

# CHARACTERS OF POSITIVE HEIGHT IN BLOCKS OF FINITE QUASI-SIMPLE GROUPS

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ABSTRACT. Eaton and Moretó proposed an extension of Brauer's famous height zero conjecture on blocks of finite groups to the case of non-abelian defect groups, which predicts the smallest non-zero height in such blocks in terms of local data. We show that their conjecture holds for principal blocks of quasi-simple groups, for all blocks of finite reductive groups in their defining characteristic, as well as for all covering groups of symmetric and alternating groups. For the proof, we determine the minimal non-trivial character degrees of Sylow  $p$ -subgroups of finite reductive groups in characteristic  $p$ . We provide some further evidence for blocks of groups of Lie type considered in cross characteristic.

## 1. INTRODUCTION

Let  $G$  be a finite group,  $p$  a prime and  $B$  a  $p$ -block of  $G$  with defect group  $D$ . The famous Height Zero Conjecture of Richard Brauer from 1955 states that all irreducible complex characters in  $B$  have the same  $p$ -part in their degree if and only if  $D$  is abelian. Recently, substantial progress on this conjecture has been made: the 'if' direction was proved by Kessar–Malle [22], and the other direction was reduced to the inductive Alperin–McKay condition for quasi-simple groups by Navarro–Späth [27]. Eaton and Moretó [7] have recently proposed an extension of this conjecture to blocks with non-abelian defect groups. Write  $\text{mh}(B)$  for the minimal non-zero height of an irreducible character in  $B$ , and similarly (by a slight abuse of notation)  $\text{mh}(D)$  for the minimal non-zero height of any irreducible character of  $D$  (and  $\text{mh} = \infty$  if there is no character of positive height).

**Conjecture** (Eaton–Moretó). *Let  $B$  be a  $p$ -block of a finite group with defect group  $D$ . Then  $\text{mh}(B) = \text{mh}(D)$ .*

Brauer's height zero conjecture is included as the claim that  $\text{mh}(B) = \infty$  if and only if  $\text{mh}(D) = \infty$ . Eaton and Moretó [7] prove that  $\text{mh}(D) \leq \text{mh}(B)$  for  $p$ -solvable groups, and they furthermore checked their conjecture for sporadic groups, for the symmetric groups and for the general linear groups  $\text{GL}_n(q)$  for the prime  $p$  dividing  $q$ .

It is the purpose of this paper to verify the conjecture in a number of further instances. Relevant test cases are certainly furnished by the blocks of nearly simple groups. Our main result is the following:

**Theorem 1.1.** *The Eaton–Moretó conjecture holds for the following blocks:*

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- (1) *the principal block of any quasi-simple group;*
- (2) *all  $p$ -blocks of quasi-simple groups of Lie type in characteristic  $p$ ;*
- (3) *all unipotent blocks of quasi-simple exceptional groups of Lie type ; and*
- (4) *all  $p$ -blocks of covering groups of an alternating or symmetric group.*

This is shown in Theorems 4.7, 3.9, 4.4 and 2.1 respectively. We also obtain further partial results for groups of Lie type in non-defining characteristic, see Proposition 4.1. The case of groups of Lie type in their defining characteristic seems particularly relevant for the conjecture since there  $\text{mh}(B)$  and  $\text{mh}(D)$  can take arbitrarily large positive values. As a side result of independent interest we determine the smallest non-trivial character degrees of Sylow  $p$ -subgroups of groups of Lie type in characteristic  $p$ , see Proposition 3.7.

## 2. ALTERNATING GROUPS

In this section we consider the quasi-simple coverings of alternating groups.

**Theorem 2.1.** *Let  $G$  be a symmetric or alternating group, or a Schur extension of one of these groups, and  $p$  a prime. Then  $\text{mh}(B) = \text{mh}(D) = 1$  or  $\text{mh}(B) = \text{mh}(D) = \infty$  for any  $p$ -block  $B$  of  $G$ .*

*Proof.* The case of the symmetric group is proved in [7, Thm. 4.3]. Let  $B'$  be a  $p$ -block of  $\mathfrak{A}_n$  of positive defect, and  $B$  be a  $p$ -block of  $\mathfrak{S}_n$  of  $p$ -weight  $w > 0$  with defect group  $D$  covering  $B'$ . By [26, Thm. 9.17],  $D' = D \cap \mathfrak{A}_n$  is a defect group of  $B'$ . In particular, when  $p$  is odd,  $D' = D$  and we obtain the result from the symmetric groups case using [28, Prop. 12.3]. Suppose  $p = 2$ . Then  $D'$  is a subgroup of index 2 in  $D$  isomorphic to a Sylow 2-subgroup of  $\mathfrak{A}_{2w}$  by [28, Prop. 11.2]. If  $w \leq 2$ , then  $D'$  is abelian, and so  $\text{mh}(D') = \infty = \text{mh}(B')$  by [22]. Suppose  $w \geq 3$ . By [28, Prop. 12.7], we have  $\text{mh}(B') = 1$ . We now will show that  $\text{mh}(D') = 1$ . For  $w = 3$ ,  $D'$  is isomorphic to a Sylow 2-subgroup of  $\mathfrak{A}_6$ , which has an irreducible character of degree 2, as required. Let  $w > 3$ . By Clifford theory, it is sufficient to show that  $D$  has an irreducible character  $\chi$  of degree 2 such that  $\varepsilon\chi \neq \chi$ , where  $\varepsilon$  is the restriction of the sign character of  $\mathfrak{S}_n$  to  $D$ . However,  $w > 3$  implies that  $2w \geq 8$ , and if we write  $2w = \sum_i a_i 2^i$  with  $a_i \in \{0, 1\}$ , then there is  $k \geq 3$  such that  $a_k \neq 0$ . Hence,  $D$  (which is isomorphic to a Sylow 2-subgroup of  $\mathfrak{S}_{2w}$  described, for example, in [28, p. 75]) has a direct factor isomorphic to  $P_k = \mathbb{Z}/2\mathbb{Z} \wr \cdots \wr \mathbb{Z}/2\mathbb{Z}$  ( $k$  times). Write  $\tau_k \in P_k$  for the element of  $P_k$  corresponding to the transposition  $(1\ 2)$  when  $P_k$  is viewed as a subgroup of  $\mathfrak{S}_{2^k}$ . By associativity of the wreath product, one has  $P_k = P_{k-3} \wr P_3 \simeq P_{k-3}^8 \rtimes P_3$ . Denote by  $\pi : P_k \rightarrow P_3$  the corresponding surjective morphism with kernel  $P_{k-3}^8$ . Using the GAP-system [30], we can show that  $P_3$  has an irreducible character  $\chi_0$  of degree 2 such that  $\chi_0(\tau_3) \neq 0$ . Furthermore, viewed as an element of  $\mathfrak{S}_{2^k}$ , the element  $\tau = (\tau_{k-3}, 1, 1, 1, 1, 1, 1, 1; \tau_3) \in P_k$  is a 4-cycle if  $k \geq 4$  or a 2-cycle if  $k = 3$  (in all cases, it is an odd permutation of  $D$ ), and  $\pi(\tau) = \tau_3$ . If we denote by  $\chi$  the irreducible character of  $P_k$  obtained by inflating  $\chi_0$  via  $\pi$ , then  $\chi$  has degree 2, and  $\chi(\tau) = \chi_0(\tau_3) \neq 0$ . In particular,  $\varepsilon\chi \neq \chi$ , and the result follows.

Suppose  $\tilde{G} = \tilde{\mathfrak{S}}_n$  is a Schur extension of  $G = \mathfrak{S}_n$ , and denote by  $\theta : \tilde{G} \rightarrow G$  the surjective homomorphism of groups with kernel  $Z(\tilde{G}) \simeq \mathbb{Z}/2\mathbb{Z}$ . First, assume that  $p > 2$ . Let  $w > 0$  be the  $p$ -weight of  $B$ . By [28, Prop. 13.3],  $D$  is conjugate in  $\tilde{\mathfrak{S}}_n$  to a Sylow  $p$ -subgroup  $P$  of  $\tilde{\mathfrak{S}}_{pw}$  and since  $p$  is odd,  $\theta|_P$  induces an isomorphism between  $P$  and a

Sylow  $p$ -subgroup of  $\mathfrak{S}_{pw}$ . In particular, by the proof of [7, Thm. 4.3], either  $D$  is abelian and  $\text{mh}(D) = \infty$ , or  $D$  is non-abelian and  $\text{mh}(D) = 1$ . We then conclude with [28, note p. 86]. Now, let  $B'$  be a  $p$ -block of  $\tilde{\mathfrak{A}}_n$  of  $p$ -weight  $w > 0$  and sign  $\sigma$  (where  $\sigma$  is the sign as in [28, p. 45] of the partition labelling the core of  $B'$ ), with non-abelian defect group  $D$ . Then the unique  $p$ -block of  $\tilde{\mathfrak{S}}_n$  covering  $B'$  has defect group  $D$ . Thus,  $\text{mh}(D) = 1$ . Let  $B$  be any  $p$ -block of  $\tilde{\mathfrak{S}}_m$  (for some positive integer  $m$ ) of weight  $w$  and sign  $-\sigma$ . Then, by [28, Prop. 13.19], we conclude that  $B$  and  $B'$  have the same number of characters of  $p$ -height 1. Since  $w \geq p$  (because  $D$  is non-abelian), [28, Note p. 86] gives that  $1 = \text{mh}(B) = \text{mh}(B')$ , as required.

Finally, suppose that  $p = 2$ . Write  $\tilde{G} = \tilde{\mathfrak{S}}_n$  and  $G = \mathfrak{S}_n$  (resp.  $\tilde{G} = \tilde{\mathfrak{A}}_n$ , and  $G = \mathfrak{A}_n$ ). Since  $Z(\tilde{G})$  is a 2-group, [26, Thm. 9.9(b)] gives that if  $\tilde{B}$  is a 2-block of  $\tilde{G}$  with defect group  $\tilde{D}$ , then there is a 2-block  $B$  of  $G$  with defect group  $D = \theta(\tilde{D})$ . Assume  $D$  is non-abelian. Inflating irreducible characters of  $B$  and  $D$  of degree 2 through  $\theta$ , we obtain  $\text{mh}(\tilde{B}) = 1 = \text{mh}(\tilde{D})$ . So we now consider the case that  $D$  is abelian. We remark that in this case,  $D$  is either trivial, or isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ . In particular,  $\tilde{D}$  is abelian, and  $\text{mh}(\tilde{D}) = \infty = \text{mh}(\tilde{B})$ .

