

PROJECTIVE INDECOMPOSABLE PERMUTATION MODULES

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To Pham Huu Tiep on the occasion of his 60th birthday

ABSTRACT. We investigate finite non-Abelian simple groups G for which the projective cover of the trivial module coincides with the permutation module on a subgroup and classify all cases unless G is of Lie type in defining characteristic.

1. INTRODUCTION

Let G be a finite group, p a prime and k an algebraically closed field of characteristic p . We are interested in the situation when the projective cover Φ_{1_G} of the trivial kG -module k is the permutation module on a subgroup H of G . We then say that G has property (I_p) with respect to the subgroup H . Note that in this case H is necessarily of p' -order. Thus, G has property (I_p) if there is a p' -subgroup H of G such that the endomorphism ring of the permutation module on H is a local ring. Of course, G has property (I_p) for every prime p not dividing its order (with respect to $H = G$), so the interesting case is for the prime divisors of $|G|$.

It is clear that if G has a Hall p' -subgroup H , then G has property (I_p) with respect to H . Note that the example $p = 7$ and $G \cong \mathrm{L}_2(7)$ shows that G may have property (I_p) with respect to two non-conjugate subgroups. We will also see examples showing that G may have property (I_p) with respect to a p' -subgroup H which is not a Hall p' -subgroup.

We investigate finite non-Abelian simple groups G enjoying property (I_p) for some prime p . This turns out to be quite a rare phenomenon.

Theorem 1. *Let G be a non-Abelian finite simple group with property (I_p) . Then either the pair (G, p) is classified below, or G is of Lie type in characteristic p .*

In all cases coming up in our classification in the subsequent sections either $p \leq 3$, or Sylow p -subgroups of G are of a very restricted form: they are either cyclic, or Abelian of rank 2, or an extra-special group 7^{1+2} .

The open cases are proof of the little that is known (to the authors) about decomposition numbers of groups of Lie type in defining characteristic. For the exceptional groups of small rank we do obtain complete results in Section 6, though.

Date: August 23, 2023.

2010 Mathematics Subject Classification. 20C15, 20C20, 20C30, 20C33.

Key words and phrases. permutation character, projective cover, 1-PIM.

The first author gratefully acknowledges support by the Deutsche Forschungsgemeinschaft – Project-ID 286237555– TRR 195. The authors would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme *Groups, Representations and Applications* when part of the work on this paper was undertaken. This work was supported by: EPSRC grant number EP/R014604/1.

As pointed out to us by Nick Kuhn the determination of all groups $\mathrm{GL}_n(p)$ satisfying (I_p) was a crucial step in the paper [17] on the indecomposable summands of $H^*((\mathbb{Z}/p\mathbb{Z})^n; \mathbb{F}_p)$ over the mod p Steenrod algebra which are unstable algebras.

After collecting some elementary observations in Section 2 we give the proof of Theorem 1 by dealing with the various classes of finite non-Abelian simple groups according to the classification in Sections 3–6. Our approach relies, among other things, on detailed information on their maximal subgroups of large order.

Acknowledgement: We thank Nick Kuhn for bringing [17] to our attention after seeing a first version of our paper. We are grateful to Jürgen Müller for proving that J_4 for $p = 11$ is not an example, and to the referees for their very careful reading of a previous version.

2. GENERAL CONSIDERATIONS

We collect some elementary observations on projective indecomposable permutation modules. Willems [36] has studied the 1-PIM for composite groups and obtained several reduction results which show that the main question is for non-Abelian simple groups; see in particular Lemma 2.5 below.

Lemma 2.1. *Let M be a kG -module and assume G satisfies (I_p) with respect to $H \leq G$. Then $\dim M^H \leq \dim M^L$ for any p' -subgroup $L \leq G$ and $|H|$ is maximal among all p' -subgroups of G .*

Proof. For any p' -subgroup $L \leq G$, $\mathrm{Ind}_L^G(k)$ is projective and so contains $\Phi_1 = \mathrm{Ind}_H^G(k)$ as a direct summand, thus

$$\begin{aligned} \dim M^L &= \dim \mathrm{Hom}_{kL}(k, M) = \dim \mathrm{Hom}_{kG}(\mathrm{Ind}_L^G(k), M) \\ &\geq \dim \mathrm{Hom}_{kG}(\mathrm{Ind}_H^G(k), M) = \dim M^H \end{aligned}$$

by Frobenius reciprocity. Observe that H has to be of p' -order as $\mathrm{Ind}_H^G(k)$ is projective by assumption and hence has degree divisible by $|G|_p$. \square

This means, in particular, that when G satisfies (I_p) with respect to H then on every finite G -set, it is true that H has fewest orbits among p' -subgroups of G . This was a main ingredient for the results obtained in [17].

Lemma 2.2. *Assume that G acts 2-transitively on the set of cosets of $H \leq G$. Then G has property (I_p) with respect to H for any prime p dividing G but not dividing $|H|$.*

Proof. If G acts 2-transitively on the cosets of H then $\mathrm{Ind}_H^G(1_H)$ has exactly two ordinary irreducible constituents. If H is a p' -group then this module is projective, and indecomposable since the trivial kG -module is not projective when p divides $|G|$. \square

The following is a kind of weak converse:

Lemma 2.3. *Let G be a finite group with cyclic Sylow p -subgroups. Assume that the trivial character is not connected to the exceptional node on the p -Brauer tree of G . If G has property (I_p) with respect to H then G acts 2-transitively on the set of cosets of H .*

Proof. By the well-known theory of blocks with cyclic defect, the assumption on the Brauer tree implies that the projective cover Φ_{1_G} of the trivial kG -module has just two ordinary constituents. As $\Phi_{1_G} = \text{Ind}_H^G(1_H)$ by assumption, the permutation character of G on the cosets of H has just two constituents, and so the action is 2-transitive. \square

We now note two results that are important for induction purposes, the first of which is clear:

Lemma 2.4. *Let $H \leq L \leq G$. If G has property (I_p) with respect to H then L has property (I_p) with respect to H .*

Now let $N \trianglelefteq G$ be a normal subgroup. Then by [36, Lemma 2.6] we have

$$\dim \Phi_{1_G} = \dim \Phi_{1_{G/N}} \dim \Phi_{1_N}. \quad (*)$$

Lemma 2.5. *Assume G has property (I_p) with respect to $H \leq G$. Then for any normal subgroup $N \trianglelefteq G$, G/N has property (I_p) with respect to HN/N and N has property (I_p) with respect to $N \cap H$.*

Proof. Clearly, both $HN/N \cong H/(H \cap N)$ and $H \cap N$ are p' -groups, and by $(*)$ we have

$$\dim \Phi_{1_{G/N}} \dim \Phi_{1_N} = \dim \Phi_{1_G} = |G : H| = |G/N : HN/N| |N : H \cap N|.$$

Since $\dim \Phi_{1_{G/N}} \leq |G/N : HN/N|$ and $\dim \Phi_{1_N} \leq |N : H \cap N|$ the claim follows. \square

While the converse holds, for example, for p -solvable groups, since for these, Hall p' -subgroups are the unique conjugacy class of maximal p' -subgroups, it is not true in general: $G = \text{L}_2(7)$ has property (I_7) with respect to a subgroup $H = \mathfrak{S}_4$, but $\hat{G} = \text{PGL}_2(7)$ does not satisfy (I_7) . Similarly, $G = \mathfrak{A}_7$ has property (I_5) with respect to $H = \text{L}_2(7)$, but $\hat{G} = \mathfrak{S}_7$ does not satisfy (I_5) . In both cases, the ambient p' -subgroup H of G is not stable under the outer automorphism of G induced by \hat{G} .

Corollary 2.6. *Assume G has property (I_p) . Then any non-Abelian simple composition factor of G satisfies (I_p) .*

Let us remark, though, that not every subgroup of a group satisfying (I_p) also does: while $G = \text{L}_3(4)$ has property (I_3) , its maximal subgroup \mathfrak{A}_6 does not. The next result is also mentioned and used in [17, 2.4].

Lemma 2.7. *If G satisfies (I_p) with respect to H , then no non-trivial simple kG -module has H -fixed points.*

Proof. Assume $\text{Ind}_H^G(k) = \Phi_{1_G}$. If H has non-zero fixed points on the simple kG -module S , then $\text{Hom}_{kH}(k, \text{Res}_H^G(S)) \neq 0$, so $\text{Hom}_{kG}(\text{Ind}_H^G(k), S) \neq 0$ by Frobenius reciprocity. Since the head of Φ_{1_G} is simple, and equal to the trivial kG -module k , this implies $S \cong k$. \square

We have the following consequence for the number $l(G)$ of irreducible p -Brauer characters of a group G satisfying (I_p) :

Lemma 2.8. *Let G be a finite group and p a prime. Assume G has property (I_p) with respect to $H \leq G$. Then $l(G) \leq |H|$.*

Proof. We have $|G| = \sum_S \dim S \cdot \dim P_S$, where the sum runs over the $l(G)$ isomorphism classes of simple kG -modules and P_S is the projective cover of S . But notice that $S^* \otimes P_S$ has the projective cover of the trivial module as a summand.

Hence we have $|G| \geq l(G) \dim \Phi_{1G}$. If Φ_{1G} is the permutation module on the cosets of H , then we obtain $|H| \dim \Phi_{1G} = |G| \geq l(G) \dim \Phi_{1G}$, so that $l(G) \leq |H|$, as claimed. \square

In particular the proof shows that assuming G satisfies (I_p) with respect to $H \leq G$, $|G : H| \leq \chi(1)^2$ for every $\chi \in \text{Irr}(G)$ of p -defect zero.

3. ALTERNATING GROUPS

Theorem 3.1. *The alternating group \mathfrak{A}_n , $n \geq 5$, has property (I_p) for $p \leq n$ if and only if we are in one of the cases of Table 1.*

TABLE 1. Induced 1-PIMs for alternating groups

G	H	p	$\dim \Phi_{1G}$
\mathfrak{A}_n	\mathfrak{A}_{n-1}	n	n
\mathfrak{A}_5	C_5	2	12
\mathfrak{A}_5	D_5	3	6
\mathfrak{A}_6	C_3^2	2	40
\mathfrak{A}_6	$3^2.4$	5	10
\mathfrak{A}_7	$L_3(2)$	5	15
\mathfrak{A}_8	$2^3.L_3(2)$	5	15

All entries in the table are indeed examples: a Sylow p -subgroup of \mathfrak{A}_p is cyclic, and the action on \mathfrak{A}_{p-1} is 2-transitive, this gives the infinite series. The additional examples for $n = 5, 6, 7, 8$ can easily be checked from the decomposition matrices in GAP [12]. The proof of the converse proceeds by considering the various types of subgroups H of \mathfrak{A}_n .

