ON THE INDUCTIVE ALPERIN–MCKAY AND ALPERIN WEIGHT CONJECTURE FOR GROUPS WITH ABELIAN SYLOW SUBGROUPS

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Abstract. We study the inductive Alperin–McKay conjecture, the inductive Isaacs–Navarro refinement and the inductive blockwise Alperin weight conjecture for groups of Lie type in the generic case of abelian Sylow ℓ-subgroups. We also show that the alternating groups, the Suzuki groups and the Ree groups satisfy the inductive condition necessary for Späth's reduction of the blockwise Alperin weight conjecture to the case of simple groups.

1. Introduction

This paper is a contribution towards a possible solution of two famous longstanding conjectures in the representation theory of finite groups. The McKay-conjecture (McK) postulates that for any finite group G and any prime p, the number of irreducible complex characters of degree prime to p is the same for G as for the normalizer of a Sylow p-subgroup. The Alperin weight conjecture (AWC) asserts that for any finite group G and any prime p, the number of p-modular irreducible Brauer characters of G is equal to the number of p-weights of G. Here, a p-weight is a p-subgroup $Q \leq G$ together with a p-defect zero character of $\mathrm{N}_G(Q)/Q$, up to G-conjugation. Thus both conjectures relate global representation theoretic invariants of the group to information encoded in p-local subgroups.

In the recent past, both conjectures have been reduced to certain (stronger) statements about finite simple groups [18, 25]. More precisely, it was shown that in order for (McK) or (AWC) to hold for all groups and the prime p, it is sufficient that all finite simple groups are *(McK)-good*, respectively *(AWC)-good*, for the prime p; see below for a definition of this term. Moreover, several infinite series of finite simple groups were already shown to be good for all or at least some primes.

Even more recently Späth [28, 29] succeeded in reducing the corresponding blockwise refinements of these conjectures, the Alperin–McKay conjecture (AM) and the blockwise Alperin weight conjecture (BAW), to inductive statements for the finite simple groups (and their covering and automorphism groups).

Thus, to complete the proof of (AM) and (BAW), it suffices to investigate the finite simple groups for these inductive properties. This paper contributes to this program in two ways. On the one hand side, we complete the verification of the necessary inductive conditions for four infinite series of finite simple groups:

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Theorem 1.1. *The alternating groups $\mathfrak{A}_n$, $n \geq 5$, are (BAW)-good for all primes p. The Suzuki groups ${}^2B_2(2^{2f+1})$, and the Ree groups ${}^2G_2(3^{2f+1})$ and ${}^2F_4(2^{2f+1})'$, are (AM)-, (IN)- and (BAW)-good for all primes p.*

Here, (IN) is the inductive formulation for finite simple groups due to Späth [28] of the refinement of (AM) proposed by Isaacs and Navarro which claims the existence of bijections preserving certain congruences of character degrees. In view of previous results (see [11] for the (AM)-condition for alternating groups), this essentially leaves the case of finite groups of Lie type with p not the defining characteristic.

In [22] we had succeeded in reformulating the inductive McKay condition (McK) using the machinery of d-Harish-Chandra theory into some local statement, which was subsequently shown to hold in many cases by Späth. The second purpose of this paper is to extend this approach to cover the blockwise version (AM) as well, and also to apply it to the blockwise inductive Alperin weight condition (BAW), at least in the (generic) case of abelian Sylow subgroups.

The structure of the paper is as follows. In Sections 2 and 3 we investigate an approach to the inductive conditions (AM) and (BAW) for finite groups of Lie type G in the case that ℓ is a large prime, and in particular, the Sylow ℓ-subgroups of G are abelian. We first show in Theorem 2.9 that an (AM)-bijection exists if all characters of suitable local subgroups extend, similar to the case considered in [22]. We then derive a similar condition for the existence of a (BAW)-bijection (see Corollary 3.7). Finally, in Theorem 3.8 we prove a very close relation between the two inductive conditions, at least when the ℓ-blocks of G have good basic sets and unitriangular decomposition matrices.

In Section 4 we present a result which guarantees equivariance of Jordan decomposition for Lusztig families parametrized by elements with connected centralizer, hence in particular in the case of groups with connected centre.

In Section 5 we prove Theorem 1.1 in the case of Suzuki and Ree groups, see Theorem 5.1. Section 6 contains some preparatory material on the Dade–Glauberman–Nagao correspondence with the help of which in Section 7 we complete the proof of Theorem 1.1 by dealing with the case of alternating groups. Our proof relies on previous results of Alperin and Fong [1] who determined the p-weights for symmetric groups and proved the weight conjecture in that case, and on the paper of Michler and Olsson [24] who did the same for the covering groups of symmetric and alternating groups for odd primes. The case of the prime $p = 2$ for alternating groups does not seem to have been studied before (see also the remark in Olsson [27, p. 82]).

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2. Abelian Sylow subgroups and height zero characters

Let $\mathbf{G}$ be a connected reductive linear algebraic group over the algebraic closure of a finite field and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism with (finite) group of fixed points $G := \mathbf{G}^F$. We let q denote the unique eigenvalues of F on the character group of an F-stable maximal torus of $\mathbf{G}$.

We'll also assume that no power of F induces a Suzuki- or Ree-type endomorphism on any simple factor of $\mathbf{G}$ (for the letter case, the validity of all inductive conditions will be shown in Section 5).

Furthermore, $\mathbf{G}^*$ will denote a group in duality with $\mathbf{G}$, with corresponding Steinberg endomorphism also denoted $F : \mathbf{G}^* \to \mathbf{G}^*$.

2.1. **Large primes.** We will consider primes ℓ with the following property with respect to $(\mathbf{G}, F)$:

$$(*) \qquad \text{there is a unique } e \text{ such that } \begin{cases} \ell | \Phi_e(q) \text{ and} \\ \Phi_e \text{ divides the order polynomial of } (\mathbf{G}, F) \end{cases}$$

If $\mathbf{G}$ is semisimple, for such primes ℓ we have $e = e_\ell(q)$ by [22, Cor. 5.4], where $e_\ell(q)$ denotes the multiplicative order of q modulo ℓ (respectively modulo 4 when $\ell = 2$, which will never be the case here).

An easy inspection shows:

Lemma 2.1. *Assume that ℓ satisfies (*). Then ℓ is odd, good for $\mathbf{G}$, different from 3 if $\mathbf{G}^F$ involves a factor of type ${}^3D_4$, and does not divide $|Z(\mathbf{G})^F/Z^\circ(\mathbf{G})^F|$ or $|Z(\mathbf{G}^*)^F/Z^\circ(\mathbf{G}^*)^F|$.*

For the last claim observe that ℓ satisfies (*) for $\mathbf{G}$ if and only if it does so for $\mathbf{G}^*$. The relevance of these primes is explained by the following:

Proposition 2.2. *Let $\ell \nmid q$. If ℓ satisfies (*) then the Sylow ℓ-subgroups of $\mathbf{G}^F$ are abelian. Conversely, if $\mathbf{G}$ is simple and $\mathbf{G}^F$ has abelian Sylow ℓ-subgroups, then ℓ satisfies (*).*

Proof. If ℓ satisfies (*), then a Sylow e-torus of $\mathbf{G}$ contains a Sylow ℓ-subgroup of $\mathbf{G}^F$, which is hence abelian. For the converse, we may suppose that $\ell > 2$, since the Sylow 2-subgroups of $\mathrm{SL}_2(q)$ and $\mathrm{PGL}_2(q)$, and hence of any semisimple group are non-abelian (in characteristic different from 2). Then, if (*) fails, $\ell | \Phi_e(q)$ and $\Phi_e, \Phi_{e\ell^i}$ both divide the order polynomial of $(\mathbf{G}, F)$ for some $i > 0$. If $\mathbf{G}$ is of type A_n, then $n \geq e\ell^i$ and so the automizer of a Sylow e-torus $\mathbf{S}$ contains elements of order ℓ acting non-trivially on $\mathbf{S}_\ell^F$, whence $\mathbf{G}^F$ has non-abelian Sylow ℓ-subgroups. An easy inspection shows that the same is in fact true for the other simple types as well (see also the discussion in [22, §5.10]). □

It's easily seen that the converse in Proposition 2.2 fails if $\mathbf{G}$ is just assumed to be semisimple. For example take $\mathbf{G} = \mathrm{SL}_2^3$ with F cyclically permuting the factors. Then $\mathbf{G}^F = \mathrm{SL}_2(q^3)$, and for $q \equiv 1 \pmod 3$, $\ell = 3$ divides $\Phi_1(q)$ and $\Phi_3(q)$, but the Sylow 3-subgroups of $\mathbf{G}^F$ are abelian.

Note that it may happen even for simple $\mathbf{G}$ that $\mathbf{G}^F/Z(\mathbf{G}^F)$ has abelian Sylow ℓ-subgroup even though $\mathbf{G}^F$ doesn't. Inspection of the arguments used to prove Proposition 2.2 and using the list of possible centers of simple groups $\mathbf{G}$ shows that in this case, either $\mathbf{G} = \mathrm{SL}_2$ and $\ell = 2$, or $\mathbf{G} = \mathrm{SL}_3$ and $\ell = 3$.

2.2. **ℓ-subgroups.** From now on we assume that ℓ satisfies (*), and we let $e := e_\ell(q)$.

Proposition 2.3. *Assume that the derived subgroup of $\mathbf{G}$ is simply connected.*

(a) *Let $Q \leq \mathbf{G}^F$ be an ℓ-subgroup. Then $C_{\mathbf{G}}(Q)$ is an e-split Levi subgroup.*
(b) *Let $\mathbf{L} \leq \mathbf{G}$ be an F-stable Levi subgroup and $Q := Z(\mathbf{L})_\ell^F$. Then $\mathbf{L}$ is e-split if and only if $\mathbf{L} = C_{\mathbf{G}}(Q)$.*

Proof. For (a) let $\mathbf{L} := C_{\mathbf{G}}(Q)$ and $\mathbf{Z} := Z(\mathbf{L})$. Since ℓ is good for $\mathbf{G}$ and $\mathbf{G}$ has simply connected derived subgroup, ℓ is not a torsion prime for $\mathbf{G}$ by [23, Prop. 14.15], so $\mathbf{L}$ is connected by [23, Thm. 14.16]. Furthermore, since ℓ is good, $\mathbf{L}^\circ = \mathbf{L}$ is a Levi subgroup of $\mathbf{G}$ by [15, Prop. 2.1], and thus by [15, Prop. 2.4], $\mathbf{Z}^F/\mathbf{Z}^{\circ F}$ is a subgroup of $Z(\mathbf{G})^F/Z^\circ(\mathbf{G})^F$, hence of order prime to ℓ by Lemma 2.1.

As Q is abelian by Proposition 2.2, we have $Q \le \mathbf{Z}^F$ and hence $Q \le \mathbf{Z}^{\circ F} \le \mathbf{Z}^\circ$, an F-stable torus of $\mathbf{G}$. By our assumption on ℓ, Q is contained in the Sylow e-torus $\mathbf{S} = (\mathbf{Z}^\circ)_{\Phi_e}$ of $\mathbf{Z}^\circ$. Thus

$$C_{\mathbf{G}}(Q) = \mathbf{L} = C_{\mathbf{G}}(\mathbf{Z}^\circ) \le C_{\mathbf{G}}(\mathbf{S}) \le C_{\mathbf{G}}(Q),$$

whence $\mathbf{L} = C_{\mathbf{G}}(\mathbf{S})$ is e-split by definition.

