REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 00, Pages 000–000 (Xxxx XX, XXXX)
S 1088-4165(XX)0000-0

HEIGHT 0 CHARACTERS OF FINITE GROUPS OF LIE TYPE

GUNTER MALLE

Dedicated to Professor Toshiaki Shoji on the occasion of his 60th birthday

Abstract. We give a classification of irreducible characters of finite groups of Lie type of p'-degree, where p is any prime different form the defining characteristic, in terms of local data. More precisely, we give a classification in terms of data related to the normalizer of a suitable Levi subgroup, which in many cases coincides with the normalizer of a Sylow p-subgroup. The McKay-conjecture asserts that there exists a bijection between characters of p'-degree of a group and of the normalizer of a Sylow p-subgroup. We hope that our result will constitute a major step towards a proof of this conjecture for groups of Lie type, and, in conjunction with a recent reduction result of Isaacs, Malle and Navarro, for arbitrary finite groups.

1. Introduction

Let G be a finite group and p a prime. The McKay-conjecture from the 1970th in its simplest form asserts that the number $|\mathrm{Irr}_{p'}(G)|$ of irreducible complex characters of G of degree prime to p equals the corresponding number for the normalizer $N_G(P)$ of a Sylow p-subgroup P of G. There also exists a block-wise version counting characters of height zero in blocks of G and in the normalizer of their defect groups. Moreover, there are various further deep conjectures in representation theory of finite groups which would imply the validity of the McKay-conjecture, the most elaborate ones probably being Dade's conjectures.

The McKay-conjecture was proved for solvable groups and more generally for p-solvable groups as well as for various classes of non-solvable groups, like the symmetric groups and the general linear groups over finite fields, but it remains open in general. Recently, Isaacs, Navarro and the author [15] proposed a reduction of McKay's conjecture to simple groups. More precisely, it is shown that the McKay conjecture holds for a finite group G if a certain quite complicated property holds for every simple group involved in G. In view of the classification of the finite simple groups this opens the way to a possible proof of the conjecture by checking that all simple groups have the required property. This has been done in [15] for example for the simple groups $\mathrm{PSL}_2(q)$, and in [23] for alternating and sporadic groups.

In the present paper we consider this question for the remaining non-abelian simple groups, the finite groups of Lie type.

For G a finite group of Lie type and p a prime different from the defining characteristic of G we obtain a natural parametrization of $\mathrm{Irr}_{p'}(G)$ in terms of p-local data. More precisely, let S be a Sylow d-torus of G for a suitably chosen d, and L its

2000 *Mathematics Subject Classification.* Primary 20C33, 20G40.

centralizer in G. Then the normalizer $N = \mathrm{N}_G(L)$ of L is a proper subgroup of G containing the normalizer of a Sylow p-subgroup of G. It has recently been shown by Späth [24] (see also [25]) that all characters of L extend to their inertia groups in N. With this, we obtain a character bijection $\mathrm{Irr}_{p'}(G) \longleftrightarrow \mathrm{Irr}_{p'}(N)$ of the kind needed for the inductive setup in [15]. If one assumes that the McKay-conjecture has been proved for all smaller groups, then, since N contains the normalizer of a Sylow p-subgroup of G, this also implies the McKay-conjecture for G. (This has now been shown unconditionally for groups of exceptional Lie type by Späth [25], using the results of this paper.)

In our proof we start from Lusztig's parametrization of irreducible characters of finite reductive groups. This parametrization involves characters of not necessarily connected groups. In order to handle the latter, we introduce the concept of disconnected generic groups in Section 3. In Section 4 we generalize several results on d-Harish-Chandra theory to these disconnected groups (see Theorems 4.6 and 4.8). This may be of independent interest. We then show in Theorem 5.9 that centralizers of Sylow p-subgroups in groups of Lie type are contained in the centralizers of suitable Sylow Φ_d-tori, and the same assertion holds for centralizers replaced by normalizers under very mild assumptions on p (for example $p \geq 5$ will always do) (see Theorem 5.14). This extends earlier results valid only for large primes. The proofs only involve case by case arguments for primes $p \leq 3$.

After these preparations we start the investigation of unipotent characters of height 0 in Section 6. A characterization in non-defining characteristic is given in Corollary 6.6. The corresponding statement for the defining characteristic is obtained in Theorem 6.8. Finally, in Section 7 we put together the various pieces obtained so far to prove in Theorem 7.5 a parametrization of characters of p'-degrees in terms of local data, that is to say, in terms of data attached to the normalizer of a Sylow torus. We further show in Theorem 7.8 that there is a natural bijective map between p'-degree characters of G and of the normalizer of a Sylow torus. In the last section we prove that the corresponding result for the Suzuki and Ree groups holds.

The main ingredients of our proofs are Lusztig's parametrization of irreducible characters together with an extension of d-Harish-Chandra theory to disconnected groups.

2. Characters of groups of Lie-type

In this section we recall some of Lusztig's results on the parametrization and the degrees of the irreducible complex characters of groups of Lie type.

2.1. Lusztig's parametrization. Let $\mathbf{G}$ be a connected reductive algebraic group defined over a finite field $\mathbb{F}_q$ and $F : \mathbf{G} \to \mathbf{G}$ the corresponding Frobenius endomorphism. Then $G := \mathbf{G}^F$ is a finite group of Lie type. Let $\mathbf{G}^*$ be a group in duality with $\mathbf{G}$, with corresponding Frobenius endomorphism $F^* : \mathbf{G}^* \to \mathbf{G}^*$ and group of fixed points $G^* := {\mathbf{G}^*}^{F^*}$. Then G^*-conjugacy classes of semisimple elements $s \in G^*$ are in natural bijection with geometric conjugacy classes of pairs $(\mathbf{T}, \theta)$ consisting of an F-stable maximal torus $\mathbf{T} \leq \mathbf{G}$ and an irreducible character $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ (see for example [8, Thm. 4.4.6]).

For any F-stable Levi subgroup $\mathbf{L} \le \mathbf{G}$ Deligne and Lusztig have defined linear maps

$$R_{\mathbf{L}}^{\mathbf{G}} : \mathbb{Z}\mathrm{Irr}(\mathbf{L}^F) \longrightarrow \mathbb{Z}\mathrm{Irr}(\mathbf{G}^F), \quad {}^*R_{\mathbf{L}}^{\mathbf{G}} : \mathbb{Z}\mathrm{Irr}(\mathbf{G}^F) \longrightarrow \mathbb{Z}\mathrm{Irr}(\mathbf{L}^F),$$

adjoint to each other with respect to the usual scalar product of characters. This allows to define the *Lusztig-series* of a semisimple element $s \in G^*$ as the subset

$$\mathcal{E}(G,s) := \{\chi \in \mathrm{Irr}(G) \mid \langle \chi, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle \ne 0 \text{ for some } (\mathbf{T},\theta)\}$$

of $\mathrm{Irr}(G)$, where $(\mathbf{T},\theta)$ runs over pairs in the geometric conjugacy class corresponding to s. According to Lusztig [17, Prop. 5.1] (see also [10, Prop. 13.17] for groups with connected center) these series define a natural partition

$$\mathrm{Irr}(G) = \coprod_{s \in G^*_{\mathrm{ss}}/\sim} \mathcal{E}(G,s)$$

of $\mathrm{Irr}(G)$, where the disjoint union is over semisimple elements $s \in G^*$ up to conjugation. The constituents $\mathcal{E}(G,1)$ of $R_{\mathbf{T}}^{\mathbf{G}}(1)$, where $\mathbf{T}$ runs over the F-stable maximal tori of $\mathbf{G}$, are called the *unipotent characters* of G. Lusztig's Jordan decomposition of characters now asserts that for each semisimple $s \in G^*$ there is a bijection

$$\psi_s : \mathcal{E}(G,s) \xrightarrow{1-1} \mathcal{E}(\mathrm{C}_{G^*}(s),1)$$

between the Lusztig series $\mathcal{E}(G,s)$ and the unipotent characters of $\mathrm{C}_{G^*}(s)$, such that the character degrees satisfy

$$\chi(1) = \frac{|G|_{q'}}{|\mathrm{C}_{G^*}(s)|_{q'}}\psi_s(\chi)(1) \qquad \text{for all } \chi \in \mathcal{E}(G,s) \tag{2.1}$$

(for the unicity of this bijection see [9, Th. 7.1]). Note that, if the center of $\mathbf{G}$ is not connected, the centralizers $\mathrm{C}_{\mathbf{G}^*}(s)$ in the dual group need not necessarily be connected. Then, by definition, $\mathcal{E}(\mathrm{C}_{G^*}(s),1)$ consists of the irreducible characters of $\mathrm{C}_{G^*}(s)$ whose restriction to the connected component $(\mathrm{C}_{\mathbf{G}^*}(s)^\circ)^F$ is unipotent, see also Section 3.5.

In our parametrization of characters we need to keep track of the restriction to the center $Z(G)$:

Lemma 2.2. *Let $s \in G^*$. Then $\theta|_{Z(G)}$ is the same for all pairs $(\mathbf{T},\theta)$ (consisting of an F-stable maximal torus $\mathbf{T} \le \mathbf{G}$ and an irreducible character $\theta \in \mathrm{Irr}(\mathbf{T}^F)$) in the geometric conjugacy class determined by s, and we have*

$$\chi|_{Z(G)} = \chi(1)\cdot\theta|_{Z(G)} \qquad \textit{for all } \chi \in \mathcal{E}(G,s).$$

Proof. Since $\mathbf{G}$ is reductive, $Z(\mathbf{G})$ consists of semisimple elements and is contained in every maximal torus. Since $Z(G) \le Z(\mathbf{G})$ the same is true for $Z(G)$. By [8, Prop. 4.1.3] the value of $\theta|_{Z(G)}$ is independent of $(\mathbf{T},\theta)$ in the geometric conjugacy class of s. The character formula [8, Prop. 7.5.3] now shows that

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(t) = R_{\mathbf{T}}^{\mathbf{G}}(\theta)(1)\theta(t) \qquad \text{for } t \in Z(G),$$

that is, $R_{\mathbf{T}}^{\mathbf{G}}(\theta)|_{Z(G)} = R_{\mathbf{T}}^{\mathbf{G}}(\theta)(1)\theta$. Since semisimple conjugacy classes are uniform [8, Cor. 7.5.7], the value of any $\chi \in \mathrm{Irr}(G)$ on the semisimple element $t \in Z(G)$ is determined by the values of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ on t. The claim follows. □

3. Generic disconnected groups

We recall the notions of generic groups and generic unipotent characters as introduced in Broué–Malle [1] and Broué–Malle–Michel [4], see also Broué–Malle [3]. We then generalize these notions to particular types of disconnected groups.

3.1. Generic groups. We start by recalling the notion of generic groups.

Let $\mathbf{G}$ be a connected reductive algebraic group defined over a finite field $\mathbb{F}_q$ with corresponding Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$. Let $\mathbf{T} \leq \mathbf{B}$ be an F-stable maximal torus contained in an F-stable Borel subgroup of $\mathbf{G}$. We denote by X the character group and by Y the cocharacter group of $\mathbf{T}$, and by $R \subset X$, respectively $R^\vee \subset Y$ the set of roots respectively coroots relative to $\mathbf{B}$. The Frobenius endomorphism F acts on $V := Y \otimes_{\mathbb{Z}} \mathbb{R}$ as $q\phi$, where ϕ is an automorphism of finite order stabilizing $R^\vee$. Replacing $\mathbf{B}$ by a different F-stable Borel subgroup containing $\mathbf{T}$ changes ϕ by some element in the Weyl group W of $\mathbf{G}$. Thus we may associate to $(\mathbf{G}, \mathbf{T}, F)$ the *generic reductive group* $\mathbb{G} = (X, R, Y, R^\vee, W\phi)$, where

(i) X, Y are $\mathbb{Z}$-lattices of equal finite rank endowed with a duality $\langle\, , \rangle : X \times Y \to \mathbb{Z}$,

(ii) $R, R^\vee$ are root systems in X, Y respectively, with a bijection ${}^\vee : R \to R^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in R$, and

(iii) W is the Weyl group of the root system $R^\vee$ in Y, ϕ is an automorphism of Y of finite order stabilizing $R^\vee$ (hence normalizing W).

Conversely, a generic reductive group $\mathbb{G}$, together with the choice of a prime power q, gives rise to a triple $(\mathbf{G}, \mathbf{T}, F)$ as above. We then also say that $(\mathbf{G}, F)$ has type $\mathbb{G}$. Thus a generic reductive group encodes a whole series

$$\{\mathbb{G}(q) := \mathbf{G}^F \mid q \text{ prime power}\}$$

of finite groups of Lie type.

The Ennola-dual of a generic reductive group $\mathbb{G} = (X, R, Y, R^\vee, W\phi)$ is by definition the generic group $\mathbb{G}^- := (X, R, Y, R^\vee, W(-\phi))$. Clearly, if $-1 \in W$ then $\mathbb{G}^-$ equals $\mathbb{G}$.

A Levi subgroup of $\mathbb{G} = (X, R, Y, R^\vee, W_{\mathbb{G}}\phi)$ is a generic group of the form

$$\mathbb{L} = (X, R', Y, {R'}^\vee, W_{\mathbb{L}} w\phi),$$

where $w \in W_{\mathbb{G}}$ and ${R'}^\vee$ is a $w\phi$-stable parabolic subsystem of $R^\vee$ with Weyl group $W_{\mathbb{L}}$. We call

$$W_{\mathbb{G}}(\mathbb{L}) := N_{W_{\mathbb{G}}}(\mathbb{L})/W_{\mathbb{L}}$$

the *relative Weyl group of* $\mathbb{L}$ *in* $\mathbb{G}$. A *generic torus* $\mathbb{T}$ *of* $\mathbb{G}$ is a generic group of the form

$$\mathbb{T} = (X', \emptyset, Y', \emptyset, (w\phi)|_{Y'}),$$

where Y' is a $w\phi$-stable summand of Y, and X' is the dual of Y'. The *centralizer of the torus* $\mathbb{T}$ in $\mathbb{G}$ is the Levi subgroup

$$C_{\mathbb{G}}(\mathbb{T}) = (X, R', Y, {R'}^\vee, W' w\phi),$$

where R' consists of the $\alpha \in R$ orthogonal to Y' and W' is the Weyl group of the root system ${R'}^\vee$. There is a perfect dictionary between generic Levi subgroups and tori of $\mathbb{G}$ on the one hand side, and F-stable Levi subgroups and tori of $\mathbf{G}$ on the other, if $(\mathbf{G}, \mathbf{T}, F)$ corresponds to $\mathbb{G}$, which respects centralizers, see [1, Thm. 2.1].

Attached to a generic group $\mathbb{G}$ is its *generic order* $|\mathbb{G}| \in \mathbb{Z}[X]$ (see for example [3, 3.1]), which is a monic product of cyclotomic polynomials over $\mathbb{Q}$, and satisfies $|\mathbf{G}^F| = |\mathbb{G}|(q)$ for any group $(\mathbf{G}, F)$ of type $\mathbb{G}$.

Now let $d \geq 1$ and $\Phi_d(X) \in \mathbb{Q}[X]$ the dth cyclotomic polynomial over $\mathbb{Q}$, that is, the minimal polynomial over $\mathbb{Q}$ of a primitive dth root of unity. A torus $\mathbb{T} \leq \mathbb{G}$ is called a *Φ_d-torus*, if its generic order is a power of Φ_d. A Φ_d-torus $\mathbb{T} \leq \mathbb{G}$ whose generic order contains the full Φ_d-part of $|\mathbb{G}|$ is called a *Sylow Φ_d-torus* of $\mathbb{G}$. If $\mathbf{T}$ is a corresponding F-stable torus in a corresponding reductive group $\mathbf{G}$, we will also call it a Φ_d-torus of $\mathbf{G}$. The following Sylow theorem from Broué–Malle [1] will be used frequently:

Theorem 3.2. *Let $\mathbf{G}$ be a reductive algebraic group, not necessarily connected, defined over $\mathbb{F}_q$ with Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ and $d \geq 1$. Then:*

(a) *There exist Sylow Φ_d-tori in $\mathbf{G}$.*
(b) *Any two Sylow Φ_d-tori of $\mathbf{G}$ are $(\mathbf{G}^\circ)^F$-conjugate.*
(c) *Any Φ_d-torus of $\mathbf{G}$ is contained in a Sylow Φ_d-torus.*
(d) *For any Sylow Φ_d-torus $\mathbf{S}$ of $\mathbf{G}$ we have*

$$\pm|G|_{q'}/|\mathrm{C}_G(\mathbf{S})|_{q'} \equiv |G|/|\mathrm{C}_G(\mathbf{S})| \equiv |W_G(\mathbf{S})| \pmod{\Phi_d(q)},$$

where $G := \mathbf{G}^F$ and $W_G(\mathbf{S}) := \mathrm{N}_G(\mathbf{S})/\mathrm{C}_G(\mathbf{S})$.