Proposition 3.2. *Assume that \mathfrak{A}_n , $n \geq 5$, is a minimal counter-example to Theorem 3.1 with respect to H . Then H is a transitive subgroup; in particular, $p \nmid n$.*

Proof. Let n be minimal such that $G = \mathfrak{A}_n$ satisfies (I_p) with respect to $H \leq G$ not occurring in the conclusion and assume H is intransitive. Let $M < \mathfrak{A}_n$ be a maximal intransitive subgroup containing H . Thus $M = (\mathfrak{S}_k \times \mathfrak{S}_{n-k}) \cap \mathfrak{A}_n$ for some $1 \leq k \leq n-1$. If \mathfrak{A}_n has property (I_p) then by Lemma 2.4 and Corollary 2.6 so do \mathfrak{A}_k and \mathfrak{A}_{n-k} . If $p > 5$ then by induction we must have $k, n-k \leq p \leq n$. In fact, by applying Lemma 2.3 either $n = p$ or $k = n-k = p$. The first case appears in the conclusion of Theorem 3.1, while in the second case the permutation character of \mathfrak{A}_{2p} on M contains characters from non-principal p -blocks, for example the restriction to \mathfrak{A}_{2p} of the character labelled by $(2p-1, 1)$, so this does not occur.

If $p = 5$ then we need to discuss $k, n-k \leq 8$, if $p = 3$, then $k, n-k \leq 5$, and if $p = 2$ then $k, n-k \leq 6$. These cases can be settled using GAP. \square

Proposition 3.3. *Assume that \mathfrak{A}_n , $n \geq 5$, is a minimal counter-example to Theorem 3.1 with respect to H . Then H is a primitive subgroup.*

Proof. Let n be minimal such that $G = \mathfrak{A}_n$ satisfies (I_p) with respect to $H \leq G$ not occurring in the conclusion of Theorem 3.1. By Proposition 3.2, H is transitive and $p \nmid n$. Assume H is imprimitive and let $M < \mathfrak{A}_n$ be a maximal imprimitive overgroup. Then $M = \mathfrak{S}_a \wr \mathfrak{S}_b \cap \mathfrak{A}_n$ with $ab = n$. By Corollary 2.6 and minimality we have $a, b < p$, or $p \leq 5$. Then, $a, b > 2$ as otherwise Sylow p -subgroups of G are cyclic and we may apply Lemma 2.3. Now, if $p = 2$ then $a, b \in \{3, 5\}$. The cases $n = 9, 15$ can be excluded with GAP, and for $n = 25$ the maximal p' -subgroup $H = C_5 \wr C_5$ of $M = \mathfrak{S}_5 \wr \mathfrak{S}_5 \cap \mathfrak{A}_{25}$ has smaller order than that of a Sylow 3-subgroup of \mathfrak{A}_{25} , so M cannot contain a relevant subgroup. If $p = 3$ then $a, b \in \{2, 4, 5\}$. Again, the cases $n \leq 16$ are settled using the known tables. When $n = 20$ a subgroup $F_{20} \wr D_8 \cap \mathfrak{A}_{20}$, with the Frobenius group F_{20} of order 20, has larger order than the largest 3'-subgroup of $\mathfrak{S}_4 \wr \mathfrak{S}_5$; when $n = 25$ again a Sylow 2-subgroup has larger order than the p' -subgroup $D_5 \wr D_5$ of $\mathfrak{A}_5 \wr \mathfrak{A}_5$.

If $p = 5$ then $a, b \in \{2, 3, 4, 6, 7, 8\}$, so (using GAP)

$$n \in \{21, 24, 28, 32, 36, 42, 48, 49, 56, 64\}.$$

Let $r = 19, 23, 23, 31, 31, 41, 47, 47, 53, 61$ in the respective cases. Then M is an r' -subgroup of \mathfrak{S}_n , so the permutation character on M is r -projective and hence contains the character connected to the trivial character on the r -Brauer tree. This has label

$$(18, 3), (22, 2), (22, 6), (30, 2), (30, 6), (40, 2), (46, 2), (46, 3), (52, 4), (60, 4)$$

respectively. Since none of these lies in the principal 5-block, no new cases arise.

We may now assume $2 < a, b < p$, so $n = ab \leq (p-1)^2$. Then [31, Thm 2.6 and Cor. 6.4] exhibits a constituent labelled (a^b) , not lying in the principal p -block of \mathfrak{A}_n unless $a = p-1$, so $n = b(p-1)$ with $2 < b < p$. Let r be a prime between $n/2 + 1$ and n . Then Sylow r -subgroups of \mathfrak{S}_n are cyclic. Arguing as above, the permutation character on M must contain the character connected to the trivial character on the r -Brauer tree, labelled $(r-1, n-r+1)$. This lies in the principal p -block only when $n-r+1 \equiv 0, 1 \pmod{p}$, so $b+r \equiv 0, 1 \pmod{p}$. Thus, less than b possible odd values for r are excluded. Now by [32, Cor. 3] the number of primes between x and $2x$ is at least $3x/(5 \log x)$ for $x \geq 21$. This shows there are at least b distinct primes r in our range as soon as $p \geq 13$. The smaller values of p are readily checked. \square

Proof of Theorem 3.1. Let n be minimal such that \mathfrak{A}_n is a counter-example to the theorem, with respect to the p' -subgroup H . By using the tables in GAP we may assume $n > 16$. By Propositions 3.2 and 3.3, H is a transitive and primitive subgroup. Hence, for $n > 24$ we have $|H| < 2^n$ by a result of Maróti [28, Cor.1.2]. But for $n \geq 7$ the subgroup $\mathfrak{S}_4^a \mathfrak{S}_b \cap \mathfrak{A}_n$, where $n = 4a + b$ with $0 \leq b < 4$, has order larger than 2^n . This contradicts our assumption on H , by Lemma 2.1, when $p \geq 5$ and $n > 24$. The primitive p' -subgroups of \mathfrak{A}_n , $n \leq 24$, are well known and easily seen to be too small as well.

We are left with the case $p \in \{2, 3\}$. For $p = 3$ consider suitable direct products of wreath products of the subgroup of order 20 inside \mathfrak{S}_5 . This has order larger than $n^{\sqrt{n}}$ for $n \geq 35$, and hence larger than any primitive non-3-transitive subgroup of \mathfrak{A}_n , by [28, Cor. 1.1(i)]. Note that 3-transitive groups are not 3'-groups. For $n \leq 34$ the primitive groups are available in GAP and have order smaller than that of a Sylow 2-subgroup of \mathfrak{A}_n .

For $p = 2$ the order of a Sylow 3-subgroup of \mathfrak{A}_n is larger than $n^{\sqrt{n}}$ for $n \geq 60$, so by the above cited result we only need to worry about $n \leq 59$. The odd-order (solvable) primitive

subgroups of these alternating groups have smaller order than a Sylow 3-subgroup of \mathfrak{A}_n by GAP. \square

4. SPORADIC GROUPS

Theorem 4.1. *Let G be a sporadic simple group and p a prime dividing $|G|$. Then G has property (I_p) with respect to $H \leq G$ if and only if (G, H, p) are as in Table 2.*

TABLE 2. Induced 1-PIMs for sporadic groups

G	H	p	$\dim \Phi_{1G}$
M_{11}	M_{10}	11	11
M_{22}	$L_3(4)$	11	22
M_{23}	M_{22}	23	23
HS	$U_3(5)$	11	176
Co_3	$McL.2$	23	276
J_2	$U_3(3)$	5	100
He	$S_4(4).2$	7	2058

Proof. The decomposition matrices of sporadic groups are completely known up to the Harada–Norton group HN and contained in [12]. From this the claim can be checked easily for those groups by first discarding those cases when $\dim \Phi_{1G}$ does not divide $|G|$, and in the few remaining cases, checking whether a subgroup of the required index exists. Five examples occur for cyclic Sylow p -subgroups, as described by Lemma 2.2, the other two have non-cyclic, Abelian Sylow p -subgroups.

For the larger groups with cyclic Sylow p -subgroups the real stem of the Brauer tree of the principal p -block is known [20]. The trivial character is never connected to the exceptional node, and since these groups do not possess 2-transitive permutation representations by [5, Thm 5.3], there are no further examples with cyclic Sylow p -subgroups by Lemma 2.3.

The permutation characters of the large maximal subgroups U of the remaining sporadic groups G are available in GAP. Removing those which involve characters from non-principal blocks, we are left with a small list of possible cases. For example, for $G = Ly$ we could have H contained in one of the two largest maximal subgroups, $G_2(5)$ or $3.McL.2$. But neither of these has property (I_p) , so nor does G by Lemma 2.4. With Lemma 2.1 this also shows in most cases that smaller maximal subgroups cannot contain candidate subgroups H . The cases that can not directly be ruled out like this or with the observation after Lemma 2.8 are: J_4 at $p = 11$ with U one of $2^{13}.3.M_{22}.2$ or $2^{10}.L_5(2)$, $G = B$ at $p = 2, 3$ with $U = 2.^2E_6(2).2$, and $G = M$ at $p = 2, 3$ with $U = 2.B$.

Arguing for $G := {}^2E_6(2)$ in the same way as we did for the sporadic groups we see that if it has property (I_p) , then either $p = 2$, or $p = 3$ with $H \leq U := 2^{1+20}.U_6(2)$. Now for $p = 3$ the 1-PIM $\bar{\Phi}_1$ of $\bar{U} := U_6(2)$ is known [12], and $\text{Ind}_U^G(\Phi_{1U}) = \text{Ind}_U^G(\text{Infl}_U^{\bar{U}}(\bar{\Phi}_1))$ is just Harish-Chandra induction of $\bar{\Phi}_1$, hence can be computed explicitly. It transpires that this contains the unipotent constituent labelled $\phi_{16,5}$ which does not lie in the principal 3-block [7]. So this does not yield an example for ${}^2E_6(2)$ at $p = 3$. For $p = 2$, using Lemma 2.5 and the structure information in GAP we can see that the only maximal subgroups of

G possibly containing a relevant $2'$ -subgroup H are the parabolic subgroup with Levi factor $\mathfrak{A}_5 \times \mathrm{L}_3(2)$, and two 3-local subgroups. The 3-local subgroup U normalising a 3C-element contains a subgroup of order 3^9 , larger than the $2'$ -part of either of the other two candidate subgroups. Hence we must have $H \leq U$, up to conjugation. But the permutation character of G on U contains characters not in the principal 2-block. So $p = 2$ is not possible for ${}^2E_6(2)$ either, and hence we also do not obtain examples for B or M at $p = 2$ or $p = 3$ by Lemma 2.4.