In (b), if $\mathbf{L} = C_{\mathbf{G}}(Q)$ then it is e-split by (a). Conversely, assume that $\mathbf{L}$ is e-split. Clearly, $\mathbf{L} \le \mathbf{C} := C_{\mathbf{G}}(Q)$, and $Q \le Z^\circ(\mathbf{C})_{\Phi_e} \le Z^\circ(\mathbf{L})_{\Phi_e}$. So by taking ℓ-Sylows we find $Q \le Z^\circ(\mathbf{C})_\ell^F \le Z^\circ(\mathbf{L})_\ell^F = Q$, whence $Z^\circ(\mathbf{C})_{\Phi_e} = Z^\circ(\mathbf{L})_{\Phi_e}$, and since both $\mathbf{L}$ and $\mathbf{C}$ are e-split, we conclude that $C_{\mathbf{G}}(Q) = \mathbf{C} = \mathbf{L}$ as claimed. □

Proposition 2.4. *Assume that the derived subgroup of* $\mathbf{G}$ *is simply connected. Let* $\mathbf{L} \le \mathbf{G}$ *be an e-split Levi subgroup and set* $Q := Z(\mathbf{L})_\ell^F$. *Then:*

(a) $N_{\mathbf{G}^F}(Q) = N_{\mathbf{G}^F}(\mathbf{L})$.
(b) $W_{\mathbf{G}^F}(\mathbf{L}) := N_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}^F$ *has order prime to* ℓ.
(c) *There is a direct decomposition* $\mathbf{L}^F = Q \times O^\ell(\mathbf{L}^F)$.

Proof. In (a), $Q = Z(\mathbf{L})_\ell^F$ is characteristic in $Z(\mathbf{L})^F$. Now any $g \in N_{\mathbf{G}^F}(\mathbf{L})$ normalizes $Z(\mathbf{L})_\ell$, and conjugates Q to an F-stable subgroup of $Z(\mathbf{L})_\ell$, so it normalizes Q. The other inclusion is obvious since $N_{\mathbf{G}^F}(Q) \le N_{\mathbf{G}^F}(C_{\mathbf{G}}(Q)) = N_{\mathbf{G}^F}(\mathbf{L})$ by Proposition 2.3(b).

The claim in (b) follows since an e-split Levi subgroup contains a Sylow e-torus, hence a Sylow ℓ-subgroup of $\mathbf{G}^F$ by (*).

For (c), note that $\mathbf{L} = [\mathbf{L},\mathbf{L}]Z(\mathbf{L})$ as $\mathbf{L}$ is reductive. Now $[\mathbf{L},\mathbf{L}]$ is connected, hence $\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F \cong (\mathbf{L}/[\mathbf{L},\mathbf{L}])^F \cong (Z(\mathbf{L})/Z(\mathbf{L}) \cap [\mathbf{L},\mathbf{L}])^F$ by [23, Prop. 23.2]. Let H denote the full preimage of the ℓ'-Hall subgroup of $(Z(\mathbf{L})/Z(\mathbf{L}) \cap [\mathbf{L},\mathbf{L}])^F$ in $\mathbf{L}^F$ under the canonical map above, then clearly $\mathbf{L}^F/H \cong (Z(\mathbf{L})/Z(\mathbf{L}) \cap [\mathbf{L},\mathbf{L}])_\ell^F = Z(\mathbf{L})_\ell^F = Q$. On the other hand, $[\mathbf{L},\mathbf{L}]^F \cap Q$ is trivial by Lemma 2.1 since $[\mathbf{L},\mathbf{L}]$ is semisimple, so $\mathbf{L} = H \times Q$. It is clear by construction that $H \ge O^\ell(\mathbf{L}^F)$. Since $[\mathbf{L},\mathbf{L}]$ is of simply connected type, $[\mathbf{L},\mathbf{L}]^F$ is perfect by [23, Thm. 24.17], unless one of its central factors is a solvable group. But even in that case, the only abelian factor groups have order a power of the characteristic, hence prime to ℓ. Thus $H = O^\ell(\mathbf{L}^F)$. □

Remark 2.5. In almost all cases we also have $Q = O_\ell(\mathbf{L}^F)$, the only exceptions occurring when $\mathbf{L}$ has a simple factor with solvable group of fixed points. Under our restrictions, this implies that $\ell = 3$ and $\mathbf{L}$ has an F-stable central factor $\mathbf{M}$ with $\mathbf{M}^F \cong \mathrm{SL}_2(2)$ (note that $\ell = 3$ does not satisfy (*) for $\mathrm{SU}_3(2)$).

2.3. **ℓ-blocks.** We now recall the parametrization of ℓ-blocks of $\mathbf{G}^F$ for primes satisfying (*). Here, for any finite group H and any $\chi \in \mathrm{Irr}(H)$ we write $b_H(\chi)$ for the ℓ-block of H containing χ. We freely use the notion of Lusztig series $\mathcal{E}(\mathbf{G}^F, s)$ and $\mathcal{E}_\ell(\mathbf{G}^F, s)$, see e.g. [8].

Theorem 2.6 (Cabanes–Enguehard). *Assume that* $\mathbf{G}$ *has simple derived subgroup of simply connected type, and assume that* $\ell \ge 5$ *satisfies (*). Let* B *be an* ℓ*-block of* $\mathbf{G}^F$.

Then there exists a semisimple ℓ'-element $s \in \mathbf{G}^{*F}$, unique up to $\mathbf{G}^{*F}$-conjugation, and an e-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$, unique up to $\mathbf{G}^F$-conjugation, such that

(1) $\mathrm{Irr}(B) \subseteq \mathcal{E}_\ell(\mathbf{G}^F, s)$;
(2) $s \in \mathbf{L}^{*F}$ *up to* $\mathbf{G}^{*F}$*-conjugation;*
(3) $\lambda \in \mathcal{E}(\mathbf{L}^F, s)$ *is of central ℓ-defect;*
(4) $\mathrm{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F, s)$ *is the set of constituents of* $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$;
(5) $Z(\mathbf{L})_\ell^F$ *is a defect group of B; and*
(6) *there is an inclusion of subpairs* $(1, B) \lhd (Z(\mathbf{L})_\ell^F, b_{\mathbf{L}^F}(\lambda))$.

Proof. By the result of Broué–Michel [8, Thm. 9.12] there exists a semisimple ℓ'-element $s \in \mathbf{G}^{*F}$, unique up to $\mathbf{G}^{*F}$-conjugation, such that $\mathrm{Irr}(B)$ lies in the union $\mathcal{E}_\ell(\mathbf{G}^F, s)$ of Lusztig series. Furthermore, by [7, Thm. 4.1] there exists an e-split Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ and an e-cuspidal character $\lambda \in \mathcal{E}(\mathbf{L}, \ell')$, unique up to $\mathbf{G}^F$-conjugation, such that (4) holds. By transitivity of Lusztig induction this implies that $\lambda \in \mathcal{E}(\mathbf{L}^F, s)$, whence (2).

For the statement in (5), we claim that in the notation of [7, §4] we have $\mathbf{M} = \mathbf{L}$. For this, note that since $\mathbf{G}$ has simple derived subgroup, either $\mathbf{G} = \mathbf{G}_a$ or $\mathbf{G} = \mathbf{G}_b$. In the first case $\mathbf{M} = C^\circ_{\mathbf{G}}(\mathbf{T}_\ell^F)$ for some F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$, while $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T}_{\Phi_e})$, whence $\mathbf{M} = \mathbf{L}$ is e-split by Proposition 2.3. Similarly, in the second case $\mathbf{M}^* = C^\circ_{\mathbf{G}}(Z(C_{\mathbf{L}^*}(s))_\ell^F)$ is again e-split, hence equal to $\mathbf{L}^*$, which is the smallest e-split Levi subgroup of $\mathbf{G}^*$ containing $\mathbf{M}^*$, by [7, Lemma 4.4].

Now by [7, Lemma 4.16] $Z(\mathbf{M})_\ell^F = Z(\mathbf{L})_\ell^F$ is the unique maximal abelian normal subgroup of a defect group D of B. Since the Sylow ℓ-subgroups of $\mathbf{G}^F$ are abelian, this forces $D = Z(\mathbf{L})_\ell^F$. Also, λ is of central ℓ-defect by [7, Lemma 4.11]. The last assertion is in [7, Lemma 4.13]. □

In the situation of Theorem 2.6 we write $B = b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$. We also write $\tilde{b}_{\mathbf{L}^F}(\lambda)$ for the union of blocks of $N_{\mathbf{G}^F}(\mathbf{L})$ lying above $b_{\mathbf{L}^F}(\lambda)$. We first parametrize the characters of $b_{\mathbf{L}^F}(\lambda)$.

Proposition 2.7. *Under the hypotheses of Theorem 2.6 let B be an ℓ-block of $\mathbf{G}^F$ parametrized by the e-cuspidal pair $(\mathbf{L}, \lambda)$, with defect group $D = Z(\mathbf{L})_\ell^F$. Then:*

(a) $\mathrm{Irr}(b_{\mathbf{L}^F}(\lambda)) = \{\lambda \otimes \theta \mid \theta \in \mathrm{Irr}(D)\}$.
(b) *All characters of* $\tilde{b}_{\mathbf{L}^F}(\lambda)$ *are of height zero.*
(c) *Assume that all* $\psi \in \mathrm{Irr}(b_{\mathbf{L}^F}(\lambda))$ *extend to their inertia group in* $N_{\mathbf{G}^F}(D) = N_{\mathbf{G}^F}(\mathbf{L})$. *Then there is a 1-1 correspondence between the irreducible characters of* $\tilde{b}_{\mathbf{L}^F}(\lambda)$ *and pairs* (θ, ϕ) *where* $\theta \in \mathrm{Irr}(D)$ *and* $\phi \in \mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L}, \lambda, \theta))$.

Proof. By Proposition 2.4(c) we have $\mathbf{L}^F = O^\ell(\mathbf{L}^F) \times Z(\mathbf{L})_\ell^F = O^\ell(\mathbf{L}^F) \times D$. As λ is of central defect by Theorem 2.6, its restriction to $O^\ell(\mathbf{L}^F)$ has defect zero, whence the claim in (a). The assertion in (b) follows with Proposition 2.4(b) by Clifford theory.

For (c) note that both factors in the above direct decomposition of $\mathbf{L}^F$ are characteristic, so any element in $N_{\mathbf{G}^F}(\mathbf{L})$ fixing $\lambda \otimes \theta$ fixes both of them. The claim then follows again from Clifford theory. □

2.4. **Counting characters in $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$.** In order to parametrize the characters of the block $B = b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ we use Lusztig's Jordan decomposition of characters. Assume that $s \in \mathbf{G}^{*F}$ is such that $\mathrm{Irr}(B) \subseteq \mathcal{E}_\ell(\mathbf{G}^F, s)$. By Enguehard [13, Thm. 1.6] there is a reductive group $\mathbf{G}(s)$ with a Steinberg endomorphism again denoted F, with $\mathbf{G}(s)^\circ$ dual to $C^\circ_{\mathbf{G}^*}(s)$ and $\mathbf{G}(s)/\mathbf{G}(s)^\circ$ isomorphic to $C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$, a unipotent block $B(s)$ of $\mathbf{G}(s)^F$ with defect groups isomorphic to those of B, and a height preserving bijection $\mathrm{Irr}(B) \to \mathrm{Irr}(B(s))$. If $C_{\mathbf{G}^*}(s)$ is contained in a proper Levi subgroup and connected, this is given by Lusztig's Jordan decomposition of characters.

