CHARACTERS OF ODD DEGREE

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Dedicated to Gabriel Navarro, for his fundamental contributions

Abstract. We prove the McKay conjecture on characters of odd degree. A major step in the proof is the verification of the inductive McKay condition for groups of Lie type and primes ℓ such that a Sylow ℓ-subgroup or its maximal normal abelian subgroup is contained in a maximally split torus by means of a new equivariant version of Harish-Chandra induction. Specifics of characters of odd degree, namely that most of them lie in the principal Harish-Chandra series then allow us to deduce from it the McKay conjecture for the prime 2, hence for characters of odd degree.

1. Introduction

In his 1972 note [MK72] dedicated to Richard Brauer on the occasion of his 70th birthday John McKay put forward the following conjecture, based on observations on the known character tables of finite simple groups and of symmetric groups:

For a finite simple group G, $m_2(G) = m_2(\mathrm{N}_G(S_2))$, where S_2 is a Sylow 2-group of G.

Here, for a finite group H, $m_2(H)$ denotes the number of complex irreducible characters of H of odd degree. Soon after the appearance of [MK72], this observation was generalised to arbitrary finite groups and primes. The *McKay conjecture* thus claims that for every finite group G and every prime ℓ the number of ordinary irreducible characters $\chi \in \mathrm{Irr}(G)$ with $\ell \nmid \chi(1)$ is locally determined, namely

$$|\mathrm{Irr}_{\ell'}(G)| = |\mathrm{Irr}_{\ell'}(\mathrm{N}_G(P))|,$$

where P is a Sylow ℓ-subgroup of G and $\mathrm{N}_G(P)$ denotes its normaliser in G. The main result of our paper is the proof of that conjecture for *all* finite groups and the prime 2.

Theorem 1. *Let G be a finite group. Then the numbers of odd degree irreducible characters of G and of the normaliser of a Sylow 2-subgroup of G agree.*

McKay's conjecture had a decisive influence on the development of modern representation theory of finite groups. Its prediction of how local structures like the normaliser of a Sylow ℓ-subgroup should influence the representation theory of a group gave rise to a

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whole array of stronger and farther reaching conjectures, like those of Alperin, of Broué and of Dade. Simultaneously, functors relating the representation theory of certain families of finite (nearly simple) groups with those of suitable subgroups were introduced. For example Harish-Chandra induction and its generalisation by Deligne–Lusztig are key to the parametrisation of characters of groups of Lie type.

It was Gabriel Navarro who, in his work and in his talks, insisted that the McKay conjecture 'lies at the heart of everything'. His insights led to the proof of the fundamental result [IMN07] that the McKay conjecture holds for all finite groups at a prime ℓ, if every finite non-abelian simple group satisfies a set of properties, the now so-called *inductive McKay condition*, for ℓ. (A streamlined version of this reduction was presented in [Sp13] while a novel approach to groups with self-normalising Sylow 2-subgroups was recently devised by Navarro and Tiep [NT15].) This opens the possibility to solve the conjecture through the classification of finite simple groups. Thanks to the work of several authors this inductive condition has been shown for all but seven infinite series of simple groups of Lie type S at primes ℓ different from the defining characteristic of S, see [CS15, Ma08a, Sp12].

The second main result of our paper is meant to provide an important step towards verifying the McKay conjecture in the case of odd primes, showing that the inductive McKay condition holds for most simple groups of Lie type in the maximally split case:

Theorem 2. *Let* $\mathbf{G}$ *be a simple linear algebraic group of simply connected type defined over* $\mathbb{F}_q$ *with respect to the Frobenius endomorphism* $F : \mathbf{G} \to \mathbf{G}$ *such that* $S := \mathbf{G}^F / \mathrm{Z}(\mathbf{G}^F)$ *is simple. Assume that* $\mathbf{G}^F \notin \{\mathsf{D}_{l,\mathrm{sc}}(q), \mathsf{E}_{6,\mathrm{sc}}(q)\}$ *for any prime power* q*. Then the inductive McKay condition from* [IMN07, §10] *holds for* S *and all primes* ℓ *dividing* $q-1$*.*

For many simple groups S of Lie type and primes ℓ different from the defining characteristic of S the authors had constructed a bijection satisfying some (but not all) of the required properties from the inductive McKay condition. Moreover, in the cases where the associated algebraic group has connected centre, Cabanes and the second author [CS13] could then verify the inductive McKay condition. It thus remains to deal with simple groups of Lie type coming from algebraic groups of simply connected type with disconnected centre.

One decisive ingredient in our proof is a criterion for the inductive McKay condition tailored to groups of Lie type, see [Sp12, Thm. 2.12], which we recall here in Theorem 2.1. It had already been used for groups of type A_l as well as in the defining characteristic, see [CS15] and [Sp12]. The main assumption of that theorem on the universal covering group G of a simple group S of Lie type and the prime ℓ consists of three requirements:

- the global part concerns the stabilisers in the automorphism group and the extendibility of elements in $\mathrm{Irr}_{\ell'}(G)$, see assumption 2.1(ii);
- for a suitably chosen subgroup N that has properties similar to the normaliser of a Sylow ℓ-subgroup of G, the elements of $\mathrm{Irr}_{\ell'}(N)$ have only stabilisers of specific structures and have an analogous property with respect to extendibility, see assumption 2.1(iii);
- there exists an equivariant global-local bijection between the relevant characters of certain groups containing G and N, respectively, see assumption 2.1(iv).

We successively establish those assumptions in the cases relevant to our Theorems 1 and 2. In accordance with [Ma07] we choose N to be the normaliser of a suitable Sylow d-torus. Extending earlier results of the second author we derive the required statement about the stabilisers of local characters, see Section 3. Afterwards we study the parametrisation of irreducible characters in terms of Harish-Chandra induction, control how automorphisms act on these characters and express this in terms of their labels, see Theorem 4.6. The proof requires an equivariant version of Howlett–Lehrer theory describing the decomposition of Harish-Chandra induced cuspidal characters and relies on an extendibility result of Howlett–Lehrer and Lusztig. We then prove that many characters of G have stabilisers of the structure required in the criterion.

Structure of the paper. After introducing some notation in Section 2, we start by recalling the parametrisation of characters of normalisers of Sylow d-tori for $d \in \{1, 2\}$ and describe how automorphisms act on the characters and the associated labels in Section 3.

In Section 4, after recalling the basic results on the endomorphism algebra of Harish-Chandra induced modules of G, we describe the action of outer automorphisms $\sigma \in \mathrm{Aut}(G)$ on such modules. This enables us in Theorem 5.2 to construct an equivariant local-global bijection given by Harish-Chandra induction. The aforementioned results on stabilisers of characters of the normaliser of a maximally split torus leads to a description of the stabilisers of some characters of G in a similar way, see Corollary 5.3.

The remaining part of the paper is devoted to the completion of the proof of our main Theorems 1 and 2. First we show that all necessary assumptions of Theorem 2.1 are satisfied for proving Theorem 2 and clarify which additional properties need to be proved for obtaining an even more general statement. Then, after the classification of odd degree characters of quasi-simple groups of Lie type in Theorem 7.7 which may be of independent interest, we complete the proof of Theorem 1.

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2. Background

In this section we recall the criterion from [Sp12] for the inductive McKay condition that is the main tool in the proof of our main result. Afterwards we introduce the groups of Lie type that play a central role in the paper and describe their automorphisms.

2.A. **A criterion for the inductive McKay condition.** We first introduce some notation. If a group A acts on a finite set X we denote by A_x the stabiliser of $x \in X$ in A, analogously we denote by $A_{X'}$ the setwise stabiliser of $X' \subseteq X$. For an element $a \in A$ we denote by $o(a)$ the order of a. If A acts on a group G by automorphisms, there is a natural action of A on $\mathrm{Irr}(G)$ given by

$$^{a^{-1}}\chi(g) = \chi^a(g) = \chi(g^{a^{-1}}) \quad \text{for every } g \in G,\ a \in A \text{ and } \chi \in \mathrm{Irr}(G).$$

For $P \leq G$ and $\chi \in \mathrm{Irr}(H)$ for some A_P-stable subgroup $H \leq G$, we denote by $A_{P,\chi}$ the stabiliser of χ in A_P.

We denote the restriction of $\chi \in \mathrm{Irr}(G)$ to a subgroup $H \leq G$ by $\chi\rceil_H$, while χ^G denotes the character induced from $\psi \in \mathrm{Irr}(H)$ to G. For $N \lhd G$ and $\chi \in \mathrm{Irr}(G)$ we denote by $\mathrm{Irr}(N \mid \chi)$ the set of irreducible constituents of the restricted character $\chi\rceil_N$, and for $\psi \in \mathrm{Irr}(N)$, the set of irreducible constituents of the induced character ψ^G is denoted by $\mathrm{Irr}(G \mid \psi)$. For a subset $\mathcal{N} \subseteq \mathrm{Irr}(N)$ we define

$$\mathrm{Irr}(G \mid \mathcal{N}) := \bigcup_{\chi \in \mathcal{N}} \mathrm{Irr}(G \mid \chi).$$

Additionally, for $N \lhd G$ we sometimes identify the characters of G/N with the characters of G whose kernel contains N. For a prime ℓ we let $\mathrm{Irr}_{\ell'}(G) := \{\chi \in \mathrm{Irr}(G) \mid \ell \nmid \chi(1)\}$.

The following criterion was proved in Späth [Sp12, Thm. 2.12]:

Theorem 2.1. *Let S be a finite non-abelian simple group and ℓ a prime dividing $|S|$. Let G be the maximal perfect central extension of S and Q a Sylow ℓ-subgroup of G. Assume there exist groups A, $\widetilde{G} \leq A$, $D \leq A$ and $N \lneq G$, such that with $\widetilde{N} := N\mathrm{N}_{\widetilde{G}}(Q)$ the following conditions hold:*

- (i) (1) *$G \lhd A$, $G \leq \widetilde{G}$ and $A = \widetilde{G} \rtimes D$,*
  - (2) *$\widetilde{G}/G$ is abelian,*
  - (3) *$\mathrm{C}_{\widetilde{G}\rtimes D}(G) = \mathrm{Z}(\widetilde{G})$ and $A/\mathrm{Z}(\widetilde{G}) \cong \mathrm{Aut}(G)$ by the natural map,*
  - (4) *N is $\mathrm{Aut}(G)_Q$-stable,*
  - (5) *$\mathrm{N}_G(Q) \leq N$,*
  - (6) *every $\chi \in \mathrm{Irr}_{\ell'}(G)$ extends to its stabiliser $\widetilde{G}_\chi$,*
  - (7) *every $\psi \in \mathrm{Irr}_{\ell'}(N)$ extends to its stabiliser $\widetilde{N}_\psi$.*
- (ii) *Let $\mathcal{G} := \mathrm{Irr}\left(\widetilde{G} \mid \mathrm{Irr}_{\ell'}(G)\right)$. For every $\chi \in \mathcal{G}$ there exists some $\chi_0 \in \mathrm{Irr}(G \mid \chi)$ such that*
  - (1) *$(\widetilde{G} \rtimes D)_{\chi_0} = \widetilde{G}_{\chi_0} \rtimes D_{\chi_0}$ and*
  - (2) *χ_0 extends to $(G \rtimes D)_{\chi_0}$.*
- (iii) *Let $\mathcal{N} := \mathrm{Irr}\left(\widetilde{N} \mid \mathrm{Irr}_{\ell'}(N)\right)$. For every $\psi \in \mathcal{N}$ there exists some $\psi_0 \in \mathrm{Irr}(N \mid \psi)$ such that $O := G(\widetilde{G} \rtimes D)_{N,\psi_0}$ satisfies*
  - (1) *$O = (\widetilde{G} \cap O) \rtimes (D \cap O)$ and*
  - (2) *ψ_0 extends to $(G \rtimes D)_{N,\psi_0}$.*
- (iv) *There exists a $(\widetilde{G} \rtimes D)_Q$-equivariant bijection $\widetilde{\Omega} : \mathcal{G} \longrightarrow \mathcal{N}$ with*
  - (1) *$\widetilde{\Omega}(\mathcal{G} \cap \mathrm{Irr}(\widetilde{G} \mid \nu)) = \mathcal{N} \cap \mathrm{Irr}(\widetilde{N} \mid \nu)$ for every $\nu \in \mathrm{Irr}(\mathrm{Z}(\widetilde{G}))$,*
  - (2) *$\widetilde{\Omega}(\chi\delta) = \widetilde{\Omega}(\chi)\, \delta\rceil_{\widetilde{N}}$ for every $\chi \in \mathcal{G}$ and every $\delta \in \mathrm{Irr}(\widetilde{G}|1_G)$.*

Then the inductive McKay condition from [IMN07, §10] *holds for S and ℓ.*

2.B. **Simple groups of Lie type.** We now introduce the most relevant groups and automorphisms. For the later detailed calculations it is relevant to fix them in a rather precise way. Let $\mathbf{G}$ be a simple linear algebraic group of simply connected type over an algebraic closure of $\mathbb{F}_q$. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ with maximal torus $\mathbf{T}$. Let Φ, Φ^+ and Δ

denote the set of roots, positive roots and simple roots of $\mathbf{G}$ that are determined by $\mathbf{T}$ and $\mathbf{B}$. Let $\mathbf{N} := \mathrm{N}_{\mathbf{G}}(\mathbf{T})$. We denote by W the Weyl group of $\mathbf{G}$ and by $\pi : \mathrm{N}_{\mathbf{G}}(\mathbf{T}) \to W$ the defining epimorphism. For calculations with elements of $\mathbf{G}$ we use the Chevalley generators subject to the Steinberg relations as in [GLS, Thm. 1.12.1], i.e., the elements $x_\alpha(t)$, $n_\alpha(t)$ and $h_\alpha(t)$ ($t \in \overline{\mathbb{F}}_q$ and $\alpha \in \Phi$) defined therein.

In the following we describe automorphisms of $\mathbf{G}$. Let p be the prime with $p \mid q$ and $F_0 : \mathbf{G} \to \mathbf{G}$ the *field endomorphism* of $\mathbf{G}$ given by

$$F_0(x_\alpha(t)) = x_\alpha(t^p) \quad \text{for every } t \in \overline{\mathbb{F}}_q \text{ and } \alpha \in \Phi.$$

Any length-preserving automorphism τ of the Dynkin diagram associated to Δ and hence automorphism of Φ determines a *graph automorphism* γ of $\mathbf{G}$ given by

$$\gamma(x_\alpha(t)) = x_{\tau(\alpha)}(t) \quad \text{for every } t \in \overline{\mathbb{F}}_q \text{ and } \alpha \in \pm\Delta.$$

Note that any such γ commutes with F_0.

For the construction of diagonal automorphisms of the associated finite groups of Lie type we introduce further groups: Let r be the rank of $\mathrm{Z}(\mathbf{G})$ (as abelian group) and $\mathbf{Z} \cong (\overline{\mathbb{F}}_q^\times)^r$ a torus of that rank with an embedding of $\mathrm{Z}(\mathbf{G})$. We set

$$\widetilde{\mathbf{G}} := \mathbf{G} \times_{\mathrm{Z}(\mathbf{G})} \mathbf{Z},$$

the central product of $\mathbf{G}$ with $\mathbf{Z}$ over $\mathrm{Z}(\mathbf{G})$. Then $\widetilde{\mathbf{G}}$ is a connected reductive group with connected centre and the natural map $\mathbf{G} \to \widetilde{\mathbf{G}}$ is a regular embedding, see [CE, 15.1]. Note that $\widetilde{\mathbf{B}} := \mathbf{B}\mathbf{Z}$ is a Borel subgroup of $\widetilde{\mathbf{G}}$ and $\widetilde{\mathbf{T}} := \mathbf{T}\mathbf{Z}$ is a maximal torus therein. Furthermore let $\widetilde{\mathbf{N}} := \mathrm{N}_{\widetilde{\mathbf{G}}}(\widetilde{\mathbf{T}}) = \mathbf{N}\mathbf{Z}$.