For J_4 the $11'$ -part of the order of the maximal subgroup $2^{13}.3.M_{22}.2$ is smaller than the order of the maximal subgroup $2^{10}.\mathrm{L}_5(2)$, which is not divisible by 11, so if J_4 is an example then with respect to the latter subgroup. Jürgen Müller was able to compute the 11-modular decomposition matrix of the 27-dimensional endomorphism ring of this permutation module and thus show that it has four indecomposable summands, see [30] for a description of the computational methods he employed. \square

5. GROUPS OF LIE TYPE IN NON-DEFINING CHARACTERISTIC

We now consider the simple groups of Lie type G in characteristic r for primes $p \neq r$ dividing $|G|$. Our general strategy is as follows. By Lemma 2.1 any admissible p' -subgroup H of G has order at least that of a Sylow r -subgroup of G . Most maximal subgroups of G with at least that order have been determined by Liebeck and Liebeck–Saxl. According to [25, Thm] and [24, Thm], these are either maximal parabolic subgroups, or some narrow class of subsystem subgroups. Let's first discuss the former. There are two main arguments:

- (1) The permutation characters on parabolic subgroups $P < G$ are known by Howlett–Lehrer theory to decompose as the corresponding characters in the associated Weyl groups. All of their constituents are unipotent. From the known block distribution of unipotent characters [2] we can determine whether some constituents of $\mathrm{Ind}_P^G(1_P)$ do not lie in the principal p -block of G . In that case, P cannot contain a candidate subgroup H .
- (2) Let $P = UL$ be the Levi decomposition of a parabolic subgroup P of G , with maximal normal unipotent subgroup U . If H is contained in P , then P has property (I_p) by Lemma 2.4, and so has $[L, L]/Z([L, L])$ by Corollary 2.6. At least if $q > 3$ this is a product of simple groups of Lie type of smaller rank, for which we know about validity of (I_p) by induction.

As for the exceptions in the Liebeck and Liebeck–Saxl theorems, these are either for specific, small values of q , in which case decomposition matrices in the GAP-library [12] can be used. For the remaining maximal subgroups M , we try to either exhibit a p' -order parabolic subgroup of strictly larger order, or at least of larger order than that of any p' -subgroup of M .

Let us note the following general result for certain primes:

Proposition 5.1. *Let $G = G(q)$ be simple of Lie type, not a Suzuki or Ree group. Let $p|(q-1)$ but prime to the order of the Weyl group of G . If G has property (I_p) with respect to $H \leq G$, then H is G -irreducible, that is, it is not contained in any proper parabolic subgroup of G .*

Proof. Under our assumptions on p , by [4, Prop. 8.11] the decomposition matrix of the principal p -block of G is lower triangular with the only unipotent character involved in Φ_{1_G} being the principal character. On the other hand, all constituents of the permutation character of G on a parabolic subgroup P are unipotent, so if P is proper, $\text{Ind}_P^G(1_P)$ contains non-trivial unipotent characters. Hence G cannot have property (I_p) with respect to any subgroup of a proper parabolic subgroup. \square

Example 5.2. The conclusion of Proposition 5.1 can fail when p divides the order of the Weyl group. For example, $G = L_3(4)$ has property (I_3) with respect to a subgroup $2^4.D_5$, and the latter is contained in a proper parabolic subgroup by Borel–Tits [27, Thm 26.5].

5.1. The linear groups. Our induction base is the following result, which also covers the defining characteristic (leading to items (4)–(6)):

Proposition 5.3. *Let $G = L_2(q)$, $q \geq 7$. Then G satisfies (I_p) for p dividing $|G|$ if and only if one of*

- (1) $2 < p|(q+1)$, $H = B$, $\dim \Phi_{1_G} = q+1$;
- (2) $q = 2^f$, $p = 2^f - 1$ a Mersenne prime, $H = D_{2(q+1)}$, $\dim \Phi_{1_G} = q(q-1)/2$;
- (3) $p = 2$, q is odd, $H = O^2(B)$, $\dim \Phi_{1_G} = (q-1)_2(q+1)/2$;
- (4) $G = L_2(7)$, $H = \mathfrak{S}_4$, $p = 7 = \dim \Phi_{1_G}$;
- (5) $G = L_2(11)$, $H = \mathfrak{A}_5$, $p = 11 = \dim \Phi_{1_G}$; or
- (6) $p = 2$, q is even, $H = C_{q+1}$, $\dim \Phi_{1_G} = q(q-1)$,

where $B < G$ is a Borel subgroup.

Proof. First assume that p is odd. If $p|(q+1)$ the 2-transitive permutation representation on a Borel subgroup gives (1) by Lemma 2.2. If $p|(q-1)$ the Brauer tree in [3] shows $\dim \Phi_{1_G} = (q-1)u(q+1-u)/2$, where $u = (q-1)_{p'}$. This divides $|G|$ only when $u = 1$, so q is even and $q-1$ is a p -power. Thus, by Catalan’s conjecture q is a Mersenne prime as in (2). For $p|q$ we have $\dim \Phi_{1_G} = (2^f - 1)q$, where $q = p^f$, see [3, Hauptsatz 9.4]. The p' -subgroups H of G of largest order the normalisers of non-split maximal tori, have order $q+1$, or $q < 60$ and H is one of \mathfrak{A}_4 , \mathfrak{S}_4 or \mathfrak{A}_5 (see e.g. [1, Tab. 8.7]). In the first case, $|G : H|$ is too large, while the last three cases can be checked to only lead to (4) and (5).

Now assume $p = 2$. If $q \equiv 3 \pmod{4}$ then a Borel subgroup B of order $q(q-1)/2$ has odd order, and the permutation character for the 2-transitive action on its cosets is Φ_{1_G} . If $q \equiv 1 \pmod{4}$, the decomposition matrix in [3, VIII(a)] shows that $\dim \Phi_{1_G} = (q+1)(q-1)_2/2 = |G : O^2(B)|$ and we obtain (3). Part (6) follows from the decomposition numbers in [3, Hauptsätze 7.5 and 7.9]. \square

We next extract the necessary information on large subgroups from [24].

Proposition 5.4. *Let M be a maximal subgroup of $L_n(q)$, $n \geq 3$, of order at least $q^{n(n-1)/2}$. Then M is the image in $L_n(q)$ of the intersection with $\text{SL}_n(q)$ of one of the following subgroups of $\text{GL}_n(q)$:*

- (1) a maximal parabolic subgroup;
- (2) $\text{GL}_{n/2}(q) \wr \mathfrak{S}_2$, $\text{GL}_{n/2}(q^2).2$, or $\text{Sp}_n(q)$ when n is even;
- (3) $\text{GL}_n(\sqrt{q}).2$ or $\text{GU}_n(\sqrt{q}).2$ if q is a square;
- (4) $C_{7.3}$ in $L_3(2)$; $C_{13.3}$ in $L_3(3)$; \mathfrak{A}_6 or $3^2.Q_8$ in $L_3(4)$; or \mathfrak{A}_7 in $L_4(2)$.

Proof. The main result of [24] characterises the maximal subgroups of $L_n(q)$ of order at least q^{3n} . Using [23, Tab. 3.5.A] and the known order formulas one arrives at the cases in (1)–(3). For $n \geq 7$ we have $q^{3n} \leq q^{n(n-1)/2}$, so no further examples arise. For $n \leq 6$ the additional groups can be read off from [1, Tab. 8.3–8.25]. \square

In what follows, we write $e_p(q)$ for the multiplicative order of q in the finite field \mathbb{F}_p .

Theorem 5.5. *Let $G = L_n(q)$ with $n \geq 3$ and p a prime dividing $|G|_{q'}$. Then G satisfies (I_p) with respect to some p' -subgroup H if and only if one of*

- (1) $e_p(q) = n$, $H = q^{n-1}.\mathrm{GL}_{n-1}(q)/C_d$ and $\dim \Phi_{1_G} = (q^n - 1)/(q - 1)$, where $d = \gcd(n, q - 1)$; or
- (2) $G = L_3(4)$, $p = 3$, $H = 2^4.D_5$ and $\dim \Phi_{1_G} = 126$.

Proof. We argue by induction over n . Let $e := e_p(q)$. If $e > n$ then $|G|$ is prime to p . If $e = n$ then the end-node parabolic subgroups of G have order coprime to p and the action on the cosets is 2-transitive. This gives conclusion (1) by Lemma 2.2. For $2e > n$ Sylow p -subgroups of G are cyclic. Since the trivial character is not connected to the exceptional character on the Brauer tree, we don't get examples for $e < n < 2e$ by Lemma 2.3, using that G has no such 2-transitive permutation representations by [5].

Now assume that $e \leq n/2$. Note that a Sylow r -subgroup, for $r|q$ the defining characteristic of G , has order $q^{n(n-1)/2}$. Thus, by Lemma 2.1 a candidate p' -subgroup $H \leq G$ with respect to which G satisfies (I_p) must lie in one of the subgroups listed in Proposition 5.4. First assume $H \leq P$, with P a maximal parabolic subgroup. Then a Levi factor L of P is the quotient of P by its maximal normal unipotent subgroup, and the derived subgroup of L modulo the centre has the form $\bar{L} = L_{n-k}(q) \times L_k(q)$ for some $1 \leq k \leq n/2$. If P contains an admissible H , then \bar{L} has property (I_p) , by Lemma 2.5. Thus, by induction, $k \leq n - k \leq e$ and so $k = e$, $n = 2e$, or we have $e = 1$ and either we are in case (2) or (3) of Proposition 5.3, whence $k, n - k \leq 2$, or $p = 3$, $q = 4$ and $k, n - k \leq 3$. We postpone the latter case for the moment. Now for $k = e$, $n = 2e$ the permutation character of G on P contains exactly those unipotent characters labelled by the constituents of the permutation character of $\mathfrak{S}_e \times \mathfrak{S}_e$ inside \mathfrak{S}_{2e} , by Howlett–Lehrer theory. Thus it contains the unipotent character ρ labelled by the partition $(2e - 1, 1)$. But that has only one e -hook, so ρ does not lie in the principal block by [9] and we do not get an example when $n = 2e$.

Let us now discuss the exceptional cases with $e = 1$. If $p = q - 1 \geq 7$ is a Mersenne prime, with $q = 2^f$, and $n \leq 4$ then H cannot lie in a parabolic subgroup by Proposition 5.1. Assume then that $p = 2$, and $n \leq 4$. The permutation character of $G = L_3(q)$ on its maximal parabolic subgroups contains the unipotent character labelled $(2, 1)$, while this is not a constituent of Φ_{1_G} , by [22, p. 253]. Also, the permutation character of $G = L_4(q)$ on a maximal parabolic subgroup of type $\mathrm{GL}_2(q)^2$ contains the character labelled $(2, 2)$, which does not appear in Φ_{1_G} by [22] again. Finally, if $p = 3$, $q = 4$, then $n \leq 6$. By GAP we do not get an example in $L_4(4)$ with $p = 3$. For $L_5(4)$ and $L_6(4)$ the permutation characters on the relevant parabolic subgroups involve the unipotent characters labelled $(4, 1)$, $(4, 2)$ respectively, neither of which occurs in Φ_{1_G} , by [22, p. 258/259]. Hence these exceptional cases do not propagate to further examples.