Thus, a parametrization of $\mathrm{Irr}(B)$ can be obtained from one for the unipotent block $\mathrm{Irr}(B(s))$. If $\mathbf{G}(s)$ is connected, again by Theorem 2.6 the unipotent ℓ-block $B(s)$ of $\mathbf{G}(s)^F$ is indexed by a unique $\mathbf{G}(s)^F$-class of e-cuspidal pairs $(\mathbf{L}(s), \lambda_s)$, where $\lambda_s \in \mathcal{E}(\mathbf{L}(s)^F, \ell')$, so that $B(s) = b_{\mathbf{G}(s)^F}(\mathbf{L}(s), \lambda_s)$. Now, the characters of $B(s)$ are described by the following crucial result:

Theorem 2.8 (Broué–Michel)**.** *Assume that $\mathbf{G}(s)$ is connected. Then there is an isotypie between the unipotent block $B(s)$ of $\mathbf{G}(s)^F$ and the principal ℓ-block of the semidirect product $N_{\mathbf{G}(s)^F}(\mathbf{L}(s), \lambda_s)/O^\ell(\mathbf{L}(s)^F) \cong Z^\circ(\mathbf{L}(s))^F_\ell \rtimes W_{\mathbf{G}(s)^F}(\mathbf{L}(s), \lambda_s)$. In particular, there exists a height preserving bijection*

$$\mathrm{Irr}\left(Z^\circ(\mathbf{L}(s))^F_\ell \rtimes W_{\mathbf{G}(s)^F}(\mathbf{L}(s), \lambda_s)\right) \longrightarrow \mathrm{Irr}(B(s)),$$

and thus all characters of $B(s)$ are of height zero.

Proof. Note that a prime ℓ satisfying (*) is $(\mathbf{G}, F)$-excellent, and then also $(\mathbf{G}(s), F)$-excellent, in the sense of [5, Def. 1.11]. Thus, all assertions are proved in [5, Thm. 3.1] except for the structure of the local subgroup $N_{\mathbf{G}(s)^F}(\mathbf{L}(s), \lambda_s)/O^\ell(\mathbf{L}(s)^F)$. The claimed isomorphism follows from Proposition 2.4(c), and the extension of $Z^\circ(\mathbf{L}(s))^F_\ell$ by the relative Weyl group $W_{\mathbf{G}(s)^F}(\mathbf{L}(s), \lambda_s)$ is split since $|W_{\mathbf{G}(s)^F}(\mathbf{L}(s), \lambda_s)|$ is prime to ℓ by Proposition 2.4(b) applied to the e-split Levi subgroup $\mathbf{L}(s)$ of $\mathbf{G}(s)$. The last claim follows from Clifford theory by Proposition 2.4. □

Thus we obtain the following criterion for the existence of a bijection suitable for the (AM) condition:

Theorem 2.9. *Assume that $\mathbf{G}$ has simple derived subgroup of simply connected type, and that $\ell \geq 5$ satisfies (*). Let $s \in \mathbf{G}^{*F}$ be such that $C_{\mathbf{G}^*}(s)$ is connected. Let $B = b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ be an ℓ-block of $\mathbf{G}^F$, where $\lambda \in \mathcal{E}(\mathbf{L}^F, s)$ is e-cuspidal. Assume that*

(**) *all $\psi \in \mathrm{Irr}(b_{\mathbf{L}^F}(\lambda))$ extend to their inertia group in $N_{\mathbf{G}^F}(\mathbf{L}, \lambda)$.*

Then there exists a height preserving bijection between the irreducible characters of B and the characters $\mathrm{Irr}(\tilde{b}_{\mathbf{L}^F}(\lambda))$ of the ℓ-block of $N_{\mathbf{G}^F}(\mathbf{L})$ above $b_{\mathbf{L}^F}(\lambda)$.

Proof. By the result of Enguehard cited above ([13, Thm. 1.6]) there is a height preserving bijection $\mathrm{Irr}(B) \to \mathrm{Irr}(B(s))$ to a unipotent block $B(s)$ of $\mathbf{G}(s)^F$, where $\mathbf{G}(s) = \mathbf{G}(s)^\circ$ is dual to $C^\circ_{\mathbf{G}^*}(s)$. By Theorem 2.8, there is a height preserving bijection

$$\mathrm{Irr}(B(s)) \xrightarrow{1-1} \mathrm{Irr}\left(Z^\circ(\mathbf{L}(s))^F_\ell \rtimes W_{\mathbf{G}(s)^F}(\mathbf{L}(s), \lambda_s)\right)$$

(and thus all characters in $\mathrm{Irr}(B)$ are of height zero), where $(\mathbf{L}(s), \lambda_s)$ is an e-cuspidal pair of $\mathbf{G}(s)$. Let $\mathbf{L}(s)^* \leq C^\circ_{\mathbf{G}^*}(s)$ be in duality with $\mathbf{L}(s)$, an e-split Levi subgroup, and $\mathbf{M} \leq \mathbf{G}$ the e-split Levi subgroup in duality with $\mathbf{M}^* := C_{\mathbf{G}^*}(Z(\mathbf{L}(s)^*)_{\Phi_e})$. Let $\mu \in \mathcal{E}(\mathbf{L}^F, s)$ be a character constructed from λ_s as in [13, Prop. 1.4.2]. Then by [13, Thm. 1.6] we have $(\mathbf{M}, \mu) = (\mathbf{L}, \lambda)$ up to $\mathbf{G}^F$-conjugation. Moreover, by the properties of Lusztig's Jordan decomposition $W_{\mathbf{G}(s)^F}(\mathbf{L}(s), \lambda_s) \cong W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$. As $\mathbf{L}(s)^F = D \times O^\ell(\mathbf{L}(s)^F)$ and

$\mathbf{L}^F = D \times O^\ell(\mathbf{L}^F)$ by Proposition 2.4, where $D = Z^\circ(\mathbf{L}(s))^F_\ell = Z^\circ(\mathbf{L}))^F_\ell$ is a defect group of B, this shows that the natural injective map $N_{\mathbf{G}(s)}(\mathbf{L}(s), \lambda_s) \to N_{\mathbf{G}}(\mathbf{L}, \lambda)$ induces an isomorphism $D \rtimes W_{\mathbf{G}(s)^F}(\mathbf{L}(s), \lambda_s) \cong D \rtimes W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$. This gives a height preserving bijection

$$\mathrm{Irr}\left(Z^\circ(\mathbf{L}(s))^F_\ell \rtimes W_{\mathbf{G}(s)^F}(\mathbf{L}(s), \lambda_s)\right) \xrightarrow{1-1} \mathrm{Irr}\left(Z^\circ(\mathbf{L})^F_\ell \rtimes W_{\mathbf{G}^F}(\mathbf{L}, \lambda)\right).$$

Then Proposition 2.7(c) allows to conclude. □

3. Radical subgroups and defect zero characters

We keep the assumptions of the previous section: let $\mathbf{G}$ be connected reductive with Steinberg endomorphism $F : \mathbf{G} \to \mathbf{G}$, ℓ a prime satisfying (*) with respect to $(\mathbf{G}, F)$, and $e = e_\ell(q)$. Now let's consider radical subgroups.

Proposition 3.1. *Assume that ℓ satisfies (*). Let $Q \leq \mathbf{G}^F$ be a radical ℓ-subgroup and $\mathbf{L} := C_{\mathbf{G}}(Q)$, $\mathbf{S} := Z^\circ(\mathbf{L})_{\Phi_e}$. Then $Q = \mathbf{S}^F_\ell$.*

Proof. By Proposition 2.3, $\mathbf{L}$ is an e-split Levi subgroup of $\mathbf{G}$, so $C_{\mathbf{G}}(Q) = \mathbf{L} = C_{\mathbf{G}}(\mathbf{S})$. Since Q is radical and abelian by Proposition 2.2, we may conclude that

$$Q = Z(C_{\mathbf{G}^F}(Q))_\ell = Z(\mathbf{L}^F)_\ell = Z(\mathbf{L})^F_\ell = Z^\circ(\mathbf{L})^F_\ell = \mathbf{S}^F_\ell.$$

□

By Proposition 2.4 we have $\mathbf{L} = Q \times O^\ell(\mathbf{L}^F)$, and $O^\ell(\mathbf{L}^F)$ is an extension of $[\mathbf{L}, \mathbf{L}]^F$ by an ℓ'-group.

Corollary 3.2. *Assume that ℓ satisfies (*). The maps*

$$\Psi : Q \mapsto C_{\mathbf{G}}(Q), \qquad \Xi : \mathbf{L} \mapsto Z^\circ(\mathbf{L})^F_\ell,$$

are mutually inverse $\mathbf{G}^F$-equivariant bijections between the set $\mathcal{Q}$ of radical ℓ-subgroups of $\mathbf{G}^F$ and the set of e-split Levi subgroups $\mathbf{L}$ of $\mathbf{G}$ for which $Z(\mathbf{L})^F_\ell = O_\ell(\mathbf{L}^F)$.

In particular, they induce natural bijections between the set $\mathcal{Q}$ of radical ℓ-subgroups of $\mathbf{G}^F$ modulo $\mathbf{G}^F$-conjugation and a certain subset of the set $\mathcal{L}_e$ of e-split Levi subgroups of $\mathbf{G}$ modulo $\mathbf{G}^F$-conjugation.

Proof. By Proposition 2.3 any (radical) ℓ-subgroup Q determines an e-split Levi subgroup $C_{\mathbf{G}}(Q)$. Conversely, if $\mathbf{L} \in \mathcal{L}_e$, then $Q := Z(\mathbf{L})^F_\ell$ is a radical ℓ-subgroup by Propositions 2.4 and 3.1. Clearly, the two maps are inverse to one another and $\mathbf{G}^F$-equivariant. □

By Remark 2.5 the map $\Psi : \mathcal{Q} \to \mathcal{L}_e$ is also surjective except possibly in certain cases when $\ell = 3$ and $\mathbf{G}$ is defined over $\mathbb{F}_2$. In these exceptions, $Z(\mathbf{L})^F_\ell$ is not a radical ℓ-subgroup, so $\mathbf{L}^F$ won't have defect zero characters. It is therefore no problem for our purpose to work with all e-split Levi subgroups of $\mathbf{G}^F$ in place of $\mathcal{L}_e$.

We'll also need to understand characters of defect zero. For H a group let

$$\mathrm{dz}(H) := \{\chi \in \mathrm{Irr}(H) \mid |H|/\chi(1) \not\equiv 0 \pmod{\ell}\}$$

denote the set of its characters of ℓ-defect zero. For $N \trianglelefteq H$ a normal subgroup and $\mu \in \mathrm{Irr}(N)$, we let $\mathrm{dz}(H|\mu)$ be the set of $\chi \in \mathrm{dz}(H)$ lying above μ.