To complete the proof, we now consider the cases of  $6.\mathfrak{A}_6$  and  $6.\mathfrak{A}_7$ , using information available in [30]. Let  $G = 6.\mathfrak{A}_6$ , and  $p \in \{2, 3\}$ . Then we can check that the  $p$ -blocks of  $G$  have either maximal defect (and have an irreducible character of height 1) or cyclic defect. Since a Sylow  $p$ -subgroup has an irreducible character of height 1, we deduce the result.

Let  $G = 6.\mathfrak{A}_7$ . If  $p = 2$ , then  $G$  has three 2-blocks with maximal defect, one 2-block with defect 8, and three 2-blocks with cyclic defect. Any 2-block with maximal defect and any Sylow 2-subgroup of  $G$  has an irreducible character of height 1. Furthermore, if  $B$  is the 2-block with defect 8, then  $B$  has a character of degree 6 (i.e., of height 1) and a defect group  $D$  of  $B$  is a non-abelian group of order 8, which has an irreducible character of degree 2. Thus,  $\text{mh}(D) = 1 = \text{mh}(B)$ . If  $p = 3$ , then the 3-blocks of  $G$  are either of maximal defect (and have characters of height 1) or of abelian defect. Since a Sylow 3-subgroup of  $G$  has a character of degree 3, the result follows.

Finally, for  $G = 3.\mathfrak{A}_6$  and  $G = 3.\mathfrak{A}_7$  the 2-blocks are either of maximal defect (and contain a character of height 1), or of abelian defect. Furthermore, a Sylow 2-subgroup of  $G$  possesses a character of degree 2. This completes the proof.  $\square$

### 3. GROUPS OF LIE-TYPE: DEFINING CHARACTERISTIC

**3.1. Character degrees.** Throughout, we consider the following setting:  $\mathbf{G}$  is a simple linear algebraic group over the algebraic closure of a finite field, and  $F : \mathbf{G} \rightarrow \mathbf{G}$  is a Steinberg endomorphism on  $\mathbf{G}$ , with finite group of fixed points  $G := \mathbf{G}^F$ . We denote by  $q$  the absolute value of all eigenvalues of  $F$  on the character group of an  $F$ -stable maximal torus of  $\mathbf{G}$ . Thus, if  $\delta$  is the smallest integer such that  $\mathbf{G}$  is split with respect to  $F^\delta$ , then  $\mathbf{G}$  is defined over the field of size  $q^\delta$ . We write  $p$  for the prime dividing  $q^\delta$ . It is well-known that all quasi-simple finite groups of Lie type, except for the finitely many exceptional covering groups and for the Tits simple group  ${}^2F_4(2)'$ , arise as a central factor group of a suitable  $G$  as above, by choosing  $\mathbf{G}$  to be of simply connected type.

We will make use of the following well-known property of character degrees of groups of Lie type, which essentially follows from the work of Lusztig:

**Proposition 3.1.** *Let  $G = \mathbf{G}^F$  be a finite quasi-simple group of Lie type and  $q$  as above. Assume that  $G$  is not of Suzuki- or Ree-type. Then the degree of any  $\chi \in \text{Irr}(G)$  is of the form*

$$\frac{1}{cd}f(q)$$

for some monic polynomial  $f \in \mathbb{Z}[X]$ , and positive integers  $c, d$ , with  $(c, q) = 1$ , where  $c$  is a divisor of  $|Z(\mathbf{G})|$  and  $d$  is only divisible by bad primes for  $\mathbf{G}$ .

A quite similar statement holds for the Suzuki and Ree groups, but since it is slightly more technical to formulate, and we will not need it here, we do not give it.

*Proof.* Let us first consider the unipotent characters of  $G$ . Here, Lusztig has given explicit formulas for their degrees: for each unipotent character  $\chi$  of  $G$  there is a polynomial  $f \in \mathbb{Z}[X]$  which is product of cyclotomic polynomials times a power  $X^a$  of  $X$  and a positive integer  $d$  which is either a power of 2 or a divisor of 120, divisible only by bad primes for  $\mathbf{G}$ , such that  $\chi(1) = \frac{1}{d}f(q)$  (see e.g. [4, §13]). (Here,  $a$  is the  $a$ -invariant of the family to which  $\chi$  belongs.) So the claim holds in this case.

In general, by Lusztig's parameterization, any irreducible character  $\chi$  of  $G$  lies in the Lusztig-series  $\mathcal{E}(G, s)$  of a semisimple element  $s \in G^*$ , where  $G^* := \mathbf{G}^{*F}$  with  $\mathbf{G}^*$  Langland's dual to  $\mathbf{G}$  with a compatible Steinberg endomorphism also denoted by  $F$ . Lusztig's Jordan decomposition then yields that

$$\chi(1) = [G^* : C_{G^*}(s)]_{p'} \psi(1)$$

for a unipotent character  $\psi$  of (the possibly disconnected group)  $C_{G^*}(s)$ , that is, an irreducible character whose restriction to the connected component  $C_{G^*}^\circ(s)$  is unipotent. Now  $C_{G^*}(s)/C_{G^*}^\circ(s)$  is isomorphic to a subgroup of the fundamental group of  $G^*$ , hence abelian, of order prime to  $q$ , isomorphic to a subgroup of  $Z(\mathbf{G})$  (see [25, Prop. 14.20]). Moreover, it is cyclic unless the quasi-simple group  $G$  is of type  $D_{2n}$ . Since the index of the connected subgroup  $C_{G^*}^\circ(s)$  in  $G^*$  is given by a monic polynomial in  $q$ , this shows the claim using the previous statement on unipotent character degrees and Clifford theory.  $\square$

**3.2. mh( $B$ ) in the defining characteristic.** We now determine the smallest positive height in blocks of groups of Lie type. Note that groups of type  $A_1$  have abelian Sylow  $p$ -subgroups, so we may exclude them here. For  $G = \mathbf{G}^F$  a quasi-simple group of Lie type in characteristic  $p$  but not of type  $A_1$ , with  $q$  as above, we define the positive integer  $m(G, p)$  by

$$p^{m(G,p)} := \begin{cases} \frac{1}{2}q & \text{if } G = B_n(q), C_n(q), F_4(q), G_2(q) \text{ with } q = 2^f > 2, \\ \frac{1}{3}q & \text{if } G = G_2(q) \text{ with } q = 3^f > 3, \\ \frac{1}{\sqrt{2}}q & \text{if } G = {}^2B_2(q^2), {}^2F_4(q^2) \text{ with } q^2 = 2^{2f+1} > 2, \\ \frac{1}{\sqrt{3}}q & \text{if } G = {}^2G_2(q^2) \text{ with } q^2 = 3^{2f+1} > 3, \\ q & \text{else.} \end{cases}$$

**Proposition 3.2.** *Let  $G = \mathbf{G}^F$  be a quasi-simple group of Lie type not of type  $A_1$ . Let  $B$  be a  $p$ -block of  $G$  of positive defect. Then  $\text{mh}(B) = m(G, p)$ .*

*Proof.* By a result of Humphreys [19] the  $p$ -blocks of  $G$  of non-zero defect are in bijection with the irreducible characters of  $Z(G)$ , and all have full defect.

Thus, the Suzuki and Ree groups have just one non-trivial block, and the claim for them can be checked from the known character tables [29, 31, 23]. Note that none of the groups  ${}^2B_2(2)$ ,  ${}^2G_2(3)$ ,  ${}^2F_4(2)$  is quasi-simple. Also, it can be checked directly from the character tables that  $\mathrm{Sp}_6(2)$ ,  $G_2(3)$  and  $F_4(2)$  have a character of height 1. So from now on we will assume that  $G$  is none of the aforementioned groups.

Let us first discuss the principal  $p$ -block  $B_0$ . Since unipotent characters have  $Z(G)$  in their kernel, they all lie in  $B_0$ . By Proposition 3.1 the degree of any unipotent character  $\chi$  of  $G$  is of the form  $\frac{1}{d}q^a f(q)$ , where  $a$  is the  $a$ -invariant of the family of the Weyl group  $W$  of  $G$  to which  $\chi$  is attached, and  $f$  is a product of cyclotomic polynomials in  $q$ .

For  $B_n(2)$  and  $C_n(2)$  with  $n > 3$ , the unipotent character parametrized by the symbol  $(0, 2 \mid n-1)$  has degree  $2(4^n - 1)(2^{n-1} + 1)(2^{n-3} + 1)/15$ , hence height 1. For the other groups  $G$  of split type, we take for  $\chi$  the unipotent principal series character of  $G$  indexed by the reflection character  $\rho$  of  $W$ . Since  $\rho$  occurs in the first exterior power of the reflection representation of  $W$ , its  $b$ -invariant equals 1 by definition. By a result of Lusztig, the  $a$ -invariant is always less or equal to the  $b$ -invariant. On the other hand the  $a$ -invariant is strictly positive for any family except the one containing the trivial character. (Alternatively, use that the reflection character is special and so  $a$ - and  $b$ -invariant agree, see e.g. [4, §12].) Thus  $\chi(1) = \frac{1}{d}q^a f(q)$ , with  $d$  divisible by bad primes only and  $f(q)$  prime to  $p$ .

We claim that this character has the desired height. This is clear for groups of type  $A$ , since there are no bad primes, and more generally if  $p$  is not a bad prime for  $\mathbf{G}$ . The explicit formulas in [4, §13] show that for types  $B_n$ ,  $C_n$  and  $F_4$  we have  $d = 2$ , for  $G_2$  we have  $d = 6$ , and  $d = 1$  for all other untwisted types, so the claim holds.

For the twisted groups of types  ${}^2A_n$ ,  ${}^2D_{2n+1}$  and  ${}^2E_6$ , the degrees of unipotent characters are obtained from those in the corresponding untwisted type by replacing  $q$  by  $-q$  in the degree polynomial, and adjusting signs (the so-called Ennola duality), so the claim follows. For type  ${}^2D_n$  with  $n$  even, the unipotent character parametrized by the symbol  $(1, n-1 \mid \emptyset)$  has degree  $q(q^n + 1)(q^{n-2} - 1)/(q^2 - 1)$ . Finally, for  ${}^3D_4(q)$  the unipotent character  $\phi''_{1,3}$  has degree  $q(q^4 - q^2 + 1)$ .