This completes the discussion of the maximal parabolic subgroups. Now assume that n is even and H is contained in (the image in G of) one of $M = \mathrm{GL}_{n/2}(q) \wr \mathfrak{S}_2$, $\mathrm{GL}_{n/2}(q^2).$

or $\mathrm{Sp}_n(q)$. Since $e \leq n/2$, the order of M is divisible by p . Induction shows that we must have $e = n/2$, again. But a recursive application of Propositions 5.4 and 5.14 shows that any p' -subgroup of M has index at least $(q^{n/2} - 1)^2/(q - 1)^2$, and thus order less than $q^{n(n-1)/2}$. Then it cannot lead to an example by Lemma 2.1. The same argument applies to the subfield subgroups in Proposition 5.4(3). For the groups $\mathrm{L}_n(2)$, $n \leq 4$, and $\mathrm{L}_3(4)$ all primes can be checked using GAP and only the case $p = 3$ for $\mathrm{L}_3(4)$ arises. \square

For later inductive purposes, let's point out the following consequence:

Corollary 5.6. *Assume $\mathrm{L}_n(q)$, $n \geq 2$, has property (I_p) , for p dividing $|\mathrm{L}_n(q)|_{q'}$, and set $e := e_p(q)$. Then either $n \leq e$, or $e = 1$ and one of $n = 2$ or $n = p = q - 1 = 3$.*

5.2. The unitary groups. For the base case of the unitary series we can again also treat the defining characteristic case (which does not lead to examples).

Proposition 5.7. *Let $G = \mathrm{U}_3(q)$, $q \geq 3$. Then G satisfies (I_p) if and only if one of*

- (1) $p|(q + 1)$, $H = O^p(B)$ where B is a Borel subgroup of G ; or
- (2) $p|(q^3 + 1)$ but $p \nmid (q + 1)$, H is a maximal parabolic subgroup, $\dim \Phi_{1_G} = q^3 + 1$.

Proof. For p dividing $q + 1$ this follows from the decomposition matrix given in [13, Thm 4.3] when $p \neq 2, 3$ and from [13, Thm 4.5] when $p = 3$. When $p = 2$ we use that the decomposition matrix of the Weyl group C_2 embeds into that of G , so the 1-PIM does contain the trivial and the Steinberg character once each, as does the permutation character on $O^2(B)$. Since the unipotent characters form a basic set, the latter must indeed be indecomposable. Now assume that $p|(q^3 + 1)$ but not $q + 1$, so in particular $p \geq 5$. Then Sylow p -subgroups of G are cyclic, and the permutation action on the cosets of a maximal parabolic subgroup is 2-transitive. We conclude using the Brauer tree from [13, Thm 4.2].

Next assume that $p|(q - 1)$ but $p \neq 2$. Again Sylow p -subgroups of G are cyclic in this case, and by [13, Thm 4.1] we have

$$\dim \Phi_{1_G} = (q - 1)(q^3 + 1 - u(q^2 + q + 1))/(2u) < q^4/6,$$

where $u = (q - 1)_{p'}$. Now H cannot be contained in a parabolic subgroup, by Proposition 5.1. But there are no subgroups of order at least $|G|/\dim \Phi_{1_G}$ that are irreducible, see [1, Tab. 8.5 and 8.6].

Finally, assume $p|q$. For $q \leq 11$ we may use [12] to verify our claim. For $q > 11$ the decomposition matrix is not known at present. But for those q , the p' -subgroups of largest order are contained in the normalisers H of maximal tori, of order $6(q + 1)^2/\mathrm{gcd}(3, q + 1)$, see [1, Tab. 8.5 and 8.6]. Direct calculation shows that the restriction of the Steinberg character of G to H contains the trivial character. Since the Steinberg character is of p -defect zero and hence does not lie in the principal p -block of G , this shows that H does not lead to an example. \square

As in the linear case we now determine relevant subgroups.

Proposition 5.8. *Let M be a maximal subgroup of $\mathrm{U}_n(q)$, $n \geq 4$, of order at least $q^{n(n-1)/2}$. Then M is the image in $\mathrm{U}_n(q)$ of the intersection with $\mathrm{SU}_n(q)$ of one of the following subgroups of $\mathrm{GU}_n(q)$:*

- (1) a maximal parabolic subgroup;

- (2) $\mathrm{GU}_d(q) \times \mathrm{GU}_{n-d}(q)$ with $1 \leq d < n/2$;
- (3) $\mathrm{GU}_{n/2}(q) \wr \mathfrak{S}_2$, $\mathrm{GL}_{n/2}(q^2).2$, or $\mathrm{Sp}_n(q)$ when n is even;
- (4) $\mathrm{GU}_n(\sqrt{q}).2$ if q is a square;
- (5) $\mathrm{SO}_n(q)$ when q, n are odd;
- (6) $\mathrm{SO}_n^\pm(q)$ when q is odd, n is even;
- (7) $3^3.\mathfrak{S}_4$ in $\mathrm{U}_4(2)$; \mathfrak{A}_7 , $\mathrm{L}_3(4).2^2$ or $2^4.\mathfrak{A}_6$ in $\mathrm{U}_4(3)$; $3^4.\mathfrak{S}_5$ in $\mathrm{U}_5(2)$; or $3.M_{22}$ or $3_1.\mathrm{U}_4(3).2$ in $\mathrm{U}_6(2)$.

Proof. This again follows from [24] using [23, Tab. 3.5.B] and [1]. \square

We now show that generically, there are no examples for unitary groups. Via Harish-Chandra theory the principal series unipotent characters of unitary groups $\mathrm{SU}_n(q)$ are in bijection with characters of its Weyl group of type B_k , $k = \lfloor n/2 \rfloor$, see [15, Prop. 4.3.6]. Recall that the irreducible characters of the Weyl group of type B_k are labelled by bi-partitions of k , see e.g. [16, 5.5.4]. The following, which is shown in [16, Prop. 6.4.7], will be used to determine the constituents of the permutation character on maximal parabolic subgroups:

Lemma 5.9. *Let W be the Weyl group of type B_n and W_d its standard parabolic subgroup of type $B_{n-d} \times A_{d-1}$, with $1 \leq d \leq n$. Then the constituents of the permutation character of W on W_d are the irreducible characters labelled by bi-partitions $(n-d+k, l; d-k-l)$ with $0 \leq k, l$ and $k+l \leq d$.*

Theorem 5.10. *Let $G = \mathrm{U}_n(q)$ with $n \geq 4$ and p a prime dividing $|G|_q$. If G satisfies (I_p) then $p = 2$ or $(p, q) = (3, 2)$.*

Proof. Throughout we may and will assume $p > 2$ and $(p, q) \neq (3, 2)$. Set $e := e_p(q)$ and $e' := 2e$ when e is odd, $e' := e/2$ when $e \equiv 2 \pmod{4}$, and $e' := e$ when $e \equiv 0 \pmod{4}$. The Sylow p -subgroups of G are cyclic when $e' > n/2$, and in this case no examples with p dividing $|G|$ can arise by Lemma 2.3 combined with [5] and the Brauer trees in [11]. So assume $e' \leq n/2$. Let $H < G$ be a p' -subgroup such that $\Phi_{1G} = \mathrm{Ind}_G^H(1_H)$. Since a Sylow r -subgroup of G , for $r|q$, has order $q^{n(n-1)/2}$, H will lie in one of the maximal subgroups M listed in Proposition 5.8. First assume M is a maximal parabolic subgroup. Then its Levi subgroups have a subquotient $\mathrm{U}_{n-2d}(q) \times \mathrm{L}_d(q^2)$ for some $1 \leq d \leq n/2$. By our results on the linear groups (Corollary 5.6) since $(p, q) \neq (3, 2)$ we must have $d \leq e$; the case $e \leq 2$, $d = 2$ cannot occur because $\mathrm{GL}_2(q^2)$ is not an example for $e = 1$ by Proposition 5.3, since $(p, q) \neq (3, 2)$.

First assume e is odd. Then $n - 2d < e' = 2e$ by induction, so $2e = e' \leq n/2 < e + d \leq 2e$, which is not possible. If $e = 2e'$ is twice an odd number, again by induction we have $d \leq e/2$, and $n - 2d < e'$ or $n - 2d = 3$ and $e = 2, 6$. We postpone the latter case for a moment. Then $2e' \leq n < 3e'$ and $d > (n - e')/2 \geq e'/2$. In this case the permutation character on M contains the unipotent character labelled by the bi-partition $\mu = (\lfloor (n-2)/2 \rfloor; 1)$, hence by the partition

$$\lambda = \begin{cases} (n-2, 2) & \text{if } n \text{ is even,} \\ (n-2, 1^2) & \text{if } n \text{ is odd,} \end{cases}$$

with 2-quotient μ . Since the e' -core of λ is not equal to the e' -core of (n) , this unipotent character does not lie in the principal p -block of G by [10], so H cannot be contained in M .

If $n - 2d = 3$ and $e = 2$ we have $n = 5$; in this case the induced 1-PIM from a parabolic subgroup of type $U_1(q) \times L_2(q^2)$ does not contain the unipotent character labelled by $(1^2; -)$ which is a constituent of the permutation character on M . If $n - 2d = 3$ and $e = 6$ we have $n = 7$ and the constituent labelled $(21; -)$ of the permutation character on M does not occur in the induced 1-PIM from $U_1(q) \times L_3(q^2)$.

Finally, if e is divisible by 4, then our assumptions and induction force $n - 2d < e$ and $d \leq e/2$ (since q^2 has order $e/2$ modulo p), so $e \leq n/2 < (2d + e)/2 \leq e$, hence no case arises. This completes the discussion of maximal parabolic subgroups.

The groups listed in Proposition 5.8(2)–(6) have order divisible by p , and their largest p' -subgroups have smaller order than a Sylow r -subgroup of G . Finally, the cases in Proposition 5.8(7) do not lead to examples by GAP. \square

We conclude our discussion of the unitary groups by dealing with the two cases left open in the previous result.