As F_0 acts on $Z(\mathbf{G})$ via $x \mapsto x^p$ for every $x \in \mathrm{Z}(\mathbf{G})$ we can extend it to a Frobenius endomorphism $F_0 : \widetilde{\mathbf{G}} \to \widetilde{\mathbf{G}}$ via

$$F_0(g, x) := (F_0(g), x^p) \quad \text{for every } g \in \mathbf{G} \text{ and } x \in \mathbf{Z}.$$

Now assume that γ is a graph automorphism of $\mathbf{G}$. If γ acts trivially on $\mathrm{Z}(\mathbf{G})$ then it extends to an automorphism of $\widetilde{\mathbf{G}}$ which we also denote by γ, via

$$\gamma(g, x) := (\gamma(g), x) \quad \text{for every } g \in \mathbf{G} \text{ and } x \in \mathbf{Z}.$$

If γ acts on $\mathrm{Z}(\mathbf{G})$ by inversion then it can be extended via

$$\gamma(g, x) := (\gamma(g), x^{-1}) \quad \text{for every } g \in \mathbf{G} \text{ and } x \in \mathbf{Z}.$$

A similar extension of γ is possible in the remaining cases. In any case F_0 and γ stabilise $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{T}}$.

Now consider a Steinberg endomorphism $F := F_0^m\gamma$, with γ a (possibly trivial) graph automorphism of $\mathbf{G}$. Then F defines an $\mathbb{F}_q$-structure on $\widetilde{\mathbf{G}}$, where $q = p^m$, and $\mathbf{B}, \mathbf{T}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{T}}$ are F-stable, so in particular $\mathbf{T}, \widetilde{\mathbf{T}}$ are maximally split tori in $\mathbf{G}$, $\widetilde{\mathbf{G}}$ respectively. We let $G := \mathbf{G}^F$. By construction the order of F_0 as automorphism of $\widetilde{G} := \widetilde{\mathbf{G}}^F$ coincides with that of F_0 as automorphism of G. The analogous statement also holds for any graph automorphism γ and the automorphisms of $\widetilde{G}$ associated with it.

Let D be the subgroup of $\mathrm{Aut}(G)$ generated by F_0 and the graph automorphisms commuting with F. Then $\widetilde{G} \rtimes D$ is well-defined and induces all automorphisms of G, see [GLS, Thm. 2.5.1]. Moreover D acts naturally on the set of F-stable subgroups of $\mathbf{G}$.

2.C. **An embedding of the group $\mathsf{D}_{l,sc}(q)$ into $\mathsf{B}_{l,sc}(q)$.** We recall an embedding of $\mathsf{D}_{l,sc}(q)$ into $\mathsf{B}_{l,sc}(q)$ given explicitly in [Sp10, 10.1] in terms of the aforementioned Chevalley generators. Let $\overline{\Phi}$ be a root system of type B_l with base $\Delta = \{\overline{\alpha}_1, \alpha_2, \ldots, \alpha_l\}$, where $\overline{\alpha}_1 = e_1$ and $\alpha_i = e_i - e_{i-1}$ $(i \geq 2)$ as in [GLS, Rem. 1.8.8]. Let $\overline{\mathbf{G}}$ be the associated simple algebraic group of simply connected type over $\overline{\mathbb{F}}_q$. In analogy to our previous terminology we denote its Chevalley generators by $\overline{x}_\alpha(t_1)$, $\overline{n}_\alpha(t_2)$ and $\overline{h}_\alpha(t_2)$ with ($\alpha \in \overline{\Phi}$, $t_1 \in \overline{\mathbb{F}}_q$ and $t_2 \in \overline{\mathbb{F}}_q^\times$).

Let $\Phi \subseteq \overline{\Phi}$ be the root system consisting of all long roots of $\overline{\Phi}$. Then the group $\langle x_\alpha(t) \mid \alpha \in \Phi,\, t \in \overline{\mathbb{F}}_q \rangle$ is a simply connected simple group over $\overline{\mathbb{F}}_q$ with the root system Φ of type D_l.

Whenever Φ is of type D_l, we identify $\mathbf{G}$ with $\langle \overline{x}_\alpha(t) \mid \alpha \in \Phi, t \in \overline{\mathbb{F}}_q \rangle$ via $\iota_{\mathsf{D}} : \mathbf{G} \to \overline{\mathbf{G}}$, $x_\alpha(t) \mapsto \overline{x}_\alpha(t)$, and choose the notation of elements in $\mathbf{G}$ such that this defines a monomorphism. Let $\zeta \in \overline{\mathbb{F}}_q$ be a primitive $(2, q-1)^2$th root of unity. The graph automorphism of $\mathbf{G}$ of order 2 coincides with the map $x \mapsto x^{\overline{n}_{e_1}(1) \prod_{i=2}^{l} \overline{h}_{e_i}(\zeta)}$, see [Sp10, Lemma 11.2], which because of $\mathrm{Z}(\mathbf{G}) = \left\langle \overline{h}_{e_1}(-1), \prod_{i=1}^{l} \overline{h}_{e_i}(\zeta) \right\rangle$ (by [GLS, Tab. 1.12.6 and Thm. 1.12.1(e)]) coincides with $x \mapsto x^{\overline{n}_{e_1}(1)\overline{h}_{e_1}(\zeta)}$.

3. Parametrisation of some local characters

In this section we prove a result on stabilisers of characters that leads to the verification of condition 2.1(iii) in the cases considered in this paper. These results enable us later in Theorem 6.3 to construct a bijection $\widetilde{\Omega} : \mathcal{G} \to \mathcal{N}$ as required in Theorem 2.1(iv).

The aim of this section is the proof of the following statement that concerns normalisers of Sylow d-tori, sometimes also called Sylow d-normalisers. Sylow d-tori were introduced in [BM92] under the name of Sylow Φ_d-tori (with Φ_d denoting the d-th cyclotomic polynomial), and play an important role in the study of height 0 characters, see [Ma07].

Theorem 3.1. *Let $d \in \{1, 2\}$, $\mathbf{S}_0$ be a Sylow d-torus of $(\mathbf{G}, F)$, $N_0 := \mathrm{N}_{\mathbf{G}}(\mathbf{S}_0)^F$, $\widetilde{N}_0 := \mathrm{N}_{\widetilde{\mathbf{G}}}(\mathbf{S}_0)^F$ and $\psi \in \mathrm{Irr}(\widetilde{N}_0)$. There exists some $\psi_0 \in \mathrm{Irr}(N_0 \mid \psi)$ such that*

(1) *$O_0 = (\widetilde{\mathbf{G}}^F \cap O_0) \rtimes (D \cap O_0)$ for $O_0 := \mathbf{G}^F(\widetilde{\mathbf{G}}^F \rtimes D)_{\mathbf{S}_0,\psi_0}$; and*

(2) *ψ_0 extends to $(\mathbf{G}^F \rtimes D)_{\mathbf{S}_0,\psi_0}$.*

This statement is related to Theorem 5.1 of [CS15], where the same assertion was proved for all positive integers d in the case that the root system of $\mathbf{G}$ is of type A_l. Accordingly we may and will assume in the following that Φ is not of type A_l.

We verify the statement in five steps mimicking the strategy applied in [CS15, Sec. 5]. First, in 3.A we replace $\mathbf{G}^F$ by an isomorphic group, then for subgroups of this group we construct in 3.B an extension map that is compatible with certain automorphisms of $\mathbf{G}^F$, which gives in 3.C a parametrisation of $\mathrm{Irr}(N_0)$. In the end, the condition 2.1(iii.1) on the structure of stabilisers is deduced from properties of characters of relative inertia groups.

By what we said before we may and will also assume throughout this section that D is non-trivial and that $\widetilde{\mathbf{G}}$ induces non-inner automorphisms on $\mathbf{G}$. Accordingly the root system Φ of $\mathbf{G}$ is of type B_l, C_l, D_l, E_6 or E_7 and $\mathrm{Z}(\mathbf{G}^F) \neq 1$, hence in particular $\mathbf{G}^F \neq {}^3\mathsf{D}_{4,\mathrm{sc}}(q)$.

3.A. **Transfer to twisted groups.** Recall the notations from Section 2. We set $V := \langle n_\alpha(\pm 1) \mid \alpha \in \Phi \rangle \leq \mathrm{N}_{\mathbf{G}}(\mathbf{T})$, and $H := V \cap \mathbf{T}$. Since a set of generators of V satisfies the braid relations, computations with the Steinberg relations show that H is an elementary abelian 2-group of rank $|\Delta|$. We define $v \in \mathbf{G}$ as

$$v := \begin{cases} \mathrm{id}_{\mathbf{G}} & \text{if } d = 1, \\ \widetilde{\mathbf{w}_0} & \text{if } d = 2, \end{cases}$$

where $\widetilde{\mathbf{w}_0}$ is the canonical representative in V of the longest element of W defined as in [Sp10, Def. 3.2].

Lemma 3.2. *The torus* $\mathbf{T}$ *contains a Sylow* d*-torus* $\mathbf{S}$ *of* $(\mathbf{G}, vF)$*. Moreover* $\mathbf{T} = \mathrm{C}_{\mathbf{G}}(\mathbf{S})$ *and* $N = TV_1$*, where* $N := \mathrm{N}_{\mathbf{G}}(\mathbf{S})^{vF}$*,* $T := \mathbf{T}^{vF}$ *and* $V_1 := V^{vF}$*.*

Proof. Let ϕ denote the automorphism induced by F on W. Comparing with the tables in [Sp74, Sect. 5 and 6] one sees that $\pi(v)\phi$ is a d-regular element of $W\phi$ in the sense of Springer, see [Sp74, Sect. 4 and 6]. Hence the centraliser of any Sylow d-torus in $\mathbf{G}$ is a torus.

According to [Sp10, Rem. 3.3 and Lemma 3.4] there exists some Sylow d-torus $\mathbf{S} \leq \mathbf{T}$ of $(\mathbf{G}, vF)$. If Φ is of classical type and $F = F_0^m$ then $TV_1 = N$ by [Sp10, Rem. 3.3(c)]. For exceptional types this was proven in [Sp09, Prop. 6.3 and 6.4].

It remains to consider the case where $\mathbf{G}^F = {}^2\mathsf{D}_{l,\mathrm{sc}}(q)$. Here for $d = 1$ one uses [Sp10, Lemma 11.2] and computes using the Steinberg relations that H^F is an elementary abelian 2-group of rank $l-1$ and that $\pi(V_1)$ is isomorphic to a Coxeter group of type B_{l-1} and hence to $\mathrm{C}_W(\phi)$. One can see analogously for $d = 2$ and hence $v = \widetilde{\mathbf{w}_0}$ that H^{vF} is an elementary abelian 2-group of rank $l-1$, and $\pi(V^{vF}) = \mathrm{C}_W(\pi(v)\phi)$ if $w_0 \in \mathrm{Z}(W)$ and hence $v \in \mathrm{Z}(V)$. If $w_0 \notin \mathrm{Z}(W)$ computations in the braid group show that $H^{vF} = H$ and $V^{vF} = V$. $\square$

Notation 3.3. Let $e := o(v)$, the order of v. In the following we denote by C_i the cyclic group of order i. Write E_1 for the subgroup $\Gamma_{\bar{K}}$ of $\mathrm{Aut}(\mathbf{G})$ from [GLS, Def. 1.15.5], a group generated by certain graph automorphisms. Let $E := \mathrm{C}_{2em} \times E_1$ act on $\widetilde{\mathbf{G}}^{F_0^{2em}}$ such that the first summand C_{2em} of E acts by $\langle F_0 \rangle$ and the second by the group generated by graph automorphisms. Note that this action is faithful. Let $\widehat{F}_0, \widehat{\gamma}, \widehat{F} \in E$ be the elements that act on $\widetilde{\mathbf{G}}^{F_0^{2em}}$ by F_0, γ and F, respectively.

Note that E stabilises N, T, V, v and hence H, V_1 and H^{vF}.

Proposition 3.4. *Let* $\mathbf{S}$ *and* N *be as in Lemma 3.2, and* $\widetilde{N} := \mathrm{N}_{\widetilde{\mathbf{G}}}(\mathbf{S})^{vF}$*. Suppose that for every* $\chi \in \mathrm{Irr}(\widetilde{N})$ *there exists some* $\chi_0 \in \mathrm{Irr}(N \mid \chi)$ *such that*

(1) $(\widetilde{N} \rtimes E)_{\chi_0} = \widetilde{N}_{\chi_0} \rtimes E_{\chi_0}$*; and*

(2) χ_0 *has an extension* $\widetilde{\chi}_0 \in \mathrm{Irr}(N \rtimes E_{\chi_0})$ *with* $v\widehat{F} \in \ker(\widetilde{\chi}_0)$.

Then the conclusion of Theorem 3.1 holds for $(\mathbf{G}, F)$ *and* d.

Proof. The statement is an analogue of [CS15, Prop. 5.3]. The proof given there is independent of the underlying type, and is based on the application of Lang's theorem using that v is D- and hence E-invariant. It relies on the fact that conjugation by a suitable element of $\mathbf{G}$ gives an isomorphism $\iota : \mathbf{G} \to \mathbf{G}$ with $\iota(\mathbf{G}^F) = \mathbf{G}^{vF}$. Via ι the automorphisms of $\mathbf{G}^F$ induced by $\widetilde{\mathbf{G}}^F \rtimes D$ coincide with the ones of $\mathbf{G}^{vF}$ induced by $\widetilde{\mathbf{G}}^{vF} E/\langle v\widehat{F}\rangle$, and $\widetilde{\mathbf{G}}^F \rtimes D \cong \widetilde{\mathbf{G}}^{vF} E/\langle v\widehat{F}\rangle$. □

3.B. **Extension maps with respect to $H_1 \lhd V_1$.** In order to verify the assumptions of Proposition 3.4 on the characters of N we label them via some so-called extension map.

Definition 3.5 (Definition 5.7 of [CS15])**.** Let $Y \lhd X$ and $\mathcal{Y} \subseteq \mathrm{Irr}(Y)$. We say that *maximal extendibility holds for* $\mathcal{Y}$ *with respect to* $Y \lhd X$ if every $\chi \in \mathcal{Y}$ extends (as irreducible character) to X_χ. Then, an *extension map for* $\mathcal{Y}$ *with respect to* $Y \lhd X$ is a map

$$\Lambda : \mathcal{Y} \to \bigcup_{Y \leq I \leq X} \mathrm{Irr}(I),$$

such that for every $\chi \in \mathcal{Y}$ the character $\Lambda(\chi) \in \mathrm{Irr}(X_\chi)$ is an extension of χ. If $\mathcal{Y} = \mathrm{Irr}(Y)$ we also say that there exists *an extension map with respect to* $Y \lhd X$.

The following is easily verified:

Lemma 3.6. *Let X be a finite group, $Y \lhd X$ and $\mathcal{Y} \subseteq \mathrm{Irr}(Y)$ an X-stable subset. Assume there exists an extension map for $\mathcal{Y}$ with respect to $Y \lhd X$. Then there exists an X-equivariant extension map for $\mathcal{Y}$ with respect to $Y \lhd X$.*

In order to prove Theorem 3.1 in the form suggested by Proposition 3.4 our next goal is to establish the following intermediate step. Recall $V_1 = V^{vF}$ and set $H_1 := H^{vF}$.

Theorem 3.7. *There exists a V_1E-equivariant extension map with respect to $H_1 \lhd V_1$.*

The proof will be given in several steps. We first consider the case when F is untwisted and $d = 1$.

Proposition 3.8. *For Φ not of type D_l there exists an extension map with respect to $H \lhd V$.*

Proof. According to [Sp09, Prop. 5.1] we can assume that Φ is of type B_l or C_l. Assume that $q = 3$. Then $V = \mathbf{N}^F$ and $H = \mathbf{T}^F$. Maximal extendibility holds with respect to $H = \mathbf{T}^F \lhd \mathbf{N}^F$ according to [HL80, Cor. 6.11] or [Sp10, Thm. 1.1].