Proof. The first three parts are in [1, Thm. 3.4]. For part (d) note that $G = G^\circ \, \mathrm{N}_G(\mathbf{S})$ by assertion (b), where $G^\circ := (\mathbf{G}^\circ)^F$, so $|G|/|\mathrm{N}_G(\mathbf{S})| = |G^\circ|/|\mathrm{N}_{G^\circ}(\mathbf{S})|$. The latter quotient is congruent to 1 modulo $\Phi_d(q)$ by [1, Thm. 3.4(4)], so we obtain the second congruence in the statement. The first congruence follows by Broué–Malle–Michel [5, Prop. 5.4]. □

The centralizers in $\mathbb{G}$ of Φ_d-tori of $\mathbb{G}$ are called *d-split Levi subgroups*; they correspond to F-stable Levi subgroups of $\mathbf{G}$ which are not necessarily contained in an F-stable parabolic subgroup. It follows from the Sylow theorems cited above that any d-split Levi subgroup contains Sylow Φ_d-tori.

The dual of $\mathbb{G} = (X, R, Y, R^\vee, W\phi)$ is by definition the generic group $\mathbb{G}^* := (Y, R^\vee, X, R, W{\phi^\vee}^{-1})$, where $\phi^\vee$ is the adjoint of ϕ. In general, the dual of a torus of $\mathbb{G}$ cannot be considered as a torus of $\mathbb{G}^*$ in a natural way. Nevertheless, for Sylow tori this is possible, as follows. Let $\mathbb{S}_d = (X', \emptyset, Y', \emptyset, w\phi)$ be a Sylow Φ_d-torus of $\mathbb{G}$. Let

$$X'' = X \cap \ker_{X \otimes \mathbb{R}} \Phi_d((w\phi)^{-1}),$$

that is, X'' is the maximal submodule of X on which all eigenvalues of $(w\phi)^{-1}$ are primitive dth roots of unity, and let Y'' be its dual. Then

$$\mathbb{S}_d^* := (Y'', \emptyset, X'', \emptyset, (w\phi^\vee)^{-1})$$

is a torus of $\mathbb{G}^*$. With this notation we have:

Lemma 3.3. *Let $\mathbb{S}_d$ be a Sylow Φ_d-torus of $\mathbb{G}$. Then:*

(a) *$\mathbb{S}_d^*$ is a Sylow Φ_d-torus of $\mathbb{G}^*$.*
(b) $\mathrm{C}_{\mathbb{G}^*}(\mathbb{S}_d^*) = \mathrm{C}_{\mathbb{G}}(\mathbb{S}_d)^*$.
(c) *$W_{\mathbb{G}^*}(\mathbb{L}^*) \cong W_{\mathbb{G}}(\mathbb{L})$ (anti-isomorphic) for $\mathbb{L} = \mathrm{C}_{\mathbb{G}}(\mathbb{S}_d)$.*

3.4. Generic unipotent characters. Recall that the unipotent characters of $\mathbf{G}^F$ are defined as the constituents of $R_\mathbf{T}^\mathbf{G}(1)$ where $\mathbf{T}$ runs over the F-stable maximal tori of $\mathbf{G}$. The work of Lusztig shows that the unipotent characters may be parametrized depending only on the type $\mathbb{G}$ of $(\mathbf{G}, F)$ and that their degrees are given by polynomials in q. That is to say, for any generic reductive group $\mathbb{G}$ there exists a set $\mathcal{E}(\mathbb{G})$ of *generic unipotent characters* and a map

$$\mathrm{Deg} : \mathcal{E}(\mathbb{G}) \longrightarrow \mathbb{Q}[X],$$

such that for any finite reductive group $\mathbf{G}^F$ of type $\mathbb{G}$, we have a bijection

$$\psi_q^{\mathbb{G}} : \mathcal{E}(\mathbb{G}) \xrightarrow{1-1} \mathcal{E}(\mathbf{G}^F, 1) \qquad \text{with } \psi_q^{\mathbb{G}}(\gamma)(1) = \mathrm{Deg}(\gamma)(q). \tag{3.1}$$

Then $\mathrm{Deg}(\gamma)$ is called the *degree polynomial* of the unipotent character γ. The results of Lusztig imply that the degree polynomials are essentially products of cyclotomic polynomials; more precisely, they have the form

$$\mathrm{Deg}(\gamma) = \frac{1}{n_\gamma} X^{a_\gamma} \prod_d \Phi_d(X)^{a(d,\gamma)}, \tag{3.2}$$

where the denominator n_γ is an integer only divisible by bad primes for $\mathbb{G}$ (see [4, Thm. 1.32]).

It follows from results of Shoji that Lusztig induction and restriction are generic on unipotent characters, that is, for any generic Levi subgroup $\mathbb{L}$ of $\mathbb{G}$ there exist linear maps

$$R_{\mathbb{L}}^{\mathbb{G}} : \mathbb{Z}\mathrm{Irr}(\mathbb{L}) \longrightarrow \mathbb{Z}\mathrm{Irr}(\mathbb{G}), \quad {}^*R_{\mathbb{L}}^{\mathbb{G}} : \mathbb{Z}\mathrm{Irr}(\mathbb{G}) \longrightarrow \mathbb{Z}\mathrm{Irr}(\mathbb{L}),$$

satisfying $\psi_q^{\mathbb{G}} \circ R_{\mathbb{L}}^{\mathbb{G}} = R_{\mathbf{L}}^{\mathbf{G}} \circ \psi_q^{\mathbb{L}}$ for all q when extending $\psi_q^{\mathbb{G}}$ linearly to $\mathbb{Z}\mathcal{E}(\mathbb{G})$ (see [4, Thm. 1.33]).

We will also later on make use of Ennola-duality which induces bijections

$$\mathcal{E}(\mathbb{L}) \to \mathcal{E}(\mathbb{L}^-), \quad \gamma \mapsto \gamma^-, \qquad \text{such that } \mathrm{Deg}(\gamma^-)(X) = \pm\mathrm{Deg}(\gamma)(-X), \tag{3.3}$$

commuting with Lusztig-induction from d-split Levi subgroups (see [4, Thm. 3.3]).

3.5. Extension to disconnected groups. Let $\tilde{\mathbf{G}}$ be a reductive algebraic group with connected component $\mathbf{G} := \tilde{\mathbf{G}}^\circ$ such that $A := \tilde{\mathbf{G}}/\mathbf{G}$ is a finite cyclic group. An element $\sigma \in \tilde{\mathbf{G}}$ is called *quasi-semisimple* if it fixes a maximal torus of $\mathbf{G}$ and a Borel subgroup containing it. Now assume that $\tilde{\mathbf{G}}$ is defined over $\mathbb{F}_q$ with corresponding Frobenius map $F : \mathbf{G} \to \mathbf{G}$ with trivial induced action on A. By Digne–Michel [11, Prop. 1.34 and 1.36] the coset $\mathbf{G}\sigma$ of a quasi-semisimple element $\sigma \in \mathbf{G}$ contains a quasi-semisimple element commuting with F which fixes an F-stable maximal torus $\mathbf{T}$ and an F-stable Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$. We may and will hence assume from now on that $\tilde{\mathbf{G}} = \mathbf{G}\langle\sigma\rangle$ with a quasi-semisimple automorphism σ as above.

Since σ stabilizes $\mathbf{T} \leq \mathbf{B}$, it acts on the root system R relative to $\mathbf{T}$ and stabilizes the set of simple roots in R. We thus have associated to $\tilde{\mathbf{G}}$ the data $\tilde{\mathbb{G}} = (\mathbb{G}, A)$, where

(i) $\mathbb{G} = (X, R, Y, R^\vee, W\phi)$ is the generic group corresponding to $(\mathbf{G}, \mathbf{T}, F)$,
(ii) A is a finite cyclic group with an action on Y which commutes with ϕ and permutes the set of simple roots in $R^\vee$ (hence stabilizes $R^\vee$).

The data $\tilde{\mathbb{G}}$ can be thought of as a generic disconnected group of type $(\tilde{\mathbf{G}}, \mathbf{T}, F)$. If $\mathbb{G}$ is semisimple, i.e., if $R^\vee$ spans Y, the action of A on Y is completely determined by its action on the set of simple roots in $R^\vee$, hence by the corresponding graph automorphism of the Dynkin diagram.

Clearly A acts on the set of tori and Levi subgroups of $\mathbb{G}$ in a natural way.

Remark 3.6. We restrict ourselves to the case where A is cyclic, since this is the only one we need later on, but the case of an arbitrary finite group of automorphisms A should fit in a similar framework.

Since σ commutes with F, we obtain a finite group $\tilde{G} := \tilde{\mathbf{G}}^F$ with $\tilde{G}/G \cong A$ generated by the image of σ, where $G = \mathbf{G}^F$ is a finite reductive group. This yields an induced action of σ on the set $\mathrm{Irr}(G)$ of complex irreducible characters of G. The results of Lusztig [17, p. 159] show that the action of graph automorphisms on unipotent characters is again generic, so may be lifted to an action on $\mathcal{E}(\mathbb{G})$. We then have (see also [19, §1]; the notation for unipotent characters is as in [8], for example):

Proposition 3.7. *If $\mathbb{G}$ is simple, any graph automorphism of $\mathbb{G}$ fixes every unipotent character of $\mathbb{G}$, except in the following two cases:*

(a) *$\mathbb{G}$ of type D_n, with n even and σ of order 2. Here, σ fixes all unipotent characters labelled by non-degenerate symbols, but it interchanges the two unipotent characters in all pairs labelled by the same degenerate symbol of defect 0 and rank n.*

(b) *$\mathbb{G}$ of type D_4, with σ of order 3. Here σ has two non-trivial orbits, both of length 3, with characters labelled by the symbols:*

$$\left\{\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}', \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}', \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \end{pmatrix}\right\}.$$

We now define the *set of generic unipotent characters of* $\tilde{\mathbb{G}}$

$$\mathcal{E}(\tilde{\mathbb{G}}) = \{([\gamma], i) \mid \gamma \in \mathcal{E}(\mathbb{G})/\langle\sigma\rangle,\ 0 \le i < e_\gamma\},$$

where $[\gamma]$ runs over the $\langle\sigma\rangle$-orbits in $\mathcal{E}(\mathbb{G})$, and e_γ is the order of the stabilizer of γ in $\langle\sigma\rangle$. Then the preceding discussion shows that $\mathcal{E}(\tilde{\mathbb{G}})$ is in bijection with those elements of $\mathrm{Irr}(\tilde{G})$ whose restriction to G contains (only) unipotent constituents.

3.8. The Suzuki- and Ree-automorphisms. The finite groups of Lie type $\mathrm{Sp}_4(2^a)$, $G_2(3^a)$ and $F_4(2^a)$ have exceptional automorphisms. These come from a symmetry of the Coxeter diagram of the Weyl group, but cannot be lifted as automorphisms of finite order to the corresponding algebraic group. If $a = 2n+1$ is odd, such an automorphism can be chosen of order 2 with group of fixed points ${}^2B_2(2^{2n+1})$, ${}^2G_2(3^{2n+1})$, ${}^2F_4(2^{2n+1})$ respectively, while if a is even it squares to a field automorphism. We call these the Suzuki- and Ree-automorphisms.

It turns out that they enjoy similar properties in their action on unipotent characters as in the setting of disconnected groups considered previously. First, in a suitable sense made precise in Broué–Malle [1, 1A], a Suzuki- or Ree-automorphism σ can be interpreted as an automorphism of the corresponding generic group $\mathbb{G}$ over a quadratic extension of $\mathbb{Z}$. Moreover, it follows from the decomposition of Deligne-Lusztig characters and considerations of principal series characters that the action of σ on unipotent characters is generic, that is, we have an induced action on $\mathcal{E}(\mathbb{G})$. More precisely, in analogy to Proposition 3.7 this action is given as follows (see [19, §1]; the notation for unipotent characters is as in [8, p. 479]):

Proposition 3.9. *In the situation described above, σ fixes every unipotent character of $\mathbb{G}$, except for the following:*

(a) *In $\mathbb{G}$ of type B_2, σ interchanges the two unipotent principal series characters labelled by the symbols*
$$\left\{\begin{pmatrix} 1 & 2 \\ 0 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & \end{pmatrix}\right\}.$$
(b) *In $\mathbb{G}$ of type G_2, σ interchanges the two unipotent principal series characters labelled by the characters $\{\phi'_{1,3}, \phi''_{1,3}\}$ of the Weyl group $W(G_2)$.*

(c) *In $\mathbb{G}$ of type F_4, σ has eight orbits of length 2, with unipotent characters labelled by*
$$\{\phi'_{8,3}, \phi''_{8,3}\},\ \{\phi'_{8,9}, \phi''_{8,9}\},\ \{\phi'_{2,4}, \phi''_{2,4}\},\ \{\phi'_{2,16}, \phi''_{2,16}\},$$
$$\{\phi'_{9,6}, \phi''_{9,6}\},\ \{\phi'_{1,12}, \phi''_{1,12}\},\ \{\phi'_{4,7}, \phi''_{4,7}\}, \{(B_2, \epsilon'), (B_2, \epsilon'')\}.$$

4. d-Harish-Chandra theory

We introduce d-Harish-Chandra series of generic unipotent characters, state some results and generalize them to disconnected generic groups.

4.1. d-Harish-Chandra series. Let $\mathbb{G}$ be a generic connected reductive group. A unipotent character $\gamma \in \mathcal{E}(\mathbb{G})$ is called *d-cuspidal* if its Lusztig restriction ${}^*R_{\mathbb{L}}^{\mathbb{G}}(\gamma)$ vanishes for every proper d-split Levi subgroup $\mathbb{L} < \mathbb{G}$. We will then also say that the unipotent character $\psi_q^{\mathbb{G}}(\gamma) \in \mathcal{E}(\mathbf{G}^F, 1)$ of a corresponding group of Lie-type $\mathbf{G}^F$ is d-cuspidal. A pair $(\mathbb{L}, \lambda)$ with $\mathbb{L}$ d-split and $\lambda \in \mathcal{E}(\mathbb{L})$ unipotent d-cuspidal is called a *d-cuspidal pair*. Again, we also use this notion for the corresponding pair $(\mathbf{L}, \psi_q^{\mathbb{L}}(\lambda))$, where $\mathbf{L} \leq \mathbf{G}$ is a d-split Levi subgroup of $\mathbf{G}$ and $\psi_q^{\mathbb{L}}(\lambda) \in \mathcal{E}(\mathbf{L}^F, 1)$ is d-cuspidal.

Note that $W_G(\mathbf{L}^F) := N_G(\mathbf{L})/\mathbf{L}^F$ acts on the unipotent characters of $\mathbf{L}^F$. Thus, for $(\mathbf{L}, \lambda)$ a d-cuspidal pair of $\mathbf{G}$ we may define its *relative Weyl group* as
$$W_G(\mathbf{L}^F, \lambda) := \mathrm{N}_G(\mathbf{L}, \lambda)/\mathbf{L}^F.$$
The results of the previous two sections show that this is a generic object, depending only on the types of $\mathbf{G}$ and $\mathbf{L}$. That is, there exists an action of $W_{\mathbb{G}}(\mathbb{L})$ on $\mathcal{E}(\mathbb{L})$, with stabilizer $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ of λ isomorphic to $W_G(\mathbf{L}^F, \lambda)$ for all choices of $(\mathbf{G}, F)$. It turns out that $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ is a finite complex reflection group in all cases (see [3, Prop. 9.3]).