Proposition 5.11. *Let $G = U_n(2)$ with $n \geq 4$. Then G satisfies (I_3) if and only if $n \leq 7$. Here, $H = [2^{(n^2-n-4)/2}].D_5$*

Proof. The claim for $n = 4, 5, 6$ can be checked using GAP. Now let $G = U_7(2)$. The Harish-Chandra induction Ψ of the two 1-PIMs of the parabolic subgroups of types $GU_5(2)$ and $GL_3(4)$ to G have the same ordinary constituents and hence agree. We claim they are indecomposable and thus G satisfies (I_3) . By Lemma 2.5 it is sufficient to show the analogous statement for $GU_7(2)$. Here, by [9, Thm (8A)] the unipotent characters form a basic set for the unipotent blocks. The Harish-Chandra restriction of any proper non-zero subcharacter of Ψ to the two types of maximal parabolic subgroups does not decompose as a non-negative integral linear combination of projectives, so indeed Ψ is indecomposable.

Now assume $n \geq 8$. We argue that a putative p' -subgroup H cannot be contained inside any of the subgroups listed in Proposition 5.8. Let first P be a parabolic subgroup, of type $GU_{n-2d}(2)GL_d(4)$, for some $1 \leq d \leq n/2$. By induction and Theorem 5.5 we then have $d \leq 3$ and $n - 2d \leq 7$, so $n \leq 13$. By Lemma 5.9, for $n = 8$ and $n = 9$ the permutation character on P contains the unipotent character labelled by the bi-partition $(31; -)$ which is not a constituent of the induced 1-PIM from a parabolic subgroup of type $GL_4(4)$, for $n = 10, 11$ it contains $(41; -)$ which is not in the 1-PIM induced from $GL_5(4)$, and for $n = 12, 13$ the character labelled $(42; -)$ has this property.

For the non-parabolic subgroups in Proposition 5.8 we can argue as in the generic case in Theorem 5.10. \square

Proposition 5.12. *Let $G = U_n(q)$ with $n \geq 4$. Then G satisfies (I_2) if and only if $n \leq 5$. Here, $H = O^2(B)$, for B a Borel subgroup of G .*

Proof. First let $n = 4$. Here the Levi factors of both maximal parabolic subgroups have property (I_2) by Proposition 5.3, as does the subgroup $S_4(q)$. In view of Corollary 2.6, to show that G has property (I_2) it suffices to prove this for $\tilde{G} := GU_4(q)$. Now the 1-PIM of the Levi factor $GL_2(q^2)$ contains both unipotent characters once, and thus its Harish-Chandra induction Ψ to \tilde{G} contains all unipotent characters, with the multiplicity given by the character degrees in the Weyl group $W(B_2)$ [15, Thm 3.2.27]. On the other hand, the decomposition matrix of the Weyl group embeds into the decomposition matrix of \tilde{G} ; since $W(B_2)$ is a 2-group, its 1-PIM is the regular representation. This means that the

1-PIM of \tilde{G} contains exactly the same unipotent constituents as Ψ . Since Harish-Chandra induction sends unipotent blocks to unipotent blocks [15, Prop. 3.3.20], \tilde{G} has a unique unipotent 2-block [7], and the unipotent characters form a basic set for the unipotent blocks [9], this shows that indeed $\Psi = \Phi_{1_{\tilde{G}}}$, so \tilde{G} satisfies (I_2) .

For $n = 5$ the exactly same argument as for $n = 4$ shows that again $H = O^2(B)$ is an admissible subgroup. For $n \geq 6$ first consider maximal parabolic subgroups. For P of type $\mathrm{GU}_{n-2d}(q)\mathrm{GL}_d(q^2)$ we have, by induction, $n - 2d \leq 5$ and $d \leq 2$, so $n \leq 9$. Here, for $n = 6, 7$ the unipotent constituent labelled by the bi-partition $(21; -)$ of the permutation character on P does not occur in the induced 1-PIM of the parabolic subgroup of type $\mathrm{GL}_3(q^2)$, and for $n = 8, 9$, it is the constituent $(2^2; -)$ which is not in the induced 1-PIM from $\mathrm{GL}_4(q^2)$.

This deals with parabolic overgroups of a possible subgroup H . The arguments for the non-parabolic maximal subgroups in Proposition 5.8 are as in the generic case. \square

5.3. The symplectic groups. We next discuss the induction base for symplectic groups.

Proposition 5.13. *Let $G = \mathrm{S}_4(q)$, $q \geq 3$, and p a prime dividing $|G|_{q'}$. Then G satisfies (I_p) if and only if $p = 2$, $H = O^2(B)$, $\dim \Phi_{1_G} = (q - 1)_2^2(q + 1)^2(q^2 + 1)$, where B is a Borel subgroup of G .*

Proof. It follows from the decomposition matrix given in [33, Thm 3.1] that we get the stated example for $p = 2$. Now assume that p is odd. For $p|(q^2 + 1)$ the Sylow p -subgroups are cyclic, so we can conclude with Lemma 2.3, using that $\mathrm{S}_4(q)$ does not have a corresponding 2-transitive permutation representation for $q > 2$, by [5]. If $p|(q + 1)$, the decomposition matrix in [34, Thm 4.2] shows that $\dim \Phi_{1_G} = (q + 1)(q^3 + 1)$, which does not divide $|G|$. Finally, when $p|(q - 1)$, then a putative p' -subgroup H cannot lie inside a parabolic subgroup, by Proposition 5.1. Any other subgroup, by [1, Tab. 8.12 and 8.13] has p' -subgroups of order smaller than q^4 , and so cannot contain a suitable H . \square

Proposition 5.14. *Let M be a maximal subgroup of $\mathrm{S}_{2n}(q)$, $n \geq 3$, of order at least q^{n^2} . Then M is the image in $\mathrm{PGL}_{2n}(q)$ of one of the following subgroups of $\mathrm{Sp}_{2n}(q)$:*

- (1) a maximal parabolic subgroup;
- (2) $\mathrm{Sp}_{2d}(q) \times \mathrm{Sp}_{2n-2d}(q)$ with $1 \leq d < n/2$;
- (3) $\mathrm{Sp}_n(q) \wr \mathfrak{S}_2$ or $\mathrm{Sp}_n(q^2).2$ when n is even
- (4) $\mathrm{GL}_n(q).2$ or $\mathrm{GU}_n(q).2$;
- (5) $\mathrm{Sp}_{2n}(\sqrt{q})$ if q is a square;
- (6) $\mathrm{GO}_{2n}^{\pm}(q)$ if q is even;
- (7) $\mathrm{Sp}_2(q^3).3$ or $\mathrm{Sp}_2(q) \wr \mathfrak{S}_3$ when $n = 3$;
- (8) $G_2(q)$ when $n = 3$ and q is even;
- (9) $2.J_2$ in $\mathrm{S}_6(5)$ or $\mathrm{S}_6(9)$; \mathfrak{S}_{10} in $\mathrm{S}_8(2)$; or \mathfrak{S}_{14} in $\mathrm{S}_{12}(2)$.

Proof. The maximal subgroups of $\mathrm{Sp}_{2n}(q)$ of order at least q^{6n} are described in [24]. More details about these can be found in [23, Tab. 3.5.C], and we arrive at (1)–(7). For $n \geq 6$ we always have $q^{n^2} \geq q^{6n}$. For $n \leq 5$ the relevant subgroups can be read off from [1, Tab. 8.28–8.65]. \square

In the proof of the following result we make use of the description of blocks and Brauer trees in classical groups by Fong and Srinivasan [10, 11].

Theorem 5.15. *Let $G = S_{2n}(q)$ with $n \geq 3$ and p a prime dividing $|G|_q$. Then G satisfies (I_p) with respect to some p' -subgroup H if and only if one of*

- (1) $q = 2$, $e_p(2) = 2n$, $H = \text{GO}_{2n}^+(2)$, $\dim \Phi_{1_G} = 2^{n-1}(2^n + 1)$; or
- (2) $q = 2$, $e_p(2) = n$ is odd, $H = \text{GO}_{2n}^-(2)$, $\dim \Phi_{1_G} = 2^{n-1}(2^n - 1)$.

Proof. We argue by induction over n . Let $e := e_p(q)$. We set $e' := 2e$ if e is odd, and $e' := e$ otherwise. The order of G is coprime to p if $e' > 2n$. The Sylow p -subgroups of G are cyclic if $n < e' \leq 2n$. The only 2-transitive actions of G are, by [5], the ones given in (1) and (2) with $q = 2$, and they lead to examples by Lemma 2.2. As the trivial character is never connected to the exceptional node on the corresponding Brauer tree [11], there are no further examples in the cyclic Sylow case by Lemma 2.3.

Now assume that $e' \leq n$. We discuss the various maximal overgroups M of p' -subgroups H of G . Note that a maximal unipotent subgroup of G has size q^{n^2} so it suffices to consider the groups occurring in Proposition 5.14. Let P be a maximal parabolic subgroup of G . Then G has a Levi factor with subquotient $\bar{L} = S_{2n-2d}(q) \times L_d(q)$, for some $1 \leq d \leq n$. By Corollary 2.6, both factors must satisfy (I_p) . Hence, by induction, Corollary 5.6 applied to the $L_d(q)$ -factor imposes that $d \leq e$, or $e = 1$ and either $d = 2$, or $d = p = 3$, $q = 4$, and similarly, by Proposition 5.13, the $S_{2n-2d}(q)$ -factor forces one of

- $2n - 2d < e'$,
- $2n - 2d = e'$, $q = 2$,
- $2n - 2d = 2$, or
- $2n - 2d = 4$, $p = 2$.

First assume $2(n - d) \leq e'$. The constituents of the permutation character on P are described in Lemma 5.9. For $d \leq e$, if e and n are both odd, the constituent labelled by the bi-partition $((n - 2 + e)/2, (n - e)/2; 1)$ does not lie in the principal p -block, if e is odd and n is even, the one labelled $((n - 1 + e)/2, (n + 1 - e)/2; -)$ is not in the principal block; if e is even, then $e \leq n$, and the constituent labelled $(n - e/2; e/2)$ is outside the principal block ([10]). On the other hand, if $d = p = 3$, $q = 4$, then $G = S_6(4)$, and P has type $\text{GL}_3(4)$. Here, Harish-Chandra induction of the trivial character contains the unipotent character labelled $(3; -)$, while this is not a constituent of the 1-PIM by [35, Thm 2.1].

Next, assume $2n - 2d = e'$, $q = 2$ and $d = 2$, $e = 1$. Then $G = S_6(2)$, which can be discarded using GAP.