By assumption q is odd. Then the isomorphism types of H and V are independent of q since V and H can be described as finitely presented groups whose relations are independent of q, see [Ti66] and [Sp07, Lemma 2.3.1(b)]. Hence the considerations for $q = 3$ already imply the statement. □

Proposition 3.9. *For Φ not of type D_l there exists a VE_1-equivariant extension map with respect to $H \lhd V$.*

Proof. If Φ has no graph automorphism Proposition 3.8 together with Lemma 3.6 proves that a V-equivariant extension map with respect to $H \lhd V$ exists.

If Φ is of type E_6 the generator $\widehat{\gamma}$ of E_1 corresponds to an automorphism of the associated braid group B, that acts by permuting the generators. The epimorphism $\tau : \mathsf{B} \to V$ is $\widehat{\gamma}$-equivariant. Let $\mathrm{r} : W \to \mathsf{B}$ be the map from [GP, 4.1.1] and w_0 the longest element in W. Conjugation with $\mathrm{r}(w_0) = \mathrm{w}_0$ acts on $\mathbf{B}$ like $\widehat{\gamma}$ by [GP, Lemma 4.1.9], analogously conjugating by $\widetilde{\mathbf{w}_0}$, which is the image of $\mathrm{r}(w_0)$ under the natural epimorphism from $\mathbf{B}$ to V, acts on V like $\widehat{\gamma}$. Hence the automorphism induced by $\widehat{\gamma}$ on V is an inner automorphism and hence any V-equivariant extension map is also VE_1-equivariant. □

Proposition 3.10. *If Φ is of type D_l there exists a VE_1-equivariant extension map with respect to $H \lhd V$.*

Proof. First let us consider the case where Φ is of type $\mathsf{D}_l \neq \mathsf{D}_4$. Let $\iota_{\mathsf{D}} : \mathbf{G} \to \overline{\mathbf{G}}$ be the embedding from 2.C. Then $\iota_{\mathsf{D}}(V) \leq \overline{V} := \langle \overline{n}_\alpha(\pm 1) \mid \alpha \in \overline{\Phi} \rangle$ and $\iota_{\mathsf{D}}(H) = \overline{H} := \langle \overline{h}_\alpha(\pm 1) \mid \alpha \in \overline{\Phi} \rangle$. Note that $\overline{H} = H$ and hence $V_\lambda \leq \overline{V}_\lambda$ for every $\lambda \in \mathrm{Irr}(H) = \mathrm{Irr}(\overline{H})$. Let Λ_{B} be the $\overline{V}$-equivariant extension map with respect to $\overline{H} \lhd \overline{V}$ from Proposition 3.8.

As explained in 2.C, $\gamma(x) = x^{\overline{n}_{e_1}(1)\overline{h}_{e_1}(\zeta)}$ for every $x \in \mathbf{G}$, where ζ is some primitive 8th root of unity. (Note that because of our initial reductions we can assume that $2 \nmid q$.) Let $\zeta' \in \overline{\mathbb{F}}_q$ be a primitive 8th root of unity and $t := \prod_{i=1}^l h_{e_i}(\zeta')$. For $n \in V$ we have

$$[t, n] = \prod_{j \in J} h_{e_j}(\zeta'^2)$$

for a set $J \subseteq \{1, \ldots, l\}$ with $2 \mid |J|$. This proves $[t, V] \subseteq H$ and hence $V^t = V$.

Hence there is a well-defined extension map Λ_0 given by

$$\Lambda_0(\lambda) = (\Lambda_{\mathsf{B}}(\lambda)\rceil_{V_\lambda})^t \qquad \text{for all } \lambda \in \mathrm{Irr}(H).$$

Since Λ_{B} is $\overline{V}$-equivariant, Λ_0 is $\overline{V}^t$-equivariant. The element $\overline{n}_{\alpha_1}(1)^t = \overline{n}_{\alpha_1}(1) h_{\alpha_1}(\zeta'^2)$ and γ induce the same automorphism on V, according to 2.C. Hence Λ_0 is $V\langle\gamma\rangle$-equivariant.

Now assume that Φ is of type D_4. According to the above considerations there exists some $V\langle\widehat{\gamma}_2\rangle$-equivariant extension map with respect to $H \lhd V$ for some $\widehat{\gamma}_2 \in E_1$ of order 2. For the proof it is sufficient to show maximal extendibility for some VE_1-transversal $\mathbb{T} \subset \mathrm{Irr}(H)$ with respect to $H \lhd VE_1$. We may choose $\mathbb{T}$ such that for each $\lambda \in \mathbb{T}$ some Sylow 2-subgroup of $(VE_1)_\lambda$ is contained in $(V\langle\widehat{\gamma}_2\rangle)_\lambda$.

Recall that H is a 2-group. According to [I, Thm. 6.26], every $\lambda \in \mathbb{T}$ extends to $(VE_1)_\lambda$ if λ extends to the full preimage of a Sylow 2-subgroup of $(VE_1)_\lambda/H$, which is itself a Sylow 2-subgroup of $(VE_1)_\lambda$. By the choice of $\mathbb{T}$ a Sylow 2-subgroup of $(VE_1)_\lambda$ is contained in $(V\langle\widehat{\gamma}_2\rangle)_\lambda$ for every $\lambda \in \mathbb{T}$. By the above λ has a $(V\langle\widehat{\gamma}_2\rangle)$-invariant extension to V_λ. Since $(V\langle\widehat{\gamma}_2\rangle)_\lambda/V_\lambda$ is cyclic, λ extends to $(V\langle\gamma_2\rangle)_\lambda$ by [I, Cor. 11.22], and hence to a Sylow 2-subgroup of $(VE_1)_\lambda$. This proves the claim. □

In the next step we construct extension maps in the case where the Frobenius endomorphism is twisted. Recall $V_1 = V^{vF}$ and $H_1 = H^{vF}$.

Lemma 3.11. *Let F_0, m and γ be defined as in 2.B. Assume that Φ is of type D_l, $v = \mathrm{id}_{\mathbf{G}}$ and $F = \gamma F_0^m$. Then there exists a V_1E_1-equivariant extension map with respect to $H_1 \lhd V_1$.*

Proof. By the proof of Lemma 3.2, V_1/H_1 is isomorphic to $\mathrm{C}_W(\gamma)$.

Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be a base of Φ. For a positive integer i and elements $x, y \in V$ let $\mathrm{prod}(x, y, i)$ be defined by

$$\mathrm{prod}(x, y, i) = \underbrace{x \cdot y \cdot x \cdot y \cdots}_{i}.$$

Following [Ti66] the group V coincides with the extended Weyl group of $\mathbf{G}$ that is the finitely presented group generated by $n_i = n_{\alpha_i}(1)$ and $h_i = n_i^2$, subject to the relations

$$h_ih_j = h_jh_i, \qquad h_i^2 = 1,$$
$$\mathrm{prod}(n_i, n_j, m_{ij}) = \mathrm{prod}(n_j, n_i, m_{ij}), \qquad h_i^{n_j} = h_j^{A_{i,j}} h_i \text{ for all } 1 \leq i, j \leq l,$$

where m_{ij} is the order of $s_{\alpha_i}s_{\alpha_j}$ in W and $(A_{i,j})$ is the associated Cartan matrix, see [Sp07, Lemma 2.3.1(b)] for more details.

Assume that Δ is chosen such that the graph automorphism γ of order 2 permutes α_1 and α_2. Using straightforward calculations one sees that the elements $n_2' := n_{\alpha_1}(-1)n_{\alpha_2}(-1)$ and $n_i' := n_{\alpha_i}(-1)$ for $i > 2$ satisfy the defining relations of an extended Weyl group of type B_{l-1}. As the orders of the groups coincide, they are isomorphic. Together with Proposition 3.8 this implies the existence of the required extension map.

Note that according to Lemma 3.6 the extension map can be chosen to be V_1-equivariant. Since by definition γ acts trivially on V_1 the extension map is also V_1E_1-equivariant. □

Proof of Theorem 3.7. The statement follows from the existence of a V_1E_1-equivariant extension map since $\widehat{F}_0$ acts trivially on V.

If Φ is of type E_6 the claim is implied by [Sp09, Lemma 8.2]. In the remaining cases Propositions 3.9 and 3.10, and Lemma 3.11 imply the statement if $d = 1$.

If $d = 2$ and Φ is of type B_l, C_l or E_7 the proof of [Sp09, Lemma 6.1] shows that $v \in \mathrm{Z}(V)$. Hence $H = H^{vF} = H_1$ and $V = V^{vF} = V_1$. Then Proposition 3.8 yields the claim.

The only remaining case is when Φ is of type D_l and $d = 2$. Computations in V show that either $V_1 = V$ or $V_1 = \mathrm{C}_V(\gamma)$ and then the statement about the maximal extendibility follows from the observations made for $d = 1$ in Proposition 3.10 and Lemma 3.11. Hence there exists a V_1E_1-equivariant extension map with respect to $H_1 \lhd V_1$ in all cases. □

We next state a lemma helping to construct extensions with specific properties.

Lemma 3.12. *Let $\lambda \in \mathrm{Irr}(H_1)$ and $\widetilde{\lambda} \in \mathrm{Irr}(V_{1,\lambda})$ a $(V_1E)_\lambda$-invariant extension of λ. Then $\widetilde{\lambda}$ has an extension $\widehat{\lambda} \in \mathrm{Irr}((V_1E)_\lambda)$ with $\widehat{\lambda}(v\widehat{F}) = 1$.*

Proof. Recall $E_1 = \langle\widehat{\gamma}\rangle \leq E$ when $\mathbf{G}^F \neq \mathsf{D}_{4,\mathrm{sc}}(q)$ and $E_1 = \langle\widehat{\gamma}_2, \widehat{\gamma}_3\rangle$ otherwise, with $\widehat{\gamma}_i$ of order i. Note that $\widehat{F}_0$ is central in $V_1 \rtimes E$, i.e., $V_1E = (V_1 \rtimes E_1) \times \langle\widehat{F}_0\rangle$. Since all Sylow subgroups of E_1 are cyclic, $\widetilde{\lambda}$ extends to a character ψ of $(V_1E_1)_{\widetilde{\lambda}}$. Note that $(V_1E_1)_{\widetilde{\lambda}} = (V_1E_1)_\lambda$.

Recall that $\mathbf{G}^F \neq {}^3\mathsf{D}_{4,\mathrm{sc}}(q)$ and $v\widehat{F} = v\kappa\widehat{F}_0^m$ for some $\kappa \in E_1$. We have $o(\widehat{F}_0^m) = 2o(v)$ by the definition of E and $o(v\kappa) \mid (2o(v))$ since v and κ commute. Accordingly there exists some character $\epsilon \in \mathrm{Irr}(\langle \widehat{F}_0\rangle)$ with $\psi(1)\epsilon(\widehat{F}_0^m) = \psi(v\kappa)^{-1}$. The character $\widehat{\lambda} = \psi \times \epsilon$ is an extension of $\widetilde{\lambda}$ with the required properties. □

3.C. **Parametrisation of** $\mathrm{Irr}(N)$**.** For the later understanding of the characters of $\mathrm{Irr}(N)$ we construct an extension map with respect to $T \lhd N$. Recall $N := \mathrm{N}_{\mathbf{G}}(\mathbf{S})^{vF}$ and $T := \mathrm{C}_{\mathbf{G}}(\mathbf{S})^{vF} = \mathbf{T}^{vF}$.

Corollary 3.13. *There exists an extension map Λ with respect to $T \lhd N$ such that*

(1) *Λ is $N \rtimes E$-equivariant; and*
(2) *for every $\lambda \in \mathrm{Irr}(T)$, there exists some linear $\widetilde{\lambda} \in \mathrm{Irr}\left((N \rtimes E)_\lambda \mid \Lambda(\lambda)\right)$ with $\widetilde{\lambda}(v\widehat{F}) = 1$.*

Note that the existence of Λ (without the properties required here) is known from [HL80, Cor. 6.11] for $d = 1$, and from [Sp09] and [Sp12].

Proof. According to Lemma 3.2 we have $N = TV_1$. Let Λ_0 be the V_1E-equivariant extension map with respect to $H_1 \lhd V_1$ from Theorem 3.7. We obtain an NE-equivariant extension map Λ by sending $\lambda \in \mathrm{Irr}(T)$ to the common extension of λ and $\Lambda_0(\lambda\rceil_{H_1})\rceil_{V_{1,\lambda}}$. According to the proof of [Sp09, Lemma 4.3], Λ is then well-defined.

For proving (2) let $\lambda \in \mathrm{Irr}(T)$. Then $\lambda_0 := \lambda\rceil_{H_1}$ extends to some $\widetilde{\lambda}_0 \in \mathrm{Irr}((V_1E)_{\lambda_0})$ with $\widetilde{\lambda}_0(v\widehat{F}) = 1$ by Lemma 3.12. According to the proof of [Sp09, Lemma 4.3] there exists a unique common extension $\widetilde{\lambda}$ of $\Lambda(\lambda)$ and $\widetilde{\lambda}_0\Big\rceil_{(V_1E)_\lambda}$ to $(NE)_\lambda$. Then $\widetilde{\lambda}(v\widehat{F}) = 1$. □

For later use we describe the action of $\widetilde{N}E$ on the extension map Λ from Corollary 3.13. Recall $\widetilde{T} := \widetilde{\mathbf{T}}^{vF}$ and $\widetilde{N} := \mathrm{N}_{\widetilde{\mathbf{G}}}(\mathbf{S})^{vF}$. In the following we set $W(\lambda) := N_\lambda/T$ for $\lambda \in \mathrm{Irr}(T)$ and $W(\widetilde{\lambda}) := N_{\widetilde{\lambda}}/T$ for $\widetilde{\lambda} \in \mathrm{Irr}(\widetilde{T})$.

Proposition 3.14. *Let $\lambda \in \mathrm{Irr}(T)$, $\widetilde{\lambda} \in \mathrm{Irr}(\widetilde{T}|\lambda)$, $x \in \widetilde{N}E$, and Λ the extension map from Corollary 3.13. Then the character $\delta \in \mathrm{Irr}(W(\lambda)^x)$ with $\delta\Lambda(\lambda^x) = \Lambda(\lambda)^x$ satisfies $\ker(\delta) \geq W(\widetilde{\lambda}^x)$.*

Proof. Observe that δ is well-defined by [I, Cor. 6.17]. Since Λ is NE-equivariant δ associated with x is trivial whenever $x \in NE$. For $x \in \widetilde{T}$ we have $\lambda^x = \lambda$ so δ has the stated property. Taking those two results together we obtain the claim. □

As mentioned earlier the extension map constructed above is key to a labelling and understanding of the characters of $\mathrm{Irr}(N)$.

Proposition 3.15. *Let Λ be the extension map from Corollary 3.13 with respect to $T \lhd N$. Then the map*

$$\Pi : \mathcal{P} = \{(\lambda, \eta) \mid \lambda \in \mathrm{Irr}(T),\, \eta \in \mathrm{Irr}(W(\lambda))\} \longrightarrow \mathrm{Irr}(N), \quad (\lambda, \eta) \longmapsto (\Lambda(\lambda)\eta)^N,$$

is surjective and satisfies

(1) $\Pi(\lambda,\eta)=\Pi(\lambda',\eta')$ *if and only if there exists some* $n\in N$ *such that* ${}^n\lambda=\lambda'$ *and* ${}^n\eta=\eta'$.
(2) ${}^\sigma\Pi(\lambda,\eta)=\Pi({}^\sigma\lambda,{}^\sigma\eta)$ *for every* $\sigma\in E$.
(3) *Let* $t\in\widetilde T$, *and* $\nu_t\in\mathrm{Irr}(N_\lambda)$ *be the linear character given by* ${}^t\Lambda(\lambda)=\Lambda(\lambda)\nu_t$. *Then* $N_{\widetilde\lambda}=\ker(\nu_t)$ *for any* $\widetilde\lambda\in\mathrm{Irr}(\langle T,t\rangle\,|\lambda)$. *For* $\widetilde\lambda_0\in\mathrm{Irr}(\widetilde T|\lambda)$ *the map* $\widetilde T\to\mathrm{Irr}(N_\lambda/N_{\widetilde\lambda_0})$ *given by* $t\mapsto\nu_t$ *is surjective, and* ${}^t\Pi(\lambda,\eta)=\Pi(\lambda,\eta\nu_t)$.