For any $d \geq 1$ and any d-cuspidal pair $(\mathbb{L}, \lambda)$ define the *d-Harish-Chandra series* $\mathcal{E}(\mathbb{G}, (\mathbb{L}, \lambda))$ in $\mathcal{E}(\mathbb{G})$ as the set of constituents of $R_{\mathbb{L}}^{\mathbb{G}}(\lambda)$. By Broué–Malle–Michel [4, Thm. 3.2], the set of unipotent characters of $\mathbb{G}$ is then partitioned into its d-Harish-Chandra series
$$\mathcal{E}(\mathbb{G}) = \coprod_{(\mathbb{L},\lambda)/\sim} \mathcal{E}(\mathbb{G}, (\mathbb{L}, \lambda)),$$
where $(\mathbb{L}, \lambda)$ runs over the d-cuspidal pairs in $\mathbb{G}$ modulo conjugation. Furthermore, for each d-cuspidal pair $(\mathbb{L}, \lambda)$, there is a bijection
$$\rho(\mathbb{L}, \lambda) : \mathcal{E}(\mathbb{G}, (\mathbb{L}, \lambda)) \xrightarrow{1-1} \mathrm{Irr}(W_{\mathbb{G}}(\mathbb{L}, \lambda)) \tag{4.1}$$
between its d-Harish-Chandra series and the irreducible characters of its relative Weyl group $W_{\mathbb{G}}(\mathbb{L}, \lambda)$.

In order to describe the behaviour of character degrees under the map $\rho(\mathbb{L}, \lambda)$, we need to note the following d-analogue of Howlett-Lehrer-Lusztig theory:

Theorem 4.2. *Let $\mathbb{G}$ be a generic group, $(\mathbb{L}, \lambda)$ a d-cuspidal pair in $\mathbb{G}$. Then for any $\phi \in \mathrm{Irr}(W_{\mathbb{G}}(\mathbb{L}, \lambda))$ there exists a rational function $D_\phi(X) \in \mathbb{Q}(X)$ with zeros and poles only at roots of unity or zero, satisfying*

(i) $D_\phi(\zeta_d) = \phi(1)/|W_{\mathbb{G}}(\mathbb{L})|$, *where* $\zeta_d := \exp(2\pi i/d)$, *and*

(ii) $\mathrm{Deg}(\gamma) = \pm|\mathbb{G}|_{X'}/|\mathbb{L}|_{X'}\mathrm{Deg}(\lambda)D_{\rho(\mathbb{L},\lambda)(\gamma)}$ *for all* $\gamma \in \mathcal{E}(\mathbb{G}, (\mathbb{L}, \lambda))$.

In the case $d = 1$ this is a consequence of the Howlett–Lehrer–Lusztig theory of Hecke algebras of induced cuspidal representations. (The rational function D_ϕ is then the inverse of the Schur element of ϕ.) In the general case, there also exists a cyclotomic Hecke algebra attached to the complex reflection group $W_{\mathbb{G}}(\mathbb{L}, \lambda)$. The rational functions $D_\phi(X)$ should now be suitable specializations of inverses of the Schur elements of this cyclotomic Hecke algebra with respect to a certain canonical trace form. The latter statement is conjectured to be true in general (see [2]). It has been proved for all but finitely many types, but a general proof is not known at present. Nevertheless, the existence of the rational functions $D_\phi(X)$ satisfying (i) and (ii) above has been verified in [20, Folg. 3.16 and 6.11] for groups of classical type, and in [21, Prop. 5.2], [22, Prop. 7.1] for those of exceptional type.

4.3. d-Harish-Chandra theory for disconnected groups. We need to generalize the previously mentioned results from [4, Th. 3.2] and Theorem 4.2 to the case of disconnected groups and also to extensions with automorphisms of Suzuki and Ree type. So let $\tilde{\mathbb{G}} = (\mathbb{G}, \langle\sigma\rangle)$ be a generic disconnected group as introduced in 3.5. The following result is well-known in the case $d = 1$ of cuspidal unipotent characters.

Proposition 4.4. *Let $\mathbb{G}$ be simple and $(\mathbb{L}, \lambda)$ be a d-cuspidal pair for $\mathbb{G}$.*

(a) *There exists $w \in W_{\mathbb{G}}$ such that $\mathbb{L}$ is $w\sigma$-stable.*

(b) *If $\mathbb{L}$ is a proper Levi subgroup, we can choose w such that $(\mathbb{L}, \lambda)$ is $w\sigma$-stable.*

Proof. Clearly we may assume that σ acts non-trivially. It is sufficient to show that σ fixes the $W_{\mathbb{G}}$-orbit of $\mathbb{L}$, respectively of $(\mathbb{L}, \lambda)$. Since $\mathbb{G}$ is simple, it is of type A_{n-1}, D_n or E_6. For A_{n-1}, for D_n with n odd and for E_6, σ acts like $-w_0$ on the root system $R^\vee$, where w_0 is the longest element of the Weyl group $W_{\mathbb{G}}$. Thus, $w_0\sigma$ acts like -1 on $R^\vee$, commuting with W, and the claim follows.

Now assume that $\mathbb{G}$ has type D_n with n even. There the explicit description of d-split Levi subgroups having a d-cuspidal character in [4, 3.A] shows that these are uniquely determined up to $W_{\mathbb{G}}$-conjugacy for example by their generic order. In particular, their $W_{\mathbb{G}}$-orbits must be σ-fixed, showing (a). For (b), let's first consider the case where σ has order 3, so $n = 4$. Then for all d-cuspidal pairs with $\mathbb{L} < \mathbb{G}$, $\mathbb{L}$ is a torus, so $\lambda = 1$, hence the assertion is obvious. For σ of order 2, by Proposition 3.7, the only unipotent characters not invariant under graph automorphisms are those labelled by degenerate symbols. The d-split Levi subgroups occurring in d-cuspidal pairs are direct products of a torus with a simple group of the same type as $\mathbb{G}$, possibly twisted (see [4, 3.A]). But it is readily checked that if $\mathbb{L}$ is a proper Levi subgroup of $\mathbb{G}$ of type D_n or ${}^2D_n$, then any pair of characters labelled by the same degenerate symbol is fused under $W_{\mathbb{G}}(\mathbb{L})$, so there is only one orbit under $W_{\mathbb{G}}$ already. □

Example 4.5. Note that part (a) of the previous statement is not true for arbitrary d-split Levi subgroups, i.e., those not necessarily containing d-cuspidal characters. Indeed, let $\mathbb{G}$ be of type D_4. Then $\mathbb{G}$ has three non-conjugate 1-split Levi subgroups of type GL_3, obtained by removing the three different end nodes of the Dynkin diagram. Two, respectively all three of them are fused under the graph automorphism of order 2 respectively 3. Also, the assumption that $\mathbb{G}$ be simple is clearly necessary.

In part (b), the assumption that $\mathbb{L}$ is proper is necessary by Proposition 3.7.

In the situation of Proposition 4.4, we have shown that there exists $w_1 \in W_{\mathbb{L}}$ such that $(\mathbb{L}, \langle w_1 w\sigma\rangle)$ is a disconnected group. By our previous remarks, this induces an action of $w_1 w\sigma$ on the set of generic unipotent characters $\mathcal{E}(\mathbb{L})$. Thus, we obtain an action on the set of cuspidal pairs modulo conjugation, which we call the action of $\tilde{\mathbb{G}}$. We define $W_{\tilde{\mathbb{G}}}(\mathbb{L},\lambda)$ as the extension of $W_{\mathbb{G}}(\mathbb{L},\lambda)$ by the stabilizer of λ under this action, so that $W_{\tilde{\mathbb{G}}}(\mathbb{L},\lambda) \cong \mathrm{N}_{\tilde{G}}(\mathbf{L},\lambda)/\mathbf{L}$. This also makes sense for not necessarily simple generic groups $\mathbb{G}$.

We can now show that d-Harish-Chandra theory extends to disconnected groups.

Theorem 4.6. *Let $\tilde{\mathbb{G}}$ be a generic disconnected group and $d \geq 1$.*

(a) *There is a partition*

$$\mathcal{E}(\tilde{\mathbb{G}}) = \coprod_{(\mathbb{L},\lambda)/\sim} \mathcal{E}(\tilde{\mathbb{G}}, (\mathbb{L},\lambda))$$

into d-Harish-Chandra series, where $(\mathbb{L},\lambda)$ runs over the d-cuspidal pairs in $\mathbb{G}$ modulo conjugation by $\tilde{\mathbb{G}}$.

(b) *For each d-cuspidal pair $(\mathbb{L},\lambda)$, there is a bijection*

$$\rho(\mathbb{L},\lambda) : \mathcal{E}(\tilde{\mathbb{G}}, (\mathbb{L},\lambda)) \xrightarrow{1-1} \mathrm{Irr}(W_{\tilde{\mathbb{G}}}(\mathbb{L},\lambda)).$$

(c) *For each $\phi \in \mathrm{Irr}(W_{\tilde{\mathbb{G}}}(\mathbb{L},\lambda))$ there exists a rational function $D_\phi(X) \in \mathbb{Q}(X)$ with zeros and poles only at roots of unity or zero, satisfying $D_\phi(\zeta_d) = \pm\phi(1)$, where $\zeta_d := \exp(2\pi i/d)$, and*

$$\mathrm{Deg}(\gamma) = \mathrm{Deg}(\lambda) D_{\rho(\mathbb{L},\lambda)(\gamma)}$$

for all $\gamma \in \mathcal{E}(\mathbb{G}, (\mathbb{L},\lambda))$.

Proof. Part (c) follows from Theorem 4.2 once the first two parts have been established. Now first assume that $\mathbb{G}$ is simple. Then $\tilde{\mathbb{G}}$ is an extension of $\mathbb{G}$ by a graph automorphism σ of the Dynkin diagram of $\mathbb{G}$. If σ acts trivially on $\mathbb{G}$, the statement follows from the corresponding one in the connected group [4, Thm. 3.2]. If σ acts by an inner automorphism on the Weyl group, then by Proposition 3.7 all generic unipotent characters of $\mathbb{G}$ are σ-stable, all d-cuspidal pairs are σ-stable by Proposition 4.4 and $W_{\tilde{\mathbb{G}}}(\mathbb{L},\lambda)$ is isomorphic to the direct product of $W_{\mathbb{G}}(\mathbb{L},\lambda)$ with $\langle\sigma\rangle$. Thus again the result follows from the corresponding result in the connected case.

It remains to consider $\mathbb{G}$ of type D_n, n even. The desired statement follows from the case of connected groups unless one of the following cases occurs:

(a) $W_{\tilde{\mathbb{G}}}(\mathbb{L},\lambda)$ is not the direct product of $W_{\mathbb{G}}(\mathbb{L},\lambda)$ by $\langle\sigma\rangle$, or
(b) $\mathcal{E}(\mathbb{G}, (\mathbb{L},\lambda))$ contains characters not invariant under σ.

(Note that by Proposition 4.4 it cannot happen for $\mathbb{L}$ proper that two classes of d-cuspidal pairs $(\mathbb{L},\lambda)$ are fused.)

Let first $\mathbb{G}$ be untwisted, that is, ϕ be trivial. We recall from [4, 3.A] the description of d-cuspidal pairs $(\mathbb{L},\lambda)$ in groups of type D_n. Let $e := d/\gcd(d,2)$. If d is odd, $\mathbb{L}$ is the direct product of a torus with generic order $(x^e-1)^a$ by a group of type D_r, with $n = ea+r$, and λ is labelled by a symbol Λ which is an e-core. If d is even, $\mathbb{L}$ is the direct product of a torus with generic order $(x^e+1)^a$ with a group of type D_r or ${}^2D_r$, with $n = ea+r$, and λ is labelled by an e-cocore Λ. The relative Weyl group of $(\mathbb{L},\lambda)$ is then the imprimitive complex reflection group

$$W_{\mathbb{G}}(\mathbb{L},\lambda) = G(2e,s,a), \qquad \text{where } s = \begin{cases} 2 & \text{if } \Lambda \text{ is degenerate,} \\ 1 & \text{otherwise.} \end{cases}$$

We can now rephrase our condition (b): by Proposition 3.7 it occurs if and only if $\mathcal{E}(\mathbb{G},(\mathbb{L},\lambda))$ contains unipotent characters labelled by degenerate symbols. But note that the e-core (respectively the e-cocore) of a degenerate symbol is again degenerate. So case (b) happens precisely if Λ is degenerate, that is, if the relative Weyl group is of type $G(2e,2,a)$. But then $W_{\tilde{\mathbb{G}}}(\mathbb{L},\lambda) = G(2e,1,a)$, hence case (b) occurs only if (a) is true.

Thus let us assume that $(\mathbb{L},\lambda)$ is d-cuspidal, λ is labelled by a degenerate symbol and $W_{\tilde{\mathbb{G}}}(\mathbb{L},\lambda) = G(2e,1,a)$. But then the pairs of characters in $\mathcal{E}(\mathbb{G},(\mathbb{L},\lambda))$ corresponding to a degenerate symbol are labelled in the bijection $\rho(\mathbb{L},\lambda)$ precisely by those pairs of characters of $G(2e,2,a)$ which fuse in $G(2e,1,a)$. Thus the result follows in this case.

In the case where ϕ is non-trivial of order 2 neither the case (a) above occurs by [4, 3.1], nor does case (b) by Proposition 3.7.

For $\mathbb{G}$ of type D_4 and σ of order 3, in any d-cuspidal pair $(\mathbb{L},\lambda)$ either $\mathbb{L}$ is a torus, or $\mathbb{L} = \mathbb{G}$ (see [4, Table 3]). In the latter case, λ is a d-cuspidal character of $\mathbb{G}$ and the theorem holds trivially. For the other relevant cases we have listed the Levi subgroups and their relative Weyl groups in Table 1, up to Ennola-duality to reduce the number of entries (see also Remark 4.7(b)). Here, $G(4,2,2)$ and G_i are Shephard and Todd's notation for certain imprimitive respectively primitive complex reflection groups. The split extension $W(D_4)\colon 3$ is a normal subgroup of the reflection group $W(F_4)$ of index 2. (This can be seen from the fact that the normalizer of the subgroup of type D_4 in F_4 contains the graph automorphism of order 3.) The assertion can now be checked from this table in conjunction with Proposition 3.7 by direct verification.

TABLE 1. d-cuspidal pairs for D_4

d	$\mathbb{L}$	$W_{\mathbb{G}}(\mathbb{L},\lambda)$	$W_{\mathbb{G}:2}(\mathbb{L},\lambda)$	$W_{\mathbb{G}:3}(\mathbb{L},\lambda)$	$W_{\mathbb{G}:\mathfrak{S}_3}(\mathbb{L},\lambda)$
1	Φ_1^4	$W(D_4)$	$W(B_4)$	$W(D_4)\colon 3$	$W(F_4)$
3	$\Phi_1^2\Phi_3$	Z_6	$Z_6 \times Z_2$	$Z_6 \times Z_3$	$Z_6 \times \mathfrak{S}_3$
4	Φ_4^2	$G(4,2,2)$	$G(4,1,2)$	G_6	G_8

The general case may easily be reduced to the case of simple generic groups. $\square$

Remark 4.7. (a) We expect that the previous theorem is the consequence of a statement on Lusztig-induction of unipotent characters, as in the case of connected groups, see [4, Thm. 3.2]. But it seems that at present not enough is known on the decomposition of Lusztig-induction in the disconnected case in order to verify this.