Proposition 3.3. *Let $Q \le \mathbf{G}^F$ be a radical ℓ-subgroup, $\mathbf{L} := C_{\mathbf{G}}(Q)$ and $\lambda \in \mathrm{dz}(\mathbf{L}^F/Q)$ a character of ℓ-defect zero. Assume that λ extends to its inertia group $N_{\mathbf{G}^F}(\mathbf{L},\lambda)$. Then $\mathrm{dz}(N_{\mathbf{G}^F}(Q)/Q|\lambda)$ is in 1-1 correspondence with $\mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L},\lambda))$.*

Proof. Note that $\mathbf{L}$ is an e-split Levi subgroup of $\mathbf{G}$ by Proposition 2.3. Using the fact that $N_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}^F$ has order prime to ℓ and that $N_{\mathbf{G}^F}(Q) = N_{\mathbf{G}^F}(\mathbf{L})$ by Proposition 2.4, the claim follows by an application of Clifford theory. □

Proposition 3.4. *Let $\mathbf{L} \le \mathbf{G}$ be an e-split Levi subgroup and $\lambda \in \mathrm{dz}(\mathbf{L}^F/O_\ell(\mathbf{L}^F))$ of ℓ-defect zero. Then there exists a semisimple ℓ'-element $s \in \mathbf{L}^{*F}$ with $\lambda \in \mathcal{E}(\mathbf{L},s)$. If moreover $\ell \ge 5$ then λ is e-cuspidal.*

Proof. We may view λ as a character of $\mathbf{L}^F$. Then by Lusztig's Jordan decomposition of characters there exists a semisimple element $s \in \mathbf{L}^{*F}$ with $\lambda \in \mathcal{E}(\mathbf{L}^F,s)$, and a unipotent character ψ of $C := C_{\mathbf{L}^{*F}}(s)$ with $\lambda(1) = |\mathbf{L}^{*F} : C|_{p'}\,\psi(1)$. By an elementary property of characters we have that $\psi(1)$ divides $|C : Z(C)|$, so $\lambda(1)_\ell \le |\mathbf{L}^{*F} : Z(C)|_\ell$. In particular, if λ is of central defect, then $|Z(C) : Z(\mathbf{L}^{*F})|$ has to be prime to ℓ. Thus, the ℓ-part of s lies in $Z(\mathbf{L}^{*F})$. On the other hand, as a character of $\mathbf{L}^F/O_\ell(\mathbf{L}^F)$, λ must be trivial on $Z(\mathbf{L})_\ell^F$, whence the ℓ-part of s in $Z(\mathbf{L}^{*F})$ must be trivial, so s is an ℓ'-element.

Since ℓ divides $\Phi_e(q)$, for $e = e_\ell(q)$, $\lambda(1)$ must be divisible by the full $\Phi_e(q)$-part of $|\mathbf{L}^F/Z(\mathbf{L})^F|$. Thus the same is true for the unipotent character ψ of C, which is then e-cuspidal by [4, Prop. 2.9]. So λ satisfies condition (U) in [7, §1.3], so condition (J) by [7, Prop. 1.10], whence it is e-cuspidal by [7, Thm. 4.2]. (This latter reference requires that $\ell \ge 5$.) □

Proposition 3.5. *Assume that $\ell \ge 5$ satisfies (*). Let $Q \le \mathbf{G}^F$ be a radical ℓ-subgroup, $\mathbf{L} := C_{\mathbf{G}}(Q)$ and $N := N_{\mathbf{G}^F}(Q)$. Assume that all $\lambda \in \mathrm{dz}(\mathbf{L}^F/O_\ell(\mathbf{L}^F))$ extend to their inertia groups in N. Then there is a bijection between $\mathrm{dz}(N/Q|\lambda)$ and the set of pairs*

$$\{(\lambda,\phi) \mid \lambda \in \mathrm{Irr}(\mathbf{L}^F/O_\ell(\mathbf{L}^F)) \text{ } e\text{-cuspidal},\ \phi \in \mathrm{dz}(W_{\mathbf{G}^F}(\mathbf{L},\lambda))\}$$

modulo N-conjugation.

Proof. First note that under our assumptions, $\mathbf{L} = C_{\mathbf{G}}(Q)$ is an e-split Levi subgroup of $\mathbf{G}$, by Proposition 3.1. Now note that $\lambda \in \mathrm{dz}(\mathbf{L}^F/O_\ell(\mathbf{L}^F))$ is e-cuspidal by Proposition 3.4, and conversely any e-cuspidal $\lambda \in \mathrm{Irr}(\mathbf{L}^F/O_\ell(\mathbf{L}^F))$ is of ℓ-defect zero again by (*). Furthermore, $N = N_{\mathbf{G}^F}(Q) = N_{\mathbf{G}^F}(\mathbf{L})$ by Proposition 3.1(c), so $N/Q = N_{\mathbf{G}^F}(\mathbf{L})/O_\ell(\mathbf{L}^F)$. The claim is now an immediate consequence of Clifford theory. □

We now discuss the ℓ-modular Brauer characters of $\mathbf{G}^F$.

Proposition 3.6. *Assume that $\ell \ge 5$ satisfies (*). Let $B = b_{\mathbf{G}^F}(\mathbf{L},\lambda)$ be an ℓ-block of $\mathbf{G}^F$. Then there is a bijection*

$$\mathrm{IBr}(B) \xrightarrow{1-1} \mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L},\lambda)),$$

where $W_{\mathbf{G}^F}(\mathbf{L},\lambda) := N_{\mathbf{G}^F}(\mathbf{L},\lambda)/\mathbf{L}$ is the relative Weyl group of the e-cuspidal pair $(\mathbf{L},\lambda)$.

Proof. By [7, Thm. 4.1(b)], $\mathcal{E}(\mathbf{G}^F,s) \cap B$ is in 1-1 correspondence with $\mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L},\lambda))$. By Lemma 2.1 a result of Geck–Hiss [15, Cor. 4.3] applies to show that $\mathcal{E}(\mathbf{G}^F,s) \cap B$ is a basic set for the block B, so in particular $|\mathrm{IBr}(B)| = |\mathcal{E}(\mathbf{G}^F,s) \cap B|$. □

Corollary 3.7. *Let Q be a radical ℓ-subgroup of $\mathbf{G}^F$. Under the assumptions of Proposition 3.5 there exists a (BAW)-bijection for $\mathbf{G}^F$ above Q (see the beginning of §6 for a description of this term).*

Proof. By Proposition 3.1, $\mathbf{L} := C_{\mathbf{G}^F}(Q)$ is an e-split Levi subgroup of $\mathbf{G}$, and for any $\lambda \in \mathrm{dz}(\mathbf{L}^F/Q)$ we have bijections

$$\begin{aligned} \mathrm{dz}(N_{\mathbf{G}^F}(Q)|\lambda) &\overset{1-1}{\longleftrightarrow} \mathrm{dz}(N_{\mathbf{G}^F}(\mathbf{L},\lambda)|\lambda) && \text{by Corollary 3.2}\\ &\overset{1-1}{\longleftrightarrow} \mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L},\lambda)) && \text{by Proposition 3.5}\\ &\overset{1-1}{\longleftrightarrow} \mathrm{IBr}(b_{\mathbf{G}^F}(\mathbf{L},\lambda)) && \text{by Proposition 3.6} \end{aligned}$$

since for $\lambda \in \mathrm{dz}(\mathbf{L}^F/Q)$, Q is a defect group for the block $b_{\mathbf{G}^F}(\mathbf{L},\lambda)$. On the other hand, all ℓ-blocks of $\mathbf{G}^F$ arise in this way by Theorem 2.6. The claim follows. □

Thus we arrive at the following connection between the inductive (AM) and inductive (BAW) conditions in the case of large primes:

Theorem 3.8. *Assume that $\mathbf{G}$ has simple derived subgroup of simply connected type, and that $\ell \geq 5$ satisfies (*). Let $Q \leq \mathbf{G}^F$ be a radical ℓ-subgroup, $\mathbf{L} := C_{\mathbf{G}}(Q)$, $B = b_{\mathbf{G}^F}(\mathbf{L},\lambda)$ an ℓ-block of $\mathbf{G}^F$ where $\lambda \in \mathcal{E}(\mathbf{L}^F, s)$ is e-cuspidal and satisfies condition (**), and $s \in \mathbf{G}^{*F}$ is an ℓ'-element. Assume that the decomposition matrix of B is unitriangular with respect to $\mathrm{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F, s)$ and that the (AM)-bijection from Theorem 2.9 can be chosen to be $\mathrm{Aut}(G)$-equivariant. Then the (BAW)-bijection in Corollary 3.7 can also be chosen to be $\mathrm{Aut}(G)$-equivariant.*

Proof. As $\mathrm{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F, s)$ is a basic set for the block B, the unitriangularity of the decomposition matrix of B yields a natural $\mathrm{Aut}(G)$-equivariant bijection $\mathrm{IBr}(B) \overset{1-1}{\leftrightarrow} \mathrm{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F, s)$. Restricting the $\mathrm{Aut}(G)$-equivariant bijection $\mathrm{Irr}(B) \to \mathrm{Irr}(\tilde{b}_{\mathbf{L}^F}(\lambda))$ from Theorem 2.9 to the subset $\mathrm{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F, s)$ gives an equivariant bijection between $\mathrm{Aut}(G)$-stable subsets

$$\begin{aligned} \mathrm{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F, s) &\overset{1-1}{\longleftrightarrow} \mathrm{Irr}(B(s)) \cap \mathcal{E}(\mathbf{L}(s)^F, 1)\\ &\overset{1-1}{\longleftrightarrow} \mathrm{Irr}(W_{\mathbf{G}(s)^F}(\mathbf{L}(s),\lambda_s))\\ &\overset{1-1}{\longleftrightarrow} \mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L},\lambda))\\ &\overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\tilde{b}_{\mathbf{L}^F}(\lambda)|\lambda). \end{aligned}$$

Since ℓ satisfies (*), all characters of $W_{\mathbf{G}^F}(\mathbf{L},\lambda)$ are of defect zero, so by Proposition 3.5 all characters of $\tilde{b}_{\mathbf{L}^F}(\lambda)$ above λ are of defect zero. Thus the last set in the displayed bijections is just $\mathrm{dz}(N_{\mathbf{G}^F}(Q)|\lambda)$. □

Remark 3.9. Unitriangularity of the decomposition matrices of finite groups of Lie type has been proved in a variety of cases; for example for $\mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q)$ (see [14]), for the other classical groups at linear primes ℓ (see [16]), and for various cases in exceptional type groups of small rank. It is conjectured to hold in general.

4. Equivariance for groups with connected center

In this section we study the action of automorphisms on characters of finite reductive groups with connected center (see also [9, Thm. 2.1] for a version of Theorem 4.1). For this, let $\mathbf{G}$ be connected reductive with connected center, with a Steinberg endomorphism $F:\mathbf{G}\to\mathbf{G}$. Assume given a collection $F_1,\ldots,F_r:\mathbf{G}\to\mathbf{G}$ of isogenies commuting with F. Thus, the F_i could also be Steinberg endomorphisms, or graph automorphisms of $\mathbf{G}$.