Proof. The arguments from [CS15, Prop. 5.11] can be transferred to prove the statement. Straightforward considerations show that the map in 3.15(3) is surjective. □

3.D. **Maximal extendibility with respect to** $W(\widetilde\lambda)\lhd W(\lambda)$**.** Our aim in this subsection is to show that maximal extendibility holds with respect to $W(\widetilde\lambda)\lhd \mathrm{N}_{W_1E}(W(\widetilde\lambda))$ for every $\widetilde\lambda\in\mathrm{Irr}(\widetilde T)$ with $W_1:=\pi(N)$, where $W(\widetilde\lambda):=N_{\widetilde\lambda}/T$. Two less general results are known in particular cases: For $d=1$ maximal extendibility is known to hold with respect to $W(\widetilde\lambda)\lhd W(\lambda)$ where $\lambda=\widetilde\lambda\rceil_T$, see Proposition 3.16 below. Proposition 5.12 of [CS15] shows the analogue for arbitrary positive integers d assuming that the underlying root system is of type A_l.

The statement in Theorem 3.17 plays a crucial role in proving Theorem 3.18 via the parametrisation of characters of N given above.

We start by rephrasing the old result known for $d=1$.

Proposition 3.16. *Assume that* $d=1$. *Let* $\lambda\in\mathrm{Irr}(T)$ *and* $\widetilde\lambda\in\mathrm{Irr}(\widetilde T|\lambda)$. *Then maximal extendibility holds with respect to* $W(\widetilde\lambda)\lhd W(\lambda)$.

Proof. The quotient $W(\lambda)/W(\widetilde\lambda)$ is abelian and for every $\eta\in\mathrm{Irr}(W(\lambda))$ every character $\eta_0\in\mathrm{Irr}(W(\widetilde\lambda)\mid\eta)$ has multiplicity one in the restriction $\eta\rceil_{W(\widetilde\lambda)}$, see [B, 13.13(a)]. This implies the statement. □

Theorem 3.17. *Let* $\lambda\in\mathrm{Irr}(T)$, $\widetilde\lambda\in\mathrm{Irr}(\widetilde T|\lambda)$ *and* $W_1:=\pi(N)$. *Then every* $\eta_0\in\mathrm{Irr}(W(\widetilde\lambda))$ *has an extension* $\kappa\in\mathrm{Irr}(\mathrm{N}_{W_1E}(W(\widetilde\lambda))_{\eta_0})$ *with* $v\widehat F\in\ker(\kappa)$.

Proof. We first prove that maximal extendibility holds with respect to $W(\widetilde\lambda)\lhd\mathrm{N}_{W_1E}(W(\widetilde\lambda))$. Let $(\widetilde{\mathbf G}^*,\widetilde{\mathbf T}^*,v'F^*)$ be the dual to $(\widetilde{\mathbf G},\widetilde{\mathbf T},vF)$ constructed as in [DM, Def. 13.10].

Note that because of our particular choice of v the automorphism on W induced by vF coincides with a graph automorphism ϕ' on W. The character $\widetilde\lambda$ corresponds to a semisimple element $s\in(\widetilde{\mathbf T}^*)^{v'F^*}$ of the dual group $(\mathbf G^*,v'F^*)$. Let $R(\widetilde\lambda)$ be the Weyl group of $\mathrm{C}_{\widetilde{\mathbf G}^*}(s)$. Since $\mathrm{C}_{\widetilde{\mathbf G}^*}(s)$ is connected, $R(\widetilde\lambda)$ is a reflection group. We have $W(\widetilde\lambda)=\mathrm{C}_{R(\widetilde\lambda)}(v'F^*)=\mathrm{C}_{R(\widetilde\lambda)}(\phi')$. Accordingly $W(\widetilde\lambda)$ is a reflection group and the ϕ'-orbits on the roots of $R(\widetilde\lambda)$ form a root system, which we denote by $\Phi(\lambda)$, see [MT, Thm. C.5]. Straightforward calculations show that $\Phi(\lambda)$ is already determined by λ.

The group $K:=\mathrm{N}_{W_1E}(W(\widetilde\lambda))$ acts on $W(\widetilde\lambda)$, $R(\widetilde\lambda)$ and $\Phi(\lambda)$ by conjugation. Let Δ be a base of $\Phi(\lambda)$. Then by the properties of root systems $K=W(\widetilde\lambda)\,\mathrm{Stab}_K(\Delta)$, even $K=W(\widetilde\lambda)\rtimes\mathrm{Stab}_K(\Delta)$, where $\mathrm{Stab}_K(\Delta)$ denotes the stabiliser of Δ in K.

First let us prove that maximal extendibility holds with respect to $W(\widetilde{\lambda}) \rtimes \mathrm{Aut}(\Delta)$, where $\mathrm{Aut}(\Delta)$ is the group of length-preserving automorphisms of Δ.

Whenever Δ is indecomposable the statement is true, since then all Sylow subgroups of $\mathrm{Aut}(\Delta)$ are cyclic. If $\Delta = \Delta_1 \sqcup \ldots \sqcup \Delta_r$ with isomorphic indecomposable systems Δ_i, the group $\mathrm{Aut}(\Delta)$ is isomorphic to the wreath product $\mathrm{Aut}(\Delta_1) \wr \mathrm{S}_r$. Since maximal extendibility holds with respect to $H^r \lhd H \wr \mathrm{S}_r$ for any group H according to [H, Thm. 25.6] we see that maximal extendibility holds with respect to $W(\widetilde{\lambda}) \rtimes \mathrm{Aut}(\Delta)$ in that case. Since maximal extendibility holds for $H_1 \times H_2 \lhd G_1 \times G_2$ whenever it holds for $H_1 \lhd G_1$ and $H_2 \lhd G_2$ the above implies that maximal extendibility holds with respect to $W(\widetilde{\lambda}) \lhd W(\widetilde{\lambda}) \rtimes \mathrm{Aut}(\Delta)$.

Now let $C := \mathrm{C}_K(\Delta)$. By definition $C \lhd K$. Let $\overline{K} := K/C$. Then maximal extendibility holds with respect to $W(\widetilde{\lambda}) \lhd K$ if it holds with respect to $\overline{R} := W(\widetilde{\lambda})C/C \lhd \overline{K}$. We see that $\overline{S} := \mathrm{Stab}_K(\Delta)/C$ is a subgroup of $\mathrm{Aut}(\Delta)$ and by the above maximal extendibility holds with respect to $W(\widetilde{\lambda}) \lhd W(\widetilde{\lambda}) \rtimes \overline{S}$. But this implies maximal extendibility with respect to $W(\widetilde{\lambda}) \lhd K$. This proves the first part of the claim.

We finish by constructing the required extension κ. Let $\eta_0 \in \mathrm{Irr}(W(\widetilde{\lambda}))$. Recall E_1 from 3.3. By the above η_0 extends to some $\widehat{F}_0$-stable $\kappa_1 \in \mathrm{Irr}(\mathrm{N}_{W_1 E_1}(W(\widetilde{\lambda}))_{\eta_0})$. Since $\pi(v\widehat{F}\widehat{F}_0^{-m}) \in \mathbf{Z}(W_1 E)$ and $\mathrm{N}_{W_1 E}(W(\widetilde{\lambda}))_{\eta_0} = \mathrm{N}_{W_1 E_1}(W(\widetilde{\lambda}))_{\eta_0} \times \langle \widehat{F}_0 \rangle$ the considerations from the proof of Lemma 3.12 ensure the existence of κ, as required. □

3.E. **Consequences.** The previous considerations allow us also to conclude that the considered characters of N have the structure stated in Proposition 3.4.

Theorem 3.18. *For every $\chi \in \mathrm{Irr}(\widetilde{N})$ there exists some $\chi_0 \in \mathrm{Irr}(N|\chi)$ with the following properties:*

(1) $(\widetilde{N} \rtimes E)_{\chi_0} = \widetilde{N}_{\chi_0} \rtimes E_{\chi_0}$*; and*
(2) χ_0 *has an extension* $\widetilde{\chi}_0 \in \mathrm{Irr}(N \rtimes E_{\chi_0})$ *with* $v\widehat{F} \in \ker(\widetilde{\chi}_0)$.

Proof. Let $\chi_1 \in \mathrm{Irr}(N|\chi)$ and $(\lambda, \eta) \in \mathcal{P}$ with $\chi_1 = \Pi(\lambda, \eta)$ for the map Π from Proposition 3.15.

Let $\widetilde{\lambda} \in \mathrm{Irr}(\widetilde{T}|\lambda)$ and $\eta_0 \in \mathrm{Irr}(W(\widetilde{\lambda}))$ such that $\eta \in \mathrm{Irr}(W(\lambda)|\eta_0)$. By Clifford correspondence there exists a unique character $\eta_1 \in \mathrm{Irr}(W(\lambda)_{\eta_0}|\eta_0)$ such that $\eta = \eta_1^{W(\lambda)}$. Now since $W(\lambda)/W(\widetilde{\lambda})$ is abelian and as by Proposition 3.16 maximal extendibility holds with respect to $W(\widetilde{\lambda}) \lhd W(\lambda)$, the character η_1 is an extension of η_0.

Let $W_1 := \pi(N)$. According to Theorem 3.17 there exists an $\mathrm{N}_{W_1 E}(W(\widetilde{\lambda}))_{\eta_0}$-invariant extension $\widetilde{\eta}_0 \in \mathrm{Irr}(W(\lambda)_{\eta_0})$ of η_0. The character $\eta' := (\widetilde{\eta}_0)^{W(\lambda)}$ is irreducible. Hence $\chi_0 := \Pi(\lambda, \eta')$ is a well-defined character of N.

We show that χ_0 is $\widetilde{N}$-conjugate to χ_1: Since the map $\widetilde{T} \to \mathrm{Irr}(W(\lambda)/W(\widetilde{\lambda}))$, $t \mapsto \nu_t$, from Proposition 3.15(3) is surjective, there exists $t \in \widetilde{T}$ such that $\eta' = \eta\nu_t$. This proves ${}^t\chi = {}^t\Pi(\lambda, \eta) = \Pi(\lambda, \eta\nu_t) = \chi_0$.

For analysing the stabiliser of χ_0 let $t \in \widetilde{T}$ and $e \in E$ such that $\chi_0^{te} = \chi_0$. Then there exists some $n \in N$ such that $(\lambda, \eta') = (\lambda^{ne}, (\eta')^{ne}\nu_t)$. Without loss of generality n can be chosen such that $\pi(n)e \in \mathrm{N}_{W_1E}(W(\widetilde{\lambda}))_{\eta_0}$. By the choice of $\widetilde{\eta}_0$, $\pi(n)e$ stabilises $\widetilde{\eta}_0$, hence $(\lambda^{ne}, (\eta')^{ne}) = (\lambda, \eta')$ and $\chi_0^e = \chi_0$. This proves the equation in (1).

By Corollary 3.13 the character $\Lambda(\lambda)$ has an extension $\widetilde{\lambda}$ to $(NE)_{\Lambda(\lambda)\eta'}$ with $\widetilde{\lambda}(v\widehat{F}) = 1$. On the other hand the character $\widetilde{\eta}_0$ can be chosen to have an extension $\widehat{\eta}_0$ to $\mathrm{N}_{W_1E}(W(\widetilde{\lambda}))_{\eta_0}$ with $v\widehat{F} \in \ker(\widehat{\eta}_0)$, see Theorem 3.17. We denote by κ_1 the lift of $\widehat{\eta}_0\rceil_{\mathrm{N}_{W_1E}(W(\widetilde{\lambda}))_{\eta_0,\lambda}}$ to $(NE)_{\eta_0,\lambda}$. Then $\kappa_2 := (\kappa_1)^{(NE)_{\lambda,\eta}}$ is irreducible with $v\widehat{F} \in \ker(\kappa_2)$. The character $(\widetilde{\lambda}\kappa_2)^{(NE)_{\chi_0}}$ is an extension of χ_0 with the required properties. □

Via Proposition 3.4 the above proves Theorem 3.1. For later two further consequences of our considerations are important. First we give the following interpretation of Theorem 3.1.

Lemma 3.19. *Let $\mathbf{S}_0$, N_0, $\widetilde{N}_0$ and O_0 be defined as in Theorem 3.1. For $\psi_0 \in \mathrm{Irr}(N_0)$ the following are equivalent:*

(i) $O_0 = (\widetilde{\mathbf{G}}^F \cap O_0) \rtimes (D \cap O_0)$.

(ii) $(\widetilde{\mathbf{G}}^F D)_{\mathbf{S}_0,\psi_0} = \widetilde{N}_{0,\psi_0}(\mathbf{G}^F \rtimes D)_{\mathbf{S}_0,\psi_0}$.

Proof. By the definition of O_0 one deduces (ii) from (i) by considering the stabiliser of $\mathbf{S}_0$:

$$\begin{aligned}(\widetilde{\mathbf{G}}^F D)_{\mathbf{S}_0,\psi_0} &= (O_0)_{\mathbf{S}_0} = \Big((\widetilde{\mathbf{G}}^F \cap O_0) \rtimes (D \cap O_0)\Big)_{\mathbf{S}_0} = \\ &= \Big((\widetilde{\mathbf{G}}^F \cap O_0)(\mathbf{G}^F D \cap O_0)\Big)_{\mathbf{S}_0} = (\widetilde{\mathbf{G}}^F \cap O_0)_{\mathbf{S}_0}(\mathbf{G}^F D \cap O_0)_{\mathbf{S}_0} = \\ &= \widetilde{N}_{0,\psi_0}(\mathbf{G}^F D)_{\mathbf{S}_0,\psi_0}.\end{aligned}$$

Here, recall that by [BM92, Thm. 3.4] all Sylow d-tori of $(\mathbf{G}, F)$ are $\mathbf{G}^F$-conjugate and the $(D \cap O_0)$-conjugates of $\mathbf{S}_0$ are Sylow d-tori.

Multiplying the equation in (ii) with $\mathbf{G}^F$ gives $O_0 = (\widetilde{\mathbf{G}}^F \cap O_0)((\mathbf{G}^F \rtimes D) \cap O_0)$. Since $\mathbf{G}^F \leq (\mathbf{G}^F \rtimes D) \cap O_0$ this gives (i). □

Proposition 3.20. *Let $\mathbf{S}_0$, N_0 and $\widetilde{N}_0$ be defined as in Theorem 3.1. Let $\widetilde{C}_0 := \mathrm{C}_{\widetilde{\mathbf{G}}^F}(\mathbf{S}_0)$. Then there exists some $\mathrm{N}_{\widetilde{\mathbf{G}}^F D}(\mathbf{S}_0)$-equivariant extension map $\widetilde{\Lambda}$ with respect to $\widetilde{C}_0 \lhd \widetilde{N}_0$, such that in addition $\widetilde{\Lambda}(\widetilde{\lambda}\,\delta\rceil_{\widetilde{C}_0}) = \widetilde{\Lambda}(\widetilde{\lambda})\,\delta\rceil_{\widetilde{N}_0}$ for every $\widetilde{\lambda} \in \mathrm{Irr}(\widetilde{C}_0)$ and $\delta \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F | 1_{\mathbf{G}^F})$.*

Proof. The considerations from the proof of [CS15, Cor. 5.14] can be transferred: Applying the isomorphism ι from Proposition 3.4 shows that it is sufficient to verify that there exists some NE-equivariant extension map $\widetilde{\Lambda}$ with respect to $\widetilde{T} \lhd \widetilde{N}$, such that in addition $\widetilde{\Lambda}(\widetilde{\lambda}\,\delta\rceil_T) = \widetilde{\Lambda}(\widetilde{\lambda})\,\delta\rceil_{\widetilde{N}}$ for every $\widetilde{\lambda} \in \mathrm{Irr}(\widetilde{T})$ and $\delta \in \mathrm{Irr}(\widetilde{\mathbf{G}}^{vF} | 1_{\mathbf{G}^{vF}})$. Let Λ be the NE-equivariant extension map with respect to $T \lhd N$ from Corollary 3.13 and

$$\widetilde{\Lambda} : \mathrm{Irr}(T) \to \bigcup_{T \leq I \leq N} \mathrm{Irr}(I)$$

be the map sending $\widetilde{\lambda} \in \mathrm{Irr}(\widetilde{T})$ to the unique common extension of $\widetilde{\lambda}$ and $\Lambda(\lambda)\rceil_{N_{\widetilde{\lambda}}}$ where $\lambda := \widetilde{\lambda}\Big\rceil_T$. Then $\widetilde{\Lambda}$ is well-defined according to [Sp09, Lemma 4.3] and has the required properties. □

The next statement is later applied to verify assumption 2.1(i.7) in the considered cases.

