covering group of a simple group of Lie type in odd characteristic with $|Z(S)|$ odd. By Proposition 2.2, S is not an exceptional covering group, and there is $\mathbf{G}$, F as introduced above such that $S = G/Z$, for $G = \mathbf{G}^F$ and $Z \le Z(G)$. Let $\chi \in \mathrm{Irr}_{2'}(S)$. Then by [2, Thm. 7.5] there is a semisimple element $s \in \mathbf{G}^{*F}$ in the dual group which is 2-central such that $\chi \in \mathcal{E}(G, s)$. Now σ_e permutes the various Lusztig series according to its action on the parametrising semisimple element (see [9, Lemma 3.4]); hence this permutation action is determined already by the (order of the) 2-part of s.

Now first assume that G is of exceptional type. Then any odd degree character of G lies in the principal series of G by [4, Thm. 7.7], above a linear character λ, say. Moreover, by Table 1 we see that the centralisers of all 2-central semisimple elements s are connected. As argued in the proof of Proposition 3.1, λ has the same order as s. Thus, if the exponent of $Z(P^*)$ is bounded above by the exponent e of P/P', then all $\chi \in \mathrm{Irr}_{2'}(G)$ are σ_e-fixed. It now follows from Propositions 3.1(b) and 3.2 that Conjecture 2 holds with $e = 1$ when G is not of type E_6. In the latter case, we need to check the invariance property under 2-automorphisms. Let $A \le \mathrm{Aut}(G)$ be a 2-group of automorphisms of G. Let $t \in \mathbf{G}^F$ be an A-invariant involution with centraliser of type D_5, and similarly $t^* \in \mathbf{G}^{*F}$. (More specifically, choose t, t^* over the prime field and inside the centraliser of a graph automorphism of order 2.) Then $C_G(t)$ contains a Sylow 2-subgroup P of G, and $C_{\mathbf{G}^*}(t^*)^F$ contains a Sylow 2-subgroup P^* of G^* (see Table 1). Now field automorphisms act on the torus

$$\mathbf{T}_1 = Z^\circ(C_{\mathbf{G}}(t)) \cong C_{\mathbf{G}}(t)/[C_{\mathbf{G}}(t), C_{\mathbf{G}}(t)]$$

by p-powers, and similarly on the torus $\mathbf{T}_1^* = Z^\circ(C_{\mathbf{G}^*}(t^*))$, while the graph automorphism acts by inversion on $\mathbf{T}_1$ as well as on $\mathbf{T}_1^*$. Thus, if A fixes all characters of P/P' of order at most 2^e, then it also fixes all such elements in $Z(P^*)$. Thus σ_e fixes all characters in the corresponding Lusztig series. This shows the claim in this case.

Next assume that $\mathbf{G}$ is of classical type B_n, C_n, D_n or ${}^2D_n$. If χ lies in the Lusztig series of a 2-central element $s \in \mathbf{G}^{*F}$ with connected centraliser, the claim follows with $e = 1$ using Propositions 3.1(a) and 3.2. The 2-central elements of $\mathbf{G}^*$ with disconnected centraliser are listed in [4, Table 1] for $q \equiv -1 \pmod 4$; for $q \equiv 1 \pmod 4$ it can be obtained by Ennola twisting this list, see Table 2 (and also [3, Table 2]).

Let $(\mathbf{L}, \lambda)$ denote the Harish-Chandra source of χ. (By [4, Thm. 7.7], unless we are in one special situation when $\mathbf{G}$ is of type C_n with $n \ge 1$, χ lies in the principal series of G.)