(b) The action of σ on $R^\vee$ determines $\sigma \in \mathrm{GL}(Y \otimes \mathbb{R})$ only up to scalars. Different choices lead to isoclinic groups $W_{\mathbb{G}}\langle\sigma\rangle$. For the entries in Table 1 we have chosen σ such that $W_{\tilde{\mathbb{G}}}(\mathbb{L}, \lambda)$ becomes a reflection group whenever possible; this choice may differ for the various cosets.

(c) The complex reflection group G_6 showing up as relative Weyl group in Table 1 for the principal 4-series does not occur as relative Weyl group in any connected group (see [2, Tabelle 3.6]).

For $\mathbb{G}$ with an automorphism of Suzuki- or Ree-type as in Section 3.8, the list of d-cuspidal pairs $(\mathbb{L}, \lambda)$ up to Ennola-duality, with $\mathbb{L} < \mathbb{G}$, is given in Table 2 (with the same understanding as in Remark 4.7(b)). From this it is easily verified that the analogue of Proposition 4.4(a) holds in the present situation. On the other hand, in $\mathbb{G}$ of type F_4 there exist two $W_{\mathbb{G}}$-orbits of 4-cuspidal pairs $(\mathbb{L}, \lambda_i)$, $i = 1, 2$, with $\mathbb{L}$ of type B_2, where σ interchanges λ_1 with λ_2; thus Proposition 4.4(b) is no longer true in this case.

Table 2. d-cuspidal pairs for Suzuki- and Ree-automorphisms

$\mathbb{G}$	d	$\mathbb{L}$	$W_{\mathbb{G}}(\mathbb{L}, \lambda)$	$W_{\tilde{\mathbb{G}}}(\mathbb{L}, \lambda)$	remark
B_2	1	Φ_1^2	$W(B_2)$	$G(8,8,2)$	
	4	Φ_4	Z_4	Z_8	
G_2	1	Φ_1^2	$W(G_2)$	$G(12,12,2)$	
	3	Φ_3	Z_6	Z_{12}	
F_4	1	Φ_1^4	$W(F_4)$	$W(F_4){:}\,2$	
	1	$\Phi_1^2.B_2$	$W(B_2)$	$G(8,8,2)$	
	3	Φ_3^2	G_5	G_{14}	
	4	Φ_4^2	G_8	G_9	
	4	$\Phi_4.B_2$	Z_4	Z_4	2 $W_{\mathbb{G}}$-orbits
	8	Φ_8	Z_8	$Z_8 \times Z_2$	
	12	Φ_{12}	Z_{12}	Z_{24}	

Nevertheless, from Table 2 it is easy to check that d-Harish-Chandra theory also extends to this situation:

Theorem 4.8. *The analogue of Theorem 4.6 continues to hold for automorphisms of Suzuki- and Ree-type.*

Remark 4.9. (Cf. Remark 4.7(c)) The complex reflection group G_{14} showing up as relative Weyl group in Table 2 does not occur as relative Weyl group in any connected group (see [2, Tabelle 3.6]).

5. Centralizers and normalizers of Sylow subgroups

In this section we collect several structural results about centralizers and normalizers of Sylow p-subgroups.

5.1. Properties of Φ_d-tori. Let p be a prime, q an integer prime to p. We define $e_p(q)$ to be the multiplicative order of q modulo p if $p \neq 2$, respectively

$$e_2(q) = \begin{cases} 1 & \text{if } q \equiv 1 \pmod 4, \\ 2 & \text{if } q \equiv -1 \pmod 4. \end{cases}$$

The following elementary number theoretic fact is well-known:

Lemma 5.2. *Let $q \geq 1$, p a prime not dividing q, and e the multiplicative order of q mod p.*

(a) *We have $p|\Phi_f(q)$ if and only if $f = ep^i$ for some $i \geq 0$.*
(b) *If $p^2|\Phi_f(q)$ then $f = e_p(q)$.*

Proof. By assumption we have $q^e \equiv 1 \pmod p$. If $p|\Phi_f(q)$ then $q^f \equiv 1 \pmod p$, hence $e|f$ by the definition of e. Now note that $\Phi_{et}(X)$ divides $\Phi_t(X^e)$. Hence $p|\Phi_{et}(q)$ implies $p|\Phi_t(1)$, so p divides the norm of $1-\zeta$ for some primitive tth root of unity ζ. This forces t to be a power of p (see [30, p.12]), proving one direction in (a).

Now first assume that $p > 2$, hence $e = e_p(q)$. By the first part we know that, if $p^2|\Phi_f(q)$ then $f = ep^i$ for some $i \geq 0$. Write $q^e = 1 + cp^a$ with $p \nmid c$, $a \geq 1$, then

$$q^{ep^j} = (1+cp^a)^{p^j} \equiv 1 + cp^{a+j} \pmod{p^{a+j+1}}$$

for all $j \geq 1$. Thus, if p^a is the precise power dividing $q^e - 1$, then p^{a+j} is the precise power dividing $q^{ep^j} - 1$. In particular, $(q^{ep^j} - 1)/(q^{ep^{j-1}} - 1)$ is divisible by p, but not by p^2. By the first part the only cyclotomic factor of $(q^{ep^j} - 1)/(q^{ep^{j-1}} - 1)$ which can account for this divisibility is $\Phi_{ep^j}(q)$. Thus $\Phi_f(q)$ is divisible by p, but not by p^2 if $i > 0$. This completes the proof of (a) and (b) for odd p.

If $p = 2$ then $\Phi_{2^i}(q) = q^{2^{i-1}} + 1$ for $i \geq 1$. Clearly, this is even, and it is only divisible by 4 if $i = 1$ and $e_2(q) = 2$. $\square$

We prove two results showing that certain abelian p-subgroups lie in Φ_d-tori: one for abelian subgroups of maximal order, the other for those of maximal rank.

Recall that a generic group $\mathbb{H} = (X, R, Y, R^\vee, W\phi)$ is called *semisimple* if the orthogonal complement of R in Y is zero.

Proposition 5.3. *Let $\mathbb{H} = (X, R, Y, R^\vee, W\phi)$ be semisimple, $p \neq 2$ a prime dividing $|\mathbb{H}|(q)$, and $\mathbb{S}$ a torus of $\mathbb{H}$ such that $|\mathbb{S}|(q)_p$ is maximal possible in $\mathbb{H}$. Then:*

(a) *$\mathbb{S}$ contains a Sylow Φ_e-torus of $\mathbb{H}$, where $e = e_p(q)$.*
(b) *If moreover $p \nmid o(\phi)$, then the Sylow p-subgroup of $\mathbb{S}(q)$ is contained in the $\mathbb{F}_q$-points of the Sylow Φ_e-torus of $\mathbb{S}$.*

Proof. This is immediate if p does not divide the order of $W\langle\phi\rangle$, since then $\Phi_e(X)$ is the only cyclotomic polynomial dividing $|\mathbb{H}|$ whose value at q is divisible by p, see [1, Cor. 3.13].

We next consider the case that p does not divide the order of ϕ. Assume, if possible, that $\mathbb{S}$ does not contain a Sylow Φ_e-torus. Embed $\mathbb{S}$ into a maximal torus $\mathbb{T}$ of $\mathbb{H}$. Since $p|\Phi_e(q)$, $\mathbb{T}$ does not contain a Sylow Φ_e-torus either. Let $w\phi \in W\phi$ such that $\mathbb{T}$ is obtained from the maximally split torus $\mathbb{T}_0 = (X, \emptyset, Y, \emptyset, \phi)$ of $\mathbb{H}$ by twisting with $w\phi$. Then the generic order of $\mathbb{T}$ is given by $|\mathbb{T}| = f_{w\phi}(X)$, where $f_{w\phi}$ is the characteristic polynomial of $w\phi$ in the natural reflection representation on Y. Write the generic order of $|\mathbb{T}|$ as a product of cyclotomic polynomials $|\mathbb{T}| = \prod_d \Phi_d(X)^{a(d)}$. Let $k \geq 0$ be minimal subject to

$$a(d) = 0 \text{ if } p^{k+1}|d \qquad \text{and} \qquad p^k \equiv 1 \pmod{o(\phi)}$$

and consider the maximal torus $\mathbb{T}'$ of $\mathbb{H}$ obtained from $\mathbb{T}_0$ by twisting with $w'\phi := (w\phi)^{p^k} \in W\phi$. By looking at the characteristic polynomial of $w'\phi$ we see that

$$|\mathbb{T}'| = \prod_d \Phi_d(X)^{a(d)'} \qquad \text{where } a(d)' = \begin{cases} 0 & \text{if } p|d, \\ \sum_{i\geq 0} \varphi(p^i)a(dp^i) & \text{if } p\not| d, \end{cases}$$

with the Euler φ-function. By Lemma 5.2(a) the only cyclotomic factor of $|\mathbb{T}'|(q)$ divisible by p is $\Phi_e(q)$, and hence $|\mathbb{T}'|(q)_p = \Phi_e(q)_p^{a(e)'}$. Clearly this is maximal if $\mathbb{T}'$ contains a Sylow Φ_e-torus. Since $\mathbb{T}$ does not contain a Sylow Φ_e-torus, there exists some $i > 0$ with $a(ep^i) > 0$. But then by Lemma 5.2(b) the p-part of $|\mathbb{T}'|(q)$ is larger than $|\mathbb{T}|(q)_p$, a contradiction. This argument also shows that in this case the Sylow Φ_e-torus of $\mathbb{S}$ contains the full p-part of $|\mathbb{S}|(q)$, proving (b).

We now consider the general case of (a). If $\mathbb{H}$ is simple and $p|o(\phi)$, then necessarily $\mathbb{H}$ is of type ${}^3D_4$ with ϕ of order $p = 3$, since $p \neq 2$. In this case the assertion is easily checked from the known order formulae. This completes the proof for $\mathbb{H}$ simple.

Next assume that

$$Y = Y_1 \oplus \ldots \oplus Y_a \qquad \text{and} \qquad W = W_1 \times \ldots \times W_a$$

are ϕ-invariant decompositions of Y, W respectively, into $a \geq 2$ isomorphic direct summands, respectively factors permuted transitively by ϕ, and where W_i acts trivially on Y_j for $i \neq j$. Thus $\mathbb{H}$ is the lifting of scalars $\mathbb{H}_1^{(a)}$ of a simple generic group

$$\mathbb{H}_1 = (X_1, R_1, Y_1, R_1^\vee, W_1\phi^a)$$

with $|\mathbb{H}(x)| = |\mathbb{H}_1(x^a)|$ (see [4, (1.3)]). Since e is minimal with $p|\Phi_e(q)$, $e_1 := e/\gcd(e,a)$ is minimal with $p|\Phi_{e_1}(q^a)$. If $\mathbb{S}$ is a torus as in the statement, then there exists a torus $\mathbb{S}_1$ of $\mathbb{H}_1$ such that $\mathbb{S}$ is contained in $\mathbb{S}_1^{(a)}$. By the first part of the proof, $\mathbb{S}_1$ contains a Sylow Φ_{e_1}-torus of $\mathbb{H}_1$. But then $\mathbb{S}_1^{(a)}$ in turn contains a Sylow Φ_e-torus of $\mathbb{H}$. By the maximality assumption on $\mathbb{S}$, this must be contained in $\mathbb{S}$, as claimed.

If $\mathbb{H}$ is a direct product of simple factors, then the result follows from the corresponding one for the individual factors. Finally note that the generic orders of tori are insensitive to the isogeny type, so we may assume that the derived group of $\mathbb{H}$ is of simply connected type. Then $\mathbb{H}$ is a direct product as before. This completes the proof. □

Let's note the following consequence:

Corollary 5.4. *If $\mathbb{H}$ is semisimple and $p > 2$ divides $|\mathbb{H}|(q)$ then Φ_e divides $|\mathbb{H}|$, where $e = e_p(q)$.*

Remark 5.5. By Lemma 5.2 it is clear that in the situation of Corollary 5.4, $|\mathbb{H}|$ is divisible by Φ_d for some $d = ep^i$, $i \geq 0$. But note that even for semisimple groups it is not true in general that if Φ_d divides $|\mathbb{H}|$ then Φ_e divides $|\mathbb{H}|$ for all divisors $e|d$. An example is $\mathbb{H}$ of type ${}^3D_4$ whose generic order is divisible by Φ_{12} but not by Φ_4.

The same group also gives a example that Proposition 5.3 (b) may fail when $p|o(\phi)$: for $q \equiv 1 \pmod 3$ the maximal torus $\mathbb{S}$ of order $\Phi_1^2\Phi_3$ has maximal possible 3-part, but the order of ${}^3D_4$ is divisible by $\Phi_1^2\Phi_3^2$, so $\mathbb{S}(q)$ does not contain a Sylow 3-subgroup of ${}^3D_4(q)$.

Both Proposition 5.3 and Corollary 5.4 fail for arbitrary generic groups; as an example take the torus of generic order Φ_3 for $\mathbb{H}$, $p = 3$ and q such that $e_3(q) = 1$.

We need the following observation from [1, Cor. 3.13]:

Proposition 5.6. *Let $\mathbb{T}$ be a Φ_d-torus of order Φ_d^a, q a prime power and p a prime dividing $|\mathbb{T}|(q)$. Then the Sylow p-subgroup of $\mathbb{T}(q)$ is homocyclic of type Z_k^a, where $k = p^b$ is the precise power of p dividing $\Phi_d(q)$.*

Proposition 5.7. *Let $\mathbb{H} = (X, R, Y, R^\vee, W\phi)$ be semisimple, $p\not|2o(\phi)$ a prime dividing $|\mathbb{H}|(q)$, and $\mathbb{S}$ a torus of $\mathbb{H}$ such that $\mathbb{S}(q)$ has maximal possible p-rank. Then $\mathbb{S}$ contains a Sylow Φ_e-torus $\mathbb{S}_e$ of $\mathbb{H}$, where $e = e_p(q)$, and the Sylow p-subgroup of $\mathbb{S}(q)$ is contained in $\mathbb{S}_e(q)$.*

Proof. This is immediate if p does not divide the order of $W\langle\phi\rangle$, since then $\Phi_e(X)$ is the only cyclotomic polynomial dividing $|\mathbb{H}|$ whose value at q is divisible by p, see [1, Cor. 3.13].

The general case follows easily with a variation of the first part of the proof of Proposition 5.3 using the structure result of Proposition 5.6. □

Note that, in contrast to Proposition 5.3, the previous statement does not hold without the restriction that $p\not|o(\phi)$: in $\mathbb{H}$ of type ${}^3D_4$, with $q \equiv 1 \pmod 3$, both a Sylow Φ_1- and a Sylow Φ_3-torus contain an elementary abelian 3-subgroup of maximal rank.

5.8. Centralizers of Sylow subgroups. We now show that centralizers of Sylow subgroups are strongly related to centralizers of Sylow tori. The proof of this result would be much simpler if we would assume that the underlying group is of simply-connected type, but unfortunately we will need the result just in the opposite case, the adjoint one.

Theorem 5.9. *Let $\mathbf{H}$ be simple, defined over $\mathbb{F}_q$ with corresponding Frobenius map $F : \mathbf{H} \to \mathbf{H}$, let $H := \mathbf{H}^F$ and p a prime divisor of $|H|$, not dividing q. Then every semisimple element $g \in H$ which centralizes a Sylow p-subgroup of H lies in a torus containing a Sylow Φ_e-torus of $\mathbf{H}$, where $e = e_p(q)$. In particular, g centralizes a Sylow Φ_e-torus.*

Proof. Let $g \in H$ be semisimple, centralizing a Sylow p-subgroup of H, and set $\mathbf{C} := \mathrm{C}_{\mathbf{H}}(g)$. We proceed in several steps.

(1) Assume for a moment that the index $(\mathbf{C} : \mathbf{C}^\circ)$ is prime to p. Then $C := (\mathbf{C}^\circ)^F$ contains a Sylow p-subgroup of H. The generic orders of $\mathbf{H}$ and $\mathbf{C}^\circ$ are both monic polynomials, with the second dividing the first. Hence, all cyclotomic polynomials Φ_i with $p|\Phi_i(q)$ dividing the generic order of $\mathbf{H}$ also divide the generic order of $\mathbf{C}^\circ$, to the same power. In particular, a Sylow Φ_e-torus of $\mathbf{C}^\circ$ is a Sylow Φ_e-torus of $\mathbf{H}$. Thus, g centralizes a Sylow Φ_e-torus of $\mathbf{H}$. By [8, Prop. 3.5.1] we even have $g \in \mathbf{C}^\circ$, so g is contained in a maximal torus containing a Sylow Φ_e-torus.