We have thus in all cases exhibited a (unipotent) character of the asserted height. We next claim that the given characters have minimal  $p$ -height among unipotent characters. This is again clear for types  $A_n$  and  ${}^2A_n$  by Proposition 3.1. For the exceptional types it follows by inspection from the lists in [4, §13]. For  $B_n, C_n, D_n$  and  ${}^2D_n$  the formulas in loc. cit. show that the  $a$ -value of a unipotent character is always at least the exponent of 2 in the denominator of the degree. This settles the case for unipotent characters.

Now let  $\chi$  be an arbitrary character of  $G$  of positive height. By the degree formula in Proposition 3.1 its height is the same as that of some unipotent character of the centralizer of some semisimple element of the dual group  $G^*$ . But unipotent characters of products are just obtained as products of the unipotent characters of the factors. Moreover, any centralizer of a semisimple element in a group of simply laced type is again of simply laced type. Thus the previous result on unipotent characters shows that in all types the non-zero heights of non-unipotent characters are not smaller than those of unipotent characters. This completes the proof for the principal block.

Now let  $\psi$  denote a non-trivial character of  $Z(G)$  and  $B_\psi$  the corresponding  $p$ -block of  $G$ . Note that by the first part since  $|Z(G)| > 1$  we are in the case where  $p^{\text{mh}(B_0)} = q$ . For each type we display in Table 1 the centralizer of a semisimple element  $s$  in the dual group  $G^*$  such that all elements in the Lusztig series  $\mathcal{E}(G, s)$  have central character  $\psi$ , hence lie in  $B_\psi$ .

TABLE 1. Semisimple elements

$G$	$C_{G^*}(s)$	conditions
$\text{SL}_n(q)$	$\text{GL}_{n-1}(q)$	$(n, q-1) > 1$
$\text{SU}_n(q)$	$\text{GU}_{n-1}(q)$	$(n, q+1) > 1$
$\text{Spin}_{2n+1}(q)$	$C_{n-1}$	$q$ odd
$\text{Sp}_{2n}(q)$	$\text{GO}_{2n}^\pm(q)$	$q$ odd
$\text{Spin}_{2n}^\pm(q)$	$B_{n-1}$	$q$ odd
$E_6(q)$	$D_5$	$q \equiv 1 \pmod{3}$
${}^2E_6(q)$	${}^2D_5$	$q \equiv 2 \pmod{3}$
$E_7(q)$	$E_6$	$q \equiv 1 \pmod{4}$
$E_7(q)$	${}^2E_6$	$q \equiv 3 \pmod{4}$

Let  $\chi$  be the character in  $\mathcal{E}(G, s)$  corresponding under Lusztig's Jordan decomposition to the unipotent character  $\chi'$  of  $C_{G^*}(s)$  constructed above, with  $\chi'(1)_p = q$ . But then we also have  $\chi(1)_p = q$ . The proof is complete.  $\square$

*Remark 3.3.* It is shown in [7, Thm. C] that under Dade's projective conjecture we always have  $\text{mh}(B) \geq \text{mh}(D)$ . Thus, assuming that deep conjecture, the part of the above proof in which we show that the given characters have minimal  $p$ -height could be omitted in view of the subsequent Proposition 3.7.

**3.3.  $\text{mh}(D)$  in the defining characteristic.** We now turn to the heights in a Sylow  $p$ -subgroup. Let  $\mathbf{T}$  be an  $F$ -stable maximal torus of  $\mathbf{G}$  contained in an  $F$ -stable Borel subgroup  $\mathbf{B}$  of  $\mathbf{G}$  and let  $\mathbf{U}$  be the unipotent radical of  $\mathbf{B}$ . We denote by  $\Phi$  the root system of  $\mathbf{G}$  with respect to  $\mathbf{T}$ , and by  $\Phi^+$  the set of positive roots of  $\Phi$  with respect to  $\mathbf{B}$ . Write  $\Delta$  for the corresponding simple roots. For any  $\alpha \in \Phi^+$ , we denote by  $\mathbf{X}_\alpha$  the corresponding root subgroup in  $\mathbf{U}$  normalized by  $\mathbf{T}$ , and we choose an isomorphism  $x_\alpha : \overline{\mathbb{F}}_p^+ \rightarrow \mathbf{X}_\alpha$ . Now  $F$  acts on the subgroups  $\mathbf{X}_\alpha$ , which induces an action of  $F$  on  $\Phi$  and on  $\Delta$ , and that we extend by linearity to the space  $V = \mathbb{R}\Phi$ . The resulting map is  $q\phi$ , where  $\phi$  is an automorphism of  $V$  of finite order. We recall the setup from [25, §23]. Write  $\pi : V \rightarrow V^\phi$  for the projection onto the subspace  $V^\phi$ . We define an equivalence relation  $\sim$  on  $\Sigma = \pi(\Phi)$  by setting  $\pi(\alpha) \sim \pi(\beta)$  if and only if there is some positive  $c \in \mathbb{R}$  such that  $\pi(\alpha) = c\pi(\beta)$ , and we let  $\widehat{\Phi}$  be the set of equivalence classes under this relation. For  $\alpha \in \Phi$ , we write  $\widehat{\alpha}$  for the class of  $\pi(\alpha)$  and define

$$(1) \quad \theta : \Phi^+ \rightarrow \widehat{\Phi}, \quad \alpha \mapsto \widehat{\alpha}.$$

Recall that  $\Phi_{\widehat{\alpha}} = \theta^{-1}(\widehat{\alpha})$  is an  $F$ -stable set of positive roots of the root system  $\pm\Phi_{\widehat{\alpha}}$  (see [12, Thm. 2.4.1]). We sometimes also write  $\Phi_{\pi(\alpha)}$  for  $\Phi_{\widehat{\alpha}}$ . From now on, we assume

that we made the same choices as in [12, Rem. 1.12.10]. For  $\alpha \in \Phi^+$ , we define  $\mathbf{X}_{\hat{\alpha}} = \prod_{\beta \in \Phi_{\hat{\alpha}}} \mathbf{X}_{\beta}$ , and  $X_{\hat{\alpha}} = \mathbf{X}_{\hat{\alpha}}^F$  the corresponding root subgroup of  $G$ . In the following, to simplify the notation, we sometimes also denote  $X_{\hat{\alpha}}$  by  $X_{\pi(\alpha)}$ . The different possible roots subgroups are described in [12, Table 2.4]. For any  $\alpha \in \Phi^+$ , when  $\Phi_{\hat{\alpha}}$  is of type  $A_1$  (resp.  $A_1 \times A_1$ ), we label the elements of  $X_{\hat{\alpha}}$  by  $x_{\hat{\alpha}}(t)$  for  $t \in \mathbb{F}_q$  (resp.  $t \in \mathbb{F}_{q^2}$ ). When  $\Phi_{\hat{\alpha}}$  is of type  $A_2$ , we label the elements of  $X_{\hat{\alpha}}$  by  $x_{\hat{\alpha}}(t, u)$  with  $t, u \in \mathbb{F}_{q^2}$  such that  $t^{q+1} = \epsilon(u + u^q)$ , where  $\epsilon$  is a sign depending only on  $\hat{\alpha}$ . For any representative  $\gamma$  of  $\hat{\alpha}$ , we also denote  $x_{\hat{\alpha}}$  by  $x_{\gamma}$ . Furthermore, by [12, Thm. 2.3.7],  $P = \prod_{\{\hat{\alpha}, \alpha \in \Phi^+\}} X_{\hat{\alpha}}$  is a Sylow  $p$ -subgroup of  $G$ .

Now, we consider  $\mathbf{G} = \mathrm{SL}_3(\overline{\mathbb{F}}_p)$ , endowed with the  $\mathbb{F}_{q^2}$ -structure corresponding to the Steinberg map  $F$  acting by raising all entries of a matrix to the  $q^2$ -th power. Let  $\{\alpha_1, \alpha_2\}$  be a system of simple roots of  $\mathbf{G}$ . Then  $\Phi^+ = \{\alpha, \beta, \alpha + \beta\}$ , and by [12, Thm. 2.4.5(b)(1)], we have  $[x_{\alpha}(u), x_{\beta}(v)] = x_{\alpha+\beta}(\epsilon uv)$  for all  $u, v \in \mathbb{F}_{q^2}$ , where  $\epsilon$  is a sign independent of  $u$  and  $v$ . Define  $Y_{\beta} = x_{\beta}(\mathbb{F}_q)$ . In the following, we will need to understand the characters degrees of the group  $Y = X_{\alpha}Y_{\beta}X_{\alpha+\beta}$  of order  $q^5$ .

**Lemma 3.4.** *The group  $Y$  has  $q^3$  linear characters and  $q^3 - q$  irreducible characters of degree  $q$ .*

*Proof.* Using the commutator relations, every elements of  $Y$  can be uniquely written as  $x_{\alpha}(u)x_{\beta}(v)x_{\alpha+\beta}(w)$  for  $u, w \in \mathbb{F}_{q^2}$  and  $v \in \mathbb{F}_q$ . Thus  $|Y| = q^5$ . Now, we describe the conjugacy classes of  $Y$ . Using the commutator relations, we can show that if  $v \neq 0$ , then for every  $u \in \mathbb{F}_{q^2}$ , the class of  $x_{\alpha}(u)x_{\beta}(v)$  is  $\{x_{\alpha}(u)x_{\beta}(v)x_{\alpha+\beta}(w) \mid w \in \mathbb{F}_{q^2}\}$ , which has  $q^2$  elements. There are  $q^2(q-1)$  such classes. Let  $u \in \mathbb{F}_{q^2}^{\times}$ . Then the set  $\{x_{\alpha}(u)x_{\alpha+\beta}(w) \mid w \in \mathbb{F}_{q^2}\}$  is the union of  $q$  classes of  $Y$  of size  $q$ . Finally, there are  $q^2$  central classes with representative in  $X_{\alpha+\beta}$ . Hence,  $Y$  has  $2q^3 - q$  classes and  $|\mathrm{Irr}(Y)| = 2q^3 - q$ . Since  $[Y, Y] = X_{\alpha+\beta}$ , we deduce that  $Y$  has  $q^3$  linear characters, and so

$$(2) \quad \sum_{\chi \in \mathrm{Irr}(Y), \chi(1) \neq 1} \chi(1)^2 = q^2(q^3 - q).$$

Moreover, the subgroup  $X_{\alpha}X_{\alpha+\beta}$  is abelian of index  $q$ . Thus, for any  $\chi \in \mathrm{Irr}(Y)$ , one has  $\chi(1) \leq q$ , and we derive the result from Relation (2).  $\square$

**Lemma 3.5.** *Let  $G = \mathbf{G}^F$  be a quasi-simple group of Lie type, with  $q$  as above. Assume that  $p \notin \{2, 3\}$ . Let  $\alpha', \beta' \in \Phi^+$  be such that  $\alpha = \pi(\alpha')$  and  $\beta = \pi(\beta')$  are linearly independent, and consider  $Q := \langle X_{\hat{\alpha}}, X_{\hat{\beta}} \rangle$ . If  $\chi$  is a character of  $Q$  with  $\chi(1) < q$ , then  $[Q, Q] \leq \ker(\chi)$ .*

*Proof.* Let  $\chi$  be a character of  $Q$  such that  $\chi(1) < q$ . Denote by  $R$  the root subsystem generated by  $\alpha$  and  $\beta$ . If  $R$  is of type  $A_1 \times A_1$  then  $Q$  is abelian and the claim is obvious, so we may exclude that case from now on.