Corollary 3.21. *For the groups N_0 and $\widetilde{N}_0$ from Theorem 3.1 maximal extendibility holds with respect to $N_0 \lhd \widetilde{N}_0$.*

Proof. Like in the proof of the preceding proposition the isomorphism ι from the proof of Proposition 3.4 allows us to prove the statement by establishing that maximal extendibility holds with respect to $N \lhd \widetilde{N}$. Let $\widetilde{\Lambda}$ be the extension map with respect to $\widetilde{T} \lhd \widetilde{N}$ from (the proof of) Proposition 3.20. Then every character $\widetilde{\psi} \in \mathrm{Irr}(\widetilde{N})$ is of the form $(\widetilde{\Lambda}(\widetilde{\lambda})\eta_0)^{\widetilde{N}}$ for some $\widetilde{\lambda} \in \mathrm{Irr}(\widetilde{T})$ and $\eta_0 \in \mathrm{Irr}(W(\widetilde{\lambda}))$. Thus

$$\begin{aligned}\widetilde{\psi}\big\rceil_N = (\widetilde{\Lambda}(\widetilde{\lambda})\eta_0)^{\widetilde{N}}\big\rceil_N &= \Big((\widetilde{\Lambda}(\widetilde{\lambda})\eta_0)\big\rceil_{N_{\widetilde{\lambda}}} \Big)^N = \Big(\Lambda(\lambda)\rceil_{N_{\widetilde{\lambda}}} \eta_0 \Big)^N = \\ &= \Big((\Lambda(\lambda)\rceil_{N_{\widetilde{\lambda}}} \eta_0)^{N_\lambda} \Big)^N = \Big(\Lambda(\lambda)(\eta_0^{N_\lambda}) \Big)^N,\end{aligned}$$

where $\lambda := \widetilde{\lambda}\big\rceil_T$. According to Theorem 3.17 maximal extendibility holds with respect to $W(\widetilde{\lambda}) \lhd W(\lambda)$. Since $W(\lambda)/W(\widetilde{\lambda})$ is abelian, $\eta_0^{N_\lambda}$ and hence $\widetilde{\psi}\big\rceil_N$ is multiplicity-free. This proves the statement. □

4. The action of $\mathrm{Aut}(G)$ on Harish-Chandra induced characters

The aim of this section is to verify that assumptions of Theorem 2.1 concerning the characters of G are satisfied. For this we describe the action of $\mathrm{Aut}(G)$ on Harish-Chandra induced characters in terms of their parameters. Thus we first have to recall how one obtains the parametrisation of those characters. We follow here the treatment of the subject given in [C, Chap. 10], which is based on the results of [HL80] and [HL83].

We consider the following slightly more general setting. Let G be a finite group with a split BN-pair of characteristic p. We write $W = N/(N \cap B)$ for the Weyl group of G, which we assume to be of crystallographic type. Then there is a root system Φ attached to W and we let Δ denote a base of Φ corresponding to the simple reflections of W. We write $s_\alpha \in W$ for the reflection along the root $\alpha \in \Phi$.

Let $P \leq G$ be a standard parabolic subgroup with standard Levi subgroup L and Levi decomposition $P = U \rtimes L$. Let $N(L) := (N_G(L) \cap N)L$. We choose and fix once and for all an $N(L)$-equivariant extension map Λ for $L \lhd N(L)$, which exists according to [Ge93] and [L, Thm. 8.6].

Let λ be an irreducible cuspidal character of L. Via the Levi decomposition λ can be inflated to a character of P. Let M be a left $\mathbb{C}P$-module affording λ and denote by ρ the corresponding representation. Let $\mathfrak{F}(\rho)$ be the vector space of $\mathbb{C}$-linear maps $f : \mathbb{C}G \to M$ satisfying

$$f(px) = \rho(p)f(x) \quad \text{for all } p \in P \text{ and } x \in \mathbb{C}G.$$

This vector space becomes a $\mathbb{C}G$-module via

$$(g \star f)(x) = f(xg) \quad \text{for all } g \in G,\ f \in \mathfrak{F}(\rho) \text{ and } x \in \mathbb{C}G. \tag{4.1}$$

We denote by $\mathrm{R}_L^G(\lambda)$ the character of G afforded by this module. It is known that $\mathrm{R}_L^G(\lambda)$ only depends on λ, not on the choice of P or of ρ (see [DD93, HL94]). The set of constituents of $\mathrm{R}_L^G(\lambda)$ is called the Harish-Chandra series above (L,λ) and will be denoted by $\mathcal{E}(G,(L,\lambda))$. The union of Harish-Chandra series associated with $N\cap B$ and its characters is called the *principal series of* G.

4.A. **Actions of automorphisms on the standard basis.** Let σ be an automorphism of G stabilising P, L, and the BN-pair. Recall that σ acts on the class functions on G via $\chi\mapsto {}^\sigma\chi$, where ${}^\sigma\chi(g)=\chi(\sigma^{-1}(g))$ for all $g\in G$. It is immediate from the definitions that $(L,{}^\sigma\lambda)$ is again a cuspidal pair of G.

The character ${}^\sigma\mathrm{R}_L^G(\lambda)$ is afforded by the $\mathbb{C}G$-module ${}^\sigma\mathfrak{F}(\rho)$ obtained from the vector space $\mathfrak{F}(\rho)$ together with the G-action

$$(g\star_\sigma f)(x)=f(x\sigma^{-1}(g)) \quad \text{for all } g\in G,\ f\in\mathfrak{F}(\rho), \text{ and } x\in\mathbb{C}G. \tag{4.2}$$

One easily sees that $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ and $\mathrm{End}_{\mathbb{C}G}({}^\sigma\mathfrak{F}(\rho))$ can be canonically identified via ${}^\sigma B(f):=B(f)$ for $B\in\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ and $f\in\mathfrak{F}(\rho)$. Let $\mathfrak{F}({}^\sigma\rho)$ be the coinduced module associated to ${}^\sigma\rho$ defined as above. Then $\iota:{}^\sigma\mathfrak{F}(\rho)\to\mathfrak{F}({}^\sigma\rho)$ given by

$$f\mapsto {}^\sigma f \text{ with } {}^\sigma f(x)=f(\sigma^{-1}(x)) \text{ for all } x\in\mathbb{C}G$$

defines a $\mathbb{C}G$-module isomorphism. Moreover $B\mapsto\iota\circ B\circ\iota^{-1}$ for $B\in\mathrm{End}_{\mathbb{C}G}({}^\sigma\mathfrak{F}(\rho))$ induces an isomorphism from $\mathrm{End}_{\mathbb{C}G}({}^\sigma\mathfrak{F}(\rho))$ to $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}({}^\sigma\rho))$. We denote by $\widehat{\iota}:\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))\to\mathrm{End}_{\mathbb{C}G}({}^\sigma\mathfrak{F}(\rho))\to\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}({}^\sigma\rho))$ the composed isomorphism.

Since we are interested in the irreducible constituents of $\mathrm{R}_L^G(\lambda)$ and $\mathrm{R}_L^G({}^\sigma\lambda)$, which are parametrised by the isomorphism classes of irreducible modules of $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ and of $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}({}^\sigma\rho))$ respectively, see [C, Prop. 10.1.2], we will need to compute $\widehat{\iota}(B)$ for some elements $B\in\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$. We start by determining $\widehat{\iota}$ on a natural basis of $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$.

With $N(L)=(N_G(L)\cap N)L$ let $W_G(L):=N(L)/L$, the *relative Weyl group* of L in G, and set $W(\lambda):=N(L)_\lambda/L$. For $w\in W(\lambda)$ we denote by $\dot w\in N(L)$ a once and for all chosen preimage under the natural map. We let $\Phi_L\subseteq\Phi$ denote the root system of W_L, with simple system $\Delta_L\subseteq\Delta$.

Let $\widetilde\rho$ be an extension of ρ to $N(L)_\lambda$ affording the extension $\Lambda(\lambda)$ from our chosen equivariant extension map Λ. For $w\in W_G(L)$ let $\mathrm{B}_{w,\rho}\in\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ be defined by

$$(\mathrm{B}_{w,\rho}f)(x)=\widetilde\rho(\dot w)f(\dot w^{-1}e_Ux) \quad \text{for all } f\in\mathfrak{F}(\rho) \text{ and } x\in\mathbb{C}G,$$

where $e_U:=\frac{1}{|U|}\sum_{u\in U}u$ is the idempotent associated to the unipotent radical U of P. Note that $\mathrm{B}_{w,\rho}$ is independent of the actual choice of $\dot w$.

Analogously we define $\mathrm{B}_{w,{}^\sigma\rho}$ by using the extension $\widetilde\rho'$ of ${}^\sigma\rho$ affording $\Lambda({}^\sigma\lambda)$. Note that $\widetilde\rho'$ and ${}^\sigma(\widetilde\rho)$ then may differ. We denote by $\delta_{\lambda,\sigma}\in\mathrm{Irr}(W({}^\sigma\lambda))$ the character of $N(L)_{{}^\sigma\lambda}$ with $\delta_{\lambda,\sigma}\Lambda({}^\sigma\lambda)={}^\sigma\Lambda(\lambda)$. This character is well-defined by [I, Cor. 6.17].

For $w\in W(\lambda)$ let $\mathrm{B}_{w,{}^\sigma\rho}\in\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}({}^\sigma\rho))$ be defined via

$$(\mathrm{B}_{w,{}^\sigma\rho}f)(x)=\widetilde\rho'(\dot w)f(\dot w^{-1}e_Ux) \quad \text{for all } f\in\mathfrak{F}({}^\sigma\rho) \text{ and } x\in\mathbb{C}G.$$

Lemma 4.1. *For all $w\in W(\lambda)$ we have $\widehat{\iota}(\mathrm{B}_{w,\rho})=\delta_{\lambda,\sigma}(\sigma(w))\,\mathrm{B}_{\sigma(w),{}^\sigma\rho}$.*

Proof. Indeed, for $f \in \mathfrak{F}({}^\sigma\rho)$ and $x \in \mathbb{C}G$ we have

$$\widehat{\iota}(\mathrm{B}_{w,\rho})(f)(x) = \widetilde{\rho}(\dot{w}) f(\sigma(\dot{w}^{-1} e_U \sigma^{-1}(x)),$$

which agrees with

$$\delta_{\lambda,\sigma}(\sigma(w))\,\mathrm{B}_{\sigma(w),{}^\sigma\rho}(f)(x) = \widetilde{\rho}(\dot{w}) f(\sigma(\dot{w})^{-1} e_U x)$$

as $\sigma(e_U) = e_U$. □

4.B. **The decomposition of $W(\lambda)$.** In order to transfer our results to the $\mathrm{T}_{w,\rho}$-basis of the endomorphism algebra we need to recall the semi-direct product decomposition of $W(\lambda)$, see [HL80, Sec. 2 and 4]. Define

$$\hat{\Omega} := \{\alpha \in \Phi \setminus \Phi_L \mid w(\Delta_L \cup \{\alpha\}) \subseteq \Delta \text{ for some } w \in W\},$$

and for $\alpha \in \hat{\Omega}$ set $v(\alpha) := w_0^L w_0^\alpha$, where w_0^L, w_0^α are the longest elements in W_L, $\langle W_L, s_\alpha\rangle$ respectively. Then let $\Omega := \{\alpha \in \hat{\Omega} \mid v(\alpha)^2 = 1\}$. Note that Ω is σ-invariant. For $\alpha \in \Omega$ let L_α denote the standard Levi subgroup of G corresponding to the simple system $\Delta_L \cup \{\alpha\}$. Then L is a standard Levi subgroup of L_α. We write $p_{\alpha,\lambda} \geq 1$ for the ratio between the degrees of the two different constituents of $\mathrm{R}_L^{L_\alpha}(\lambda)$. Let

$$\Phi_\lambda := \{\alpha \in \Omega \mid s_\alpha \in W(\lambda),\, p_{\alpha,\lambda} \neq 1\},$$

a root system with set of simple roots $\Delta_\lambda \subseteq \Phi_\lambda \cap \Phi^+$, and let $R(\lambda) := \langle s_\alpha \mid \alpha \in \Phi_\lambda\rangle$ its Weyl group. Then $W(\lambda)$ satisfies $W(\lambda) = R(\lambda) \rtimes C(\lambda)$, where the group $C(\lambda)$ is the stabiliser of Δ_λ in $W(\lambda)$, see [C, Prop. 10.6.3].

Lemma 4.2. *We have $p_{\alpha,\lambda} = p_{\sigma(\alpha),{}^\sigma\lambda}$ for all $\alpha \in \Phi_\lambda$ and hence $R({}^\sigma\lambda) = \sigma(R(\lambda))$ and $C({}^\sigma\lambda) = \sigma(C(\lambda))$.*

Proof. By definition we have ${}^\sigma\mathrm{R}_L^{L_\alpha}(\lambda) = \mathrm{R}_L^{L_{\sigma(\alpha)}}({}^\sigma\lambda)$ since σ stabilises U. This implies $p_{\alpha,\lambda} = p_{\sigma(\alpha),{}^\sigma\lambda}$ by its definition. □

For $w \in W$ we set $\mathrm{ind}(w) := |U_0 \cap (U_0)^{w_0 w}|$, where U_0 is the unipotent radical of B and $w_0 \in W$ is the longest element. Also, for $\alpha \in \Delta_\lambda$ a simple root of Φ_λ we define $\epsilon_{\alpha,\lambda} \in \{\pm 1\}$ by

$$\mathrm{B}_{s_\alpha,\rho}^2 = \frac{1}{\mathrm{ind}(s_\alpha)}\,\mathrm{id} + \epsilon_{\alpha,\lambda}\frac{p_{\alpha,\lambda} - 1}{\sqrt{\mathrm{ind}(s_\alpha) p_{\alpha,\lambda}}}\mathrm{B}_{s_\alpha,\rho} \tag{4.3}$$

(see [C, Prop. 10.7.9]). Here, the square root is always taken positive.