(2) Now first assume that $\mathbf{H}$ is an arbitrary connected reductive group and the order of g is prime to p. By [29, Cor. 2.16] the exponent of $\mathbf{C}/\mathbf{C}^\circ$ divides the order of g, so is prime to p. Hence the claim follows from the previous consideration.

(3) Next assume that g has p-power order. Let $\pi : \tilde{\mathbf{H}} \to \mathbf{H}$ denote the universal covering of $\mathbf{H}$. Again by [29, Cor. 2.16] the factor group $\mathbf{C}/\mathbf{C}^\circ$ is isomorphic to a subgroup of the fundamental group $\ker(\pi)$, hence we are done by step (1) if $|\ker(\pi)|$ is prime to p. This argument works for arbitrary semisimple groups.

Consulting the list of possible orders of $|\ker(\pi)|$ for simple groups we see that the only cases not covered by the above argument are $\mathbf{H}$ of type A_{n-1} with p dividing n, $\mathbf{H}$ of type B_n, C_n, D_n or E_7 with $p = 2$, and $\mathbf{H}$ of type E_6 with $p = 3$.

If $p = 2$ and $\mathbf{H}$ is of type B_n, C_n, or E_7, then $|\mathbf{C}/\mathbf{C}^\circ| \leq |\ker(\pi)| \leq 2$. By the definition of $e = e_2(q)$, $\Phi_e(q)$ is divisible by 4. Hence, if $\mathbf{C}^\circ$ does not contain a Sylow Φ_e-torus, then $\mathbf{C}/\mathbf{C}^\circ$ is also divisible at least by 4, a contradiction.

Now let $\mathbf{H}$ be of type D_n and $p = 2$. By [26, Cor. II.5.19] a Sylow 2-subgroup S of H is contained in the normalizer of some maximal torus of H. If H is untwisted and $q \equiv 1 \pmod 4$ (so $e = 1$), then $S \leq T.W(D_n)$, where T is the Sylow Φ_1-torus. Since $W(D_n)$ acts faithfully on the homocyclic subgroup C_4^n of T, every 2-central element of S is already contained in the maximal torus T. Similarly, for $q \equiv 3 \pmod 4$ (so $e = 2$) and n even, we have $S \leq T.W(D_n)$ where T is a Sylow Φ_2-torus of H. If $q \equiv 3 \pmod 4$ and n is odd, we have $S \leq T.W(B_{n-1})$ where now T contains a Sylow Φ_2-torus of H. We may now argue as before. The same reasoning applies if H is of twisted type.

For $\mathbf{H}$ of type A_{n-1} we may again use the explicit knowledge of the Sylow structure of H. First assume that H is untwisted. Then, a Sylow p-subgroup of H is contained in the normalizer of a Sylow Φ_e-torus T of order Φ_e^a of H, where $0 \leq n - ea < e$, with centralizer quotient a wreath product $C_e \wr \mathfrak{S}_a$. Here, the C_e-factors act as field automorphisms of $\mathbb{F}_{q^e}/\mathbb{F}_q$ on cyclic subgroups of T of order $\Phi_e(q)$, while $\mathfrak{S}_a$ permutes the coordinates. In particular the centralizer quotient acts faithfully on the Sylow p-subgroup of T, and we may conclude as before.

For $\mathbf{H}$ of type E_6 with $p = 3$, with H untwisted, the only class of 3-central 3-elements has centralizer of type $A_2(q)^3.3$ if $q \equiv 1 \pmod 3$ (so $e = 1$), respectively ${}^2A_2(q)A_2(q^2)$ if $q \equiv -1 \pmod 3$ (so $e = 2$), by [14, Tab. 4.7.3A]. Both contain a Sylow Φ_e-torus. The center of a Sylow 3-subgroup of $A_2(q)^3.3$ respectively ${}^2A_2(q)A_2(q^2)$ does not contain elements of order 9. The case for twisted H is analogous.

(4) Finally, in the general case, let $g = g_1 g_2$ be the decomposition of g into its p- and p'-parts. By step (3) the connected component $\mathbf{C}_1^\circ$ of the centralizer $\mathbf{C}_1 = \mathrm{C}_{\mathbf{H}}(g_1)$ of g_1 contains a Sylow Φ_e-torus of $\mathbf{H}$. Since $\mathrm{C}_{\mathbf{H}}(g) \geq \mathrm{C}_{\mathbf{C}_1^\circ}(g_2)$, the result now follows from (2) applied to the element g_2 of the reductive group $\mathbf{C}_1^\circ$. $\square$

5.10. Normalizers of Sylow subgroups. We now turn to an important property of Sylow normalizers. Let $\mathbf{H}$ be connected reductive, defined over $\mathbb{F}_q$. See [1, Prop. 3.12] for a version of the following result:

Proposition 5.11. *Let* $\mathbf{S}$ *be a Sylow* Φ_e*-torus of* $\mathbf{H}$*. Then* $\mathrm{N}_H(\mathbf{S})$ *controls* H*-fusion in* $\mathrm{C}_H(\mathbf{S})$*.*

Proof. Let $g_1, g_2 \in \mathrm{C}_H(\mathbf{S})$ be conjugate in H, so $g_2 = g_1^g$ for some $g \in H$. Let $\mathbf{C} := \mathrm{C}_{\mathbf{H}}(g_2)^\circ$ and write $\bar{\ }: \mathbf{C} \to \mathbf{C}/R_u(\mathbf{C})$ for the natural epimorphism onto the connected reductive group $\bar{\mathbf{C}} = \mathbf{C}/R_u(\mathbf{C})$. Then $\bar{\mathbf{S}}$ and $\bar{\mathbf{S}}^g$ are Sylow Φ_e-tori of $\bar{\mathbf{C}}$, hence conjugate by some $c \in \bar{\mathbf{C}}^F$ by the Sylow Theorem 3.2 for Φ_e-tori. Thus $\mathbf{S}^c R_u(\mathbf{C}) = \mathbf{S}^g R_u(\mathbf{C})$, so that $\mathbf{S}^c, \mathbf{S}^g$ are F-stable maximal tori of the solvable connected group $\mathbf{S}^c R_u(\mathbf{C})$. Hence they are conjugate by some $v \in R_u(\mathbf{C})^F$. In conclusion, $g(cv)^{-1} \in H$ normalizes $\mathbf{S}$, so $g = ncv$ for some $n \in \mathrm{N}_H(\mathbf{S})$, and $g_2 = g_1^g = g_1^n$ as claimed. $\square$

Let $\sigma : \mathbf{H} \to \mathbf{H}$ be an onto endomorphism of $\mathbf{H}$ commuting with F. Then σ stabilizes $H = \mathbf{H}^F$, and its restriction to H is an element of $\mathrm{Aut}(H)$. If $\mathbf{H}$ is simple, then by Steinberg's classification of automorphisms of finite groups of Lie type [27, Th. 30 and 36] every element of $\mathrm{Aut}(H)$ arises in this way. We can therefore consider any automorphism of H as an automorphism (of abstract groups) of $\mathbf{H}$ commuting with F. This gives sense to the following statement:

Proposition 5.12. *Let* $\mathbf{H}$ *be simple,* $\mathbf{S}_e$ *a Sylow* Φ_e*-torus of* $\mathbf{H}$. *Then* $\mathrm{Aut}(H) = \mathrm{N}_{\mathrm{Aut}(H)}(\mathbf{S}_e).H$.

Proof. Let $\sigma \in \mathrm{Aut}(\mathbf{H})$ commuting with F. By [28, 7.1] σ permutes the F-stable tori of $\mathbf{H}$, and it preserves the generic orders. Thus, σ permutes the Φ_e-tori of $\mathbf{H}$. The claim now follows from the Sylow theorem for Φ_e-tori by a Frattini-argument. $\square$

We now show that normalizers of Sylow subgroups are strongly related to normalizers of Sylow tori. Here, in contrast to the situation for Theorem 5.9, the proof would be much simpler if we would assume that the underlying group has connected center, but unfortunately we will need the result just in the opposite case, the simply-connected one. Recall that a subgroup is called *toral* if it is contained in some torus.

Proposition 5.13. *Assume that* $\mathbf{H}$ *is simple. Let* $p \not| 2o(\phi)$, $P \leq \mathbf{H}^F$ *a toral elementary abelian* p*-subgroup of maximal possible rank. Then* P *is contained in a unique Sylow* Φ_e*-torus of* $\mathbf{H}$.

Proof. By assumption there exists an F-stable torus $\mathbf{T}$ of $\mathbf{H}$ such that $P \leq \mathbf{T}^F$. Clearly all Sylow Φ_e-tori containing P lie in $\mathbf{L} := \mathrm{C}_{\mathbf{H}}(P)^\circ$. Clearly, $\mathbf{T} \leq \mathbf{L}$, so $P \leq \mathbf{L}$ and hence $P \leq Z(\mathbf{L})$.

First assume that p is good for $\mathbf{H}$. Then by repeated application of [12, Prop. 2.1] we see that $\mathbf{L}$ is a proper Levi subgroup of $\mathbf{H}$. We claim that $|Z(\mathbf{L})/Z(\mathbf{L})^\circ|$ is prime to p. If $|Z(\mathbf{H})/Z(\mathbf{H})^\circ|$ is prime to p this follows as in [10, Lemma 13.14]. In the general case observe that if $|Z(\mathbf{L})/Z(\mathbf{L})^\circ|$ is divisible by p, then $\mathbf{L}$, being connected, has at least one simple factor $\mathbf{L}_1$ with the same property. Going through the list of simple groups $\mathbf{L}_1$ one sees that the p-rank of $|Z(\mathbf{L}_1)/Z(\mathbf{L}_1)^\circ|$ is at most one, but Φ_e divides the generic order of $\mathbf{L}_1$ at least twice. Since the rank of P is at least equal to the precise power of Φ_e dividing the generic order of $\mathbf{H}$, no such factor can arise, and the claim is proved. Hence P is an elementary abelian p-subgroup of the torus $Z(\mathbf{L})^\circ$ of $\mathbf{H}$. By Proposition 5.7 this implies that $Z(\mathbf{L})^\circ$ contains a Sylow Φ_e-torus of $\mathbf{H}$, the only Sylow Φ_e-torus of $\mathbf{L}$. The claim follows in this case.

If p is bad for $\mathbf{H}$, that is, if $p = 3$ for the exceptional types, or $p = 5$ for E_8, one sees by inspection (using for example [14, Tab. 4.7.1]) that $\mathbf{L}$ is a proper Levi subgroup of $\mathbf{H}$. Then the previous argument applies. $\square$

The example of ${}^3D_4(q)$ with $p = 3$, $q \equiv 1 \pmod 3$, given after Proposition 5.7 shows that the assumption $p \not| o(\phi)$ in Proposition 5.13 is again necessary.

Theorem 5.14. *Let* $\mathbf{H}$ *be simple, defined over* $\mathbb{F}_q$ *with corresponding Frobenius map* $F : \mathbf{H} \to \mathbf{H}$ *and* $H := \mathbf{H}^F$. *Let* $p \neq 2$ *be a prime divisor of* $|H|$, *not dividing* q

and S_p a Sylow p-subgroup of H. Then there exists a Sylow Φ_e-torus $\mathbf{S}_e$ of $\mathbf{H}$ with $\mathrm{N}_H(S_p) \le \mathrm{N}_H(\mathbf{S}_e)$, where $e = e_p(q)$, unless $p = 3$ and one of

(a) $H = \mathrm{SL}_3(q)$ *with* $q \equiv 4, 7 \pmod 9$, *or*
(b) $H = \mathrm{SU}_3(q)$ *with* $q \equiv 2, 5 \pmod 9$, *or*
(c) $H = G_2(q)$ *with* $q \equiv 2, 4, 5, 7 \pmod 9$.

Proof. If p is large enough, that is, if p does not divide $|W\langle\phi\rangle|$, then this is implicit in [4, Thm. 5.24]. In the more general case that the Sylow p-subgroup of H is abelian, this is shown in [6, Thm. 2.1(3)].

If $p \ge 5$, then by the conceptual argument in Cabanes [7, Thm. 4.4] any Sylow p-subgroup S_p of H contains a unique maximal elementary abelian toral p-subgroup P. By Proposition 5.13 this is contained in a unique Sylow Φ_e-torus $\mathbf{S}_e$ of $\mathbf{H}$. Thus any $g \in \mathrm{N}_H(S_p)$ stabilizes P, hence stabilizes $\mathbf{S}_e$, so lies inside $\mathrm{N}_H(\mathbf{S}_e)$.

For $p = 3$ a case-by-case analysis from Gorenstein–Lyons [13, (10-2)] shows that when $\mathbf{H}$ is not of type A_2, G_2, or D_4 with $o(\phi) = 3$, then again S_p contains a unique maximal elementary abelian toral p-subgroup, and we may conclude as before.

Explicit computation shows that the conclusion continues to hold if $\mathbf{H} = \mathrm{PGL}_3$, if $H = \mathrm{SL}_3(q)$ with $q \not\equiv 4, 7 \pmod 9$, and if $H = \mathrm{SU}_3(q)$ with $q \not\equiv 2, 5 \pmod 9$. Also, if $G_2(q)$ contains elements of order 9, the normalizer of a Sylow 3-subgroup is contained in the normalizer of a Φ_e-torus. For $H = {}^3D_4(q)$, $3 \nmid q$, the description of local subgroups in [16] shows that the normalizer in H of a Sylow 3-subgroup is contained in a subgroup $U = (C_{q^2+q+1} \circ \mathrm{SL}_3(q)).3$ respectively $U = (C_{q^2-q+1} \circ \mathrm{SU}_3(q)).3$. If $q \equiv \pm 1 \pmod 9$ then by the case of $\mathrm{SL}_3(q)$ respectively $\mathrm{SU}_3(q)$ above, the normalizer of a Sylow 3-subgroup is contained in the normalizer of a Φ_e-torus of U. If $q \equiv 2, 4, 5, 7 \pmod 9$, the Sylow 3-subgroup of U is contained in a subgroup of $\mathrm{GL}_3(q)$ respectively $\mathrm{GU}_3(q)$ of $3'$-index. In particular the Sylow 3-normalizer is contained in the Sylow 3-normalizer of $\mathrm{GL}_3(q)$ respectively $\mathrm{GU}_3(q)$, which in turn is contained in the normalizer of a Φ_e-torus. This leaves the stated exceptions. □

Remark 5.15. Note that the assumption in Theorem 5.14 is necessary: the normalizer of a Sylow 2-subgroup of $\mathrm{SL}_2(q)$, $q \equiv 3, 5 \pmod 8$, is $\mathrm{SL}_2(3)$ which is not contained in the normalizer of a Sylow Φ_1- or Φ_2-torus. Also, in $\mathrm{SL}_3(q)$, $q \equiv 4, 7 \pmod 9$, as well as in $\mathrm{SU}_3(q)$, $q \equiv 2, 5 \mod 9$, the normalizer of a Sylow 3-subgroup is an extraspecial group 3^{1+2} extended by the quaternion group of order 8, which again does not lie in the normalizer of any maximal torus.