(a) We start by considering the case of untwisted groups. Then  $F$  acts trivially on  $\Phi$ , and  $\pi = \mathrm{Id}$ .

(a1) Suppose first that  $R$  is of type  $A_2$ . By [12, Thm. 1.12.1],  $Q = X_{\alpha}X_{\beta}X_{\alpha+\beta}$  and  $[x_{\alpha}(u), x_{\beta}(v)] = x_{\alpha+\beta}(\epsilon uv)$  for all  $u, v \in \mathbb{F}_q$ , where  $\epsilon$  is a sign depending only on  $\alpha$  and  $\beta$ . Thus,  $Z(Q) = [Q, Q] = X_{\alpha+\beta}$ . Furthermore, for any  $g \notin X_{\alpha+\beta}$  and  $c \in X_{\alpha+\beta}$ , we derive from the commutator relation that there is  $t \in Q$  such that  $[g, t] = c$ . Hence, by [20,

Thm. 7.5], the degrees of the irreducible characters of  $Q$  are 1 or  $q$ . Since  $\chi(1) < q$ , it follows that  $\chi$  is a sum of linear characters. In particular,  $[Q, Q]$  lies in the kernel of  $\chi$ , as required. In the following, we denote the group  $Q$  considered here by  $Q_{A_2}$ .

(a2) Assume that  $R$  is of type  $B_2$ , and  $R^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$ . By the commutator relations [12, Thm. 1.12.1],  $Y = X_\alpha X_{\alpha+\beta} X_{2\alpha+\beta}$  is a subgroup of  $Q$  isomorphic to  $Q_{A_2}$ , because  $p \neq 2$ ; an isomorphism from  $Q_{A_2}$  to  $Y$  is given by  $x_\alpha(u)x_\beta(v)x_{\alpha+\beta}(w) \mapsto x_\alpha(u)x_{\alpha+\beta}(v)x_{2\alpha+\beta}(2w)$ . In particular, the argument for type  $A_2$  can be applied to  $\text{Res}_Y^Q(\chi)$ . This implies that  $X_{2\alpha+\beta} \leq \ker(\chi)$ . Since  $X_{2\alpha+\beta} \triangleleft Q$ ,  $\chi$  factorizes through the quotient  $\overline{Q} = Q/X_{2\alpha+\beta}$ . By the commutator relations,  $\overline{Q} = \overline{X}_\alpha \overline{X}_\beta \overline{X}_{\alpha+\beta}$ , where  $\overline{X}_\gamma = \{\overline{x}_\gamma(t) \mid t \in \mathbb{F}_q\}$  is isomorphic to  $X_\gamma$  for  $\gamma \in \{\alpha, \beta, \alpha + \beta\}$ , and satisfies  $[\overline{x}_\alpha(u), \overline{x}_\beta(v)] = \overline{x}_{\alpha+\beta}(\epsilon uv)$  for all  $u, v \in \mathbb{F}_q$  and some sign  $\epsilon$  depending only on  $\alpha$  and  $\beta$  (see [12, Thm. 1.12.1(b)]). Hence,  $\overline{Q}$  is isomorphic to  $Q_{A_2}$ , and the argument above gives  $X_{\alpha+\beta} \leq \ker(\chi)$ . Since  $[Q, Q] = X_{\alpha+\beta} X_{2\alpha+\beta}$ , the result follows. In the following, we denote by  $Q_{B_2}$  the group  $Q$  considered here.

(a3) Assume that  $R$  is of type  $G_2$ , and  $R^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}$ . Since  $p \neq 3$ , we deduce from [12, Thm. 1.12.1] that the subgroup  $Y = X_\alpha X_{2\alpha+\beta} X_{3\alpha+\beta}$  is isomorphic to  $Q_{A_2}$  (with  $Q_{A_2} \rightarrow Y$  defined by  $x_\alpha(u)x_\beta(v)x_{\alpha+\beta}(w) \mapsto x_\alpha(u)x_{2\alpha+\beta}(v)x_{3\alpha+\beta}(3w)$ ), and as above, we obtain  $X_{3\alpha+\beta} \leq \ker(\chi)$ . Similarly,  $X_{3\alpha+2\beta} \leq \ker(\chi)$ . Note that  $X_{3\alpha+\beta} X_{3\alpha+2\beta}$  is normal in  $Q$ , and  $\overline{Q} = Q/X_{3\alpha+\beta} X_{3\alpha+2\beta}$  is isomorphic to  $Q_{B_2}$  by the commutator relations. Since  $\chi$  factorizes through  $\overline{Q}$ , and because  $p \neq 2$ , we deduce in a similar way that  $\ker(\chi)$  contains  $[\overline{Q}, \overline{Q}] = \overline{X}_{\alpha+\beta} \overline{X}_{2\alpha+\beta}$ . The result follows.

(b) Now, we consider the case of twisted groups. Suppose that  $R$  is of type  $A_2$ . By [12, Thm. 2.4.5],  $Q$  is isomorphic to the untwisted group  $Q_{A_2}$  over  $\mathbb{F}_{q^2}$ . In particular, the minimal degree of a non-linear character of  $Q$  is  $q^2$  and the result follows as above.

(b1) Suppose that  $R$  is of type  $B_2$ . Then there are two possibilities. Either  $Q$  is a Sylow  $p$ -subgroup of  ${}^2A_3(q)$  or of  ${}^2A_4(q)$ . We first assume that  $\mathbf{G}$  is of type  $A_3$ . Let  $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$  be a system of simple roots of  $\mathbf{G}$ . Without loss of generality, we can assume that  $\alpha = \pi(\alpha_1)$  and  $\beta = \pi(\alpha_2)$ . We have  $\Phi_{\theta(\alpha_1)} = \{\alpha_1, \alpha_3\}$ ,  $\Phi_{\theta(\alpha_2)} = \{\alpha_2\}$ ,  $\Phi_{\theta(\alpha_1+\alpha_2)} = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}$  and  $\Phi_{\theta(\alpha_1+\alpha_2+\alpha_3)} = \{\alpha_1 + \alpha_2 + \alpha_3\}$ , where  $\theta$  is the map defined in (1) above. In particular, the root subgroups  $X_\alpha$  and  $X_{\alpha+\beta}$  are isomorphic to  $\mathbb{F}_{q^2}$ , and  $X_\beta$  and  $X_{2\alpha+\beta}$  to  $\mathbb{F}_q$ . Consider  $L = Y_\alpha Y_{\alpha+\beta} X_{2\alpha+\beta}$ , where  $Y_\alpha = x_\alpha(\mathbb{F}_q)$  and  $Y_{\alpha+\beta} = x_{\alpha+\beta}(\mathbb{F}_q)$ . By [12, Thm. 2.4.5(b)(2)],  $L$  is a subgroup of  $Q$  and we have  $[x_\alpha(u), x_{\alpha+\beta}(v)] = x_{2\alpha+\beta}(2\epsilon uv)$  for all  $u, v \in \mathbb{F}_q$ . In particular,  $L$  is isomorphic to  $Q_{A_2}$  (because  $p \neq 2$ ) and we conclude as above that  $X_{2\alpha+\beta} \leq \ker(\chi)$ . Note that  $X_{2\alpha+\beta}$  is normal in  $Q$ , and by [12, Thm. 2.4.5(b)(3)] the quotient  $\overline{Q} = Q/X_{2\alpha+\beta}$  is isomorphic to  $\overline{X}_\alpha \overline{X}_\beta \overline{X}_{\alpha+\beta}$  with the commutator relation  $[\overline{x}_\alpha(u), \overline{x}_\beta(v)] = \overline{x}_{\alpha+\beta}(\epsilon u^q v)$ . In particular,  $\overline{Q}$  is isomorphic to the group  $Y$  of Lemma 3.4. An isomorphism between  $Y$  and  $\overline{Q}$  is given by  $x_\alpha(u)x_\beta(v)x_{\alpha+\beta}(w) \mapsto \overline{x}_\alpha(u)\overline{x}_\beta(v)\overline{x}_{\alpha+\beta}(\epsilon w^q)$ . So, the degrees of the irreducible characters of  $\overline{Q}$  are 1 or  $q$ , and the same argument as above gives that  $X_{\alpha+\beta} \leq \ker(\chi)$ , as required.