Lemma 4.3. *If $R({}^\sigma\lambda) \leq \ker(\delta_{\lambda,\sigma})$ then $\mathrm{ind}(\sigma(s_\alpha)) = \mathrm{ind}(s_\alpha)$ and $\epsilon_{\sigma(\alpha),{}^\sigma\lambda} = \epsilon_{\alpha,\lambda}$ for all $\alpha \in \Delta_\lambda$.*

Proof. Let $\alpha \in \Delta_\lambda$ and set $s := s_\alpha$, $\alpha' := \sigma(\alpha)$, $s' := s_{\alpha'}$, $\lambda' := {}^\sigma\lambda$, $\rho' := {}^\sigma\rho$. Applying $\widehat{\iota}$ to Equation (4.3) we obtain

$$\widehat{\iota}(\mathrm{B}_{s,\rho}^2) = \frac{1}{\mathrm{ind}(s)}\,\mathrm{id} + \epsilon_{\alpha,\lambda}\frac{p_{\alpha,\lambda} - 1}{\sqrt{\mathrm{ind}(s) p_{\alpha,\lambda}}}\widehat{\iota}(\mathrm{B}_{s,\rho}).$$

Now $p_{\alpha',\lambda'} = p_{\alpha,\lambda}$ by Lemma 4.2, and since σ stabilises U_0 and w_0 we also have $\mathrm{ind}(s') = \mathrm{ind}(s)$. Then Lemma 4.1 yields

$$\delta_{\lambda,\sigma}(s')^2\,\mathrm{B}^2_{s',\rho'} = \frac{1}{\mathrm{ind}(s')}\,\mathrm{id} + \epsilon_{\alpha,\lambda}\frac{p_{\alpha',\lambda'}-1}{\sqrt{\mathrm{ind}(s')p_{\alpha',\lambda'}}}\delta_{\lambda,\sigma}(s')\,\mathrm{B}_{s',\rho'}.$$

Since $s' \in \sigma(R(\lambda)) = R({}^\sigma\lambda)$ the assumption $R(\lambda') \leq \ker(\delta_{\lambda,\sigma})$ allows us to simplify this to

$$\mathrm{B}^2_{s',\rho'} = \frac{1}{\mathrm{ind}(s')}\,\mathrm{id} + \epsilon_{\alpha,\lambda}\frac{p_{\alpha',\lambda'}-1}{\sqrt{\mathrm{ind}(s')p_{\alpha',\lambda'}}}\,\mathrm{B}_{s',\rho'}.$$

The claim follows by comparison with (4.3) for $\mathrm{B}_{s',\rho'}$. □

Now for $\alpha \in \Delta_\lambda$ set $\mathrm{T}_{s_\alpha,\rho} := \epsilon_{\alpha,\lambda}\sqrt{\mathrm{ind}(s_\alpha)p_{\alpha,\lambda}}\,\mathrm{B}_{s_\alpha,\rho}$; for $w \in R(\lambda)$ with a reduced expression $w = s_1\cdots s_r$ with $s_i = s_{\alpha_i}$ simple reflections (so $\alpha_i \in \Delta_\lambda$) let $\mathrm{T}_{w,\rho} := \mathrm{T}_{s_1,\rho}\cdots\mathrm{T}_{s_r,\rho}$; for $w \in C(\lambda)$ define $\mathrm{T}_{w,\rho} := \sqrt{\mathrm{ind}(w)}\,\mathrm{B}_{w,\rho}$, and then for $w \in W(\lambda)$ with $w = w_1w_2$ where $w_1 \in C(\lambda)$, $w_2 \in R(\lambda)$, let $\mathrm{T}_{w,\rho} := \mathrm{T}_{w_1,\rho}\mathrm{T}_{w_2,\rho}$. This does not depend on the choice of reduced expressions, see [C, Prop. 10.8.2]. Then we have:

Proposition 4.4. *If $R({}^\sigma\lambda) \leq \ker(\delta_{\lambda,\sigma})$ then for all $w \in W(\lambda)$ we have*

$$\widehat{\iota}(\mathrm{T}_{w,\rho}) = \delta_{\lambda,\sigma}(\sigma(w))\,\mathrm{T}_{\sigma(w),{}^\sigma\rho}.$$

Proof. First assume that $w = s_\alpha =: s$ for some $\alpha \in \Delta_\lambda$. Then

$$\begin{aligned}\widehat{\iota}(\mathrm{T}_{s,\rho}) &= \widehat{\iota}\big(\epsilon_{\alpha,\lambda}\sqrt{\mathrm{ind}(s)p_{\alpha,\lambda}}\,\mathrm{B}_{s,\rho}\big)\\ &= \epsilon_{\alpha,\lambda}\sqrt{\mathrm{ind}(s)p_{\alpha,\lambda}}\,\widehat{\iota}(\mathrm{B}_{s,\rho}) = \epsilon_{\alpha,\lambda}\sqrt{\mathrm{ind}(s)p_{\alpha,\lambda}}\;\delta_{\lambda,\sigma}(s')\mathrm{B}_{s',{}^\sigma\rho}\end{aligned}$$

by Lemma 4.1, where $s' = \sigma(s)$, $\alpha' = \sigma(\alpha)$. From Lemmas 4.2 and 4.3 we know $p_{\alpha',\lambda'} = p_{\alpha,\lambda}$, $\mathrm{ind}(s') = \mathrm{ind}(s)$ and $\epsilon_{\alpha',\lambda'} = \epsilon_{\alpha,\lambda}$. So indeed

$$\widehat{\iota}(\mathrm{T}_{s,\rho}) = \delta_{\lambda,\sigma}(s')\,\epsilon_{\alpha',\lambda'}\sqrt{\mathrm{ind}(s')p_{\alpha',\lambda'}}\,\mathrm{B}_{s',{}^\sigma\rho} = \delta_{\lambda,\sigma}(s')\mathrm{T}_{s',{}^\sigma\rho}.$$

Next, if $w \in C(\lambda)$ then

$$\widehat{\iota}(\mathrm{T}_{w,\rho}) = \sqrt{\mathrm{ind}(w)}\,\widehat{\iota}(\mathrm{B}_{w,\rho}) = \delta_{\lambda,\sigma}(\sigma(w))\sqrt{\mathrm{ind}(w)}\mathrm{B}_{\sigma(w),{}^\sigma\rho} = \delta_{\lambda,\sigma}(\sigma(w))\,\mathrm{T}_{\sigma(w),{}^\sigma\rho}.$$

In the general case, let $w \in W(\lambda)$ with $w = w_1w_2$ where $w_1 \in C(\lambda)$, and $w_2 \in R(\lambda)$ has a reduced expression $w_2 = s_1\cdots s_r$. Then by the above we get

$$\begin{aligned}\widehat{\iota}(\mathrm{T}_{w,\rho}) &= \widehat{\iota}(\mathrm{T}_{w_1,\rho})\widehat{\iota}(\mathrm{T}_{s_1,\rho})\cdots\widehat{\iota}(\mathrm{T}_{s_r,\rho})\\ &= \delta_{\lambda,\sigma}(\sigma(w_1))\Big(\prod_{i=1}^r \delta_{\lambda,\sigma}(\sigma(s_i))\Big)\,\mathrm{T}_{\sigma(w_1),{}^\sigma\rho}\mathrm{T}_{\sigma(s_1),{}^\sigma\rho}\cdots\mathrm{T}_{\sigma(s_r),{}^\sigma\rho}\\ &= \delta_{\lambda,\sigma}(\sigma(w_1s_1\cdots s_r))\,\mathrm{T}_{\sigma(w_1),{}^\sigma\rho}\mathrm{T}_{\sigma(s_1\cdots s_r),{}^\sigma\rho} = \delta_{\lambda,\sigma}(\sigma(w))\,\mathrm{T}_{\sigma(w),{}^\sigma\rho}\end{aligned}$$

as claimed. □

4.C. **Central-primitive idempotents of** $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$**.** Next we describe the central primitive idempotents of $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$. It is well-known (see e.g. [C, Prop. 10.9.2]) that $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ is a symmetric algebra with symmetrising trace defined by the linear map $\tau_\rho : \mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho)) \to \mathbb{C}$ with

$$\tau_\rho(\mathrm{T}_{w,\rho}) = \begin{cases} 1 & w = 1, \\ 0 & w \neq 1. \end{cases}$$

Let us denote by $\{\mathrm{T}^\vee_{w,\rho}\}$ the basis dual to $\{\mathrm{T}_{w,\rho}\}$ with respect to the bilinear form associated with τ_ρ. Thus

$$\mathrm{T}^\vee_{w,\rho} = p_{w,\lambda}^{-1}\mathrm{T}_{w^{-1},\rho} \qquad \text{for } w \in W(\lambda)$$

where $p_{w,\lambda} := \prod_{\alpha\in\Phi^+_\lambda,\, w(\alpha)<0} p_{\alpha,\lambda}$ (see [C, pp. 348–349] or [GP, 8.1.1]). Note that via $\widehat{\iota}$, τ_ρ defines a symmetrising trace $\tau_{^\sigma\rho}$ on $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(^\sigma\rho))$ with

$$\tau_{^\sigma\rho}(\mathrm{T}_{w,^\sigma\rho}) = \begin{cases} 1 & w = 1, \\ 0 & w \neq 1. \end{cases}$$

Let M be a simple $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$-module and η its character. It can be considered as a submodule of $e_{\eta,\rho}\mathfrak{F}(\rho)$, where

$$e_{\eta,\rho} := \frac{1}{c_{\eta,\rho}} \sum_{w\in W(\lambda)} \eta(\mathrm{T}_{w,\rho})\, \mathrm{T}^\vee_{w,\rho}$$

denotes the central-primitive idempotent of $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ corresponding to M (see [GP, 7.2.7(c)]). Here, $c_{\eta,\rho}$ is the Schur element associated to η as in [GP, Thm. 7.2.1].

Proposition 4.5. *Assume that* $R(^\sigma\lambda) \leq \ker(\delta_{\lambda,\sigma})$*. There exists a simple* $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(^\sigma\rho))$*-module with character* η' *such that*

$$\eta'(\mathrm{T}_{\sigma(w),^\sigma\rho}) = \delta^{-1}_{\lambda,\sigma}(w)\, \eta(\mathrm{T}_{w,\rho}) \quad \textit{for all } w \in W(\lambda).$$

The $\mathbb{C}G$*-modules* $e_{\eta,\rho}\,\mathfrak{F}(\rho)$ *and* $(e_{\eta',^\sigma\rho}\,\mathfrak{F}_{^\sigma\rho})^\sigma$ *are isomorphic.*

Proof. Since $\widehat{\iota}$ is an isomorphism of algebras we see from Proposition 4.4 that if $R(^\sigma\lambda) \leq \ker(\delta_{\lambda,\sigma})$ then

$$\widehat{\iota}(e_{\eta,\rho}) = \frac{1}{c_{\eta,\rho}} \sum_{w\in W(\lambda)} \eta(\mathrm{T}_{w,\rho})\, \delta_{\lambda,\sigma}(\sigma(w^{-1}))\, \mathrm{T}^\vee_{\sigma(w),^\sigma\rho}$$

is a central-primitive idempotent of $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho^\sigma))$. Let η' be the character of the associated simple $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(^\sigma\rho))$-module. Then comparison of coefficients between $\widehat{\iota}(e_{\eta,\rho})$ and

$$e_{\eta',^\sigma\rho} = \frac{1}{c_{\eta',^\sigma\rho}} \sum_{w\in W(^\sigma\lambda)} \eta'(\mathrm{T}_{w,^\sigma\rho})\, \mathrm{T}^\vee_{w,^\sigma\rho}$$

at $w = 1$ gives

$$\frac{\eta(\mathrm{T}_{1,\rho})}{c_{\eta,\rho}} = \frac{\eta'(\mathrm{T}_{1,^\sigma\rho})}{c_{\eta',^\sigma\rho}}.$$

Since $\mathrm{T}_{1,\rho}$ is the identity element of $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$, and η, η' have the same degree, this implies $c_{\eta,\rho} = c_{\eta',{}^\sigma\rho}$. Then comparison of coefficients at arbitrary $w \in W(\lambda)$ gives the first statement. The second is also clear as $\widehat{\iota}$ is a $\mathbb{C}G$-module isomorphism. □

4.D. **The generic algebra $\mathcal{H}$.** We next analyse in more detail the bijection between $\mathrm{Irr}(\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho)))$ and $\mathrm{Irr}(W(\lambda))$ using the approach presented in [HL83, Sec. 4]. The main idea is to introduce a generic algebra over a polynomial ring $\mathbb{C}[u_\alpha \mid \alpha \in \Delta_\lambda]$. One specialisation then gives the endomorphism algebra $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ and another specialisation gives the group algebra $\mathbb{C}W(\lambda)$. Application of these specialisations to the irreducible characters defines a parametrisation of the constituents of $\mathrm{R}_L^G(\lambda)$ by $\mathrm{Irr}(W(\lambda))$.

Let $\mathbf{u} = (u_\alpha \mid \alpha \in \Delta_\lambda)$ be indeterminates with $u_\alpha = u_\beta$ if and only if α and β are conjugate under $W(\lambda)$. Let K be an algebraic closure of the quotient field of the Laurent series ring $A_0 = \mathbb{C}[\mathbf{u}^{\pm 1}]$, and let A be the integral closure of A_0 in K. Let $\mathcal{H}$ be the free A-module with basis $\{a_w \mid w \in W(\lambda)\}$. According to [HL83, 4.1] one can define a unique A-bilinear associative multiplication on $\mathcal{H}$ such that for all $x \in C(\lambda)$, $w \in W(\lambda)$ and $\alpha \in \Delta_\lambda$ one has

$$\begin{aligned} a_x a_w &= a_{xw} \text{ and } a_w a_x = a_{wx}, \\ a_{s_\alpha} a_w &= \begin{cases} a_{s_\alpha w} & \text{if } w^{-1}\alpha \in \Phi_\lambda^+, \\ u_\alpha a_{s_\alpha w} + (u_\alpha - 1)a_w & \text{if } w^{-1}\alpha \notin \Phi_\lambda^+, \end{cases} \\ a_w a_{s_\alpha} &= \begin{cases} a_{w s_\alpha} & \text{if } w\alpha \in \Phi_\lambda^+, \\ u_\alpha a_{w s_\alpha} + (u_\alpha - 1)a_w & \text{if } w\alpha \notin \Phi_\lambda^+. \end{cases} \end{aligned}$$

Any homomorphism $f : A \to \mathbb{C}$ induces a right A-module structure on the field $\mathbb{C}$, so we obtain from $\mathcal{H}$ a $\mathbb{C}$-algebra $\mathcal{H}^f := \mathbb{C} \otimes_A \mathcal{H}$ with $\mathbb{C}$-vector space basis $\{1 \otimes a_w \mid w \in W(\lambda)\}$. The structure constants of $\mathcal{H}^f$ are obtained from the ones of $\mathcal{H}$ by applying f.

By [HL83, 4.2] the morphisms $f_0, g_0 : A_0 \to \mathbb{C}$ defined by $f_0(u_\alpha) = p_{\alpha,\lambda}$ and $g_0(u_\alpha) = 1$ for $\alpha \in \Delta_\lambda$ can be extended to morphisms $f, g : A \to \mathbb{C}$. Then $\mathcal{H}^f$ is isomorphic to $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ via $1 \otimes a_w \mapsto \mathrm{T}_{w,\rho}$ and $\mathcal{H}^g$ is isomorphic to $\mathbb{C}W(\lambda)$ via $1 \otimes a_w \mapsto w$.

By [HL83, 4.7] the map $\eta \mapsto \eta^f$ with $\eta^f(1 \otimes a_w) := f(\eta(a_w))$ defines a bijection between the set of K-characters associated to simple $K \otimes_A \mathcal{H}$-modules and the characters associated to simple $\mathcal{H}^f$-modules. The analogous result holds for $\mathcal{H}^g$. This combines to give a bijection between $\mathrm{Irr}(W(\lambda))$ and $\mathrm{Irr}(\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho)))$ and thus provides a labelling of the irreducible constituents of $\mathrm{R}_L^G(\lambda)$ by $\mathrm{Irr}(W(\lambda))$: for $\eta \in \mathrm{Irr}(W(\lambda))$ we denote by $\mathrm{R}_L^G(\lambda)_\eta$ the irreducible character of G occurring in $e_{\eta',\rho^f}\mathfrak{F}(\rho)$, where η' is the K-character of $\mathcal{H}$ with $\eta'^g = \eta$.

Together with Proposition 4.5 this proves:

Theorem 4.6. *If $R({}^\sigma\lambda) \le \ker(\delta_{\lambda,\sigma})$, then for $\eta \in \mathrm{Irr}(W(\lambda))$ we have*

$${}^\sigma(\mathrm{R}_L^G(\lambda)_\eta) = \mathrm{R}_L^G({}^\sigma\lambda)_{\eta'}$$

with $\eta' := {}^\sigma\eta\delta_{\lambda,\sigma}^{-1}$.

4.E. **Uniqueness of parametrisation.** So far, our parametrisation of constituents of $\mathrm{R}_L^G(\lambda)$ and hence also the assertion of Theorem 4.6 both depend on the choice of the parabolic subgroup P containing L, although the character $\mathrm{R}_L^G(\lambda)$ itself is independent of that choice, see [DD93, HL94]. The following result extends [MG82, Thm. 2.12] from the case of a torus to an arbitrary Levi subgroup. It allows us to compare the different parametrizations of the constituents of $\mathrm{R}_L^G(\lambda)$ by $\mathrm{Irr}(W(\lambda))$ for different parabolic subgroups P containing L.