5.16. Normalizers of Sylow 2-subgroups. The statement of the previous Theorem remains true for the prime $p = 2$, with a single family of exceptions. This can be proved using so-called fundamental A_1-subgroups. For this, let $\mathbf{H}$ be a simple algebraic group defined over a finite field $\mathbb{F}_q$ of odd order, with corresponding Frobenius endomorphism $F : \mathbf{H} \to \mathbf{H}$, and $H := \mathbf{H}^F$. Let $\mathbf{B}$ be an F-stable Borel subgroup and $\mathbf{T} \subset \mathbf{B}$ an F-stable maximal torus. Denote by Σ the set of roots of $\mathbf{T}$ with respect to $\mathbf{B}$. For a long root $\alpha \in \Sigma$ the subgroup $\langle \mathbf{X}_\alpha^F, \mathbf{X}_{-\alpha}^F \rangle$ generated by the F-fixed points of the long root subgroup $\mathbf{X}_\alpha$ and its opposite is called a *fundamental A_1-subgroup* of H. Let $\mathcal{A}$ be a maximal set of commuting A_1-subgroups of H, and $M := \langle A \in \mathcal{A} \rangle$. Let $R := \mathrm{N}_H(M)$. Then we have the following crucial result of Aschbacher (see [14, Thm. 4.10.6]):

Theorem 5.17. *In the above notation, R contains the normalizer of a Sylow 2-subgroup of H. Furthermore, R permutes the $A \in \mathcal{A}$ transitively. The kernel R_0 of* *this action is a product $R_0 = MTL$, where $T = \mathbf{T}^F$ is a maximally split torus and L is either trivial or a short root A_1-subgroup (possibly defined over an extension field).*

As we will see, it follows from this description that in most cases Sylow 2-normalizers are contained in normalizers of maximal tori. Set $e := e_2(q)$. We first consider the case where $\mathbf{H}$ is of type A_1. The subgroup structure of $\mathrm{PGL}_2(q)$ and of $\mathrm{PSL}_2(q)$ is well-known. It follows from this that Sylow 2-subgroups of H are self-normalizing and hence normalize a maximal torus of H with maximal 2-part, that is, a Sylow Φ_e-torus, except when $H = \mathrm{SL}_2(q)$ and $q \equiv 3, 5 \pmod 8$. (In the latter case the Sylow 2-normalizer are isomorphic to $\mathrm{SL}_2(3)$ and are *not* contained in the normalizer of any maximal torus of H):

Lemma 5.18. *Let $\mathbf{H}$ be of type A_1. Then the Sylow 2-subgroups of H are self-normalizing and hence contained in the normalizer of a Sylow Φ_e-torus of $\mathbf{H}$, unless $H = \mathrm{SL}_2(q)$ with $q \equiv 3, 5 \pmod 8$.*

We now obtain the analogue of Theorem 5.14 for $p = 2$:

Theorem 5.19. *Let $\mathbf{H}$ be a simple algebraic group defined over a field of odd order q, with corresponding Frobenius endomorphism $F : \mathbf{H} \to \mathbf{H}$, and $H := \mathbf{H}^F$. Let $e := e_2(q)$. Then the normalizer of a Sylow 2-subgroup of H is contained in the normalizer of a Sylow Φ_e-torus of $\mathbf{H}$, unless $H = \mathrm{Sp}_{2n}(q)$ with $n \geq 1$ and $q \equiv 3, 5 \pmod 8$.*

Proof. In the notation introduced above, let S be a Sylow 2-subgroup of R. By Theorem 5.17 we have $\mathrm{N}_H(S) = \mathrm{N}_R(S)$. Let $\bar{R} := R/R_0$. Then $\mathrm{N}_R(S)R_0/R_0 \subseteq \mathrm{N}_{\bar{R}}(\bar{S})$ where $\bar{S} := SR_0/R_0$. By inspection of the cases in [14, Table 4.10.6], $\bar{R}$ has self-normalizing Sylow 2-subgroups, so that $\mathrm{N}_{\bar{R}}(\bar{S}) = \bar{S}$. Hence $\mathrm{N}_R(S) \subseteq SR_0$, and then in fact $\mathrm{N}_R(S) \subseteq S\mathrm{N}_{R_0}(S \cap R_0)$.

We next claim that $\tilde{N} := \mathrm{N}_{R_0}(S \cap R_0)/Z(R_0)$ is a 2-group. Then

$$\mathrm{N}_{R_0}(S \cap R_0) = (S \cap R_0)Z(R_0) = (S \cap R_0) \times Z(R_0)_{2'},$$

so $\mathrm{N}_R(S) \subseteq S \times \mathrm{C}_{Z(R_0)}(S)_{2'}$. Since the latter is a supersolvable group, it is contained in the normalizer of a maximal torus of H by [26, Th. II.5.16(b)]. By Proposition 5.20 below, it normalizes a Sylow Φ_e-torus.

In order to prove the claim, let

$$\tilde{R}_0 := R_0/Z(R_0), \ \tilde{A}^* := A^*T/Z(R_0) \text{ for } A^* \in \mathcal{A}, \ \tilde{L} := LT/Z(R_0),$$

so that $\tilde{R}_0$ is the subdirect product of $\tilde{L}$ with the $\tilde{A}^*$ for $A^* \in \mathcal{A}$. Also, denote $\tilde{S} := (S \cap R_0)/Z(R_0)$. Now assume that $g \in \tilde{N}$ has odd order. Since R_0 normalizes all $A^* \in \mathcal{A}$, g normalizes the Sylow 2-subgroup $S^* := \tilde{S} \cap \tilde{A}^*$ of $\tilde{A}^*$, and it normalizes a Sylow 2-subgroup S_0 of $\tilde{L}$. We will show that, except when $H = \mathrm{Sp}_{2n}(q)$ with $q \equiv 3, 5 \pmod 8$, $\tilde{A}^*$ and $\tilde{L}$ both have self-normalizing Sylow 2-subgroups. Then necessarily $g \in Z(\tilde{R}_0) = 1$ and our claim is proved.

First assume that $\mathbf{H}$ contains an F-stable subgroup $\mathbf{H}_1$ of type A_2 generated by long root subgroups. Since all long root subgroups are conjugate, we may assume that $A^* \leq \mathbf{H}_1$. Then $A^*(H_1 \cap T)$ is the Levi complement of a maximal parabolic subgroup of H_1, hence isomorphic to $\mathrm{GL}_2(q)$ or $\mathrm{GU}_2(q)$. Thus, $\tilde{A}^* \cong \mathrm{PGL}_2(q)$. It follows from Lemma 5.18 that its Sylow 2-subgroup is self-normalizing, as claimed.

The only cases without an F-stable subgroup of type A_2 generated by long root subgroups are for $\mathbf{H}$ of type C_n with $n \geq 1$. Here, the fundamental subgroups are isomorphic to $\mathrm{SL}_2(q)$, hence have self-normalizing Sylow 2-subgroups except when $q \equiv 3, 5 \pmod 8$ by Lemma 5.18. In the case of adjoint type, it follows from the Steinberg relations that $\tilde{A}^* \cong \mathrm{PGL}_2(q)$, so the Sylow 2-subgroups are self-normalizing.

The cases with $L \neq 1$ are listed in [14, Table 4.10.6]: then H is of type B_n with n odd, of type ${}^2D_n$ with n even, of type G_2, or of type ${}^3D_4$. Here, it can be seen by direct calculations inside R with the Steinberg relations that T acts by outer diagonal automorphisms on L and thus, as above, the Sylow 2-subgroups of $\tilde{L}$ are self-normalizing. □

We also record the following assertion in the case $p = 2$, where now $\mathbf{H}$ is an arbitrary connected reductive group:

Proposition 5.20. *Let $\mathbf{H}$ be a connected reductive group defined over $\mathbb{F}_q$, q odd, with corresponding Frobenius map $F : \mathbf{H} \to \mathbf{H}$ and $H := \mathbf{H}^F$. Let $e = e_2(q)$ and $\mathbf{S}_e$ a Sylow Φ_e-torus of $\mathbf{H}$. Then $\mathrm{N}_H(\mathbf{S}_e)$ contains a Sylow 2-subgroup of H.*

Proof. Let $\mathbf{T}$ be some F-stable maximal torus in the centralizer of $\mathbf{S}_e$. Since $\mathbf{S}_e$ is the unique Sylow e-torus of $\mathbf{T}$, clearly $\mathrm{N}_H(\mathbf{T}) \leq \mathrm{N}_H(\mathbf{S}_e)$. So it suffices to see that $\mathrm{N}_H(\mathbf{T})$ contains a Sylow 2-subgroup.

The group $\mathbf{H}$ has an F-stable decomposition $\mathbf{H} = \mathbf{H}_1 \cdots \mathbf{H}_n \mathbf{Z}$ with $\mathbf{H}_i$ simple and $\mathbf{Z}$ a central torus. First assume that $\mathbf{Z} = 1$ and F permutes the $\mathbf{H}_i$ transitively. Thus $H \cong \mathbf{H}_1^{F^n}$ is the group of fixed points of a simple algebraic group, and the assertion can be checked from the order formulae and the fact that $|\mathrm{N}_H(\mathbf{T})| = |\mathbf{T}^F||W(\mathbb{T})|$, where $W(\mathbb{T})$ is the centralizer in $W\phi$ of ϕ or $w_0\phi$, with the longest element w_0 of W.

In general, change notation so that now the $\mathbf{H}_i$ are minimal F-stable products of simple factors of $\mathbf{H}$. Then clearly

$$\mathrm{N}_H(\mathbf{T}) = \mathrm{N}_{H_1}(\mathbf{T} \cap \mathbf{H}_1) \cdots \mathrm{N}_{H_n}(\mathbf{T} \cap \mathbf{H}_n)\mathbf{Z}^F,$$

with $H_i := \mathbf{H}_i^F$, so the claim follows from the previous case. □

This allows to state the following general property:

Proposition 5.21. *Let $\mathbf{H}$ be simple, defined over $\mathbb{F}_q$ with corresponding Frobenius map $F : \mathbf{H} \to \mathbf{H}$ and $H := \mathbf{H}^F$. Let p be a prime divisor of $|H|$, not dividing q, and $e = e_p(q)$. Then the normalizer $\mathrm{N}_H(\mathbf{S}_e)$ of a Sylow Φ_e-torus $\mathbf{S}_e$ of $\mathbf{H}$ contains a Sylow p-subgroup of H.*

Proof. For $p \geq 3$ this is in [14, Th. 4.10.2], for $p = 2$ it was shown in Proposition 5.20. □

6. Unipotent height 0 characters

In this section we start our investigation of characters of p-height 0 by considering unipotent characters.

6.1. Specializations of cyclotomic polynomials. We need to study specializations of cyclotomic polynomials at integers and roots of unity. Let q be a prime power, p a prime not dividing q. Let ζ_e denote a primitive $e := e_p(q)$th root of unity over $\mathbb{Q}$, and $N_e : \mathbb{Q}[\zeta_e] \to \mathbb{Q}$ the norm of $\mathbb{Q}[\zeta_e]/\mathbb{Q}$.

Lemma 6.2. *Let p, q, e be as above. Let $f(X) \in \mathbb{Q}[X]$ be a product of cyclotomic polynomials not divisible by $\Phi_e(X)$. Then $N_e(f(q)/f(\zeta_e)) \equiv 1 \pmod p$.*

Proof. Our assumption on p, q, e implies that $q \equiv \zeta_e \pmod{\mathfrak{p}}$ in $\mathbb{Q}[\zeta_e]$ for some prime ideal $\mathfrak{p}$ of the ring of integers of $\mathbb{Q}[\zeta_e]$ containing p.

Clearly it is sufficient to verify the assertion for linear polynomials $f(X) = X - \zeta$, where ζ is a root of unity. Let $\mathfrak{P}$ be a prime divisor of $q - \zeta_e$ in $K := \mathbb{Q}[\zeta, \zeta_e]$ lying above $\mathfrak{p}$. Let $\nu_{\mathfrak{P}}$ denote the extension of the p-valuation from $\mathbb{Q}_p$ to the p-adic field $K_{\mathfrak{P}}$. If $\zeta\zeta_e^{-1}$ does not have p-power order, then $\zeta - \zeta_e$ is a unit in $K_{\mathfrak{P}}$, see for example Washington [30, p.12], hence prime to p, and $q - \zeta \equiv \zeta_e - \zeta \pmod{\mathfrak{P}}$ are both prime to p. Our claim follows.

If $\zeta\zeta_e^{-1}$ has order p^a, $a \geq 1$, then

$$\nu_{\mathfrak{P}}(\zeta - \zeta_e) = 1/\phi(p^a)$$

with the Euler ϕ-function (see [30, p.9]). On the other hand, $\Phi_e(q)$ is divisible by p, respectively by 4 if $p = 2$, so

$$\nu_{\mathfrak{P}}(q - \zeta_e) \geq \begin{cases} 1/\phi(e) & \text{for } p > 2, \\ 2/\phi(e) & \text{for } p = 2. \end{cases}$$

But if $p > 2$ then $e|(p-1)$ implies $\phi(e) < \phi(p^a)$, while for $p = 2$ we have $\phi(e)/2 < \phi(p^a)$. Thus the first term on the right hand side of

$$\frac{q - \zeta}{\zeta_e - \zeta} = \frac{q - \zeta_e}{\zeta_e - \zeta} + 1$$

has positive $\mathfrak{P}$-valuation, and the result follows. □

As an immediate consequence we obtain:

Corollary 6.3. *Let $f(X) = rf_1(X)/f_2(X)$ with $r \in \mathbb{Q}^\times$, $f_1(X), f_2(X) \in \mathbb{Q}[X]$ products of cyclotomic polynomials not divisible by $\Phi_e(X)$. Then $N_e(f(q)/f(\zeta_e)) \equiv 1 \pmod{p}$. In particular, $N_e(f(q)/f(\zeta_e))$ is a rational number with numerator and denominator prime to p.*

6.4. p-heights of unipotent characters in non-defining characteristic. We need to study p-heights of unipotent characters of a not necessarily connected reductive group $\mathbf{H}$ defined over $\mathbb{F}_q$ with Frobenius endomorphism $F : \mathbf{H} \to \mathbf{H}$ and group of fixed points $H := \mathbf{H}^F$. This is most conveniently done in the setting of disconnected generic groups as recalled in Section 3. We start with a property of cuspidal pairs of connected groups:

Proposition 6.5. *Assume that $\mathbf{H}$ is connected. Let p be a prime, $p \not| q$, and $e = e_p(q)$. Let $\chi \in \mathcal{E}(H, 1)$ be e-cuspidal. If the Sylow Φ_e-tori of $\mathbf{H}$ are not central, then $\chi(1)$ is divisible by p.*

Proof. By [4, Prop. 2.4] the assumptions that χ is e-cuspidal and that the Sylow Φ_e-tori of $\mathbf{H}$ are not central imply that Φ_e also divides the degree polynomial $\mathrm{Deg}(\chi)$ of χ. Since the degree polynomial of a unipotent character is a product of cyclotomic polynomials times a rational integer whose denominator only involves bad primes for $\mathbf{H}$ (see 3.4(3.2)) and, by assumption, p divides $\Phi_e(q)$, we are done whenever p is good for $\mathbf{H}$.

By the properties of Lusztig induction $R_{\mathbf{L}}^{\mathbf{H}}$ the e-cuspidal characters of a product of groups are the exterior tensor products of the e-cuspidal characters of the factors, thus it suffices to prove the result for simple groups $\mathbf{H}$, and, by our previous argument, for bad primes p.

First assume that $e = 1$. The 1-cuspidal characters are the cuspidal characters in the usual sense; their degree polynomial is divisible by the full Φ_1-part of the generic order of $\mathbf{H}$ (see [4, Prop. 2.4]). On the other hand, Lusztig has shown that the denominators occurring in degree polynomials of unipotent characters all divide the order of a certain finite group attached to $\mathbf{H}$. In the following table we list the relevant data. The first row lists those simple types $\mathbf{H}$ for which cuspidal unipotent characters exist, where $s \geq 1$ is an integer. The second gives the precise power of Φ_1 dividing the generic order of $\mathbf{H}$, hence a lower bound on the exponent of p in the cyclotomic part of the degree of cuspidal unipotent characters. The third row gives the order of the finite group controlling the denominators.