Then each F_i induces an automorphism of $\mathbf{G}^F$, again denoted F_i, and hence also naturally acts on characters of $\mathbf{G}^F$ via $F_i(\chi)(g)=\chi({F_i}^{-1}(g))$ for $\chi\in\mathrm{Irr}(\mathbf{G}^F)$, $g\in\mathbf{G}^F$. We also denote by $F_i:\mathbf{G}^*\to\mathbf{G}^*$ corresponding isogenies on the dual group $\mathbf{G}^*$, commuting with $F:\mathbf{G}^*\to\mathbf{G}^*$. (Note that even if F is given as part of the duality, these F_i on $\mathbf{G}^*$ are only unique up to inner automorphisms induced by elements of $\mathbf{T}^{*F}$). Since $\mathbf{G}$ has connected center, all centralizers of semisimple elements in $\mathbf{G}^*$ are connected.

In this situation we have the following result, essentially due to Digne–Michel [12]:

Theorem 4.1. *Assume that $\mathbf{G}$ has connected center, and let $F:\mathbf{G}\to\mathbf{G}$ be a Steinberg endomorphism, $F_i:\mathbf{G}\to\mathbf{G}$ $(1\le i\le r)$ isogenies commuting with F, and set $\Gamma:=\langle F_1,\ldots,F_r\rangle$. Then for $s\in\mathbf{G}^{*F}$ we have:*
(a) $F_i(\mathcal{E}(\mathbf{G}^F,s))=\mathcal{E}(\mathbf{G}^F,{F_i}^{-1}(s))$ *for all i.*
(b) *If $s\in\mathbf{G}^{*\Gamma}$ then there exists a Γ-equivariant Jordan–Lusztig bijection*

$$\Psi_s:\mathcal{E}(\mathbf{G}^F,s)\longrightarrow\mathcal{E}(C_{\mathbf{G}^{*F}}(s),1).$$

(c) *If s is F_i-invariant, F_i is a split Frobenius endomorphism of $\mathbf{G}$ and there exists a common positive power of F and F_i, then F_i acts trivially on $\mathcal{E}(\mathbf{G}^F,s)$.*

Proof. The first assertion is in [6, Prop. 1] (observe that the proof given there also applies when F_i is just an isogeny). For (b), first note that when $s\in\mathbf{G}^{*\Gamma}$, then $\mathcal{E}(\mathbf{G}^F,s)$ is Γ-stable by (a). We now consider the Jordan decomposition as given in [12, Thm. 7.1]. Let $\chi\in\mathcal{E}(\mathbf{G}^F,s)$. If χ is not one of the exceptions in [12, Prop. 6.3], then it is uniquely determined by its multiplicities in the various Deligne–Lusztig characters, and these are respected by the action of the isogenies F_i. This already covers the case of classical groups. In the remaining cases, χ lies in a family corresponding to the group $\mathfrak{S}_3$, and there are exactly two elements in that family with that property. In that situation [12, Thm. 7.1(v)] specifies a bijection, which again is Γ-equivariant since all F_i preserve the subspace spanned by cuspidal representations (since they preserve the set of split Levi subgroups) which occurs in the defining property of that bijection.

The last assertion is due to Lusztig [19, Prop. 2.20]. □

5. Suzuki and Ree groups

We verify the inductive Alperin-McKay condition (AM) (as introduced by Späth [28, Def. 7.2]), the inductive Isaacs–Navarro refinement (IN) on congruences for character degrees (see [28, Def. 3.1]), and the inductive blockwise Alperin weight condition (BAW) (see [29]) for the very twisted simple groups of Lie type, that is, for the Suzuki groups ${}^2B_2(q^2)$ and the Ree groups ${}^2G_2(q^2)$ and ${}^2F_4(q^2)$ and thus prove one part of Theorem 1.1 from the introduction. Several partial results had already been obtained by B. Späth and others (see the references in the subsequent proof).

Theorem 5.1. *Let S be any of the simple groups ${}^2B_2(2^{2f+1})$, ${}^2G_2(3^{2f+1})'$ or ${}^2F_4(2^{2f+1})'$. Then S satisfies the inductive Alperin-McKay condition (AM), the inductive Isaacs–Navarro refinement (IN) and the inductive blockwise Alperin weight condition (BAW) for all primes.*

Proof. Note that for all three families of groups, except for ${}^2F_4(2)'$, the case of defining characteristic was dealt with by Späth in [28, Cor. B, Thm. 4.4] and [29, Thm. B, Cor. 6.3], so for these we only need to worry about non-defining primes ℓ. Furthermore, note that all groups considered here have cyclic outer automorphism group, so the verification of the three inductive conjectures essentially reduces to finding (McK) and (AWC)-bijections that are compatible with block induction, by [28, Lemma 8.1] and [29, Prop. 6.2].

First assume that $S = {}^2B_2(2^{2f+1})$ with $f \geq 1$. The inductive McKay-condition was shown to hold in [18, Thm. 16.1], in particular also for the exceptional covering groups of ${}^2B_2(8)$. It is immediate to check that the bijections given there respect ℓ-blocks and also satisfy the (IN)-congruences (see the remark after the proof of Thm. 17.1 in loc. cit.), so that both (AM) and (IN) follow. Also, (BAW) was shown to hold by Späth [29, Cor. 6.3].

Now let $S = {}^2G_2(3^{2f+1})'$ with $f \geq 0$. Again, the inductive McKay-condition was checked to hold in [18, Thm. 17.1], respectively in [18, Thm. 15.3] when $f = 0$, and the given bijections do respect blocks and the (IN)-congruences. Now note that for all primes $\ell \neq 3$, any ℓ-blocks of S is either of full defect or of defect zero. In all these cases, the (McK)-bijections are compatible with blocks by [9, Thm. 6.3], so the (AM)-condition follows. The inductive Alperin weight condition (BAW) was verified in [29, Prop. 6.4].

Next, let $S = {}^2F_4(2)'$, with $\mathrm{Aut}(S) = {}^2F_4(2)$. Here, (BAW) has been verified in [3] (see also [25, Prop. 6.1] for (AWC) at $p = 2$). For all prime divisors ℓ of $|S|$, all non-principal ℓ-blocks of S are of defect zero. The relevant data for the principal blocks B_0 of S and B_0' of $\mathrm{Aut}(S) = S.2$ and the normalizers of Sylow ℓ-subgroups S_ℓ are collected in Table 1.

Table 1. Principal ℓ-blocks in ${}^2F_4(2)'$

ℓ	$\lvert\mathrm{Irr}_0(B_0)\rvert$	$\mathrm{Irr}_0(N_S(S_\ell))$	$\lvert\mathrm{Irr}_0(B_0')\rvert$	$\mathrm{Irr}_0(N_{\mathrm{Aut}(S)}(S_\ell))$
2	8	1^8	16	1^{16}
3	9	$1^4, 2, 4^4$	9	$1^4, 2^3, 8^2$
5	16	$1^6, 2^6, 3^2, 24^2$	20	$1^4, 2^6, 3^4, 4^2, 24^4$
13	8	$1^6, 6^2$	13	$1^{12}, 12$

In the table, n^a indicates that there are a characters of degree n.

The validity of (AM) follows from the data in this table, and the statement of (IN) can also be checked easily from the known character table of S.

Finally, we consider $S = {}^2F_4(q^2)$, where $q^2 = 2^{2f+1}$ with $f \geq 1$, so $S = \mathbf{G}^F$ for an algebraic group $\mathbf{G}$ of type F_4 with a very twisted Steinberg endomorphism $F : \mathbf{G} \to \mathbf{G}$. Here, there are two essentially distinct cases: for $\ell = 3$, the Sylow subgroups are non-abelian, while for $\ell > 3$, they are abelian. In the latter case, replacing cyclotomic polynomials over $\mathbb{Q}$ by cyclotomic polynomials Φ over $\mathbb{Q}(\sqrt{2})$, respectively by $X^2 - 1$, the prime ℓ satisfies condition (*) from Section 2. The same arguments as there then show that the analogues of Propositions 2.3 and 2.4 (for Φ-split Levi subgroups instead of e-split ones) continue to hold. The ℓ-blocks of S have been determined in Malle [20] and from this the assertions from Theorem 2.6 can be seen to hold in this case as well, and thus the analogue of Proposition 2.7 is true.

Table 2. Φ-split Levi subgroups in ${}^2F_4(q^2)$

Φ	$\mathbf{L}^F$	$N_{\mathbf{G}^F}(\mathbf{L})$	$\mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L}))$
$\tilde\Phi_1$	$\tilde\Phi_1^2$	$\mathbf{L}^F : D_8$	$1^4, 2^3$
	$\tilde\Phi_1 \times {}^2B_2(q^2)$	$\tilde\Phi_1 : 2 \times {}^2B_2(q^2)$	1^2
Φ_4	Φ_4^2	$\mathbf{L}^F : \mathrm{GL}_2(3)$	$1^2, 2^3, 3^2, 4$
	$\Phi_4 \times \mathrm{L}_2(q^2)$	$\Phi_4 : 2 \times \mathrm{L}_2(q^2)$	1^2
Φ_8'	${\Phi_8'}^2$	$\mathbf{L}^F : 4\mathfrak{S}_4$	$1^4, 2^6, 3^4, 4^2$
	$\Phi_8' \times {}^2B_2(q^2)$	$\Phi_8' : 4 \times {}^2B_2(q^2)$	1^4
Φ_{12}	Φ_{12}	$\mathbf{L}^F : 6$	1^6
Φ_{24}'	Φ_{24}'	$\mathbf{L}^F : 12$	1^{12}

In the table, Φ_i denotes the ith cyclotomic polynomial evaluated at q,
$\tilde\Phi_1 = q^2 - 1$, $\Phi_8' = q^2 \pm \sqrt{2}q \pm 1$, $\Phi_{24}' = q^4 \pm \sqrt{2}q^3 + q^2 \pm \sqrt{2}q + 1$,
D_8 is the dihedral group of order 8, and $4\mathfrak{S}_4$ denotes a central extension of 4 by $\mathfrak{S}_4$.

We have listed the Φ-split Levi subgroups $\mathbf{L}$ of $(\mathbf{G}, F)$ for all cyclotomic polynomials Φ dividing $|{}^2F_4(q^2)|$ (at least) twice in Table 2. Here, the last column contains the character degrees of $W_{\mathbf{G}^F}(\mathbf{L})$ with multiplicities. By the information given in Malle [21, Prop. 1.2,1.3] one sees that all these Levi subgroups $\mathbf{L}$ have the property that $N_{\mathbf{G}^F}(\mathbf{L})$ is a split extension of $\mathbf{L}^F$ by $W_{\mathbf{G}^F}(\mathbf{L})$, and $W_{\mathbf{G}^F}(\mathbf{L})$ acts trivially except on some central direct factor. Thus all characters of $\mathbf{L}^F$ extend to their inertia groups in $N_{\mathbf{G}^F}(\mathbf{L})$. On the other hand, it is shown by Himstedt [17] that $\mathcal{E}(S, s)$ forms a basic set for $\mathcal{E}_\ell(S, s)$ for all semisimple ℓ'-elements $s \in \mathbf{G}^{*F} \cong S$ and that the decomposition matrix is unitriangular on that basic set. The blockwise Alperin weight conjecture for S now follows as in Corollary 3.7.