First assume that G is of type C_n. Let $s \in \mathbf{G}^{*F}$ be 2-central with disconnected centraliser. Let $\mathbf{M} \le \mathbf{G}$ be a Levi subgroup of type C_{n-1} containing $\mathbf{L}$. From the description

TABLE 2. Disconnected centralisers of 2-central elements, $q \equiv 1 \pmod 4$

$\mathbf{G}^{*F}$	$C_{\mathbf{G}^*}(s)^F$	conditions
$B_n(q)_{\mathrm{ad}}$	$B_{n-d}(q) \cdot D_d(q).2$	$1 \leq d \leq n$
$C_n(q)_{\mathrm{ad}}$	$C_{n/2}(q) \wr 2$	$n = 2^f$
$D_n(q)_{\mathrm{ad}}$	$(D_d(q) \cdot D_{n-d}(q)).2$	$1 \leq d < n/2,\ d(n-1) \equiv 0\,(2)$
	$(D_{n/2}(q) \cdot D_{n/2}(q)).2^2$	$n = 2^f$
${}^2D_n(q)_{\mathrm{ad}}$	$(D_d(q) \cdot {}^2D_{n-d}(q)).2$	$2 \leq d \leq n-1,\ d \neq n/2$

given in [1, Rem. 4.5.4], s lies in $\mathbf{M}^{*F}$ (up to conjugation), and it has disconnected centraliser in $\mathbf{M}^*$ as well, unless $C_{\mathbf{G}^*}(s)$ is of type B_{n-1} with n odd. In the latter case, we take for $\mathbf{M}$ instead a Levi subgroup of type $C_{n-2} \times A_1$. Let δ be a non-trivial outer diagonal automorphism of $\mathbf{G}$ normalising $\mathbf{M}$. Then δ has orbits of length 2 on $\mathcal{E}(G, s)$ as well as on $\mathcal{E}(M, s)$ as s has disconnected centraliser. By the Howlett–Lehrer comparison theorem there is a bijection $\chi \mapsto \chi^0$ between the Harish-Chandra series above $(\mathbf{L}, \lambda)$ and the irreducible characters of the relative Weyl group $W_G(\mathbf{L}, \lambda)$ such that Harish-Chandra restriction ${}^*R_{\mathbf{M}}^{\mathbf{G}}(\chi)$ decomposes as the ordinary restriction of the associated characters of $W_G(\mathbf{L}, \lambda)$. As χ has odd degree, the character χ^0 of $W_G(\mathbf{L}, \lambda)$ has odd degree (as in [4, Lemma 7.9]). Thus there is some constituent ψ^0 of $\chi^0|_{W_M(\mathbf{L},\lambda)}$ of odd degree with odd multiplicity and such that $\delta(\psi)^0$ has even multiplicity. So ${}^*R_{\mathbf{M}}^{\mathbf{G}}(\chi)$ has a constituent ψ of odd degree with odd multiplicity such that $\delta(\psi)$ has even multiplicity. Assume that χ is not σ_e-fixed. As Galois automorphisms commute with Harish-Chandra restriction (in fact, with Lusztig restriction, as can be seen for example from the character formula), ψ cannot be σ_e-fixed either. But by Schur's character table for $\mathrm{Sp}_2(q) = \mathrm{SL}_2(q)$, the characters lying in the Lusztig series of an involution only involve the square root of $q^* = (-1)^{(q-1)/2}q$, and this is σ_1-invariant as q is odd. Thus, so is χ by induction.

Next assume that G is of type B_n, $n \geq 3$. By Table 2 there exist 2-central elements s in $\mathbf{G}^{*F}$ with disconnected centraliser only when n is a 2-power. Let $\mathbf{M} \leq \mathbf{G}$ be a split Levi subgroup of type $B_{n/2} \times A_1^{n/4}$. Then $s \in \mathbf{M}^{*F}$ (up to conjugation), and it has disconnected centraliser there as well. By what we just proved before, the characters of $B_1(q) = \mathrm{Sp}_2(q)$ lying in the Lusztig series of s are σ_1-fixed and so again we may conclude by induction.