Theorem 4.7. *Let $n \in N(L)$. Assume that the parametrisation of the constituents of $\mathrm{R}_L^G(\lambda)$ and $\mathrm{R}_L^G({}^n\lambda)$ is obtained using the same parabolic subgroup P of G with $L \leq P$ and extensions of λ and ${}^n\lambda$ given by an $N(L)$-equivariant extension map. Then*

$$\mathrm{R}_L^G(\lambda)_\eta = \mathrm{R}_L^G({}^n\lambda)_{{}^n\eta},$$

where ${}^n\eta \in \mathrm{Irr}(W({}^n\lambda))$ is the character with ${}^n\eta({}^nx) = \eta(x)$ for $x \in W(\lambda)$.

Proof. Write w for the image of n in W. Note that by multiplying n by elements of L we may assume that w fixes Δ_L, and also that w preserves the set of positive roots Φ_λ^+ (by multiplying with a suitable element from $R(\lambda)$). By [C, 10.1.3], for $v \in W$ the map

$$\theta_v : \mathfrak{F}(\rho) \to \mathfrak{F}({}^v\rho), \quad \theta_v(f)(x) := f(\dot{v}e_U x) \quad \text{for } f \in \mathfrak{F}(\rho),\, x \in G,$$

is a homomorphism of G-modules. Moreover, it is invertible by [C, 10.5.1, 10.5.3]. Now let $v \in W(\lambda)$ and set $v' := wvw^{-1}$. It then follows by [C, 10.7.5] that

$$\theta_w\theta_v = \sqrt{\frac{\mathrm{ind}(wv)}{\mathrm{ind}(w)\,\mathrm{ind}(v)}}\theta_{wv} \quad \text{and} \quad \theta_{v'}\theta_w = \sqrt{\frac{\mathrm{ind}(v'w)}{\mathrm{ind}(w)\,\mathrm{ind}(v')}}\theta_{v'w},$$

so that

$$\theta_w\theta_v\theta_w^{-1} = \sqrt{\frac{\mathrm{ind}(v')}{\mathrm{ind}(v)}}\theta_{v'}.$$

Now we have that $\mathrm{B}_{v,\rho} = \widetilde{\rho}(\dot{v}) \circ \theta_v$, and that $\theta_w \circ \widetilde{\rho}(\dot{v}) = \widetilde{\rho}(\dot{v}) \circ \theta_w$ by the argument given in the proof of [C, Prop. 10.2.4]. Thus

$$\begin{aligned}\theta_w \circ \mathrm{B}_{v,\rho} \circ \theta_w^{-1} =& \theta_w \circ \widetilde{\rho}(\dot{v}) \circ \theta_v \circ \theta_w^{-1} = \widetilde{\rho}(\dot{v}) \circ \theta_w \circ \theta_v \circ \theta_w^{-1} \\ =& \widetilde{\rho}(\dot{v}) \circ \sqrt{\frac{\mathrm{ind}(v')}{\mathrm{ind}(v)}}\theta_{v'} = \sqrt{\frac{\mathrm{ind}(v')}{\mathrm{ind}(v)}}{}^n\widetilde{\rho}(\dot{v'}) \circ \theta_{v'} = \sqrt{\frac{\mathrm{ind}(v')}{\mathrm{ind}(v)}}\mathrm{B}_{v',{}^n\rho}.\end{aligned}$$

Comparing the quadratic polynomials satisfied by $\mathrm{B}_{s_\alpha,\rho}$ and $\mathrm{B}_{{}^ws_\alpha,{}^n\rho}$ we see that $\epsilon_{\alpha,\lambda}$ and $\epsilon_{w(\alpha),{}^w\lambda}$ agree for $\alpha \in \Delta_\lambda$. Also, as conjugation by n does not change the degrees of the two constituents of $\mathrm{R}_L^{L_\alpha}(\lambda)$ we have $p_{\alpha,\lambda} = p_{w(\alpha),{}^w\lambda}$. Thus, the isomorphism of G-modules θ_w sends the standard generators $\mathrm{T}_{v,\rho}$ of $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ to the generators $\mathrm{T}_{{}^nv,{}^n\rho}$ of $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}({}^n\rho))$. (This can also be seen by use of the explicit isomorphism given by [HL94, Thm. 2.3].) It then follows from our construction of the central primitive idempotents and the specialisation argument as in the proof of Theorem 4.6 that conjugation by n sends the character of $\mathrm{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ parametrised by $\eta \in \mathrm{Irr}(W(\lambda))$ to the character parametrised by ${}^n\eta \in \mathrm{Irr}(W({}^n\lambda))$. □

5. The stabilisers of some Harish-Chandra induced Characters

Using Proposition 3.15 for characters lying in the principal Harish-Chandra series together with the results from Section 4 we can show that Harish-Chandra induction induces an equivariant map between certain local characters and suitable characters of $G = \mathbf{G}^F$. For this we determine the stabilisers of characters $\chi \in \mathrm{Irr}(G)$ lying in a Harish-Chandra series $\mathcal{E}(G, (L, \lambda))$ where L is a maximally split torus.

In order to assure the assumptions made in Section 4 we first describe the groups $R(\lambda)$. Recall that $\mathbf{B}^F$ and $\mathrm{N}_{\mathbf{G}}(\mathbf{T})^F$ form a split BN-pair in G, see [MT, Thm. 24.10] for example. Let Φ_1 be the associated root system of G and $X_\alpha \leq G$ ($\alpha \in \Phi_1$) the associated root subgroups. In the following we use freely the notation around Harish-Chandra induction introduced in the previous section.

Lemma 5.1. *Assume that* $\mathbf{G}$ *is not of type* A_l. *Let* $\lambda \in \mathrm{Irr}(\mathbf{T}^F)$, $\alpha \in \Phi_1$ *and let* Φ_λ *be defined as in 4.B. Then* $\alpha \in \Phi_\lambda$ *if and only if* $\lambda(\mathbf{T}^F \cap \langle X_\alpha, X_{-\alpha}\rangle) = 1$. *Moreover* $R(\lambda) \leq W(\widetilde{\lambda})$ *for any* $\widetilde{\lambda} \in \mathrm{Irr}(\widetilde{\mathbf{T}}^F|\lambda)$.

Proof. Assume first that F is a standard Frobenius endomorphism. Since $T = \mathbf{T}^F$ is a torus the integer $p_{\alpha,\lambda}$ from Section 4.B is determined inside the standard Levi subgroup $L_\alpha := \langle T, X_{\pm\alpha}\rangle$. As $\mathbf{G}$ is of simply connected type, the group $K := \langle X_{\pm\alpha}\rangle$ is isomorphic to $\mathrm{SL}_2(q)$. Moreover $T_0 := \mathbf{T} \cap \langle X_{\pm\alpha}\rangle = \left\langle h_\alpha(t) \mid t \in \mathbb{F}_q^\times \right\rangle$. Let $\lambda_0 := \lambda\rceil_{T_0}$.

From the situation in $\mathrm{SL}_2(q)$ we know that $\mathrm{R}_{T_0}^K(\lambda_0)$ splits into two constituents of different degrees if λ_0 is trivial. Hence in that case $p_{\alpha,\lambda} \neq 1$. If λ_0 is not trivial then either the character $\mathrm{R}_{T_0}^K(\lambda_0)$ and hence $\mathrm{R}_T^{L_\alpha}(\lambda)$ is irreducible or it is the sum of two characters of the same degree.

The above argument remains valid for twisted Frobenius endomorphisms, possibly replacing $\mathrm{SL}_2(q)$ by $\mathrm{SL}_2(q^m)$ with m the order of the graph automorphism induced by F.

Let $\widetilde{\lambda} \in \mathrm{Irr}(\widetilde{\mathbf{T}}^F|\lambda)$. If $\alpha \in \Phi_\lambda$ and $s_\alpha \in \pi(\mathrm{N}_G(\mathbf{T}))$ is the reflection associated with α the Steinberg relations imply

$$[\widetilde{\mathbf{T}}^F, s_\alpha] \subseteq \mathbf{T}^F \cap \langle X_\alpha, X_{-\alpha}\rangle\,.$$

Together with the above we conclude that $s_\alpha \in W(\widetilde{\lambda})$ if $\alpha \in \Phi_\lambda$. Since $R(\lambda)$ is generated by the elements s_α ($\alpha \in \Phi_\lambda$) this proves the statement. □

Theorem 5.2. *Let* T *be a maximally split torus of* $(\mathbf{G}, F)$. *Let* $N := \mathrm{N}_G(\mathbf{S})$ *and* $\widetilde{N} := \mathrm{N}_{\widetilde{G}}(\mathbf{S})$ *where* $\mathbf{S}$ *is a torus of* $\mathbf{G}$ *such that* $\mathrm{C}_G(\mathbf{S}) = T$. *Then there exists an* $\widetilde{N}D$*-equivariant bijection*

$$\mathrm{Irr}_{\mathrm{cusp}}(N) \longrightarrow \bigcup_{\lambda \in \mathrm{Irr}_{\mathrm{cusp}}(T)} \mathcal{E}(G, (T, \lambda)),$$

where $\mathrm{Irr}_{\mathrm{cusp}}(T)$ *is the set of cuspidal characters of* T, $\mathrm{Irr}_{\mathrm{cusp}}(N) := \mathrm{Irr}(N|\,\mathrm{Irr}_{\mathrm{cusp}}(T))$ *and* $\mathcal{E}(G, (T, \lambda))$ *denotes the set of constituents of* $\mathrm{R}_T^G(\lambda)$.

Proof. In Proposition 3.15 we gave a parametrisation of the set $\mathrm{Irr}_{\mathrm{cusp}}(N)$.

As T is a maximally split torus we have $N = \mathbf{N}^F$. Then $\mathrm{Irr}_{\mathrm{cusp}}(T) = \mathrm{Irr}(T)$ and hence $\mathrm{Irr}_{\mathrm{cusp}}(N) = \mathrm{Irr}(N)$. Let Λ be the ND-equivariant extension map from Corollary 3.13

applied with $d = 1$ and $v = 1$. Then Proposition 3.15 yields a map

$$\Pi : \mathcal{P} \longrightarrow \mathrm{Irr}(N), \qquad (\lambda, \eta) \longmapsto (\Lambda(\lambda)\eta)^{\mathbf{N}^F},$$

with $\mathcal{P} = \{(\lambda, \eta) \mid \lambda \in \mathrm{Irr}(T),\, \eta \in \mathrm{Irr}(W(\lambda))\}$. On the other hand let

$$\Pi' : \mathcal{P} \longrightarrow \bigcup_{\lambda \in \mathrm{Irr}_{\mathrm{cusp}}(T)} \mathcal{E}(G, (T, \lambda)), \qquad (\lambda, \eta) \longmapsto \mathrm{R}_T^G(\lambda)_\eta,$$

where $\mathrm{R}_T^G(\lambda)_\eta$ is defined using Λ. The maps Π and Π' induce bijections between the set of N-orbits in $\mathcal{P}$ and the characters in $\mathrm{Irr}_{\mathrm{cusp}}(N)$ and $\bigcup_{\lambda \in \mathrm{Irr}_{\mathrm{cusp}}(T)} \mathcal{E}(G, (T, \lambda))$ respectively, see Proposition 3.15(1) for Π and Theorem 4.7 for the statement about Π'. Accordingly the concatenation $\Pi' \circ \Pi^{-1}$ gives the required bijection

$$\mathrm{Irr}_{\mathrm{cusp}}(N) \to \bigcup_{\lambda \in \mathrm{Irr}_{\mathrm{cusp}}(T)} \mathcal{E}(G, (T, \lambda)), \qquad \Pi(\lambda, \eta) \mapsto \Pi'(\lambda, \eta).$$

Now the action of $\widetilde{N}D$ on $\mathrm{Irr}(N)$ has been described in Proposition 3.15 in terms of the associated labels. Analogously Theorem 4.6 determines the action of $\widetilde{T}D$ on the sets $\mathcal{E}(G, (T, \lambda))$ in terms of the associated labels. (Note that by Proposition 3.14 the map Λ satisfies the requirements made in Theorem 4.6.) Comparing the induced actions on $\mathcal{P}$ we see that the bijection is $\widetilde{N}D$-equivariant. □

Corollary 5.3. *Let (T, λ) be a cuspidal pair as in Theorem 5.2. Then for every character $\chi_0 \in \mathcal{E}(G, (T, \lambda))$ there exists some $\widetilde{G}$-conjugate χ such that*

$$(\widetilde{G}D)_\chi = \widetilde{G}_\chi D_\chi.$$

Proof. Via the bijection from Theorem 5.2 the character χ_0 corresponds to a character $\psi_0 \in \mathrm{Irr}_{\mathrm{cusp}}(N)$. Some $\widetilde{N}$-conjugate ψ of ψ_0 satisfies $(\widetilde{N}D)_\psi = \widetilde{N}_\psi (ND)_\psi$, see Theorem 3.1 together with Lemma 3.19. Since the bijection from Theorem 5.2 is $\widetilde{N}D$-equivariant, this implies

$$(\widetilde{G}D)_\chi = G(\widetilde{N}D)_\chi = G(\widetilde{N}D)_\psi = G(\widetilde{N}_\psi (ND)_\psi) = \widetilde{G}_\chi (ND_\chi) = \widetilde{G}_\chi D_\chi$$

where χ corresponds to ψ via Theorem 5.2, so is $\widetilde{G}$-conjugate to χ_0. □

6. Towards the inductive McKay condition

In this section we collect the previous results to prove Theorem 2 on the inductive McKay condition for primes ℓ with $d_\ell(q) = 1$, whenever $\mathbf{G}^F \notin \{\mathsf{D}_{l,\mathrm{sc}}(q), \mathsf{E}_{6,\mathrm{sc}}(q)\}$. Here $d_\ell(q)$ denotes the order of q modulo ℓ if $\ell > 2$, respectively the order of q modulo 4 if $\ell = 2$. We describe under which additional assumptions this result can be extended to primes with $d_\ell(q) = 2$ and to the missing types.

We first collect some cases in which the assertion of Theorem 2 had already been proven.

Proposition 6.1. *Assume that $S := \mathbf{G}^F / \mathrm{Z}(\mathbf{G}^F)$ is simple, and let ℓ be a prime dividing $|S|$. The inductive McKay condition holds for S and ℓ if one of the following is satisfied:*

- *$\ell = p$,*

- $\mathrm{Z}(\mathbf{G}^F) = 1$,
- Φ *is of type* A_l, *or*
- $\ell = 2$ *and* $\mathbf{G}^F = \mathsf{C}_{l,\mathrm{sc}}(q)$, *where q is an odd power of an odd prime.*

Proof. The case where ℓ is the defining characteristic has been settled in [Sp12, Thm. 1.1]. The case $\mathrm{Z}(\mathbf{G}^F) = 1$ has been considered in [CS13, Thm. A, Prop. 5.2]. If Φ is of type A_l the statement follows from [CS15, Thm. A]. According to [Ma08b, Thm. 4.11] the inductive McKay condition is satisfied for S and $\ell = 2$ whenever $\mathbf{G}^F = \mathsf{C}_{l,\mathrm{sc}}(q)$ for an odd power q of an odd prime. □

Proposition 6.2. *Let $(\mathbf{G}, F)$ be as in Section 2 and ℓ a prime different from the defining characteristic with $d = d_\ell(q) \in \{1, 2\}$. Assume that $\mathbf{G}^F$ and ℓ are not as in Proposition 6.1. Let $\mathbf{S}$ be a Sylow d-torus of $(\mathbf{G}, F)$. Then Assumption 2.1(i) is satisfied for $G := \mathbf{G}^F$, $\widetilde{G} := \widetilde{\mathbf{G}}^F$, D, $N := \mathrm{N}_{\mathbf{G}^F}(\mathbf{S})$ and some Sylow ℓ-subgroup Q of G such that $\mathrm{N}_{\widetilde{\mathbf{G}}^F}(\mathbf{S}) = \mathrm{N}_{\widetilde{\mathbf{G}}^F}(Q)N$.*

Proof. We can argue as in the proof of Lemma 7.1 of [CS15]. According to [Ma07, Thms. 5.14 and 5.19] since $\mathbf{G}^F$ and ℓ are not as in Proposition 6.1 there exists some Sylow ℓ-subgroup Q of $\mathbf{G}^F$ with $\mathrm{N}_{\mathbf{G}^F}(Q) \leq \mathrm{N}_{\mathbf{G}^F}(\mathbf{S}) \lneq \mathbf{G}^F$. Since all Sylow d-tori of $(\mathbf{G}, F)$ are $\mathbf{G}^F$-conjugate, we can conclude that $\mathrm{N}_{\mathbf{G}^F}(\mathbf{S})$ is $\mathrm{Aut}(\mathbf{G}^F)_Q$-stable, see also [CS13, Sect. 2.5].