The outer automorphism group of S is cyclic, generated by the split Frobenius endomorphism of $\mathbf{G}$ with respect to $\mathbb{F}_2$ restricted to S. Now note that $\mathbf{G}$ has connected center. Thus, by Theorem 4.1 an irreducible character $\chi \in \mathcal{E}(S, s)$ is fixed by a field automorphism γ if and only if γ fixes the S-class of the semisimple element s. Similarly, locally, if γ stabilizes the e-cuspidal character $\lambda \in \mathcal{E}(\mathbf{L}^F, s)$, then it fixes all characters of $N_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ lying above it, since the field automorphisms commute with the action of the Weyl group of $\mathbf{G}^F$, and $N_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ is generated over $\mathbf{L}^F$ by elements from the Weyl group. Since $\mathrm{Out}(S)$ is cyclic, all invariant characters extend. The inductive blockwise Alperin weight condition (BAW) follows for $\ell > 3$.

In Table 3 we describe the non-trivial 3-blocks of ${}^2F_4(q^2)$ and their defect groups according to [20] in terms of semisimple $3'$-elements $s \in S$, and give their numbers of 3-modular irreducibles according to [17, §4]. Here $S_3(\mathbf{G}^F)$ denotes a Sylow 3-subgroup of $\mathbf{G}^F$. The fifth column lists the number of conjugacy classes of $3'$-element with that particular centralizer, hence the number of 3-blocks of that type.

TABLE 3. 3-blocks of positive defect in ${}^2F_4(q^2)$

	$C_{\mathbf{G}^F}(s)$	$D(B)$	$\lvert\mathrm{IBr}(B)\rvert$	No. of such blocks
1	$\mathbf{G}^F$	$S_3(\mathbf{G}^F)$	9	1
2	$\Phi_4 \times \mathrm{L}_2(q^2)$	$3^a \times 3^a$	2	$(b-1)/2$
3	Φ_4^2	$3^a \times 3^a$	1	$(b-1)(b-11)/48$
4	$\tilde{\Phi}_1 \times \mathrm{L}_2(q^2)$	3^a	2	$(q^2-2)/2$
5	$\tilde{\Phi}_1\Phi_4$	3^a	1	$(b-1)(q^2-2)/4$
6	Φ_{12}	3	1	$(q^4-q^2-2)/18$

Here, 3^a is the precise power of 3 dividing Φ_4, $b := (\Phi_4)_{3'} = \Phi_4/3^a$.

The radical 3-subgroups and their normalizers were determined by An [2, §2–3]. We have reproduced the results in Table 4, where the fourth column contains the character degrees of 3-defect zero with their multiplicities and the first column indicates the corresponding 3-block in $\mathbf{G}^F$.

TABLE 4. Non-trivial radical 3-subgroups in ${}^2F_4(q^2)$

block	R	$N_{\mathbf{G}^F}(R)/R$	$\mathrm{dz}(N_{\mathbf{G}^F}(R)/R)$	condition
1	$S_3(\mathbf{G}^F)$	2×2	$1, 1, 1, 1$	$a \geq 2$
1	3_+^{1+2}	$\mathrm{SL}_2(3)$	3	$a \geq 2$
1	3_+^{1+2}	$\mathrm{SL}_2(3).2$	$3, 3$	$a \geq 2$
1	3_+^{1+2}	8.2	$1, 1, 1, 1, 2, 2, 2$	$a = 1$
1	$3^a \times 3^a$	$((q^2+1)_{3'})^2.\mathrm{GL}_2(3)$	$3, 3$	
2			$24, 24 \quad (\times(b-1)/2)$	
3			$48 \quad (\times(b-1)(b-11)/48)$	
4	3^a	$\mathrm{L}_2(q^2) \times ((q^2+1)_{3'}).2$	$\Phi_4, \Phi_4 \quad (\times(q^2-2)/2)$	$a \geq 2$
5			$2\Phi_4 \quad (\times(b-1)(q^2-2)/4)$	$a \geq 2$
4	3	$\mathrm{U}_3(q^2).2$	$\tilde{\Phi}_1\Phi_4\Phi_{12}, \tilde{\Phi}_1\Phi_4\Phi_{12} \quad (\times(q^2-2)/2)$	$a = 1$
5			$2\tilde{\Phi}_1\Phi_4\Phi_{12} \ (\times(b-1)(q^2-2)/4)$	$a = 1$
6			$2\tilde{\Phi}_1\Phi_4^2 \quad (\times(q^4-q^2-2)/18)$	

It follows from this already that the ordinary Alperin weight conjecture holds for S at for $\ell = 3$. We now consider the action of $\mathrm{Out}(S)$, which in this case is cyclic and consists only of field automorphisms. For the principal block B_0 of S, the decomposition matrix between the nine 3-modular irreducible characters and a suitable basic set of ordinary characters is unitriangular by [17, Cor. 4.3]. Now the ordinary characters in this basic set are either unipotent or lie in the Lusztig series of the 3-central 3-element of S. In particular, all characters in this basic set are invariant under all field automorphisms of S. Thus, the nine modular irreducibles in $\mathrm{IBr}(B_0)$ are invariant, and hence extend. On the other hand, the radical 3-subgroups listed in Table 4 are normalized by field automorphisms, and these stabilize the irreducible characters in the first five lines of

Table 4. Since $\mathrm{Out}(S)$ is cyclic, this already gives the required equivariance properties for B_0.

Each non-principal 3-block B of S lies in the union of Lusztig series $\mathcal{E}_3(S,s)$ of a certain semisimple $3'$-element $1 \neq s \in S$. On the other hand, it is easily seen that the characters of 3-defect zero of the corresponding normalizers of radical subgroups are naturally indexed by the same class of semisimple elements. Since in all cases the elements of $\mathcal{E}(S,s)$ form a basic set for B by [17, §4] and are invariant under field automorphisms, the action of field automorphisms is completely determined by their action on the label s of B. This gives the required equivariance in all cases. (In fact, for the blocks of types 2 and 3, as well as those of types 4 and 5 when $a \geq 2$, the situation is the same as for primes $3 \neq \ell | \Phi_4$.)

The inductive (McK)-condition for ${}^2F_4(2^{2f+1})$, $f \geq 1$, $\ell \neq 2$, was verified in [9]. For the blocks of maximal defect, this implies the (AM)-condition by [9, Thm. 6.3]. This leaves the ℓ-blocks for primes $\ell | (q^4-1)$. The 3-blocks of positive defect of G are listed in Table 3. For all of these, the ordinary Alperin-McKay conjecture was already verified in [20]. It is immediate from this to check that there exist bijections which are moreover equivariant with respect to field automorphisms. The same applies for the ℓ-blocks with non-maximal and non-trivial defect for the other primes ℓ dividing (q^4-1). As S has trivial Schur multiplier and its outer automorphism group is generated by field automorphisms, this completes the verification of (AM) for S. Then (IN) follows from this by the remarks after [22, Thm. 8.5]. □

6. Dade–Glauberman–Nagao correspondence

In this section we collect some preparatory material for the proof that alternating groups satisfy the inductive blockwise Alperin weight condition (BAW). For G a finite group and p a fixed prime, we write $\mathrm{IBr}(G)$ for the set of p-modular irreducible Brauer characters of G, and

$$\mathrm{dz}(G) := \{\chi \in \mathrm{Irr}(G) \mid \gcd(|G|/\chi(1), p) = 1\}$$

for the set of irreducible p-defect zero characters. For $\chi \in \mathrm{dz}(G)$ we let χ^0 denote its restriction to p'-classes. Recall that a p-subgroup $Q \leq G$ is called *p-radical* if $\mathrm{N}_G(Q)/Q$ has no non-trivial normal p-subgroup.

We now explain the condition needed for a finite simple group S to be (AWC)-good (see [25, §3]). Fix a covering group G of S, with cyclic center $Z = Z(G)$ of order prime to p (so G is quasi-simple and $S \cong G/Z$), and a faithful character $\lambda \in \mathrm{Irr}(Z)$. Then for every p-radical subgroup Q of G there should exist subsets $\mathrm{IBr}(G|Q,\lambda) \subseteq \mathrm{IBr}(G|\lambda)$ with the following properties:

(1) $\mathrm{IBr}(G|Q_1,\lambda) = \mathrm{IBr}(G|Q_2,\lambda)$ if and only if Q_1, Q_2 are G-conjugate,
(2) $\mathrm{IBr}(G|\lambda)$ is the disjoint union of the $\mathrm{IBr}(G|Q,\lambda)$, where Q runs over a set of representatives for the G-classes of p-radical subgroups of G, and
(3) for all $a \in A := C_{\mathrm{Aut}(G)}(Z)$ we have $\mathrm{IBr}(G|Q,\lambda)^a = \mathrm{IBr}(G|Q^a,\lambda)$.

Moreover, for all Q there should exist bijections

$$^* : \mathrm{IBr}(G|Q,\lambda) \longrightarrow \mathrm{dz}(\mathrm{N}_G(Q)/Q,\lambda),$$

such that

(4) $(\varphi^a)^* = (\varphi^*)^a$ for all $\varphi \in \mathrm{IBr}(G|Q,\lambda)$, $a \in A$.

Furthermore, for any p-radical subgroup Q of G and any $\varphi \in \mathrm{IBr}(G|Q,\lambda)$, let A_0 be the subgroup of A stabilizing Q and φ, and let $\tilde{G}$ be a group containing G as a normal subgroup such that $Z \le Z(\tilde{G})$ and the group of automorphisms of G induced by $\mathrm{N}_{\tilde{G}}(Q)$ is exactly A_0. We then require that:

(5) $C := \mathrm{C}_{\tilde{G}}(G)$ is abelian,

(6) $\mathrm{IBr}(C|\lambda)$ contains a $\tilde{G}$-invariant character γ,

(7) we have equality of 2-cocycles $[\varphi\gamma]_{\tilde{G}/CG} = [(\varphi^*)^0\gamma]_{\mathrm{N}_{\tilde{G}}(Q)/C\mathrm{N}_G(Q)}$.

We then say that (G,λ) *is (AWC)-good for p.* If this holds for all covering groups G of S, then we say that S *is (AWC)-good for p.* If $\mathrm{Out}(G)$ is cyclic and moreover the bijection * satisfies that

(8) φ lies in the induced to G of the block of φ^*

for all $\varphi \in \mathrm{IBr}(G)$, then we have that S *is (BAW)-good for p* (see [29, Lemma 6.1]).

Remark 6.1. Let's note the following special cases: if $\tilde{G}$ is the semidirect product of G with $A_0/\bar{\mathrm{N}}_G(Q)$ (where $\bar{\mathrm{N}}_G(Q)$ denotes the image of $\mathrm{N}_G(Q)$ in $G/Z(G)$), then $C = \mathrm{C}_{\tilde{G}}(G) = Z(\tilde{G}) = Z(G) = Z$ in (5) is abelian, and (6) is satisfied with $\gamma = \lambda$. If moreover $A_0/\mathrm{N}_G(Q) \cong \tilde{G}/G$ is cyclic, then both characters in (7) extend to their inertia groups, so the associated cocycles are trivial and hence equal. So (5)–(7) are automatically satisfied under these conditions.

For B a block of G we write $l(B) = |\mathrm{IBr}(B)|$ for the number of irreducible Brauer characters in B.