For $\mathbf{G}$ of (twisted or untwisted) type D_n, we take $\mathbf{M}$ a split Levi subgroup of (twisted or untwisted) type D_{n-1}, respectively of type $D_{n-2} \times A_1$ when $C_{\mathbf{G}^*}(s)$ is of type D_{n-1} with n odd. Then a similar consideration applies except for the isolated elements s with centraliser $(D_{n/2}(q) \cdot D_{n/2}(q)).2^2$ in untwisted type D_n for $n = 2^f$ a power of 2. In the latter case take split Levi subgroups $\mathbf{M}_i$, $i = 1, 2$, of $\mathbf{G}$ representing the two non-conjugate A_{n-1}-subsystems. Then s has disconnected centraliser in $\mathbf{M}_i$, with group of components of order 2, and the two groups of components together generate the group of components of $C_{\mathbf{G}^*}(s)$. Arguing as before, for $\chi \in \mathcal{E}(G, s) \cap \mathrm{Irr}_{2'}(G)$ we find $\psi_i \in \mathcal{E}(M_i, s)$ of odd degree below χ. Then χ is distinguished from its images under diagonal automorphisms by the multiplicities of the ψ_i and $\delta_i(\psi_i)$ in ${}^*R_{\mathbf{M}_i}^{\mathbf{G}}(\chi)$, where δ_i is an outer diagonal automorphism of $\mathbf{M}_i$. As $\mathbf{M}_i$ is of type A_{n-1} with $n = 2^f$, the characters in $\mathcal{E}(M_i, s)$ are σ_1-fixed, (as we will see below), and hence so is χ.

Finally, assume that $\mathbf{G}$ is of type A_{n-1}, $n \geq 3$, so $G = \mathrm{SL}_n(\epsilon q)$ with $\epsilon \in \{\pm 1\}$. Here, by [1, Table 4.5.1], 2-central elements with disconnected centraliser exist in $\mathbf{G}^{*F}$ only when

$n = 2^f$ is a 2-power. In this case, with $\mathbf{M}$ a split Levi subgroup of type $A^2_{n/2-1}$ we may reduce inductively to the case of $A_1(q) = C_1(q)$ treated above to conclude that all characters in these Lusztig series are σ_1-invariant. In all other cases, using Propositions 3.1 and 3.2 in conjunction with Theorem 3.3 we conclude except for the question of 2-automorphisms when $4|(q-\epsilon)$ and n is not a power of 2. But then we may argue as in the case of E_6. $\square$

This completes the proof of Conjecture 2 for the prime $p = 2$. Theorem 1 immediately follows from this by virtue of [7, Thm. 5.7].

References

[1] D. Gorenstein, R. Lyons, R. Solomon, *The Classification of the Finite Simple Groups.* American Mathematical Society, 40, Number 3. Providence, RI, 1998.

[2] G. Malle, Height 0 characters of finite groups of Lie type. *Represent. Theory* **11** (2007), 192–220.

[3] G. Malle, Cuspidal characters and automorphisms. *Advances Math.* **320** (2017), 887–903.

[4] G. Malle, B. Späth, Characters of odd degree. *Ann. of Math. (2)* **184** (2016), 869–908.

[5] J. Michel, The development version of the CHEVIE package of GAP3. *J. Algebra* **435** (2015), 308–336.

[6] G. Navarro, The McKay conjecture and Galois automorphisms. *Ann. of Math. (2)* **160** (2004), 1129–1140.

[7] G. Navarro, P.H. Tiep, Sylow subgroups, exponents and character values. Preprint, 2018.

[8] A. Schaeffer Fry, Galois automorphisms on Harish-Chandra series and Navarro's self-normalizing Sylow 2-subgroup conjecture. *Trans. Amer. Math. Soc.*, to appear. ArXiv: 1707.03923.

[9] A. Schaeffer Fry, J. Taylor, On self-normalising Sylow 2-subgroups in type A. *J. Lie Theory* **28** (2018), 139–168.

FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany.
Email address: malle@mathematik.uni-kl.de