H	B_{s^2+s}	C_{s^2+s}	D_{4s^2}	${}^2D_{(2s+1)^2}$	G_2	${}^3D_4$	F_4	E_6	${}^2E_6$	E_7	E_8
$\mathrm{rk}(H)$	s^2+s	s^2+s	$(2s)^2$	$(2s+1)^2$	2	2	4	6	4	7	8
denom	2^s	2^s	2^{2s-1}	2^{2s}	6	2	24	6	6	2	120

(This table can be extracted from [8, 13.7], for example.) A quick look shows that in any case, the entry in the second row is superior to the exponent of any (bad) prime occurring in the factorization of the number in the third row. Thus the statement follows in this case.

If $e = 2$, then we may use Ennola duality (see (3.3)) to pass from 2-cuspidal characters of $\mathbb{H}$ to 1-cuspidal characters of $\mathbb{H}^-$. Thus we may conclude by the case $e = 1$.

Now bad primes satisfy $p \leq 5$. For $p \in \{2, 3\}$ we always have $e = e_p(q) \leq 2$ by definition, so only $p = 5$ with $e = 4$ remains. The prime 5 is only bad for simple groups of type E_8. Again by [4, Prop. 2.4], the degree polynomial of any 4-cuspidal unipotent character of E_8 is divisible by Φ_4^4. Since all denominators are divisors of 120, which contains 5 only to the first power, we again obtain divisibility by 5. $\square$

We now return to the case of a general, not necessarily connected group $\mathbf{H}$.

Corollary 6.6. *A unipotent character $\chi \in \mathcal{E}(H, 1)$ has p-height 0 if and only if χ lies in the e-Harish-Chandra series of (L, λ), where $L := \mathrm{C}_H(\mathbf{S}_e)$ is the centralizer of a Sylow Φ_e-torus and $\lambda \in \mathcal{E}(L, 1)$, and both λ and $\rho(L, \lambda)(\chi) \in W_H(L, \lambda)$ are of p-height 0.*

Proof. By Theorem 4.6 χ lies in the e-Harish-Chandra series of some e-cuspidal pair (L, λ) of $(\mathbf{H}^\circ)^F$. By Proposition 6.5 the Sylow Φ_e-tori of L are central if χ has height 0, thus $L = \mathrm{C}_H(\mathbf{S}_e)$. But then all unipotent characters of L are e-cuspidal. According to Theorem 4.6(c) we have the degree formula $\mathrm{Deg}(\gamma) = \mathrm{Deg}(\lambda) D_\phi$ with $\phi = \rho(L, \lambda)(\chi)$. Here the polynomials $\mathrm{Deg}(\gamma)$ and $\mathrm{Deg}(\lambda)$ are products of cyclotomic polynomials not divisible by Φ_e. Hence D_ϕ is a rational function satisfying the assumptions of Corollary 6.3. Then

$$\chi(1) = \mathrm{Deg}(\gamma)(q) = \mathrm{Deg}(\lambda)(q) D_\phi(q) = \lambda(1) D_\phi(q).$$

Again by Theorem 4.2(c) and Corollary 6.3 we have

$$D_\phi(q) \equiv D_\phi(\zeta_e) = \pm\phi(1) \pmod{p}.$$

Thus, χ is of p-height 0 if and only if both λ and $\phi = \rho(L, \lambda)(\chi) \in W_H(L, \lambda)$ are of p-height 0. $\square$

6.7. p-heights of unipotent characters in defining characteristic. Let now $\mathbf{H}$ be a connected reductive group defined over $\mathbb{F}_q$ with Frobenius endomorphism $F : \mathbf{H} \to \mathbf{H}$ and group of fixed points $H := \mathbf{H}^F$, and p the prime dividing q.

Theorem 6.8. *Assume that* $\mathbf{H}$ *is simple. Let* $\mathcal{E}(H,1)_{p'}$ *denote the set of unipotent characters of* H *of degree prime to* p. *Then one of the following holds:*

(1) $|\mathcal{E}(H,1)_{p'}| = 1$, *or*
(2) H *is of type* B_n $(n \geq 2)$, C_n $(n \geq 3)$, G_2 *or* F_4, $q = 2$, *and* $|\mathcal{E}(H,1)_{p'}| = 5$, *or*
(3) H *is of type* G_2, $q = 3$, *and* $|\mathcal{E}(H,1)_{p'}| = 7$.

Proof. By formula 3.4(3.2), the degree $\gamma(1)$ of a unipotent character $\gamma \in \mathcal{E}(H,1)$ is divisible by p unless $\frac{1}{n_\gamma} q^{a_\gamma}$ is prime to p. Lusztig's results show that $a_\gamma > 0$ if $\gamma \neq 1_H$. Thus if γ is of p'-degree, we must necessarily have $q = p$, since character degrees are integers. Moreover p is a bad prime for $\mathbf{H}$. The maximal possible denominators n_γ have been described in the proof of Proposition 6.5. It ensues that either $a_\gamma = 1$ or $p = 2$. For $\mathbf{H}$ of type A_n or ${}^2A_n$ there are no bad primes. For $\mathbf{H}$ of exceptional type the explicit formulae for the degrees of unipotent characters [8, 13.8] show that only cases (2) and (3) arise.

For the other (classical) groups $p = 2$ is the only bad prime. The unipotent characters in type B_n are parametrized by symbols of rank n and odd defect (see for example [8, 13.8]). We may assume that not both rows in the symbol contain 0. Then the formulae show that $2^{a_\gamma}/n_\gamma$ is even unless the symbol has at most three entries, and the second smallest entry is equal to 1. The case of one entry leads to the trivial character, and there are four symbols of rank n with three entries such that the two smallest entries are $0, 1$. The parametrization and the degrees of unipotent characters in type C_n are the same as for B_n.

The unipotent characters in types D_n and ${}^2D_n$ are parametrized by symbols of rank n and even defect. Again the explicit formulae show that the degrees are even unless the symbol has at most two entries. But then the denominator actually turns out to be 1, and we only get the trivial character. □

Another way to formulate the previous result is the following. A finite reductive group H over $\mathbb{F}_q$, q a power of p, has unipotent characters of p-height 0 other than the trivial character if and only if the Dynkin diagram of H has at least a p-fold bond, and $q = p$.

7. Height 0 characters of groups of Lie type

In this section we use Lusztig's classification of the irreducible complex characters of groups of Lie type, together with the theory of generic groups and generic unipotent characters in order to classify the height 0 characters for primes p different from the defining characteristic.

7.1. Characters of p-height 0. Let $\mathbf{G}$ be a connected reductive algebraic group defined over $\mathbb{F}_q$ with respect to the Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$, $\mathbf{G}^*$ a group in duality with $\mathbf{G}$ with corresponding Frobenius endomorphism $F^* : \mathbf{G} \to \mathbf{G}^*$, and $G := \mathbf{G}^F$, $G^* := \mathbf{G}^{*F^*}$ their groups of fixed points. Let p be a prime dividing $|G|$ but not dividing q and let $e = e_p(q)$. By the discussion in Section 2.1 any $\chi \in \mathrm{Irr}(G)$ lies in a unique Lusztig series $\mathcal{E}(G,s)$ such that $\chi(1) = (G : \mathrm{C}_{G^*}(s))_{q'} \psi_s(\chi)(1)$. Now $\chi(1)$ is prime to p if and only if both $(G : \mathrm{C}_{G^*}(s))_{q'}$ and $\psi_s(\chi)(1)$ are. Thus, height 0 characters of G may be parametrized as follows:

Proposition 7.2. *Let* p *be a prime,* $p \nmid q$, *and* S_p *a Sylow* p*-subgroup of* G^*. *Then* $\chi \in \mathrm{Irr}(G)$ *has* p*-height 0 if and only if*

(i) *$\chi \in \mathcal{E}(G, s)$ with $s \in \mathrm{C}_{G^*}(S_p)$, and*
(ii) *$\psi_s(\chi) \in \mathcal{E}(\mathrm{C}_{G^*}(s), 1)$ has p-height 0.*

We thus obtain the following natural parametrization of characters of height 0:

Proposition 7.3. *The irreducible complex characters of G of p-height 0 are in bijection with triples (s, λ, ϕ), where*

(i) *$s \in G^*$ is semisimple centralizing a Sylow p-subgroup of G^*, modulo conjugation,*
(ii) *λ is a unipotent height 0 character of $L := \mathrm{C}_M(\mathbf{S}_e^*)$ up to $W_M(L)$-conjugation, where $M = \mathrm{C}_{G^*}(s)$ and $\mathbf{S}_e^*$ is a Sylow Φ_e-torus of M, and*
(iii) *$\phi \in \mathrm{Irr}(W_M(L, \lambda))$ is of p-height 0.*

Proof. By Proposition 7.2, to $\chi \in \mathrm{Irr}(G)$ of height 0 there is attached a semisimple element $s \in G^*$ centralizing a Sylow p-subgroup, modulo conjugation. Furthermore, the unipotent character $\chi' := \psi_s(\chi) \in \mathcal{E}(\mathrm{C}_{G^*}(s), 1)$ of $\mathrm{C}_{G^*}(s)$ has height 0. The assertion now follows from Corollary 6.6 applied to $\mathrm{C}_{G^*}(s)$. □

Additionally, we have the following congruence of character degrees:

Proposition 7.4. *In the notation of Proposition 7.3 above, assume that $\chi \in \mathrm{Irr}_{p'}(G)$ corresponds to the triple (s, λ, ϕ). Then*

$$\chi(1) \equiv \pm \frac{|G|_{q'}}{|L|_{q'}|W_M(L, \lambda)|} \lambda(1)\phi(1) \pmod{\Phi_e(q)}.$$

In particular, this congruence holds modulo p.

Proof. By Lusztig's Jordan decomposition of characters (2.1) we have

$$\chi(1) = \frac{|G|_{q'}}{|M|_{q'}} \psi_s(\chi)(1).$$

since by definition $M = \mathrm{C}_{G^*}(s)$. By Theorem 4.2(ii) the degree of $\psi_s(\chi)$ satisfies

$$\psi_s(\chi) = \pm \frac{|M|_{q'}}{|L|_{q'}} \lambda(1)\, D_{\rho(L,\lambda)(\psi_s(\chi))}$$

and by Theorem 4.2(i) we have

$$D_{\rho(L,\lambda)(\psi_s(\chi))} \equiv \phi(1)/|W_M(L, \lambda)| \pmod{\Phi_e(q)}.$$

This yields the desired conclusion. Note that by definition $p|\Phi_e(q)$. □

This congruence can also be seen as being a consequence of the main result of [4] which states that for large prime divisors of $\Phi_e(q)$ there exists a perfect isometry between the unipotent characters of G and characters of the normalizer of a Sylow Φ_e-torus.

Let us now fix a Sylow Φ_e-torus $\mathbf{S}_e$ of $\mathbf{G}$, and let $\mathbf{C} := \mathrm{C}_{\mathbf{G}}(\mathbf{S}_e)$. Then by Lemma 3.3 we may identify the dual group $\mathbf{C}^*$ with the centralizer $\mathrm{C}_{\mathbf{G}^*}(\mathbf{S}_e^*)$. We also set $\mathbf{N} := \mathrm{N}_{\mathbf{G}}(\mathbf{S}_e)$ and $\mathbf{N}^* := \mathrm{N}_{\mathbf{G}^*}(\mathbf{S}_e^*)$. Clearly, $\mathbf{C}$, $\mathbf{N}$ and $\mathbf{C}^*$ and $\mathbf{N}^*$ are F- respectively F^*-stable. Let $C := \mathbf{C}^F$, $C^* := \mathbf{C}^{*F}$, $N := \mathbf{N}^F$ and $N^* := \mathbf{N}^{*F}$.

Theorem 7.5. *Assume that $\mathbf{G}$ (and hence $\mathbf{G}^*$) is simple. Then the irreducible complex characters of G of p-height 0 are in bijection with triples (s, λ, ϕ), where*

(i) *$s \in C^*$ is semisimple centralizing a Sylow p-subgroup of N^*, modulo N^*-conjugation,*

(ii) $\lambda \in \mathcal{E}(\mathrm{C}_{C^*}(s), 1)$ *is unipotent of p-height 0, up to* $\mathrm{C}_{N^*}(s)$*-conjugation, and*
(iii) $\phi \in \mathrm{Irr}(W_{N^*}(s, \lambda))$ *is of p-height 0.*

Proof. This is a rephrasement of Proposition 7.3. Indeed, by Theorem 5.9 applied to $\mathbf{H} = \mathbf{G}^*$ every semisimple element of G^* which centralizes a Sylow p-subgroup of G^* also centralizes a Sylow Φ_e-torus $\mathbf{S}_e^*$ of G^*. Moreover, by Proposition 5.11 the normalizer $N^* = \mathrm{N}_{G^*}(\mathbf{S}_e^*)$ controls fusion of semisimple elements in $C^* = \mathrm{C}_{G^*}(\mathbf{S}_e^*)$, and it contains a Sylow p-subgroup of G^* by Proposition 5.21. Thus, Proposition 7.3(i) is equivalent to (i) above.

Secondly, in (ii) we have

$$L = \mathrm{C}_M(\mathbf{S}_e^*) = \mathrm{C}_{G^*}(\mathbf{S}_e^*) \cap M = C^* \cap \mathrm{C}_{G^*}(s) = \mathrm{C}_{C^*}(s).$$

Also, $\mathbf{S}_e^*$ is the unique Sylow Φ_e-torus in its centralizer $\mathrm{C}_M(\mathbf{S}_e^*)$, so by the Sylow theorem for Φ_e-tori, $\mathrm{N}_M(L) = \mathrm{N}_M(\mathrm{C}_M(\mathbf{S}_e^*)) = \mathrm{N}_M(\mathbf{S}_e^*)$. But

$$\mathrm{N}_M(\mathbf{S}_e^*) = \mathrm{N}_{G^*}(\mathbf{S}_e^*) \cap M = N^* \cap \mathrm{C}_{G^*}(s) = \mathrm{C}_{N^*}(s),$$

so

$$W_M(L) = \mathrm{N}_M(\mathbf{L})/L = \mathrm{C}_{N^*}(s)/\mathrm{C}_{C^*}(s).$$

Thus, Proposition 7.3(ii) gives (ii).

Finally, $\mathrm{N}_M(L, \lambda)$ is the stabilizer of the pair (s, λ) in N^*. Hence, condition (iii) is just Proposition 7.3(iii). $\square$

Remark 7.6. It follows from the above that the blocks of full p-defect of G are just the union

$$\bigcup_{s \in \mathrm{C}_{G^*}(\mathbf{S}_e)} \; \bigcup_{t \in \mathrm{C}_{G^*}(s)_p} \mathcal{E}(G, st).$$

Now consider the height 0 characters of the Sylow centralizer C.

Proposition 7.7. *The irreducible characters of C of p'-degree are in bijection with pairs (s, λ) where*

(i) *$s \in C^*$ is semisimple centralizing a Sylow p-subgroup of C^*, modulo C^*-conjugation,*
(ii) *λ is a unipotent p'-character of $M = \mathrm{C}_{C^*}(s)$ up to M-conjugation.*

The inertia factor group in N of $\chi \in \mathrm{Irr}_{p'}(C)$ parametrized by (s, λ) is isomorphic to $W_{N^*}(s, \lambda)$.

Proof. Being the centralizer of a torus in $\mathbf{G}$, $\mathbf{C}$ is connected reductive, so our previous results apply to C. According to Proposition 7.3 a character $\chi \in \mathrm{Irr}(C)$ is of p'-degree if it lies in the Lusztig series of some $s \in C^*$ centralizing a Sylow p-subgroup of C^* and $\psi_s(\chi) \in \mathcal{E}(M, 1)$ has p'-degree, where $M = \mathrm{C}_{C^*}(s)$. By Corollary 6.6 such characters are in one-to-one correspondence with pairs (λ, ϕ), where λ is a unipotent p'-character of $L := \mathrm{C}_M(\mathbf{S}_e^*)$ up to $W_M(L)$-conjugation and $\phi \in \mathrm{Irr}(W_M(L, \lambda))$ is of p'-degree. But $M \leq C^* = \mathrm{C}_{G^*}(\mathbf{S}_e^*)$, so $L = \mathrm{C}_M(\mathbf{S}_e^*) = M$. We conclude that $N_M(L) = N_M(M) = M$, $W_M(L, \lambda) = 1$, and hence $\phi = 1$. This proves the first assertion.