An important ingredient in the Navarro–Tiep reduction is the Dade–Glauberman–Nagao correspondence. We'll make use of the following consequence of it whose proof is implicit in [25, Thm. 5.1] and which was kindly communicated to us by Gabriel Navarro:

Proposition 6.2. *Let G be a finite group with a normal subgroup N of index p. There is a bijection between the set of G-invariant N-classes of p-weights of N and the set of G-classes of p-weights (Q,γ) of G with $Q \not\le N$, which sends p-weights of N in a given block b of N to p-weights of G in blocks covering b.*

Proof. Define a map Ξ from the set of G-invariant N-classes of p-weights of N to the set of G-classes of p-weights (Q,μ) of G with $Q \not\le N$ as follows: Suppose that (P,γ) is an N-weight whose N-class is G-invariant. Thus, $G = N\,\mathrm{N}_G(P)$ and γ is an $\mathrm{N}_G(P)$-invariant defect zero character of $\mathrm{N}_N(P)/P$. By the Dade–Glauberman–Nagao correspondence (see for example [25, §4]) there exists a complement Q/P of $\mathrm{N}_N(P)/P$ in $\mathrm{N}_G(P)/P$. Since $Q \cap \mathrm{N}_N(P) = P = Q \cap N$ we have $\mathrm{N}_G(Q) \le \mathrm{N}_G(P)$, and the Dade–Glauberman–Nagao correspondence gives a unique irreducible character of $\mathrm{C}_{\mathrm{N}_N(P)/P}(Q)$. Since the latter group is isomorphic to $\mathrm{N}_G(Q)/Q$, this naturally gives a defect zero character γ^* of $\mathrm{N}_G(Q)/Q$ and hence a G-weight (Q,γ^*) whose class is denoted $\Xi(P,\gamma)$.

It remains to show that this construction yields a complete set of representatives for the G-classes of weights (Q,μ) with $Q \not\le N$. First suppose that two G-weights (Q_i,γ_i^*), $i = 1,2$, obtained from N-weights (P_i,γ_i) as above are conjugate by $g \in G$. As $Q_i \cap N = P_i$ by construction, we have $P_1^g = P_2$. By the construction of γ_i^* it follows that $\gamma_2^* = (\gamma_1^*)^g = (\gamma_1^g)^*$, and then uniqueness gives that $\gamma_2 = \gamma_1^g$.

Now let (Q,μ) be a G-weight with $Q \not\leq N$, and set $P = N \cap Q$. Then $\mathrm{N}_G(Q)/Q$ is naturally isomorphic to $\mathrm{N}_N(Q)/P$, so $\mu|_{\mathrm{N}_N(Q)}$ is a Q-invariant defect zero character of $\mathrm{N}_N(Q)/P$. As $\mathrm{N}_G(Q) \leq \mathrm{N}_G(P)$ and $\mathrm{N}_N(Q)/P = \mathrm{C}_{\mathrm{N}_N(P)/P}(Q)$, by the Dade–Glauberman–Nagao correspondence $\mu|_{\mathrm{N}_N(Q)}$ equals γ^* for some Q-invariant defect zero character γ of $\mathrm{N}_N(P)/P$. So Ξ is also surjective.

For a p-weight (P,γ) of N, let b denote its block in N. By the definition of the Dade–Glauberman–Nagao correspondence the image (Q,γ^*) of (P,γ) lies in the unique block B of $N_G(P)$ that covers b (B is the Brauer correspondent of b). □

This will be used as follows:

Corollary 6.3. *Let G be a quasi-simple group with $\sigma \in \mathrm{Aut}(G)$ of prime order p generating $A := C_{\mathrm{Out}(G)}(Z(G))$. Assume the following:*

(1) The ordinary blockwise Alperin weight conjecture holds for the semidirect product $\tilde{G} = G.A$ and the prime p.

(2) The number of A-invariant characters in $\mathrm{IBr}(G)$ and of A-invariant G-classes of p-weights of G agree.

Then (G,λ) is (BAW)-good for p for all faithful $\lambda \in \mathrm{Irr}(Z(G))$.

Proof. Let w be the number of G-orbits of p-weights of G, w_1 the number of A-invariant G-orbits of p-weights, and $w_2 = w - w_1$. Similarly, let b_1 be the number of A-invariant irreducible Brauer characters of G and $b_2 = |\mathrm{IBr}(G)| - b_1$. Let $\tilde{w}$ be the number of classes of p-weights of $\tilde{G}$, $\tilde{w}_1$ the number of such for which the radical p-subgroups are not contained in G and $\tilde{w}_2 = \tilde{w} - \tilde{w}_1$. Finally, let $\tilde{b}_1$ be the number of irreducible Brauer characters of $\tilde{G}$ which remain irreducible upon restriction to G and $\tilde{b}_2 = |\mathrm{IBr}(\tilde{G})| - \tilde{b}_1$.

Clearly we have $b_1 = \tilde{b}_1$ and $b_2 = 2\tilde{b}_2$. Assumption (1) gives $\tilde{b}_1 + \tilde{b}_2 = \tilde{w}_1 + \tilde{w}_2$, while (2) says that $b_1 = w_1$. By Proposition 6.2 we also have $b_1 = \tilde{w}_1$. Finally, it is obvious that $w_2 = 2\tilde{w}_2$. In conclusion we get $w_1 = b_1$, $w_2 = b_2$, and then clearly an A-equivariant bijection exists. Conditions (5)–(7) for (AWC)-goodness are satisfied by our Remark 6.1 above.

According to Proposition 6.2 these bijections satisfy the block compatibility condition for (BAW). □

7. Blocks of alternating groups

The blockwise version of Alperin's weight conjecture for symmetric groups and odd primes p has been proved by Alperin and Fong [1]. We show how to derive from this the fact that the p-blocks of the alternating groups are (BAW)-good, thus completing the proof of Theorem 1.1.

For this we need to recall some results from [1] and from a paper of Olsson [26]. The ordinary irreducible characters of $\mathfrak{S}_n$ are naturally labelled by partitions of n, and two characters lie in the same p-block if and only if their labels have the same p-core. Thus the p-blocks of $\mathfrak{S}_n$ are naturally labelled by p-cores κ of partitions $\kappa \vdash m$ with $m \leq n$ such that $n - m \equiv 0 \pmod{p}$; we write $B(\kappa)$ for the corresponding block. The integer $w := (n-|\kappa|)/p$ is called the *weight* of $B(\kappa)$. The number of complex irreducible characters in $B(\kappa)$ is then given by

$$|\mathrm{Irr}(B(\kappa))| = k(p-1, w)$$

where $k(e,w)$ is the number of multipartitions $(\mu_1,\ldots,\mu_e)\vdash w$ of w of length e (see [27, Prop. 11.4]). In particular, a character of $\mathfrak{S}_n$ is of p-defect zero if and only if it is labelled by a p-core.

We fix a prime p. For $c\geq 1$ let A_c denote the elementary abelian p-group of order p^c in its regular permutation representation. For a sequence $\mathbf{c}=(c_1,\ldots,c_t)$ of integers, let $A_{\mathbf{c}}:=A_{c_1}\wr A_{c_2}\wr\cdots\wr A_{c_t}$ the iterated wreath product, naturally embedded as a transitive subgroup of $\mathfrak{S}_{p^d}$, where $d=c_1+\ldots+c_t$. These groups are called *p-basic subgroups.* Note that

$$\mathrm{N}_{\mathfrak{S}_{p^d}}(A_{\mathbf{c}})/A_{\mathbf{c}}\cong\mathrm{GL}_{c_1}(p)\times\cdots\times\mathrm{GL}_{c_t}(p).$$

Then the radical p-subgroups of $G=\mathfrak{S}_n$ are direct products

$$R=R_1\times\cdots\times R_s$$

with each R_i a direct product of m_i isomorphic p-basic subgroups $A_{\mathbf{c}_i}\leq\mathfrak{S}_{r_i}$ (see [1, 2A, 2B]). Furthermore,

$$\mathrm{N}_G(R)/R=\mathfrak{S}_{r_0}\times\prod_{i=1}^{s}(N(A_{\mathbf{c}_i})/A_{\mathbf{c}_i})\wr\mathfrak{S}_{m_i}$$

where $r_0=n-\sum_i m_ir_i$. (Note that for $p=2,3$ not all such subgroups are radical, but this will not matter.)

Note that by Clifford theory the p-defect zero characters of a wreath product $T\wr\mathfrak{S}_m$ are naturally parametrized by e-tuples of partitions $(\mu_1,\ldots,\mu_e)\vdash m$ consisting of p-cores μ_i, where $e=|\mathrm{dz}(T)|$ is the number of defect zero characters of T. In the situation considered above, the defect zero characters of $\mathrm{GL}_{c_i}(p)$ are the $p-1$ extensions of the Steinberg character of $\mathrm{SL}_{c_i}(p)$ to $\mathrm{GL}_{c_i}(p)$. So the weights of G belonging to the radical subgroup R above are naturally indexed by pairs (κ,λ), where $\kappa\vdash r_0$ is a p-core indexing a defect zero character of the first factor $\mathfrak{S}_{r_0}$ of $\mathrm{N}_G(R)/R$, and λ is an s-tuple of multipartitions $\mu_i=(\mu_{i1},\ldots,\mu_{i,p-1})\vdash m_i$, $1\leq i\leq s$, consisting of p-cores.

For κ a p-core, let $W(\kappa)$ denote the set of weights of $\mathfrak{S}_n$ of the form (R,φ) where $r_0=|\kappa|$ and $\varphi=(\kappa,\lambda)$ for some λ as above. It is shown in [1, (2C)] that for any block $B(\kappa)$ of $\mathfrak{S}_n$, the corresponding weights are precisely those in $W(\kappa)$ (up to conjugation), and that $l(B(\kappa))=|W(\kappa)|$, so the blockwise version of Alperin's weight conjecture holds for $\mathfrak{S}_n$.

We claim that this descends to the p-blocks of the alternating group $\mathfrak{A}_n$. Let $B=B(\kappa)$ be a p-block of $\mathfrak{S}_n$ of weight w. It is known that B covers a unique block $\tilde{B}$ of $\mathfrak{A}_n$ (see [27, Prop. 12.2]). The block B is called *self-dual* if the p-core κ is symmetric, that is, it agrees with its conjugate partition κ'. Let σ denote the sign character of $\mathfrak{S}_n$. Then $\sigma B=B$ if and only if B is self-dual.

7.1. **Blocks of $\mathfrak{A}_n$ for odd primes.** First we deal with the case that p is odd. Assume that $B=B(\kappa)$ is not self-dual, so $\kappa\neq\kappa'$. Then B and σB are disjoint, they both cover $\tilde{B}$, and by [26, Prop. 2.8] we have

$$l(\tilde{B})=l(B)=k(p-1,w).$$

On the other hand, none of the weights (R,φ) associated to B is stable under tensoring with the sign character, since $\sigma\varphi=(\kappa',\lambda')$ for a suitable λ', so the number of weights of $\mathfrak{A}_n$ lying below $W(\kappa)$, so corresponding to $\tilde{B}$, is also $k(p-1,w)$ in this case.