Consider now the case that  $\mathbf{G}$  is of type  $A_4$ , with positive roots

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\},$$

and  $Q$  is a Sylow  $p$ -subgroup of  ${}^2A_4(q)$ . Choose notation so that  $\beta = \pi(\alpha_1)$  and  $\alpha = \pi(\alpha_2)$ . The orbits under the relation  $\sim$  are  $\{\alpha, 2\alpha\}$ ,  $\{\beta\}$ ,  $\{\alpha + \beta, 2(\alpha + \beta)\}$  and  $\{2\alpha + \beta\}$ . We choose  $\alpha, \beta, \alpha + \beta$  and  $2\alpha + \beta$  as representatives. Note that  $\Phi_\alpha$  and  $\Phi_{\alpha+\beta}$  are of type  $A_2$ ,



and  $\Phi_\beta$  and  $\Phi_{2\alpha+\beta}$  are of type  $A_1 \times A_1$ . Thus,  $X_\alpha$  and  $X_{\alpha+\beta}$  have  $q^3$  elements and are isomorphic to a Sylow  $p$ -subgroup of  ${}^2A_2(q)$ . We consider the groups  $L = X_\alpha X_{\alpha+\beta} X_{2\alpha+\beta}$ , and  $L' = X_{\alpha'} X_{\beta'} X_{\alpha'+\beta'}$ , where the  $X_\gamma$ 's for  $\gamma \in \{\alpha', \beta', \alpha' + \beta'\}$ , are the root subgroups associated to a positive root system of type  $A_2$  over  $\mathbb{F}_{q^2}$ . Define

$$\varphi : L \rightarrow L', \quad x_\alpha(t, u)x_{\alpha+\beta}(t', v)x_{2\alpha+\beta}(w) \mapsto x_{\alpha'}(t)x_{\beta'}(t')x_{\alpha'+\beta'}(w),$$

for all  $t, t', u, v \in \mathbb{F}_{q^2}$  such that  $t^{q+1} = \epsilon(u+u^q)$  and  $t'^{q+1} = \epsilon'(v+v^q)$ , where  $\epsilon$  (resp.  $\epsilon'$ ) is a sign depending only on  $\alpha$  (resp.  $\alpha + \beta$ ). Then, by [12, Table 2.4 and Thm. 2.4.5(b)(1), (c)(1)],  $\varphi$  is a surjective group homomorphism with kernel  $[X_\alpha, X_\alpha][X_{\alpha+\beta}, X_{\alpha+\beta}]$ . Furthermore, we derive from [11, Table 2.1] that the irreducible characters of  $X_\alpha$  and  $X_{\alpha+\beta}$  have degree 1 or  $q$ . Indeed, Geck [11] computed the character table of a Borel subgroup  $B$  of  ${}^2A_2(q)$ , and by Clifford theory, the degrees of the irreducible characters of the Sylow  $p$ -subgroup contained in  $B$  are the  $p$ -parts of the degrees of the irreducible characters of  $B$ . So, by the same argument as above,  $\ker(\chi)$  contains  $[X_\alpha, X_\alpha][X_{\alpha+\beta}, X_{\alpha+\beta}]$ ,  $\chi$  factorizes through  $L'$ , and we obtain that  $X_{\alpha'+\beta'}$  lies in  $\ker(\chi)$ . This proves that  $X_{2\alpha+\beta} \leq \ker(\chi)$ . Since  $X_{2\alpha+\beta} \triangleleft Q$ , we can work in the quotient  $Q/X_{2\alpha+\beta}$ , and using the relations [12, Thm. 2.4.5(b)(1), (c)(2)], the same argument gives that  $X_{\alpha+\beta} \leq \ker(\chi)$ . The result follows.

(b2) Finally, assume that  $R$  is of type  $G_2$ . Then  $Q$  is a Sylow  $p$ -subgroup of  ${}^3D_4(q)$ , and we derive the result from [13, Table A.6].  $\square$

*Remark 3.6.* Note that the argument of the previous proof also applies for  $G$  untwisted and  $R$  of type  $B_2$  with  $p \neq 2$ , and for  $G$  any twisted group such that  $R$  is not of type  $G_2$  or  $B_2$  (the case that  $Q$  is a Sylow  $p$ -subgroup of type  ${}^2A_3(q)$ ).

Recall the definition of  $m(G, p)$  from Section 3.3.

**Proposition 3.7.** *Let  $G = \mathbf{G}^F$  be a quasi-simple group of Lie type not of type  $A_1$ , and  $P$  a Sylow  $p$ -subgroup of  $G$ . Then  $\text{mh}(P) = m(G, p)$ .*

*Proof.* When  $G$  is a Suzuki or a Ree group, the result holds by [29, p. 126], [6, Lemma 5] and [15, Table 1.5]. For  $G = {}^2A_2(q)$ , we have already argued in Lemma 3.5 that the result holds by [11, Table 2.1]. For  $G = G_2(q)$  with  $q = 2^f$  or  $q = 3^f$ , we derive the claim from [10, Table I-2] and [9, Table I-2], and for  $G = {}^3D_4(q)$ , the result follows from [13, Table A.6] for  $p$  odd, and from [14, Table A.6] for  $p = 2$ . So from now on, we assume that  $G$  is none of the groups considered above and in particular  $\Sigma$  has rank at least 2.

By [18], we have  $[P, P] = \prod_{\hat{\alpha}|\alpha \notin \Delta} X_{\hat{\alpha}}$  whenever

$$(3) \quad G \notin \{B_n(2), C_n(2), G_2(3), F_4(2), {}^2B_2(2), {}^2G_2(3), {}^2F_4(2)\}.$$

First, we suppose that  $p > 3$ . Let  $\chi \in \text{Irr}(P)$  be such that  $\chi(1) < q$ . Let  $\gamma' \in \Phi^+ \setminus \Delta$ . Write  $\gamma = \pi(\gamma')$ . By the proof of [25, Cor. B.2], there are linearly independent positive roots  $\alpha, \beta \in \Sigma$  such that  $\gamma = \alpha + \beta$  and hence  $\text{ht}(\gamma) > \max\{\text{ht}(\alpha), \text{ht}(\beta)\}$ , where  $\text{ht}$  denotes the height function on the root system. In particular,  $X_\gamma$  lies in the derived subgroup of  $\langle X_\alpha, X_\beta \rangle$ . Now, applying Lemma 3.5, we conclude that  $X_\gamma \leq \ker(\chi)$ . It follows that  $[P, P] \leq \ker(\chi)$ , and  $\chi$  is a linear character of  $P$ . So, if  $\chi$  is a non-linear irreducible character of  $P$ , then  $\chi(1) \geq q$ . By Remark 3.6 this also covers the case  $p = 3$ .

Assume  $p = 2$ . Then by [1, Tables A.1–A.4], the minimal degree of a non-linear character of a Sylow 2-subgroup of  $B_2(q)$  is  $q/2$ . Suppose now that  $G = C_n(q)$  with  $q > 2$

even and  $\chi \in \text{Irr}(P)$  with  $\chi(1) < q/2$ . (Recall that  $B_n(q) \simeq C_n(q)$  by [12, Thm. 2.2.10].) Note that the previous argument does not apply directly. Indeed, whenever a long root is the sum of two short roots  $\alpha$  and  $\beta$ , then  $X_{\alpha+\beta}$  commutes with  $X_\alpha$  and  $X_\beta$ ; see [12, Thm. 1.12.1 (b)(1)]. However, [12, Rem. 1.8] implies that in a root system of type  $C_n$  with basis  $\Delta$ , every positive long root  $\gamma \notin \Delta$  belongs to a root system  $\Psi$  of type  $B_2$  with basis  $\alpha, \beta$  such that  $\alpha$  is a short root and  $\beta$  is the long root of  $\Delta$ . Restricting  $\chi$  to the subgroup  $\prod_{\delta \in \Psi^+} X_\delta$ , we deduce from the case  $B_2(q)$  that  $X_\gamma$  lies in  $\ker(\chi)$ . Furthermore, for  $\gamma$  a positive short root such that  $\gamma \notin \Delta$ , the argument for  $p > 3$  applies. It follows that  $[P, P] \leq \ker(\chi)$ , and  $\chi$  is linear.

Suppose next that  $G = {}^2A_{2n+1}(q)$  and  $p = 2$ . Then the root system  $\Sigma$  is also of type  $C_{n+1}$ . We conclude in the same way that if  $\chi$  is a non-linear character of  $P$ , then  $\chi(1) \geq q$ , remarking that the minimal degree of a non-linear character of a Sylow 2-subgroup of  ${}^2A_3(q)$  is  $q$  by [1, Tables A.6, A.8, A.9 and A.10].

Assume  $G = F_4(q)$  with  $q > 2$  even, and let  $\chi$  be an irreducible character of  $P$  such that  $\chi(1) < q/2$ . Let  $\Phi$  be the root system of  $G$  as in [12, Rem. 1.8]. Denote by

$$\Phi^+ = \left\{ \varepsilon_i (1 \leq i \leq 4), \varepsilon_i \pm \varepsilon_j (1 \leq i < j \leq 4), \frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \right\}$$

the positive roots with respect to the basis

$$\Delta = \left\{ \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \right\}.$$

Let  $i < j$ . Then the positive long root  $\varepsilon_i + \varepsilon_j$  lies in the subsystem  $\Psi$  of type  $B_2$  with basis  $\{\varepsilon_i - \varepsilon_j, \varepsilon_j\}$ . In particular, restricting  $\chi$  to  $\prod_{\delta \in \Psi^+} X_\delta$ , we conclude as in the case of type  $C_n$  in characteristic 2 that  $X_{\varepsilon_i + \varepsilon_j}$  lies in  $\ker(\chi)$ . Now, suppose  $2 \leq j \leq 4$ , and let  $2 \leq k < l \leq 4$  be such that  $\{j, k, l\} = \{2, 3, 4\}$ . Then  $\alpha = \varepsilon_k + \varepsilon_l$  and  $\beta = \frac{1}{2}(\varepsilon_1 - \varepsilon_j - \varepsilon_k - \varepsilon_l)$  are in  $\Phi^+$ , and  $\varepsilon_1 - \varepsilon_j$  lies in the root subsystem of type  $B_2$  with basis  $\{\alpha, \beta\}$ . As previously, we conclude that  $X_{\varepsilon_1 - \varepsilon_j}$  lies in  $\ker(\chi)$ . For the remaining non-simple positive roots, the argument for  $p > 3$  can be applied (note that the positive long root  $\varepsilon_2 - \varepsilon_4 = (\varepsilon_2 - \varepsilon_3) + (\varepsilon_3 - \varepsilon_4)$  is the sum of two positive long roots). This proves that  $\chi$  is linear.