The inertia factor group in N of $\chi \in \mathrm{Irr}(C)$ is isomorphic to the inertia factor group in N^* of the label (s, λ) by Lemma 3.3. Now let $g \in N^*$ stabilize the label (s, λ). Then firstly s and s^g must parametrize the same Lusztig series, so s^g is conjugate to s in C^*. That is, there exists $c \in C^*$ such that $g' := gc \in \mathrm{C}_{N^*}(s)$ normalizes $M = \mathrm{C}_{C^*}(s) = L$. Secondly, $\lambda \in \mathcal{E}(L, 1)$ must be fixed by g', so g' lies in the inertia group of λ in $\mathrm{C}_{N^*}(s)$. Hence the inertia factor group of (s,λ) in N^* equals

$$C^* I_{\mathrm{C}_{N^*}(s)}(\lambda)/C^* = W_{N^*}(s,\lambda).$$

□

This result already shows a remarkable resemblance between the parametrization of p'-degree characters of G and that of p'-degree characters of C. Using a recent extensibility result of Späth [24], this gives a natural correspondence when replacing the centralizer C^* by the normalizer N^* of a Sylow Φ_e-torus of G:

Theorem 7.8. *Assume that* $\mathbf{G}$ *is simple and let* $G := \mathbf{G}^F$. *Assume that* $p \geq 5$, *or* $p = 3$ *and* G *is not as in Theorem 5.14(a)–(c), or* $p = 2$ *and* $G \neq \mathrm{Sp}_{2n}(q)$ *with* $q \equiv 3, 5 \pmod 8$. *Let* $N = \mathrm{N}_G(\mathbf{S}_e)$. *Then:*

(a) N *contains the normalizer of a Sylow* p*-subgroup of* G, *and*
(b) *there is a natural bijection*

$$' : \mathrm{Irr}_{p'}(G) \longrightarrow \mathrm{Irr}_{p'}(N), \qquad \chi \mapsto \chi',$$

which moreover satisfies:
(c) χ *and* χ' *lie above the same character of* $Z(G)$,
(d) $\chi(1) \equiv \pm\chi'(1) \pmod p$.

Proof. Part (a) is Theorem 5.14. We show that $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(N)$ are naturally parametrized by the same set such that (a) and (b) are satisfied. Let $\theta \in \mathrm{Irr}_{p'}(C)$ and denote by $I(\theta)$ its inertia group in N. By Proposition 7.7 the inertia factor group $I(\theta)/C$ is isomorphic to $W_{N^*}(s,\lambda)$. By the main result of [24] θ extends to its inertia group $I(\theta)$ in N. The characters of $I(\theta)$ above θ are then in bijection with $\mathrm{Irr}(W_{N^*}(s,\lambda))$. Induction from $I(\theta)$ to N now gives a bijection between N-orbits in $\mathrm{Irr}_{p'}(I(\theta)|\theta)$ and $\mathrm{Irr}_{p'}(N|\theta)$, hence the characters of N above θ are in bijection with $\mathrm{Irr}(W_{N^*}(s,\lambda))$. If $\chi \in \mathrm{Irr}(N)$ lies above θ and is parametrized by $\phi \in \mathrm{Irr}(W_{N^*}(s,\lambda))$, then its degree is given by

$$\chi(1) = \theta(1)\phi(1)(N^* : I(\theta)).$$

Thus, χ has p'-degree if and only if ϕ has p'-degree and moreover $(N : I(\theta))$ is prime to p, so s centralizes a Sylow p-subgroup of N^*. Comparing with Theorem 7.5 and Proposition 7.7 we see that indeed both $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(N)$ are parametrized by the same set.

Assertion (c) now follows immediately from Lemma 2.2. For (d) assume that $\chi \in \mathrm{Irr}_{p'}(G)$ is parametrized by (s,λ,ϕ), and denote by θ a character of C parametrized by (s,λ). Then

$$\chi(1) \equiv \pm \frac{|G|_{q'}}{|\mathrm{C}_{C^*}(s)|_{q'}|W_M(L,\lambda)|}\,\lambda(1)\phi(1) \pmod{\Phi_e(q)},$$

and

$$\theta(1) \equiv \pm \frac{|C|_{q'}}{|\mathrm{C}_{C^*}(s)|_{q'}}\,\lambda(1) \pmod{\Phi_e(q)}$$

by Proposition 7.4. Furthermore, by the first part of the proof

$$\begin{aligned}\chi'(1) &= \theta(1)\phi(1)(N : I(\theta))\\ &= \theta(1)\phi(1)(\mathrm{N}_{G^*}(\mathbf{S}_e) : \mathrm{C}_{G^*}(\mathbf{S}_e))/|W_{N^*}(s,\lambda)|\\ &= \theta(1)\phi(1)(W_G(\mathbf{S}_e) : W_M(L,\lambda)|,\end{aligned}$$

so

$$\frac{\chi(1)}{\chi'(1)} \equiv \pm \frac{|G|_{q'}}{|C|_{q'}} \frac{1}{|W_G(\mathbf{S}_e)|} \equiv \pm 1 \pmod{\Phi_e(q)}$$

as claimed, since

$$|G : C|_{q'} = |G : \mathrm{C}_G(\mathbf{S}_e)|_{q'} \equiv \pm |W_G(\mathbf{S}_e)| \pmod{\Phi_e(q)},$$

by Theorem 3.2(d). □

8. Suzuki- and Ree-groups

The arguments used in the previous section to relate height 0 characters of G to height 0 characters of the normalizer of a Sylow torus do not carry over immediately to Suzuki- and Ree-groups. In this section we show that the main result remains true in these cases.

8.1. Sylow centralizers in Suzuki- and Ree-groups. Let G be one of the groups ${}^2B_2(q^2)$, ${}^2F_4(q^2)$ with $q^2 = 2^{2f+1}$, or ${}^2G_2(q^2)$ with $q^2 = 3^{2f+1}$, that is, a Suzuki- or Ree-group. With a suitable adaptation of the definition of Φ_e, the description of Sylow centralizers in Theorem 5.9 continues to hold for these groups. Here, the generic order is a product of cyclotomic polynomials over $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ respectively, see [1, 3F], where the polynomial $X^2 - 1$ is considered as an irreducible cyclotomic polynomial. For a prime divisor p of the group order, different from the defining characteristic, we let $\Phi^{(p)}$ be a cyclotomic polynomial over $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ respectively, dividing the generic order and such that p divides $\Phi^{(p)}(q)$. From the order formulas for the Suzuki and Ree groups it follows that $\Phi^{(p)}$ is uniquely determined by this condition, except when $p = 2$ for ${}^2G_2(q^2)$ or $p = 3$ for ${}^2F_4(q^2)$. In the first case, both $q^2 - 1$ and $q^2 + 1$ are even, and we let $\Phi^{(2)} := q^2 + 1$, while in the second case both q^2+1 and q^4-q^2+1 are divisible by 3 and we let $\Phi^{(3)} := q^2+1$.

Theorem 8.2. *Let $\mathbf{H}$ be simple of type B_2 or F_4 in characteristic 2, or of type G_2 in characteristic 3. Let $F : \mathbf{H} \to \mathbf{H}$ be a Frobenius map such that $H := \mathbf{H}^F$ is a Suzuki- or Ree-group. Let p be a prime divisor of $|H|$ different from the defining characteristic. Then every semisimple element $g \in H$ which centralizes a Sylow p-subgroup of H lies in a torus containing a Sylow $\Phi^{(p)}$-torus of $\mathbf{H}$, where $\Phi^{(p)}$ is the cyclotomic polynomial defined above. In particular, g centralizes a Sylow $\Phi^{(p)}$-torus.*

Proof. Let $g \in H$ be semisimple, centralizing a Sylow p-subgroup of H, and $\mathbf{C} := \mathrm{C}_{\mathbf{H}}(g)$. Then the index $(\mathbf{C} : \mathbf{C}^\circ)$ is prime to p, since simple groups of type G_2 and F_4 are of simply connected type, so all centralizers of semisimple elements are connected by [8, Thm. 3.5.6], and for type B_2 the index $(\mathbf{C} : \mathbf{C}^\circ)$ is a power of 2, hence again prime to p. So we may argue as in step (1) of the proof of Theorem 5.9. □

8.3. Sylow normalizers in Suzuki- and Ree-groups. Again, with suitable adaptations the description of Sylow normalizers in Theorem 5.14 also holds for Suzuki- and Ree-groups. For a prime divisor p of the group order, different from the defining characteristic, we let $\Phi^{(p)}$ be the cyclotomic polynomial as defined above.

Theorem 8.4. *Let $\mathbf{H}$ be simple of type B_2 or F_4 in characteristic 2, or of type G_2 in characteristic 3. Let $F : \mathbf{H} \to \mathbf{H}$ be a Frobenius map such that $H := \mathbf{H}^F$ is a*

Suzuki- or Ree-group. Let p be a prime divisor of $|H|$ different from the defining characteristic and S_p a Sylow p-subgroup of H. Then there exists a Sylow $\Phi^{(p)}$-torus $\mathbf{S}$ *of* $\mathbf{H}$ *with* $\mathrm{N}_H(S_p) \le \mathrm{N}_H(\mathbf{S})$, *unless one of*

(a) $H = {}^2G_2(3^{2f+1})$ *and* $p = 2$, *or*
(b) $H = {}^2F_4(2^{2f+1})$, $p = 3$ *and* $2^{2f+1} \equiv 2, 5 \pmod 9$.

Proof. Again, if p does not divide $|W\langle\phi\rangle|$, then this is implicit in [4, Thm. 5.24]. It only remains to consider $p = 2$ for type G_2 and $p = 3$ for type F_4. In the first case, the Sylow 2-normalizer is a split extension of an elementary abelian group of order 8 with a Frobenius group of order 21, not contained in the normalizer of any torus. In the second case, the structure of the Sylow normalizer was worked out in [18, Bem. 5], for example. It is isomorphic to $\mathrm{SU}_3(2).2$ for the congruences listed in case (b), and contained in a torus normalizer otherwise. □

Using the two results above, we obtain the following extension of Theorem 7.5:

Theorem 8.5. *Assume that G is a Suzuki- or Ree-group and that $p \ge 5$, or $p = 3$ and G is not as in Theorem 8.4(b). Then the irreducible complex characters of G of p-height 0 are in bijection with triples (s, λ, ϕ), where*

(i) $s \in C^*$ *is semisimple centralizing a Sylow p-subgroup of N^*, modulo N^*-conjugation,*
(ii) $\lambda \in \mathcal{E}(\mathrm{C}_{C^*}(s), 1)$ *is unipotent of p-height 0, up to $\mathrm{C}_{N^*}(s)$-conjugation, and*
(iii) $\phi \in \mathrm{Irr}(W_{N^*}(s, \lambda))$ *is of p-height 0.*

Similarly the assertion of Theorem 7.8 continues to hold in this situation. We note that the McKay conjecture has already been proved in [15] for all groups whose non-abelian composition factors are Suzuki groups or Ree groups in characteristic 3.

References

[1] M. Broué and G. Malle, Théorèmes de Sylow génériques pour les groupes réductifs sur les corps finis. Math. Ann. **292** (1992), 241–262.
[2] M. Broué and G. Malle, Zyklotomische Heckealgebren. Astérisque **212** (1993), 119–189.
[3] M. Broué and G. Malle, Generalized Harish-Chandra theory. Pp. 85–103 in: *Representations of reductive groups*, Cambridge University Press, Cambridge, 1998.
[4] M. Broué, G. Malle and J. Michel, Generic blocks of finite reductive groups. Astérisque **212** (1993), 7–92.
[5] M. Broué, G. Malle and J. Michel, Towards spetses. I. Transform. Groups **4** (1999), 157–218.
[6] M. Broué and J. Michel, Blocs à groupes de défaut abéliens des groupes réductifs finis. Astérisque **212** (1993), 93–117.
[7] M. Cabanes, Unicité du sous-groupe abélien distingué maximal dans certains sous-groupes de Sylow. C. R. Acad. Sci. **318** (1994), 889–894.
[8] R.W. Carter, *Finite groups of Lie type*. Wiley-Interscience, New York, 1985.
[9] F. Digne and J. Michel, On Lusztig's parametrization of characters of finite groups of Lie type. Astérisque **181-182** (1990), 113–156.
[10] F. Digne and J. Michel, *Representations of finite groups of Lie type*. London Math. Soc. Student Texts, Cambridge University Press, Cambridge, 1991.
[11] F. Digne and J. Michel, Groupes réductifs non connexes. Ann. Sci. École Norm. Sup. **27** (1994), 345–406.
[12] M. Geck and G. Hiss, Basic sets of Brauer characters of finite groups of Lie type. J. Reine Angew. Math. **418** (1991), 173–188.
[13] D. Gorenstein and R. Lyons, *The local structure of finite groups of characteristic* 2 *type*. Mem. Amer. Math. Soc. **42** (1983), no. 276.

[14] D. Gorenstein, R. Lyons and R. Solomon, *The classification of the finite simple groups, Number 3.* Mathematical Surveys and Monographs, Amer. Math. Soc., Providence, 1994.
[15] M. Isaacs, G. Malle, G. Navarro, A reduction theorem for the McKay conjecture. Invent. Math. **170** (2007), 33–101.
[16] P. Kleidman, The maximal subgroups of the Steinberg triality groups ${}^3D_4(q)$ and of their automorphism groups. J. Algebra **115** (1988), 182–199.
[17] G. Lusztig, On the representations of reductive groups with disconnected centre. Pp. 157–166 in: *Orbites unipotentes et représentations, I.* Astérisque **168**, (1988).
[18] G. Malle, Die unipotenten Charaktere von ${}^2F_4(q^2)$. Comm. Algebra **18** (1990), 2361–2381.
[19] G. Malle, Darstellungstheoretische Methoden bei der Realisierung einfacher Gruppen vom Lie Typ als Galoisgruppen. Pp. 443–459 in: *Representation theory of finite groups and finite-dimensional algebras*, Progr. Math., **95**, Birkhäuser, Basel, 1991.
[20] G. Malle, Unipotente Grade imprimitiver komplexer Spiegelungsgruppen. J. Algebra **177** (1995), 768–826.
[21] G. Malle, Degrés relatifs des algèbres cyclotomiques associées aux groupes de réflexions complexes de dimension deux. Pp. 311–332 in: M. Cabanes (ed.), *Finite reductive groups, related structures and representations*, Progress in Math. **141**, Birkhäuser, 1997.
[22] G. Malle, On the generic degrees of cyclotomic algebras. Represent. Theory **4** (2000), 342–369.
[23] G. Malle, The inductive McKay condition for simple groups not of Lie type, to appear in *Comm. Algebra.*
[24] B. Späth, Die McKay-Vermutung für quasi-einfache Gruppen vom Lie-Typ. Dissertation, TU Kaiserslautern, 2007.
[25] B. Späth, The McKay-conjecture for exceptional groups and odd primes, submitted, 2007.
[26] T. Springer and R. Steinberg, Conjugacy classes. Pp. 167–266 in: A. Borel et al. (eds.), *Seminar on algebraic groups and related finite groups*, Lecture Notes in Math. **131**, Springer, Berlin Heidelberg New York, 1970.
[27] R. Steinberg, Lectures on Chevalley groups. Mimeographed notes, Yale University, New Haven, Conn., 1968.
[28] R. Steinberg, Endomorphisms of linear algebraic groups. Memoirs of the Amer. Math. Soc. 80, Amer. Math. Soc., Providence, R.I., 1968.
[29] R. Steinberg, Torsion in reductive groups. Advances in Math. **15** (1975), 63–92.
[30] L.C. Washington, *Introduction to cyclotomic fields.* 2nd ed. Graduate Texts in Math. 83, Springer, New York, 1997.

FB Mathematik, Universität Kaiserslautern, Postfach 3049, D 67653 Kaiserslautern, Germany

E-mail address: `malle@mathematik.uni-kl.de`