Now consider the case that $2n - 2d = 2$, so $n = d + 1$. If $d \leq e$, then $e' \leq n = d + 1 \leq e + 1$, so $e = e'$ is even and equal to $n - 1$ or n . Here the permutation character on P of type $\text{Sp}_2(q)\text{GL}_{n-1}(q)$ contains the unipotent characters labelled $(n - 1; 1)$ and $(n - 3; 3)$ which do not lie in the principal p -block when $e = n$, $e = n - 1$ respectively. If $d = 2$, $e = 1$, then by Proposition 5.1 this can only give rise to an example if $p \leq 3$. For $p = 3$ the Harish-Chandra induction of the trivial character from P of type $\text{Sp}_2(q)\text{GL}_2(q)$ contains the unipotent character labelled $(3; -)$, while this is not a constituent of the 1-PIM by [35, Thm 2.1]. When $p = 2$ the 1-PIM of each simple factor of \bar{L} contains the Steinberg character, and so its Harish-Chandra induction to G contains the unipotent character labelled $(-; 2, 1)$ not lying in the principal block. Next, if $d = 3$, $p = 3$, $q = 4$, then $G = S_8(4)$. Here, the Harish-Chandra induced 1-PIM from a maximal parabolic subgroup

P of type $\mathrm{Sp}_6(q)$ (given in [35, Thm 2.1]) does not contain the unipotent character labelled $(3; 1)$, but the permutation character on P does, so H cannot be contained in P .

Finally, consider $2n - 2d = 4$, $p = 2$ and $e = 1$, so $n = d + 2$. If $d \leq e = 1$ then $n = 3$ and $G = \mathrm{S}_6(q)$. Here the induced 1-PIM from the parabolic subgroup P of type $\mathrm{Sp}_4(q)$ is not contained in the induced 1-PIM from the parabolic subgroup of type $\mathrm{GL}_3(q)$, so H cannot lie inside P . The last remaining case is now when $d = 2$, so $n = 4$, $G = \mathrm{S}_8(q)$. Here again the induced 1-PIM from the parabolic subgroup P of type $\mathrm{Sp}_4(q)\mathrm{GL}_2(q)$ is not contained in that from the parabolic subgroup of type $\mathrm{GL}_4(q)$. This completes the discussion of maximal parabolic subgroups.

Since $e' \leq n$, the groups in (2)–(8) of Proposition 5.14 have order divisible by p . Again by a recursive application of Propositions 5.4, 5.8, 5.14, 5.18 and 5.19, respectively using [1, Tab. 8.41, 8.42], their largest p' -subgroups are of order less than q^{n^2} , so they cannot contain a relevant subgroup H .

As far as the groups in (9) of Proposition 5.14 are concerned, \mathfrak{S}_{10} and \mathfrak{S}_{14} do not satisfy (I_p) for any prime divisor of their order by Theorem 3.1, and the Sylow p -subgroups of G are cyclic for all other primes. The group $2.J_2$ has property (I_p) only for $p = 5$, by Theorem 4.1, so only $G = \mathrm{S}_6(9)$ needs to be considered. But here a Borel subgroup of G has larger order than any $5'$ -subgroup of $2.J_2$. \square

5.4. The orthogonal groups.

Proposition 5.16. *Let M be a maximal subgroup of $\mathrm{O}_{2n+1}(q)$, $n \geq 3$ and q odd, of order at least q^{n^2} . Then M is the intersection with $\mathrm{O}_{2n+1}(q)$ of one of the following subgroups of $\mathrm{GO}_{2n+1}(q)$:*

- (1) a maximal parabolic subgroup;
- (2) $\mathrm{SO}_{2d+1}(q) \times \mathrm{GO}_{2n-2d}^\pm(q)$ with $0 \leq d < n$;
- (3) $\mathrm{SO}_{2n+1}(\sqrt{q}).2$ if q is a square;
- (4) $G_2(q)$ when $n = 3$; or
- (5) $2^6.\mathfrak{A}_7$, \mathfrak{S}_9 or $\mathrm{S}_6(2)$ in $\mathrm{O}_7(3)$; or $2^8.\mathfrak{A}_9$ in $\mathrm{O}_9(3)$.

Proof. This follows again from [24, Thm 4.2], where maximal subgroups of order at least q^{4n+6} are identified, in conjunction with the explicit descriptions of the generic subgroups in [23, Tab. 3.5.D] and the complete lists of maximal subgroups in [1] for $n \leq 5$. \square

Theorem 5.17. *Let $G = \mathrm{O}_{2n+1}(q)$ with $n \geq 3$ and q odd. Then G does not satisfy (I_p) for any prime p dividing $|G|_{q'}$.*

Proof. We again proceed by induction on n . The group G does not have 2-transitive permutation actions by [5], so using [11] no examples arise for cyclic Sylow p -subgroups. So setting $e = e_p(q)$ and $e' = 2e/\mathrm{gcd}(2, e)$ we have $e' \leq n$. To see that a putative p' -subgroup H can not lie inside a maximal parabolic subgroup of G we can argue as in the proof of Theorem 5.15, using that groups of types B_n and C_n have the same Weyl group and hence the same decomposition of Harish-Chandra induction.

Again the remaining possibilities listed in Proposition 5.16, except for those in item (5), have order divisible by p and do not contain large enough p' -subgroups. The group \mathfrak{A}_9 does not have property (I_p) for $p \leq 7$ by Theorem 3.1, and Sylow p -subgroups of $G = \mathrm{O}_9(3)$ are cyclic for all larger primes. The case of $G = \mathrm{O}_7(3)$ can be discarded using the known decomposition matrices in GAP. \square

In the following we set $\mathrm{GO}_{2m+1}^0(q) := \mathrm{GO}_{2m+1}(q)$. The next two results are proved in a very similar way to the earlier results of this type, using [24], [23] and [1]:

Proposition 5.18. *Let M be a maximal subgroup of $\mathrm{O}_{2n}^+(q)$, $n \geq 4$, of order at least q^{n^2-n} . Then M is the intersection with $\mathrm{O}_{2n}^+(q)$ of the image in $\mathrm{PGL}_{2n}(q)$ of one of the following subgroups of $\mathrm{GO}_{2n}^+(q)$:*

- (1) a maximal parabolic subgroup;
- (2) $\mathrm{GO}_d^\epsilon(q) \times \mathrm{GO}_{2n-d}^\epsilon(q)$ with $0 < d < 2n$, $\epsilon \in \{0, \pm\}$;
- (3) $\mathrm{Sp}_{2n-2}(q)$ when q is even;
- (4) $\mathrm{GO}_n^\pm(q) \wr \mathfrak{S}_2$ or $\mathrm{GO}_n^+(q^2)$ when n is even;
- (5) $\mathrm{GO}_n(q) \wr \mathfrak{S}_2$ or $\mathrm{GO}_n(q^2)$ when nq is odd;
- (6) $\mathrm{GL}_n(q).2$ or $\mathrm{GU}_n(q)$ when n is even;
- (7) $\mathrm{GO}_{2n}^\pm(\sqrt{q})$ when q is a square; or
- (8) \mathfrak{A}_9 in $\mathrm{O}_8^+(2)$; $\mathrm{O}_8^+(2)$ in $\mathrm{O}_8^+(3)$; or \mathfrak{A}_{16} in $\mathrm{O}_{14}^+(2)$.

Proposition 5.19. *Let M be a maximal subgroup of $\mathrm{O}_{2n}^-(q)$, $n \geq 4$, of order at least q^{n^2-n} . Then M is the intersection with $\mathrm{O}_{2n}^-(q)$ of the image in $\mathrm{PGL}_{2n}(q)$ of one of the following subgroups of $\mathrm{GO}_{2n}^-(q)$:*

- (1) a maximal parabolic subgroup;
- (2) $\mathrm{GO}_d^\epsilon(q) \times \mathrm{GO}_{2n-d}^{-\epsilon}(q)$ with $0 < d < 2n$, $\epsilon \in \{0, \pm\}$;
- (3) $\mathrm{Sp}_{2n-2}(q)$ when q is even;
- (4) $\mathrm{GO}_n^-(q^2)$ when n is even;
- (5) $\mathrm{GO}_n(q) \wr \mathfrak{S}_2$ or $\mathrm{GO}_n(q^2)$ when nq is odd;
- (6) $\mathrm{GU}_n(q)$ when n is odd; or
- (7) \mathfrak{A}_{12} in $\mathrm{O}_{10}^-(2)$; or \mathfrak{A}_{13} in $\mathrm{O}_{12}^-(2)$.

Theorem 5.20. *Let $G = \mathrm{O}_{2n}^\pm(q)$ with $n \geq 4$. Then G does not satisfy (I_p) for any prime p dividing $|G|_q$.*

Proof. We proceed as in our earlier proofs. Let $e := e_p(q)$ and $e' := 2e/\mathrm{gcd}(2, e)$. If $e' > n$ then Sylow p -subgroups of G are cyclic and we conclude with Lemma 2.3, again using [11]. Thus, $e' \leq n$. Now first assume H lies in some maximal parabolic subgroup P of $G = \mathrm{O}_{2n}^+(q)$. Then a Levi factor of P has a subquotient of the form $\bar{L} = \mathrm{O}_{2n-2d}^+(q) \times \mathrm{L}_d(q)$. First assume e is odd, so $e' = 2e$. By induction and Corollary 5.6 we have $2(n-d) < \max\{e', 5\}$ and $d \leq e$ or $d \leq 3, e = 1$. Then our inequalities force $e = 1$, and either $d = 2$, $n = 4$, or $d = p = 3$, $q = 4$ and $n \leq 5$. If $d = 2$ we have P is of type $\mathrm{SO}_4^+(q)\mathrm{GL}_2(q)$. For $p \geq 3$ the permutation character on P contains the unipotent characters labelled $(2^2; -)^\pm$, not contained in the induced 1-PIM of the parabolic subgroup of type $\mathrm{GL}_4(q)$ (known from [22]), while for $p = 2$, the latter contains the character $(2^2; -)^\pm$ with a smaller multiplicity than the former. If $d = p = 3$, P is of type $\mathrm{SO}_{2n-6}^+(q)\mathrm{GL}_3(q)$. This is not a maximal parabolic subgroup when $n = 4$. For $n = 5$ the permutation character on P contains the character labelled $(1; 3, 1)$ which is not a constituent of the 1-PIM induced from a parabolic subgroup of type $\mathrm{GL}_5(q)$. This concludes the case when e is odd.

If e is even, then induction and Corollary 5.6 give $2(n-d) < \max\{e, 5\}$ or $2(n-d) = 6$ and $e = 4$. If $e = 4$ as $d \leq e$, we have $n \leq 7$. Now the induced 1-PIM for P of type $\mathrm{SO}_6^+(q)\mathrm{GL}_d(q)$ contains the unipotent characters labelled $(3; 1)$, $(41; -)$, $(2^2 1^2; -)^\pm$, $(61; -)$ for $n = 4, 5, 6, 7$ respectively, not lying in the principal p -block. For P of type

$\mathrm{SO}_4^+(q)\mathrm{GL}_d(q)$ it again contains $(3; 1)$, $(41; -)$, $(2^2 1^2; -)^\pm$ for $n = 4, 5, 6$ respectively. If $e = 2$ then only P of type $\mathrm{SO}_4^+(q)\mathrm{GL}_2(q)$ needs to be considered. Here the character labelled $(2, 1; 1)$, not in the principal p -block, is a constituent of the permutation character. Hence we have $e \geq 6$ and $n < 3e/2 \leq 3n/2$. Here the unipotent character labelled $(n+1-e, e-1; -)$ is a constituent of the permutation character on P , but does not lie in the principal p -block.