It remains to prove that  $P$  has an irreducible character with the announced degree. First, recall from step (a1) in the proof of Lemma 3.5 that if  $G = A_2(q)$ , then  $P$  has an irreducible character of degree  $q$ . Let  $G = B_2(q)$ . If  $p \neq 2$ , then  $P/X_{2\alpha+\beta}$  is isomorphic to a Sylow  $p$ -subgroup of  $A_2(q)$ . Inflating to  $P$  a character of degree  $q$  of  $P/X_{2\alpha+\beta}$ , we conclude that  $P$  has an irreducible character of degree  $q$ . If  $p = 2$ , then  $P$  has an irreducible character of degree  $q/2$  by [10]. Similarly, if  $G = G_2(q)$  and  $p > 3$ , then  $P/X_{2\alpha+\beta}X_{3\alpha+\beta}X_{3\alpha+2\beta}$  is isomorphic to a Sylow  $p$ -subgroup of  $A_2(q)$ . Thus,  $P$  has an irreducible character of degree  $q$ . Let  $G = {}^2A_3(q)$ . If  $p \neq 2$ , then we have shown in step (b1) of the proof of Lemma 3.5 that  $P$  has a quotient isomorphic to the group  $Y$  of Lemma 3.4. In particular,  $P$  has irreducible characters of degree  $q$ . If  $p = 2$ , then we conclude with [1, Table A.6, A.8, A.9 and A.10]. Let  $G = {}^2A_4(q)$ , and  $\alpha$  be the simple short root of  $\Sigma^+$ . Then  $X = \prod_{\gamma \in \Sigma^+ \setminus \{\alpha\}} X_\gamma \triangleleft P$ , and  $P/X$  is isomorphic to  $X_\alpha$  which has an irreducible character of degree  $q$  by [11, Table 2.1] (because  $X_\alpha$  is isomorphic to a Sylow  $p$ -subgroup of  ${}^2A_2(q)$ ).

Suppose now that  $G$  is any quasi-simple group of Lie type as in Lemma 3.5 or in Remark 3.6, or  $p = 2$  and  $G = C_n(q)$  or  $F_4(q)$  with  $q > 2$ . Let  $\alpha$  and  $\beta$  be two simple roots generating a subgroup of type  $A_2$  (when  $G$  is untwisted of type  $A_n, D_n, E_6, E_7, E_8$ ) or  $B_2$  (when  $G$  is untwisted of type  $B_n, C_n, F_4$ , or  $G = {}^2A_n(q), {}^2D_n(q)$  or  ${}^2E_6(q)$ ). Let  $\Psi^+$  be the positive roots generated by  $\alpha$  and  $\beta$ . Then  $X = \prod_{\gamma \in \Phi^+ \setminus \Psi^+} X_\gamma \triangleleft P$  and  $P/X$  is isomorphic to  $\langle X_\alpha, X_\beta \rangle$ . By the above discussion,  $P$  has an irreducible character of the required degree.

Finally, we consider the exceptions in (3). The only cases to treat are  $G = C_n(2)$  and  $G = F_4(2)$ . But then  $P$  has a quotient isomorphic to a Sylow 2-subgroup of  $B_2(2)$ , which has an irreducible character of degree  $q = 2$ . This completes the proof.  $\square$

*Remark 3.8.* Kazhdan [21, Prop. 2] has shown that for large enough primes  $p$ , all character degrees of Sylow  $p$ -subgroups of finite reductive groups in characteristic  $p$  are polynomials in  $q$ . This gives another approach for the lower bound on  $\text{mh}(P)$  at least when  $p$  is large.

### 3.4. The Eaton–Moretó conjecture for quasi-simple groups of Lie type in defining characteristic.

**Theorem 3.9.** *Let  $S$  be a quasi-simple group of Lie type in characteristic  $p$  and  $B$  a  $p$ -block of  $S$  of positive defect. Then the Eaton–Moretó conjecture holds for  $B$ .*

*Proof.* The character table of  ${}^2F_4(2)'$  is known and one finds that it has a single 2-block of positive defect, which contains a character of degree 26, and its Sylow 2-subgroup has an irreducible character of degree 2. For all other cases, let us first assume that  $|Z(S)|$  is prime to  $p$ . In particular,  $S$  is not an exceptional covering group of its non-abelian composition factor. Thus,  $S$  is a quotient of some finite group  $G$  of simply connected Lie type. As any irreducible character of  $S$  can be considered as a character of  $G$ , we may assume that  $S = G$  is of simply connected type. Thus we are in the situation of Proposition 3.2. In particular  $B$  is of full defect, so a Sylow  $p$ -subgroup of  $G$  is a defect group for  $B$ . Now compare Proposition 3.2 with Proposition 3.7.

Finally, assume that  $G$  is an exceptional covering group of one of

$$\begin{aligned} &L_2(4), L_2(9), L_3(2), L_3(4), L_4(2), U_4(2), U_4(3), U_6(2), \\ &Sp_6(2), O_7(3), O_8^+(2), {}^2B_2(8), G_2(3), G_2(4), F_4(2), {}^2E_6(2) \end{aligned}$$

(see [25, Tab. 24.3]). Note that the exceptional part of the Schur multiplier always has order a power of the defining prime  $p$ . Thus, the block  $B$  lifts a block  $B'$  of  $G/O_p(G)$  of the same defect, and for that we already showed the conjecture before. Using [30] one sees that in all cases but

$$L_2(4), L_2(9), L_3(4),$$

the minimal non-zero height in  $B'$  is equal to 1, so cannot become smaller in  $B$ . The alternating groups  $L_2(4) \cong \mathfrak{A}_5$  and  $L_2(9) \cong \mathfrak{A}_6$  were already treated in Theorem 2.1. The exceptional covering groups  $2.L_3(4)$  and  $6.L_3(4)$  have faithful irreducible characters of degrees 10, respectively 6, and direct calculation shows that their Sylow 2-subgroups have twelve irreducible characters of degree 2. This completes the proof.  $\square$

## 4. GROUPS OF LIE-TYPE: CROSS CHARACTERISTIC

We now discuss some cases in groups of Lie type in cross characteristic. So here  $G$  is defined in characteristic  $r$ , and we consider  $p$ -blocks of  $G$  for primes  $p \neq r$ .

**4.1. Unipotent blocks in groups of exceptional type.** We first deal with exceptional groups of Lie type.

**Proposition 4.1.** *Let  $p$  be a prime and  $G$  be quasi-simple with  $G/Z(G)$  one of*

$${}^2B_2(q^2), {}^2G_2(q^2), G_2(q), {}^3D_4(q), {}^2F_4(q^2).$$

*Then the Eaton–Moretó conjecture holds for all  $p$ -blocks of  $G$ .*

*Proof.* Since Brauer’s height zero conjecture has been proved for blocks of quasi-simple groups with abelian defect groups (see [22]) we may assume that  $G$  has non-abelian Sylow  $p$ -subgroups. The case where  $p$  is the defining prime was already handled in Theorem 3.9, thus now assume that we are in cross characteristic. For groups  ${}^2B_2(q^2)$  and  ${}^2G_2(q^2)$  all such Sylow subgroups are abelian. For  $G_2(q)$  we only need to consider  $p = 2, 3$ . For  $p = 3$  only the principal block has non-abelian defect, and it contains characters of height 1 (see [16, §2.2,2.3]). Since the Sylow 3-subgroups are non-abelian extensions of a homocyclic abelian group with a group of order 3, they also possess characters of height 1. For  $p = 2$ , the blocks with non-abelian defect groups are described in [17]: these are the blocks denoted  $B_1, B_3, B_{1a}, B_{1b}, B_{2a}$  and  $B_{2b}$ . All of them contain characters of height 1. The defect groups are either the Sylow 2-subgroups of  $G_2(q)$ , or semidihedral, thus also possess characters of height 1.

For  ${}^3D_4(q)$  the relevant primes are again  $p = 2, 3$ . By [5, Prop. 5.4] only the principal 3-block has non-abelian defect groups, and the characters  $\chi_{5,1}$  (if  $q \equiv 1 \pmod{3}$ ) respectively  $\chi_{10,1}$  have height 1. The Sylow 3-subgroups possess an abelian normal subgroup of index 3, so have characters of height 1 as well. For the prime  $p = 2$  there exist three types of non-abelian defect groups, by [5, Prop. 5.3], and all these have characters of height 1. The principal block will be discussed in Proposition 4.5. For the other 2-blocks Jordan decomposition gives a height preserving bijection to a unipotent block of a group  ${}^{(2)}A_2(q)$ ,  $A_1(q)$  or  $A_1(q^3)$ , where the existence of characters of height 1 is easily checked. For  ${}^2F_4(q^2)$  we just need to consider  $p = 3$ . By [23, Bem. 2] only the principal 3-block has non-abelian defect groups, and it contains characters of height 1. The Sylow 3-subgroup of  $G$  is contained inside a subgroup  $SU_3(q^2)$  and thus easily seen to possess characters of height 1 as well.  $\square$

To deal with the groups of large rank, we need some information on heights in Sylow  $p$ -subgroups.

**Proposition 4.2.** *Let  $G$  be a quasi-simple group of Lie type in characteristic  $r$ , and  $p \neq r$  be a prime number. Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . If  $P$  is non-abelian, then  $P$  has an irreducible character of degree  $p$ .*

*Proof.* For  $p = 2$ , this will be proved in Proposition 4.5. Assume  $p \geq 3$ , and let  $P$  be a non-abelian Sylow  $p$ -subgroup of  $G$ . First consider the case that  $G$  is of classical type. Then, by [32], there exist integers  $a \geq 1$  and  $m \geq 1$  (because  $P$  is non-abelian) such that

the group

$$\bar{P} \simeq \mathbb{Z}/p^a\mathbb{Z} \wr \underbrace{\mathbb{Z}/p\mathbb{Z} \wr \cdots \wr \mathbb{Z}/p\mathbb{Z}}_{m \text{ times}}$$

is a quotient of  $P$ . If  $m = 1$ , then  $\bar{P}$  has an irreducible character of degree  $p$  by Clifford theory. If  $m \geq 2$ , then  $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}/p\mathbb{Z}$  is a quotient of  $\bar{P}$  that possesses an irreducible character of degree  $p$  (again by Clifford theory). This proves that  $P$  has an irreducible character of degree  $p$ , as required.

Now, suppose that  $G$  is of exceptional  $E$ - or  $F$ -type. Here we only need to consider primes  $p \leq 7$  (respectively  $p \leq 5$  when  $G = {}^2E_6(q)$ , respectively  $p = 3$  when  $G = F_4(q)$ ) because any other Sylow  $p$ -subgroups of  $G$  are abelian by [25, Thm. 25.16]. When  $p = 5$  for  $E_6(q)$ ,  ${}^2E_6(q)$  or  $E_7(q)$ , or when  $p = 7$  for  $E_7(q)$  or  $E_8(q)$ , then a Sylow  $p$ -subgroup  $P$  of  $G$  is contained in the extension of an (abelian) maximal torus by its relative Weyl group (see e.g. [25, Cor. 25.17]), and the latter has  $p$ -part of order  $p$ . So  $P$  contains an abelian normal subgroup of index  $p$ , hence  $\text{mh}(P) = 1$ . In the remaining case that  $p = 3$ , or that  $G = E_8(q)$  with  $p = 5$ , we give in Table 2 a subsystem subgroup  $H$  of  $G$  that contains a Sylow  $p$ -subgroup of  $G$  in which we may detect an irreducible character of height 1. Indeed, consider for example  $G = F_4(q)$ . Here, for  $q \equiv 1 \pmod{3}$  a Sylow 3-subgroup of  $G$  is contained in the centralizer of a 3-element, of type  $A_2^2$ . Taking the quotient of this subgroup by a normal subgroup  $\text{SL}_3(q)$  we obtain a group  $\text{PGL}_3(q)$ , whose Sylow 3-subgroup is non-abelian, hence has an irreducible character of height 1. So the same holds for  $P$ . When  $q \equiv 2 \pmod{3}$ , the analog argument goes through with the subgroup  ${}^2A_2^2A_2$  with a quotient of type  $\text{PGU}_3(q)$ . A similar argument applies for the other types.