On the other hand, if B is symmetric, then by [26, Prop. 2.13] we have

$$l(\tilde{B}) = \begin{cases} \frac{1}{2}k(p-1,w) & w \text{ odd},\\ \frac{1}{2}\left(k(p-1,w)+3k((p-1)/2,w/2)\right) & w \text{ even}.\end{cases}$$

We claim that this is also the number of weights of $\mathfrak{A}_n$ lying below $W(\kappa)$. For this note that a weight (κ,λ) is fixed by σ if and only if $\lambda=(\mu_1,\ldots,\mu_s)$ is such that for each $\mu_i=(\mu_{i1},\ldots,\mu_{i,p-1})$ we have that $\mu_{ij}=\mu'_{i,p-j}$. Clearly this is only possible if each m_i and hence also $w=\sum_i m_i r_i$ is even. Thus any such weight is already uniquely specified by the first halfs $(\mu_{i1},\ldots,\mu_{i,e})\vdash m_i/2$ of the μ_i, where $e=(p-1)/2$. But the number of such tuples of p-cores equals the number of multipartitions $(\lambda_1,\ldots,\lambda_e)\vdash w/2$, hence is given by $k(e,w/2)=k((p-1)/2,w/2)$, by the following combinatorial result from [1, (1A)]:

Lemma 7.1. *For $d\geq 0$, $e\geq 1$, $p\geq 1$, let $I=\{(d,i)\mid d\geq 0,\ 1\leq i\leq ep^d\}$. Then for any $w\geq 0$ the number of maps*

$$I\to\{\lambda\mid\lambda\ p\text{-core}\},\qquad (d,i)\mapsto\kappa_i^d,$$

with $\sum_{d,i}p^d|\kappa_i^d|=w$ equals $k(e,w)$, the number of multipartitions $(\lambda_1,\ldots,\lambda_e)\vdash w$.

All other $|W(\kappa)|-k(e,w/2)$ weights in $W(\kappa)$ come in pairs interchanged by σ. The claim then follows as

$$\begin{aligned}\frac{1}{2}(|W(\kappa)|-k(e,w/2))+2k(e,w/2) &= \frac{1}{2}(|W(\kappa)|+3k(e,w/2))\\ &= \frac{1}{2}(k(p-1,w)+3k(e,w/2)).\end{aligned}$$

7.2. **Blocks of $\mathfrak{A}_n$ for $p=2$.** Now let $p=2$. Note that the 2-weights for $\mathfrak{A}_n$ have not been determined in [24]. In order to apply the criterion in Corollary 6.3 we need to determine the radical 2-subgroups of $\mathfrak{S}_n$ contained inside $\mathfrak{A}_n$.

Lemma 7.2. *Let $R=R_1\times\cdots\times R_s$ be a radical 2-subgroup of $\mathfrak{S}_n$, with R_i a direct product of m_i isomorphic 2-basic subgroups $A_{\mathbf{c}_i}\leq\mathfrak{S}_{r_i}$. Then $R\leq\mathfrak{A}_n$ if and only if each $A_{\mathbf{c}}$ is of the form $A_{\mathbf{c}}=A_{c_1}\wr\cdots\wr A_{c_t}$ with $c_1\geq 2$.*

Proof. By construction, $R\leq\mathfrak{A}_n$ if and only if each $A_{\mathbf{c}_i}\leq\mathfrak{A}_{r_i}$. Now $A_c\leq\mathfrak{A}_{2^c}$ if and only if $c\geq 2$, and furthermore $A\leq\mathfrak{A}_m$ with m even implies $A\wr B\leq\mathfrak{A}_{km}$ for all $B\leq\mathfrak{S}_k$, so the claim follows. □

For a positive integer m, let $\pi(m)=k(1,m)$ denote the number of partitions of m.

Proposition 7.3. *Let $\kappa\vdash n-2w$ be a 2-core. The number of 2-weights (R,φ) of $\mathfrak{S}_n$ in $W(\kappa)$ with $R\leq\mathfrak{A}_n$ equals $\pi(w/2)$ if w is even, 0 else.*

Proof. For $d\geq 1$ let n_d denote the number of basic subgroups of degree 2^d, that is, the number of tuples $\mathbf{c}=(c_1,\ldots,c_t)$ of positive integers with $t\geq 1$, $c_1+\ldots+c_t=d$. Clearly, the set of such tuples for a fixed t is in bijection with the set of subsets of $\{1,\ldots,d-1\}$ of size $t-1$, so

$$n_d=\sum_{t=1}^{d}\binom{d-1}{t-1}=2^{d-1}.$$

Let m_d be the number of such tuples $\mathbf{c}$ for which $A_{\mathbf{c}} \leq \mathfrak{A}_{2^d}$. By Lemma 7.2 this is the case if and only if $c_1 \geq 2$. Again, for fixed t the set of tuples

$$\{\mathbf{c} = (c_1, \ldots, c_t) \mid c_1 \geq 2,\ c_1 + \ldots + c_t = d\}$$

is in bijection with the set of subsets of $\{1, \ldots, d-2\}$ of size $t-1$, so

$$m_d = \sum_{t=1}^{d-1} \binom{d-2}{t-1} = 2^{d-2} \quad \text{for } d \geq 2,$$

and $m_1 = 0$.

Thus, by [1, (2C)], the 2-weights (R, φ) of $\mathfrak{S}_n$ of weight w with $R \leq \mathfrak{A}_n$ (up to conjugation) are parametrized by the set of tuples $(K_1, \ldots, K_n)$ where each K_d is a tuple $(\kappa_1^d, \ldots, \kappa_{m_d}^d)$ of 2-cores with

$$\sum_{d \geq 0} \sum_{i=1}^{m_d} 2^{d-1} |\kappa_i^d| = w.$$

As we have $m_1 = 0$, such a tuple can only exist if w is even. In that case, according to Lemma 7.1, the number of such tuples equals the number of partitions of $w/2$. □

Corollary 7.4. $(\mathfrak{A}_n, 1)$ *is (BAW)-good for* $p = 2$.

Proof. Every 2-core is symmetric, so by [27, Prop. 12.2 and subsequent remarks] any 2-block B of $\mathfrak{S}_n$ covers a unique block $\tilde{B}$ of $\mathfrak{A}_n$, and it is the only block with this property. Now first assume that B has odd weight w. Then by [26, Prop. 2.17] we have

$$l(\tilde{B}) = l(B) = k(1, w) = \pi(w),$$

the number of partitions of w. In particular, any irreducible Brauer character of $\tilde{B}$ extends to $\mathfrak{S}_n$, so is $\mathfrak{S}_n$-invariant. On the other hand, we just showed that all 2-weights of $\mathfrak{A}_n$ correspond to a unique 2-weight of $\mathfrak{S}_n$, so any bijection is $\mathfrak{S}_n$-equivariant.

Secondly, if w is even then by [26, Prop. 2.17] we have

$$l(\tilde{B}) = l(B) + k(1, w/2) = \pi(w) + \pi(w/2).$$

In particular, $\pi(w) - \pi(w/2)$ irreducible Brauer characters of $\tilde{B}$ extend to $\mathfrak{S}_n$, while $2\pi(w/2)$ are not invariant. By Proposition 7.3 the same holds for the 2-weights of $\mathfrak{A}_n$. Then (BAW)-goodness for the prime 2 follows from Corollary 6.3. □

7.3. **Faithful blocks of $\hat{\mathfrak{A}}_n$ for odd primes.** The p-weights for faithful blocks of $2.\mathfrak{A}_n$ and $2.\mathfrak{S}_n$ were determined in [24, §5] for all odd primes p, and the results show that the number of $2.\mathfrak{S}_n$-invariant Brauer characters and p-weights is the same for any faithful p-block of $2.\mathfrak{A}_n$, so [29, Lemma 6.1] is applicable. This completes the proof of Theorem 1.1 for $n \neq 6, 7$.

7.4. **The groups $\mathfrak{A}_6$ and $\mathfrak{A}_7$.** In this section we handle the exceptional automorphism group of $\mathfrak{A}_6$ and the exceptional covering groups of $\mathfrak{A}_6$ and $\mathfrak{A}_7$:

Proposition 7.5. *The alternating groups* $\mathfrak{A}_6$ *and* $\mathfrak{A}_7$ *are (BAW)-good for all primes* p.

Proof. Let first $S = \mathfrak{A}_6$. Here $\mathrm{Out}(S) = C_2 \times C_2$ and $M(S) = C_6$. Since $\mathfrak{A}_6 \cong \mathrm{L}_2(9)$ is a group of Lie type in characteristic 3, (BAW) for $p = 3$ was proved in [29, Thm. B]. Next, S is (AWC)-good for $p = 2$ by [25, Prop. 6.1]. Moreover, it is easily seen that the bijection constructed in loc. cit. satisfies condition 4.1(ii)(3) in [29]. For the trivial character the necessary additional conditions for (BAW) are trivially satisfied. In all other cases, the inertia group in the group of outer automorphisms is cyclic, whence (BAW)-goodness follows by [29, Lemma 6.1].

Now let $p = 5$. The only non-trivial radical 5-subgroup (up to conjugation) is the Sylow 5-subgroup P, with $\mathrm{N}_{2.\mathfrak{A}_6}(P)/P \cong C_{12}$. Note that by [25, Cor. 7.2] and [29, Prop. 6.2] we need not consider the 3- or 6-fold coverings of S since these are only centralized by a cyclic subgroup of $\mathrm{Out}(S)$, and the Sylow 5-subgroup of S is cyclic. The relevant data for $2.\mathfrak{A}_6$ are collected in Table 5; here the first two lines correspond to characters of S, the last one to faithful characters of $2.S$. The notation for the outer automorphisms $2_1, 2_2, 2_3$ is taken from [10]. All characters extend to their inertia groups in $\mathrm{Out}(G)$. All characters of S of positive defect lie in the principal 5-block, and similarly all faithful characters of $2.S$ of positive defect lie in a unique 5-block. The same holds for the corresponding characters of $\mathrm{N}_S(P)$, respectively $\mathrm{N}_{2.S}(P)$.

TABLE 5. (BAW)-bijection in $G = 2.\mathfrak{A}_6$ for $p = 5$

P	$\mathrm{N}_G(P)/P$	$\varphi(1)$	$\mathrm{IBr}(G)$	2_1	2_2	2_3	2^2
C_5	C_2	1	1	2	2	2	4
		1	8	2	2	2	4
	C_4	$1, 1$	$4, 4$	4	1	1	2

Next let $S = \mathfrak{A}_7$, with $\mathrm{Out}(S) = C_2$ and $M(S) = C_6$. Then S is (BAW)-good for $p = 5, 7$ by [25, Cor. 7.2] and [29, Prop. 6.2]. Also, all p-blocks of $2.\mathfrak{A}_7$ satisfy the necessary conditions by our general results on alternating groups and their twofold covering above. Thus we only need to consider the faithful blocks of $G = 3.\mathfrak{A}_7$ for $p = 2$. The outer automorphism acts non-trivially on $Z(G)$, so we need not worry about equivariance here. The radical 2-subgroups P in G are (isomorphic) preimages of those in $\mathfrak{A}_7$ described above, with normalizer quotients $\mathrm{N}_G(P)/P$ and faithful defect zero characters as given in Table 6. (We have again omitted the trivial group $P = 1$ and the corresponding defect zero characters of G.) On the other hand, the faithful 2-Brauer characters of G have degrees $\{6, 6, 15, 15, 24, 24, 24, 24\}$, see [10], where the last four are of 2-defect 0. The claim follows. □

TABLE 6. (BAW)-bijection in $G = 3.\mathfrak{A}_7$ for $p = 2$

P	$\mathrm{N}_G(P)/P$	$\varphi(1)$	$\mathrm{IBr}(G)$
$C_2 \times C_2$	$C_3 \times \mathfrak{S}_3$	$2, 2$	$6, 6$
D_8	C_3	$1, 1$	$15, 15$

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