Turning to the groups of minus type, assume now P is maximal parabolic in $G = \mathrm{O}_{2n}^-(q)$, of type $\mathrm{SO}_{2n-2d}^-(q)\mathrm{GL}_d(q)$. Again first assume e is odd. Then as before this forces $e = 1$, and either $d = 2$, $n = 4$, or $d = p = 3$, $q = 4$ and $n = 5$. In the first case, the relevant maximal parabolic subgroup P has a subquotient $\mathrm{O}_4^-(q)\mathrm{L}_2(q) = \mathrm{L}_2(q^2)\mathrm{L}_2(q)$, and by Proposition 5.3 we need to have $p = 2$. In this case, the permutation character on P contains the unipotent character labelled $(2, 1; -)$, but this is not a constituent of the induced 1-PIM from a maximal parabolic subgroup of type $\mathrm{GL}_3(q)$. When $d = p = 3$, $q = 4$ and $n = 5$ then P is of type $\mathrm{O}_4^-(4)\mathrm{L}_3(4) = \mathrm{L}_2(16)\mathrm{L}_3(4)$, but this does not have property (I_3) by Proposition 5.3.

So now assume e is even. Then our conditions yield $2(n-d) < \max\{e, 5\}$. Since Sylow p -subgroups of G are still cyclic when $n = e$ we may also assume $n \geq e+1$. If $e = 2$ then P has type $\mathrm{SO}_4^-(q)\mathrm{GL}_2(q)$, and again the permutation character on P contains the character $(2, 1; -)$, which is not a constituent of the induced 1-PIM from a parabolic subgroup of type $\mathrm{GL}_3(q)$. If $n \geq 4$ then furthermore, $n \leq d + e/2 \leq 3e/2$. In this case, by Lemma 5.9 the permutation character on P contains the principal series unipotent character labelled $(e-1; n-e)$ which does not lie in the principal p -block of G .

The groups M in Proposition 5.18(2)–(7) and in Proposition 5.19(2)–(6) have order divisible by p , and thus the order of a maximal p' -subgroup of M is smaller than q^{n^2-n} , the size of a Sylow r -subgroup of G , where $r|q$. Finally, the groups \mathfrak{A}_9 , $\mathrm{O}_8^+(2)$, \mathfrak{A}_{12} and \mathfrak{A}_{13} do not have property (I_p) for $p \leq 7$, $p \leq 7$, $p \leq 11$, $p \leq 11$ respectively, and the Sylow p -subgroups of $\mathrm{O}_8^+(2)$, $\mathrm{O}_8^+(3)$, $\mathrm{O}_{10}^-(2)$ and $\mathrm{O}_{12}^-(2)$, respectively, are cyclic for all larger primes, so no examples can arise from these. \square

5.5. Groups of exceptional type. We now discuss the exceptional groups of Lie type. Structural results on their Sylow p -subgroups are given in [27, Thm 25.14], the characters in the principal blocks are described in [2] and [7]. We use Chevie [29] for the computation of Harish-Chandra induction.

Theorem 5.21. *Let G be simple of exceptional Lie type in characteristic r and p a prime dividing $|G|_{r'}$. Then G has property (I_p) if and only if it occurs in Table 3.*

TABLE 3. Exceptional groups of Lie type satisfying (I_p)

G	H	p	$\dim \Phi_{1G}$
${}^2\mathrm{B}_2(q^2)$, $q^2 \geq 8$	P	$p (q^4 + 1)$	$q^4 + 1$
${}^2\mathrm{G}_2(q^2)$, $q^2 \geq 27$	P	$2 < p (q^6 + 1)$	$q^6 + 1$
${}^2\mathrm{G}_2(q^2)$, $q^2 \geq 27$	$\mathrm{O}^2(B)$	2	$2(q^6 + 1)$
${}^2\mathrm{F}_4(2)'$	$\mathrm{L}_3(3).2$	5	1600

Proof. We discuss the various families in turn. If Sylow p -subgroups of G are cyclic, we can argue using Lemma 2.3 in conjunction with [5]. This leads to the first two entries in Table 3.

For the Suzuki groups, all Sylow subgroups for non-defining primes are cyclic. For the Ree groups ${}^2G_2(q^2)$, $q^2 = 3^{2f+1} \geq 27$, the only non-cyclic Sylow subgroups for non-defining primes are for $p = 2$. Here, Fong [8] has shown that Φ_{1G} is induced from $O^2(B)$, for B a Borel subgroup.

For the other series we make use of the result of Liebeck and Saxl mentioned earlier. For $G = G_2(q)$, $q > 2$, the relevant primes are the divisors of $q^2 - 1$. The cases $q = 3, 4$ can be dealt with via GAP, so assume $q \geq 5$. If $2 < p | (q + 1)$, then a Borel subgroup B of G has order $q^6(q - 1)^2$ prime to p . The only maximal subgroups of larger order, by [25, Tab. 1], are the two types of maximal parabolic subgroups (which both contain a Borel subgroup of G) as well as $SL_3(q).2$ and $SU_3(q).2$. The largest p' -subgroups of the latter two have order less than $|B|$ by Propositions 5.4 and 5.7, while (by Harish-Chandra theory) both parabolic subgroups have a defect zero constituent in their permutation character, so H is contained in neither, and no example can arise.

Now assume $2 < p | (q - 1)$. Here a Sylow r -subgroup U of G gives a lower bound q^6 for $|H|$. Again, the maximal subgroups $SL_3(q).2$ and $SU_3(q).2$ do not have large enough p' -subgroups, and the same holds for ${}^2G_2(q)$ (if q is an odd power of 3) and $G_2(q^{1/2})$ (if q is a square). So by [25], H must lie in a maximal parabolic subgroup, of structure $[q^5].GL_2(q)$. By Proposition 5.1 this forces $p = 3$, and then $q = 4$ by Corollary 2.6 and Proposition 5.3, which was excluded before.

Finally, for $p = 2$ the decomposition matrices in [21, §2.2.1 and 2.3.1] show that we do not get an example.

For $G = {}^3D_4(q)$, the relevant primes are the divisors of $q^6 - 1$. For $2 < p$ dividing $q^3 + 1$, a Borel subgroup B of p' -order $q^{12}(q - 1)(q^3 - 1)$ shows that H must lie in a maximal parabolic subgroup, by [25]. But their permutation characters contain constituents from non-principal blocks. For $2 < p$ dividing $q^3 - 1$ a potential subgroup H must again be contained in a maximal parabolic subgroup. In this case [14, Prop. 5.3] shows that the permutation characters properly contain the 1-PIM. For $p = 2$ we do not get an example by the decomposition matrix in [18].

For $G = {}^2F_4(q^2)'$, $q^2 = 2^{2f+1} \geq 2$, the case $q^2 = 2$ can be treated directly, leading to the stated example with $p = 5$. For $q^2 \geq 8$ we need to discuss primes p dividing $q^8 - 1$. Again, by [26], H must be contained in a maximal parabolic subgroup P . The permutation characters on both maximal parabolic subgroups contain constituents of p -defect zero, for any divisor p of $(q^2 + 1)(q^4 + 1)$. So assume $p | (q^2 - 1)$. Here [19, Thm 3.1] shows that the trivial character is the only unipotent constituent of the 1-PIM, and hence the 1-PIM is a proper summand of the permutation character on P .

For $G = F_4(q)$ the relevant primes are the divisors of $(q^6 - 1)(q^2 + 1)$. By Propositions 5.16 and 5.18 any p' -subgroup of the maximal subgroups of types $B_4(q)$, $D_4(q).$ \mathfrak{S}_3 or ${}^3D_4(q).3$ of G has order less than q^{24} , the order of a Sylow r -subgroup of G , or is parabolic, hence contained in a parabolic subgroup of G by the Borel–Tits theorem. The same holds for the subfield subgroups. So we just need to discuss parabolic subgroups. Now by Harish-Chandra theory, for any p dividing $(q^6 - 1)(q^2 + 1)$ but not $q - 1$, the permutation characters on all maximal parabolic subgroups of G have constituents in

non-principal p -blocks. So only primes dividing $q - 1$ remain. By Proposition 5.1, we may in fact assume that $p = 2$ or $p = 3$. Then by Theorems 5.5, 5.15 and 5.17 the only Levi subgroups of a maximal parabolic subgroup having property (I_p) are the ones of type $A_2 + A_1$ for $p = 3$ when $q = 4$. Now the Harish-Chandra induction of the 1-PIM from one of the two Levi subgroups of type $A_2 + A_1$ contains the unipotent character $\phi_{9,2}$ four times, while the Harish-Chandra induction of the 1-PIM from a Levi subgroup of type C_3 only contains it twice. Thus, the former cannot contain a p' -subgroup with respect to which G satisfies (I_3) . The other potential maximal parabolic subgroup is now also ruled out by application of the graph automorphism.

For $G = E_6(q)$, arguing as before we need to discuss primes dividing $q^6 - 1$. The candidates for maximal parabolic subgroups P satisfying (I_p) can be read off from Theorems 5.5 and 5.20. For P of type A_5 or $A_4 + A_1$ the only possibility is $e_p(q) = 6$, but in the first case, the Harish-Chandra induction of the 1-PIM to G contains the unipotent character $\phi_{15,5}$, and in the second, the permutation character contains $\phi_{64,4}$. Since both lie outside the principal p -block, these cases are out. For P of type $2A_2 + A_1$, the constituent $\phi_{10,9}$ in the permutation character rules out the possibility $e_p(q) = 6$, and the constituent $\phi_{81,6}$ shows we can't have $e_p(q) = 3$ or $p = 3$. This exhausts the candidate maximal parabolic subgroups. All of the non-parabolic maximal subgroups listed in [25, Tab. 1] have order divisible by any prime p for which Sylow p -subgroups of G are non-cyclic, but none of them has property (I_p) by Theorems 5.5, 5.20 and by induction, except for the subsystem subgroup of type $A_5 + A_1$ for $e_p(q) = 6$. But by Proposition 5.4 the largest p' -subgroup of the latter has order smaller than q^{36} , the order of a maximal unipotent subgroup of G .