TABLE 2. Overgroups of Sylow subgroups

$G$	$p$	$H$	$H$
$F_4(q)$	3	$A_2(q)^2$ ( $q \equiv 1 \pmod{3}$ )	${}^2A_2(q)^2$ ( $q \equiv 2 \pmod{3}$ )
$E_6(q)$	3	$A_2(q)^3$ ( $q \equiv 1 \pmod{3}$ )	$A_2(q^2) {}^2A_2(q)$ ( $q \equiv 2 \pmod{3}$ )
${}^2E_6(q)$	3	$A_2(q^2)A_2(q)$ ( $q \equiv 1 \pmod{3}$ )	${}^2A_2(q)^3$ ( $q \equiv 2 \pmod{3}$ )
$E_7(q)$	3	$E_6(q)A_1(q)$ ( $q \equiv 1 \pmod{3}$ )	${}^2E_6(q)A_1(q)$ ( $q \equiv 2 \pmod{3}$ )
$E_8(q)$	3	$E_6(q)A_2(q)$ ( $q \equiv 1 \pmod{3}$ )	${}^2E_6(q) {}^2A_2(q)$ ( $q \equiv 2 \pmod{3}$ )
	5	$A_4(q)^2$ ( $q \equiv 1 \pmod{5}$ )	${}^2A_4(q)^2$ ( $q \equiv 4 \pmod{5}$ )
	5	${}^2A_4(q^2)$ ( $q \equiv 2, 3 \pmod{5}$ )	

□

**Proposition 4.3.** *The Eaton–Moretó-conjecture holds for the principal blocks of exceptional type quasi-simple groups.*

*Proof.* Let  $G$  be quasi-simple of exceptional Lie type, and  $p$  a prime. We may assume that  $p$  is different from the defining characteristic of  $G$  by Theorem 3.9, and also that Sylow  $p$ -subgroups of  $G$  are non-abelian by [22]. Thus  $p$  divides the order of the Weyl group of  $G$ . We may further assume by Proposition 4.1 that  $G$  is of Lie rank at least 4. The case  $p = 2$  will be settled in Proposition 4.5 by a general argument. So assume that  $p \geq 3$ . We claim that  $\text{mh}(B_0) = 1 = \text{mh}(P)$  whenever  $G$  is simple, where  $P$  is a

Sylow  $p$ -subgroup of  $G$ . Then the claim automatically follows for all covering groups of  $G$ . The fact that  $\text{mh}(P) = 1$  is shown in Proposition 4.2. We go through the individual cases. The unipotent characters in the principal block are described in [8, Thm. A(c)] (see also the tables on p.349–358 in loc. cit.). For  $G = F_4(q)$  only  $p = 3$  matters. Here, the principal block contains the unipotent character  $\phi_{12,4}$  if  $q \equiv 1 \pmod{3}$ , respectively  $F_4^{II}[1]$  if  $q \equiv 2 \pmod{3}$ , of 3-defect 1. For  $G = E_6(q)$  and  $p = 3$ , the unipotent character  $\phi_{6,1}$  has height 1 and lies in the principal block when  $q \equiv 1 \pmod{3}$ , while for  $q \equiv 2 \pmod{3}$ , the character  $\phi_{20,10}$  is as required. For  ${}^2E_6(q)$ , we may take the Ennola duals of the above characters, viz.  $\phi'_{2,4}$  when  $q \equiv 2 \pmod{3}$  respectively  $\phi_{12,4}$  when  $q \equiv 1 \pmod{3}$ .

For  $E_7(q)$  we need to look at  $p = 3$ , and at  $p = 5, 7$  when  $q \equiv \pm 1 \pmod{p}$ . For  $p = 3$ , the unipotent character  $\phi_{21,3}$  is of 3-height 1 in the principal block. For  $p = 5$ , the unipotent character  $\phi_{210,6}$  has 5-height 1, for  $p = 7$  we may instead take the character  $\phi_{7,1}$ . Finally, for  $G = E_8(q)$  the relevant primes are  $p = 3, 5$ , and  $p = 7$  when  $q \equiv \pm 1 \pmod{7}$ . For  $p = 3$ , the character  $\phi_{525,12}$  has 3-height 1, while for  $p = 5, 7$  we may choose  $\phi_{35,2}$ . This completes the proof.  $\square$

**Theorem 4.4.** *The Eaton–Moretó-conjecture holds for all unipotent blocks of quasi-simple groups of exceptional Lie type.*

*Proof.* Let  $G$  be a quasi-simple exceptional group of Lie type. By Proposition 4.1 we may assume that  $G$  has Lie rank at least 4. Moreover, we can assume that  $p$  is not the defining characteristic of  $G$ , by Theorem 3.9. In addition we may assume that  $p$  divides the order of the Weyl group of  $G$ , since otherwise the Sylow  $p$ -subgroups of  $G$  are abelian (see [25, Thm. 25.14]). The unipotent blocks of  $G$  and their defect groups are described in [8]. By Proposition 4.3 we need not consider principal blocks. The only non-principal unipotent  $p$ -blocks with non-abelian defect groups are then those listed in Table 3, lying above a  $d$ -cuspidal character of a  $d$ -split Levi subgroup  $L$  as indicated in column 4 of the table.

TABLE 3. Non-principal unipotent blocks

$G$	$p$	$d$	$L$	conditions	$\chi$
$E_6(q)$	3	1	$\Phi_1^2.D_4(q)$	$q \equiv 1 \pmod{3}$	*
${}^2E_6(q)$	3	2	$\Phi_2^2.D_4(q)$	$q \equiv 2 \pmod{3}$	*
$E_7(q)$	2	1	$\Phi_1.E_6(q)$	$q \equiv 1 \pmod{4}$	*
		2	$\Phi_2.{}^2E_6(q)$	$q \equiv 3 \pmod{4}$	*
	3	1	$\Phi_1^3.D_4(q)$	$q \equiv 1 \pmod{3}$	$D_4, r$
		2	$\Phi_2^3.D_4(q)$	$q \equiv 2 \pmod{3}$	$\phi_{280,8}$
$E_8(q)$	2	1	$\Phi_1^2.E_6(q)$	$q \equiv 1 \pmod{4}$	$E_6[\theta], \phi_{2,1}$
		2	$\Phi_2^2.{}^2E_6(q)$	$q \equiv 3 \pmod{4}$	$E_8[-\theta]$
	3	1	$\Phi_1^4.D_4(q)$	$q \equiv 1 \pmod{3}$	$D_4, \phi_{12,4}$
		2	$\Phi_2^4.D_4(q)$	$q \equiv 2 \pmod{3}$	$\phi_{448,25}$

In all cases it is immediate from the description of defect groups  $D$  in [8] that  $\text{mh}(D) = 1$ . Except for the cases marked with a '\*' we have printed in the last column of Table 3 a unipotent character (in the notation of [4, §13.8]) in the relevant block of height 1. In

the first four cases, no such unipotent character exists. But it was argued in the proof of [22, Prop. 8.4] that there exists a character of height 1 in these blocks if and only if the defect groups are non-abelian. This achieves the proof.  $\square$

**4.2. Principal blocks.** We now consider principal blocks of arbitrary quasi-simple groups and prove Theorem 1.1(1).

**Proposition 4.5.** *Let  $G$  be a finite quasi-simple group. Then the principal 2-block of  $G$  satisfies the Eaton–Moretó conjecture.*

*Proof.* We go through the various possibilities for  $G$  according to the classification. For  $G/Z(G)$  sporadic, the claim was shown in [7, Thm. D]. For alternating groups and their covering groups, see Theorem 2.1. If  $G$  is of Lie type in characteristic 2, then we showed the claim in Theorem 3.9. So now assume that  $G = \mathbf{G}^F$  is of Lie type in odd characteristic. The Ree groups  ${}^2G_2(q^2)$  have abelian Sylow 2-subgroups, so we may assume that  $G$  is not very twisted. If  $G$  is of type  $A_1$ , so either  $G = \mathrm{SL}_2(q)$  or  $\mathrm{L}_2(q)$  with  $q$  odd, then the characters in the principal 2-block of  $G$  are those lying in Lusztig series indexed by 2-elements  $s$  in the dual group  $G^* = \mathrm{PGL}_2(q)$  or  $\mathrm{L}_2(q)$ , and they are of height 1 if and only if the centralizer  $C_{G^*}(s)$  has index congruent to 2 (mod 4) in  $G^*$ . Now the Sylow 2-subgroups of  $G^*$  are dihedral, thus such elements  $s$  exist if and only if these Sylow 2-subgroups are non-abelian, which is the case if and only if the Sylow 2-subgroups of  $G$  are non-abelian. These in turn are quaternion or dihedral, hence possess characters of height 1 if and only if they are non-abelian. This deals with the case of groups of type  $A_1$  (the exceptional covering groups of  $\mathrm{L}_2(9) \cong \mathfrak{A}_6$  were considered in Theorem 2.1).

In the remaining cases, let  $B_0$  denote the principal 2-block of  $G$ . Let  $q$  be defined as above, and set  $d = 1$  if  $q \equiv 1 \pmod{4}$  and  $d = 2$  if  $q \equiv 3 \pmod{4}$ . Let  $\mathbf{T}$  be an  $F$ -stable maximal torus of  $\mathbf{G}$  containing a Sylow  $\Phi_d$ -subgroup. Then the pair  $(\mathbf{T}, 1)$  is  $d$ -cuspidal, and by [8, Thm. A] the  $d$ -Harish-Chandra series  $\mathcal{E}(G, (T, 1))$  above the trivial character of  $T = \mathbf{T}^F$  lies in  $B_0$ . Moreover, by [24, Prop. 7.4], there is a bijection  $\mathrm{Irr}(W) \rightarrow \mathcal{E}(G, (T, 1))$ ,  $\phi \mapsto \chi_\phi$ , with the property that  $\chi_\phi(1) \equiv \pm\phi(1) \pmod{\Phi_d(q)}$ , where  $W = N_G(\mathbf{T})/T$  is the relative Weyl group of  $T$ . Since  $\Phi_d(q) \equiv 0 \pmod{4}$ , this shows that  $\phi \in \mathrm{Irr}(W)$  has 2-height 1 if and only if  $\chi_\phi$  has.