For $G = {}^2E_6(q)$, the maximal parabolic subgroups possibly having property (I_p) are those of type ${}^2A_5(q)$ for $p = 2$; those of type ${}^2A_5(q)$ and $A_2(q^2)A_1(q)$ for $p = 3, q = 2$; and those of type $A_2(q)A_1(q^2)$ for $p = 3, q = 4$, by Theorems 5.5, 5.10 and 5.20. The permutation characters of the parabolic subgroups in the last two cases contain the unipotent character $\phi'_{8,3}$, outside the principal 3-block by [7]. Now assume $p = 2$ and P is parabolic of type ${}^2A_5(q)$. Here, the unipotent part of the induced 1-PIM of P properly contains the unipotent part of the induced 1-PIM from a parabolic subgroup of type $A_2(q^2)A_1(q)$ and thus P cannot contain an admissible p' -subgroup. The non-parabolic subgroups in [25, Tab. 1] all have order divisible by p , but none has (I_p) by Theorems 5.10, and 5.20, except possibly the subsystem subgroup of type ${}^2A_5(q)A_1(q)$ for $(p, q) = (3, 2)$ or for $p = 2$. Here again the largest p' -subgroup of the latter has order smaller than q^{36} .

For $G = E_7(q)$, the relevant primes are the divisors p of $(q^6 - 1)(q^2 + 1)$. The maximal parabolic subgroups possibly having (I_p) are those of types $A_5 + A_1$, $A_3 + A_2 + A_1$ and $A_4 + A_2$ for $e_p(q) = 6$, and $A_3 + A_2 + A_1$ for $e_p(q) = 4$, by Theorems 5.5 and 5.20. For all of these, the permutation character has a constituent not lying in the principal p -block. All large non-parabolic maximal subgroups of G from [25, Tab. 1] have order divisible by p , but none of them has property (I_p) by our earlier results.

For $G = E_8(q)$, Sylow p -subgroups are non-cyclic for primes p such that the Euler φ -function of $e_p(q)$ is at most 4. For all maximal parabolic subgroups the permutation character contains constituents outside the principal p -block when $e_p(q) > 3$. Furthermore, by our earlier results, none of these has a Levi subgroup with (I_p) for $e_p(q) \leq 3$, so parabolic subgroups cannot contain an admissible p' -subgroup. Of the non-parabolic subgroups in [25, Tab. 1], only $H = E_8(\sqrt{q})$ might lead to an example, for $e_p(q) = 8$. But

the order of H is smaller than that of the p' -parabolic subgroup of type $A_6 + A_1$, so it cannot lead to an example by Lemma 2.1. \square

6. ON GROUPS OF LIE TYPE IN DEFINING CHARACTERISTIC

At present we are not able to settle our question for simple groups of Lie type when p is the defining characteristic. The only examples with property (I_p) we are aware of are the ones for $L_2(q)$ in Proposition 5.3(4)–(6), $\mathfrak{A}_5 \cong L_2(4) \cong L_2(5)$ for $p = 2$ and $p = 5$, $\mathfrak{A}_6 \cong S_4(2)$ for $p = 2$, $L_3(2)$ for $p = 2$, and the group $U_4(2) \cong S_4(3)$ for $p = 2$ and $p = 3$.

We would not be surprised if these turn out to be the only ones. A heuristic argument for this runs as follows. First, the Steinberg character has trivial constituents upon restriction to small enough subgroups. Observe that the bound for type 2A_n is the same as the one implied by Lemma 2.8.

Proposition 6.1. *Let $G = G(q)$ be simple of Lie type in characteristic p . Then G does not have property (I_p) with respect to any p' -subgroup of order at most q^m , with $m = m(G)$ as in Table 4.*

TABLE 4. Bounds for p' -subgroups

G	A_n, B_n, C_n, D_n	2A_n ($n \geq 2$)	G_2	${}^3D_4, F_4$	$E_6, {}^2E_6$	E_7	E_8
$m(G)$	$\max\{1, 2n - 2\}$	n	3	8	16	32	56

Proof. Let $H \leq G$. We evaluate the scalar product of the Steinberg character St of G restricted to H with the trivial character of H . The value of St on a semisimple element $s \in G$ is up to sign equal to $|C_G(s)|_p$ (see e.g. [15, Prop. 3.4.10]), which in turn equals q^N for N the number of positive roots in the root system of the underlying algebraic group $C_G(s)$, and zero on all other elements. Since centralisers of semisimple elements are maximal rank subgroups which can be seen on the extended Dynkin diagram of \mathbf{G} (see e.g. [27, §13]), it is easy to determine an upper bound m such that $|\text{St}(s)|q^m \leq \text{St}(1)$ for any $1 \neq s \in G$. For example, if G is of type B_n with $n \geq 2$, then $\text{St}(1) = q^{n^2}$ while the maximal rank subgroups with largest Sylow p -subgroup are of type $B_{n-1}B_1$, so we have $m = n^2 - (n-1)^2 - 1 = 2(n-1)$. For G of type E_8 , $\text{St}(1) = q^{120}$, and maximal Sylow p -subgroups are attained for subgroups of type E_7A_1 , giving $m = 120 - 63 - 1 = 56$. In type 3D_4 the relevant subgroups are involution centralisers of type $A_1(q^3)A_1(q)$.

Then

$$|H| \langle \text{St}|_H, 1_H \rangle \geq \text{St}(1) - (|H| - 1)\text{St}(1)/q^m.$$

Thus, $\text{St}|_H$ has trivial constituents when $|H| \leq q^m$. Since the Steinberg character is of p -defect zero, it does not lie in the principal block and thus cannot be a constituent of Φ_{1_G} . So for H a p' -subgroup, G cannot have property (I_p) with respect to H by Lemma 2.7. This also shows that the bound $m = n$ in type A_n , $n \geq 2$, can be improved since there the largest proper centraliser in $G = \text{SL}_n(q)$ (namely $\text{GL}_{n-1}(q)$) has the same relative \mathbb{F}_q -rank as G . Hence the Steinberg character takes positive values on elements of this type [15, Prop. 3.4.10] and we can neglect them in the inequality, and the next smallest centraliser gives $m = 2n - 2$. \square

As we expect that generically any p' -subgroup of G should have order bounded above by $q^{m(G)}$, Lemma 2.1 would allow to conclude.

We have implemented this approach for the exceptional groups of small rank.

Lemma 6.2. *The Suzuki groups $G = {}^2B_2(q^2)$ with $q^2 = 2^{2f+1} \geq 8$ do not satisfy (I_2) .*

Proof. According to the description by Suzuki, the largest odd order subgroups of G are cyclic of order $q^2 + \sqrt{2}q + 1$ ([1, Tab. 8.16]). The restriction of the Steinberg character of G to such a subgroup H contains the trivial character with multiplicity $q^2 - \sqrt{2}q > 0$, so G cannot have property (I_2) with respect to H by Lemma 2.7, and hence neither with respect to any other p' -subgroup by Lemma 2.1. \square

Lemma 6.3. *The Ree groups $G = {}^2G_2(q^2)$ with $q^2 = 3^{2f+1} \geq 27$ do not satisfy (I_3) .*

Proof. From [1, Tab. 8.43] it follows that the largest order $3'$ -subgroups H of G are direct products of a dihedral group of order $q^2 + 1$ with a Klein four group. An easy calculation shows that the trivial character occurs with positive multiplicity in the restriction to H of the Steinberg character of G and we conclude as in the previous case. \square

Lemma 6.4. *The groups $G = G_2(q)$ with $q = p^f \geq 3$ do not satisfy (I_p) .*

Proof. The cases with $q \leq 5$ are out using GAP. According to the lists in [1, Tab. 8.30, 8.41 and 8.42] the maximal order p' -subgroups H of G are among the torus normaliser of order $12(q + 1)$ and then $2^3.L_3(2)$, $L_2(13)$, $L_2(8)$ and $G_2(2)$. The bound obtained in Proposition 6.1 is not quite large enough to exclude these for all $q > 5$, so we refine the argument.

Namely, the only elements $s \in G$ with $|C_G(s)|_p = q^3$ are elements of order 3, so only such elements of H can contribute $-q^3$ to the scalar product. The only elements with $|C_G(s)|_p = q^2$ are involutions, but here $C_G(s)$ has type A_1^2 of \mathbb{F}_q -rank 2, so $\text{St}(s) > 0$. All other elements $1 \neq s \in G$ have $|\text{St}(s)| \leq q$. With these improved estimates it is straightforward to check that the four individual groups listed above are not relevant for $q > 5$. Finally, at most $2(q + 1)^2$ of the elements in the torus normaliser H have order 3 (unless $p = 2$ in which case we only consider a maximal odd order subgroup). Then using the above argument one sees that again H is not relevant for $q > 5$. \square

Lemma 6.5. *The groups $G = {}^3D_4(q)$ with $q = p^f$ do not satisfy (I_p) .*

Proof. Using GAP we may assume $q \geq 3$. By [1, Tab. 8.51] p' -subgroups of maximal order are among the torus normaliser $(q^2 + q + 1)^2.SL_2(3)$ (respectively an odd order subgroup $(q^2 + q + 1)^2.3$ if $p = 2$), and the p' -subgroups of $G_2(q)$. The claim follows immediately from Proposition 6.1. \square

Lemma 6.6. *The groups $G = {}^2F_4(q^2)'$ with $q^2 = 2^{2f+1} \geq 2$ do not satisfy (I_2) .*

Proof. Any odd order subgroup H of G is (of course) solvable and hence local, and lies inside torus normalisers. From [26] it follows that $|H| \leq 3(q^2 + \sqrt{2}q + 1)^2$. Using GAP to deal with the case $q^2 = 2$ we can then conclude with Proposition 6.1. \square

Lemma 6.7. *The groups $G = F_4(q)$ with $q = p^f$ do not satisfy (I_p) .*

Proof. Again the decomposition matrix of $F_4(2)$ is available in GAP and shows the dimension of the 1-PIM does not divide the group order. So we assume $q \geq 3$. Candidates for maximal p' -subgroups apart from torus normalisers are determined in [6]. All of these have orders divisible by 6, and for $q \geq 5$ they are ruled out by Proposition 6.1. The same result shows that the maximal order torus normaliser of type $(q+1)^4.W(F_4)$ is too small, using that only involutions $s \in G$ attain $|C_G(s)|_p = q^{16}$ while all other semisimple elements $1 \neq s \in G$ have $|\text{St}(s)| = |C_G(s)|_p \leq q^{10}$. \square

While we expect that with some more work the remaining exceptional groups can also be dealt with along those same lines, the bound in Proposition 6.1 is definitely too weak to handle the linear and unitary groups. It was shown in [17, Thm 1.3.2] that the only examples for $G = \text{GL}_n(p)$ in defining characteristic p are obtained from the ones listed above. This argument can be used to see that in fact there are no further examples for $G = \text{L}_n(q)$ when $\gcd(n, q-1) = 1$.

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