On the other hand by [12, 4.10.2],  $N_G(T)$  contains a Sylow  $p$ -subgroup  $P$  of  $G$ . Then  $P_T = T \cap P$  is a normal subgroup of  $P$  such that  $W_2 = P/P_T$  is isomorphic to a Sylow 2-subgroup of  $W$ . Thus, we are done if we can show that both  $W$  and  $W_2$  possess characters of height 1 if  $P$  is non-abelian. However, it is shown in [12, 4.10.2] that whenever  $p = 2$ , the torus  $\mathbf{T}$  is of type 1 or  $w_0$ , where  $w_0$  is the longest element of the Weyl group of  $\mathbf{G}$ . In particular, [25, Prop. 25.3] gives that  $W$  is isomorphic to a Weyl group.

If  $G$  is of exceptional type, the claim can now be checked from the known character tables. Suppose that  $G$  is of classical type. Then  $W$  is a classical Weyl group, and has a quotient  $W'$  isomorphic to the symmetric group  $\mathfrak{S}_m$  with  $m \geq n/2$ , where  $n$  is the rank of  $G$ . Denote by  $\pi : W \rightarrow W'$  the canonical projection, and write  $W'_2 = \pi(W_2)$ . Then  $W'_2$  is a Sylow 2-subgroup of  $W'$ , and by [7, Thm. 4.3], whenever  $m > 2$  then  $W'$  and  $W'_2$  both have a character of 2-height 1, and the result follows. This still remains true when  $m = 2$  and  $W_2$  is the dihedral group of order 8 or the symmetric group  $\mathfrak{S}_3$ .

Finally, if  $m = 1$  and hence  $S = \mathrm{L}_3(q)$  with  $d = 2$  or  $S = \mathrm{U}_3(q)$  with  $d = 1$ , then the Sylow 2-subgroups contain an abelian normal subgroup of index 2 and hence possess

characters of height 1. On the other hand, the irreducible Deligne–Lusztig characters of  $S$  of degree  $q^3 - 1$ , respectively  $q^3 + 1$ , lying in the Lusztig series of a regular element of order 8 in a torus of order  $q^2 - 1$  also have height 1 and are contained in the principal block.  $\square$

The following result will be crucial for dealing with primes  $p > 2$ :

**Lemma 4.6.** *Let  $p > 2$  be a prime and  $d$  an integer dividing  $p - 1$ . Then for a primitive  $d$ th root of unity  $\zeta$  we have*

$$\Phi_{dp^i}(\zeta)_p = p \quad \text{for all } i \geq 1,$$

and  $\Phi_m(\zeta)$  is prime to  $p$  for all  $m \neq dp^i$ .

*Proof.* The fact that  $\Phi_m(\zeta)$  is prime to  $p$  for  $m \neq dp^i$  is shown for example in [24, Lemma 5.2]. For the other assertion note that  $\Phi_{dp^i}(X)$  divides  $\Phi_{p^i}(X^d)$  and that  $\Phi_{p^i}(X) = \Phi_p(X^{p^{i-1}})$  for  $i \geq 1$ . Thus  $\Phi_{dp^i}(\zeta)$  divides  $\Phi_{p^i}(1) = \Phi_p(1) = p$ .  $\square$

**Theorem 4.7.** *The Eaton–Moretó conjecture holds for the principal blocks of all quasi-simple groups.*

*Proof.* We go through the various possibilities for  $G$  according to the classification. For  $G/Z(G)$  sporadic, the claim was shown in [7, Thm. D]. For alternating groups and their covering groups, see Theorem 2.1 above. If  $G$  is of Lie type in characteristic  $p$ , then we showed the claim in Theorem 3.9. If  $G$  is of Lie type in odd characteristic and  $p = 2$ , the assertion is in Proposition 4.5. Finally, if  $G$  is of exceptional type, we proved this in Proposition 4.3.

Thus we may assume that  $G$  is of classical type, that  $p \neq 2$  is not the defining characteristic of  $G$  and that the Sylow  $p$ -subgroups of  $G$  are non-abelian. Let  $G = \mathbf{G}^F$  where  $\mathbf{G}$  is simple of simply connected classical type. Let  $d$  denote the order of  $q$  modulo  $p$ .

As we saw in Proposition 4.2, any non-abelian Sylow  $p$ -subgroup of  $G$  has a character of  $p$ -height 1. We now will show that the same property holds for the principal  $p$ -block of  $G$ . We consider two cases.

Assume first that  $p \geq 5$ . Then the principal  $p$ -block contains the principal  $\Phi_d$ -series  $\mathcal{E}(G, d)$  of unipotent characters of  $G$  (see [22, Rem. 6.12] for references). These are distinguished by the property that their generic degree polynomial is not divisible by  $\Phi_d$ . Let  $W$  denote the relative Weyl group of the principal  $\Phi_d$ -series, that is,  $W = N_G(\mathbf{L})/L$ , where  $\mathbf{L}$  is the centralizer of a Sylow  $\Phi_d$ -torus of  $\mathbf{G}$ . Then by [2, Thm. 3.2], there is a bijection  $\text{Irr}(W) \rightarrow \mathcal{E}(G, d)$ ,  $\phi \mapsto \chi_\phi$ , with the property that  $\phi(1) = \pm \chi_\phi(\zeta_d)$ , where  $\chi_\phi(X)$  is the degree polynomial of the unipotent character  $\chi_\phi$ . Now none of these degree polynomials is divisible by  $\Phi_d$ , and then Lemma 4.6 in conjunction with [24, Lemma 5.2] shows that both  $\chi_\phi(1)$  and  $\phi(1)$  are divisible by the same power of  $p$ .

Furthermore, by [2, §3],  $W$  is either isomorphic to  $G(d', 1, m)$  for some positive integers  $m$  and  $d'$  (with  $d'$  prime to  $p$ ), or to its subgroup  $G(d', 2, m)$  of index 2. In all cases,  $W$  has a quotient isomorphic to  $\mathfrak{S}_m$ . On the other hand, by [25, Thm. 25.14] we have that  $p$  divides  $|W|$  and since  $d'$  is prime to  $p$ , it follows that  $p$  divides  $m!$ . Let  $P$  be a Sylow  $p$ -subgroup of  $\mathfrak{S}_m$ . If  $P$  is non-abelian, then by [28, Note p.74], the principal  $p$ -block of  $\mathfrak{S}_m$  contains an irreducible character  $\phi$  of  $p$ -height 1. Suppose now that  $P$  is abelian, and write  $m = a_0 + a_1 p$  for the  $p$ -adic decomposition of  $m$  (where  $0 \leq a_0, a_1 < p$ , and  $a_1 \neq 0$



because  $p \leq m$ ). Set  $\alpha_0 = a_0 + p$  and  $\alpha_1 = a_1 - 1 \geq 0$ . Then we have  $\alpha_0 + \alpha_1 p = m$  and  $\alpha_0 + \alpha_1 - a_0 - a_1 = p - 1$ . Hence, following [28, p. 43], the  $p$ -extension  $(\alpha_0, \alpha_1)$  of  $m$  has deviation 1. With the notation (5) of [28, p. 43], we have  $c_p(p, \alpha_1) \neq 0$  because  $p > \alpha_1$ . Furthermore, one has  $c_p(1, \alpha_0) = c(p, \alpha_0)$ , where  $c(p, \alpha_0)$  denotes the number of  $p$ -cores of  $\alpha_0$ . Assume  $a_0 \neq 0$ . Then the partition  $(a_0 + 1, 1^{p-1})$  has no  $p$ -hooks. It follows that  $c(p, \alpha_0) \neq 0$ . (Note that until now, we did not need the assumption  $p \neq 3$ ). Assume  $a_0 = 0$ . Then any non-hook partition is a  $p$ -core (because  $p \geq 5$ ), and we also have  $c(p, \alpha_0) \neq 0$ . In these cases, [28, Prop. 6.6] gives that  $\mathfrak{S}_m$  has an irreducible character of degree  $p$ , and so, the principal  $p$ -block of  $G$ .

Suppose now that  $p = 3$  and  $a_0 = 0$ , i.e.,  $m \in \{3, 6\}$  and  $d \in \{1, 2\}$ . If  $d' = 2$  then  $W \cong G(2, 1, m)$  or  $G(2, 2, m)$  has a character of 3-height 1, and we conclude by the same argument as above. Assume  $d' = 1$ . Thus,  $W \cong \mathfrak{S}_m$  with  $m \in \{3, 6\}$ . On the other hand, [2, §3] gives that  $G$  is of type  $A$ , isomorphic to  $\mathrm{SL}_3(q)$  or  $\mathrm{SL}_6(q)$  (resp.  $\mathrm{SU}_3(q)$  or  $\mathrm{SU}_6(q)$ ) whenever  $d = 1$  (resp.  $d = 2$ ). Suppose that  $G$  is untwisted, that is, 3 divides  $q - 1$  and  $G = \mathrm{SL}_3(q)$  or  $G = \mathrm{SL}_6(q)$  (according to  $m = 3$  or  $m = 6$ ). Let  $\mathbf{T}$  be an  $F$ -stable maximally split torus of  $\mathbf{G}$ , and  $s$  be a semisimple element of order 3 of  $\mathbf{T}^{*F}$  such that  $C_{\mathbf{G}^*}(s) = \mathbf{T}^*$ . Then the unique irreducible character of  $\mathcal{E}(G, s)$  has 3-height 1 and lies in the principal 3-block of  $G$  by [3, Main Thm.]. When  $G$  is twisted, that is  $G \cong \mathrm{SU}_3(q)$  or  $\mathrm{SU}_6(q)$  with  $q + 1$  divisible by 3, we conclude by a similar argument.

We need not consider exceptional covering groups of  $G/Z(G)$ , since the order of their centers differ from that of some non-exceptional covering group of  $G$  by a power of the defining characteristic.  $\square$

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