Maximal extendibility for $\mathbf{G}^F \lhd \widetilde{\mathbf{G}}^F$ as required in 2.1(i.6) was shown by Lusztig, see [Lu88, Prop. 10] or [CE, Thm. 15.11]. The maximal extendibility with respect to $N \lhd \widetilde{N}$ as required in 2.1(i.7) has been proven in Corollary 3.21. □

In our next step we establish the existence of a bijection as required in 2.1(iv).

Theorem 6.3. *Let ℓ be a prime such that $d = d_\ell(q) \in \{1, 2\}$. Let $\mathbf{S}$ be a Sylow d-torus of $(\mathbf{G}, F)$, $N := \mathrm{N}_G(\mathbf{S})$ and $\widetilde{N} := \mathrm{N}_{\widetilde{G}}(\mathbf{S})$. Let $\mathcal{G} := \mathrm{Irr}\left(\widetilde{G} \mid \mathrm{Irr}_{\ell'}(G)\right)$ and $\mathcal{N} := \mathrm{Irr}\left(\widetilde{N} \mid \mathrm{Irr}_{\ell'}(N)\right)$. Then there is a $(\widetilde{G} \rtimes D)_{\mathbf{S}}$-equivariant bijection*

$$\widetilde{\Omega} : \mathcal{G} \longrightarrow \mathcal{N}$$

with $\widetilde{\Omega}(\mathcal{G} \cap \mathrm{Irr}(\widetilde{G} \mid \nu)) = \mathcal{N} \cap \mathrm{Irr}(\widetilde{N} \mid \nu)$ for every $\nu \in \mathrm{Irr}(\mathrm{Z}(\widetilde{G}))$, and $\widetilde{\Omega}(\chi\delta) = \widetilde{\Omega}(\chi)\, \delta\rceil_{\widetilde{N}}$ for every $\delta \in \mathrm{Irr}(\widetilde{G} \mid 1_G)$ and $\chi \in \mathcal{G}$.

Proof. According to Corollary 3.13 there exists an $\mathrm{N}_{\widetilde{G}D}(\mathbf{S})$-equivariant extension map Λ with respect to $\mathrm{C}_{\widetilde{G}}(\mathbf{S}) \lhd \widetilde{N}$ that is compatible with multiplication by linear characters of $\widetilde{G}$. The considerations made in Section 6 of [CS15] for groups of type A_l apply in our more general situation as well. Using our map Λ the construction presented there gives the required bijection. □

Theorem 6.4. *Let $(\mathbf{G}, F)$ be as in Section 2 and ℓ a prime different from the defining characteristic of $\mathbf{G}$ with $d = d_\ell(q) \in \{1, 2\}$. Assume that $\mathbf{G}^F$ and ℓ are not as in Proposition 6.1, and that $\mathbf{G}^F$ is the universal covering group of $S = \mathbf{G}^F/\mathrm{Z}(\mathbf{G}^F)$. Then the inductive McKay condition holds for S and ℓ in any of the following cases:*

(a) *$(\mathbf{G}, F)$ is of type B_l, C_l, ${}^2\mathsf{D}_l$, ${}^2\mathsf{E}_6$ or E_7 and $d = 1$;*
(b) *$(\mathbf{G}, F)$ is of type D_l or E_6, $d = 1$, and 2.1(ii.2) holds;*

(c) $(\mathbf{G}, F)$ *is of type* B_l, C_l, ${}^2\mathsf{D}_l$, ${}^2\mathsf{E}_6$ *or* E_7, $d = 2$ *and 2.1(ii.1) holds; or*
(d) $(\mathbf{G}, F)$ *is of type* D_l *or* E_6, $d = 2$, *and 2.1(ii.1) and 2.1(ii.2) hold.*

Proof. This is proven by an application of Theorem 2.1. We successively ensure that the necessary assumptions are satisfied.

The groups $G := \mathbf{G}^F$, $\widetilde{G} := \widetilde{\mathbf{G}}^F$, D, N and Q are chosen as in Proposition 6.2 and satisfy accordingly the assumptions made in 2.1(i). For this group N the characters satisfy assumption 2.1(iii) according to Theorem 3.1.

Let $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ lie in a Harish-Chandra series $\mathcal{E}(\mathbf{G}^F, (\mathbf{T}^F, \lambda))$ for some character $\lambda \in \mathrm{Irr}(\mathbf{T}^F)$. Then 2.1(ii.1) holds for χ (after suitable $\widetilde{\mathbf{G}}^F$-conjugation) according to Corollary 5.3.

Now assume that $d = 1$. Then according to [Ma07, Prop. 7.3] each character in $\mathrm{Irr}_{\ell'}(\mathbf{G}^F)$ lies in a Harish-Chandra series $\mathcal{E}(\mathbf{G}^F, (\mathbf{T}^F, \lambda))$ for some character $\lambda \in \mathrm{Irr}(\mathbf{T}^F)$. So assumption 2.1(ii.1) holds again by Corollary 5.3. On the other hand 2.1(ii.2) clearly holds whenever D is cyclic or by assumption. If $(\mathbf{G}, F)$ of type B_l, C_l, ${}^2\mathsf{D}_l$, ${}^2\mathsf{E}_6$ or E_7 then D is cyclic.

Whenever $d \in \{1, 2\}$ the bijection from Theorem 6.3 has the properties required in 2.1(iv). Altogether this proves the above statements. □

Remark 6.5. Note that the equation given in 2.1(ii.1) has only to be checked for characters of ℓ'-degree of G that are not $\widetilde{G}$-invariant since every $\chi \in \mathrm{Irr}(G)$ with $\widetilde{G}_\chi = \widetilde{G}$ satisfies $(\widetilde{G} \rtimes D)_\chi = \widetilde{G} \rtimes D_\chi$. In particular only characters in Lusztig rational series $\mathcal{E}(G, s)$ have to be considered where the centraliser of s in the dual group $\mathbf{G}^*$ is not connected, since characters in Lusztig series corresponding to elements with connected centralisers are $\widetilde{G}$-invariant according to [Lu88, Prop. 5.1], see also [CE, Cor. 15.14].

Furthermore 2.1(ii) holds whenever $\mathrm{Aut}(S)/S$ is cyclic since then $(\widetilde{G} \rtimes D)_\chi/(G\,\mathrm{Z}(\widetilde{G}))$ is cyclic and hence coincides with $(\widetilde{G}_\chi \rtimes D_\chi)/(G\,\mathrm{Z}(\widetilde{G}))$.

Theorem 2 in the case where $\mathbf{G}^F$ is a universal covering group and $d_\ell(q) = 1$ is now part (a) of the preceding theorem. For odd primes ℓ the following completes the proof of Theorem 2. The cases where the universal covering group of a simple group S is not of the form $\mathbf{G}^F$ can be determined by Table 6.1.4 of [GLS]. Then the Schur multiplier of S is said to have a non-trivial exceptional part, see [GLS, Sec. 6.1].

Lemma 6.6. *Let S be a simple group of Lie type with a non-trivial exceptional part of the Schur multiplier and let ℓ be a prime dividing $|S|$. Assume that ℓ is the defining characteristic or $d_\ell(q) \in \{1, 2\}$. Then the inductive McKay condition holds for S and ℓ.*

Proof. If S is a Suzuki or Ree group the result is known from [IMN07, Thm. 16.1] and [CS13, Thm. A]. Otherwise $S \cong \mathbf{G}^F/Z(\mathbf{G}^F)$ for some pair $(\mathbf{G}, F)$ as in Section 2. If ℓ is the defining characteristic of $\mathbf{G}$ the statement follows from [Sp12, Thm. 1.11]. If $(\mathbf{G}, F)$ is of type A_l, ${}^2\mathsf{A}_l$, F_4, or G_2 then the claim is known by [CS13, Thm. A] and [CS15, Thm. A].

In the other cases the considerations from [CS15, Sec. 7] can be transferred: the inductive McKay condition holds for S if it holds for any pair (S, Z) in the sense of [CS15, Def. 7.3], where Z is a cyclic ℓ'-quotient of the Schur multiplier of S, see [CS15, Lemma 7.4(a)].

Taking into account [Ma08a, Thm. 1.1] it is sufficient to prove the claim in the cases where Z is a quotient of the non-exceptional Schur multiplier of S.

According to [GLS, Table 6.1.3] all Sylow subgroups of D are cyclic and every automorphism of $\mathbf{G}^F$ is induced by $\widetilde{\mathbf{G}}^F D$ for groups $\widetilde{\mathbf{G}}^F$ and D defined as in Section 2. Further in those cases any Sylow subgroup of the outer automorphism group of S is cyclic and hence 2.1(ii) holds. The proofs of Theorem 6.4 and [Sp12, Thm. 2.12] imply that the inductive McKay condition holds for (S, Z) in those missing cases if $d = d_\ell(q) \in \{1, 2\}$. □

7. The McKay conjecture for $\ell = 2$

In this section we prove Theorem 1 from the introduction. Let $\mathbf{G}$ be a connected reductive linear algebraic group over an algebraically closed field of characteristic p and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism defining an $\mathbb{F}_q$-structure on $\mathbf{G}$ such that $G := \mathbf{G}^F$ has no component of Suzuki or Ree type.

7.A. **Degree polynomials.** We begin by defining degree polynomials for the irreducible characters of $G = \mathbf{G}^F$, which play a crucial role in our arguments. These are probably known to (some) experts, but we have not been able to find a convenient reference. Let $\mathbb{G}$ be the complete root datum of $(\mathbf{G}, F)$. Then there is a monic integral polynomial $|\mathbb{G}| \in \mathbb{Z}[X]$ such that $|\mathbf{G}^{F^m}| = |\mathbb{G}|(q^m)$ for all natural numbers m prime to the order of the automorphism induced by F on the Weyl group W of $\mathbf{G}$ (see [BM92, 1C]). The same then also holds for any connected reductive F-stable subgroup of $\mathbf{G}$, like maximal tori or connected components of centralisers. Furthermore, by work of Lusztig the unipotent characters of any finite reductive group with complete root datum $\mathbb{G}$ are parameterised uniformly, and the degree of a unipotent character χ of $\mathbf{G}^F$ is given by $f_\chi(q)$ for a suitable polynomial $f_\chi \in \mathbb{Q}[X]$ depending only on the parameter of χ, see [BMM, §1B]. Now let $\chi \in \mathrm{Irr}(G)$ be arbitrary. Then χ lies in the Lusztig series $\mathcal{E}(G, s)$ of a semisimple element s of the dual group $G^* = \mathbf{G}^{*F}$, and Lusztig's Jordan decomposition of characters gives a bijection

$$\Psi : \mathcal{E}(G, s) \to \mathcal{E}(\mathrm{C}_{G^*}(s), 1) \quad \text{such that} \quad \chi(1) = |G^* : \mathrm{C}_{G^*}(s)|_{p'}\, \Psi(\chi)(1), \tag{7.1}$$

where $\mathcal{E}(\mathrm{C}_{G^*}(s), 1)$ denotes the unipotent characters of $\mathrm{C}_{G^*}(s)$, see [DM, Thm. 13.25]. While this bijection is not defined canonically, the formula in loc. cit. for scalar products with Deligne–Lusztig characters shows that its uniform projection is, and hence in particular so is the correspondence of degrees. Moreover, by the description in [Lu88, Prop. 5.1], the multiplicities of unipotent characters of $\mathrm{C}^\circ_{G^*}(s)$ in those of $\mathrm{C}_{G^*}(s)$ are determined by the complete root datum of $(\mathbf{G}, F)$, hence generic, so there exist well-defined degree polynomials f_ψ for the unipotent characters ψ of the possibly disconnected group $\mathrm{C}_{G^*}(s)$. Thus, denoting by $|\mathbb{G}_s|$ the order polynomial of $\mathrm{C}_{G^*}(s)$, we can define from (7.1) the *degree polynomial* $f_\chi := (|\mathbb{G}|/|\mathbb{G}_s|)_{X'} f_{\Psi(\chi)} \in \mathbb{Q}[X]$ of χ.

Lemma 7.1. *Let $G = \mathbf{G}^F$ be as above. If $\chi \in \mathrm{Irr}(G)$ lies in the Harish-Chandra series of a cuspidal character of a Levi subgroup of G of semisimple $\mathbb{F}_q$-rank r then its degree polynomial f_χ is divisible by $(X-1)^r$.*

Proof. Let $\mathbf{L} \leq \mathbf{G}$ be an F-stable Levi subgroup such that χ lies in the Harish-Chandra series $\mathcal{E}(G,(L,\lambda))$, where $L = \mathbf{L}^F$. Let $\mathbf{T} \leq \mathbf{L}$ be an F-stable maximal torus of $\mathbf{G}$, with Sylow 1-torus $\mathbf{T}_1 \leq \mathbf{T}$. Then $\mathbf{M} = \mathrm{C}_{\mathbf{G}}(\mathbf{T}_1)$ is a (1-split) Levi subgroup of $\mathbf{G}$, and

$$\langle \mathrm{R}_T^G(\theta), \chi\rangle = \langle \mathrm{R}_M^G(\mu), \chi\rangle$$

with $\mu = \mathrm{R}_T^M(\theta)$ a (virtual) character of $M = \mathbf{M}^F$, where $T = \mathbf{T}^F$. Thus, if $\langle \mathrm{R}_T^G(\theta), \chi\rangle \neq 0$ then by disjointness of Harish-Chandra series we must have $\mathbf{M} \geq \mathbf{L}$ up to conjugation, whence $\mathbf{T}_1 \leq \mathrm{Z}(\mathbf{M})_{\Phi_1} \leq \mathrm{Z}(\mathbf{L})_{\Phi_1}$, where $\mathrm{Z}(\mathbf{M})_{\Phi_1}$ and $\mathrm{Z}(\mathbf{L})_{\Phi_1}$ denote the Sylow 1-torus of the groups $\mathrm{Z}(\mathbf{M})$ and $\mathrm{Z}(\mathbf{L})$.

Now we have $\chi(1) = \langle \mathrm{reg}_G, \chi\rangle$, where the regular character reg_G of G is given by

$$\mathrm{reg}_G = \frac{1}{|W|} \sum_{w\in W} |G\colon T_w|_{p'}\, \mathrm{R}_{T_w}^G(\mathrm{reg}_{T_w}),$$

where W is the Weyl group of $\mathbf{G}$, $T_w = \mathbf{T}_w^F$ is a maximal torus of G of type w, and reg_{T_w} denotes the regular character of T_w (see [DM, Cor. 12.14]). Let W_0 denote the set of elements $w \in W$ satisfying $\dim(\mathbf{T}_w)_{\Phi_1} \leq \dim(\mathrm{Z}(\mathbf{L})_{\Phi_1})$. Our above considerations then yield

$$\chi(1) = \frac{1}{|W|} \sum_{w\in W_0} |G\colon T_w|_{p'}\, \langle \mathrm{R}_{T_w}^G(\mathrm{reg}_{T_w}), \chi\rangle.$$

(Note that this is generic, as by [L, Thm. 4.23] the multiplicities $\langle \mathrm{R}_{T_w}^G(\mathrm{reg}_{T_w}), \chi\rangle$ only depend on the unipotent Jordan correspondent of χ.) For any F-stable reductive subgroup $\mathbf{H}$ of $\mathbf{G}$ we write in the following $\mathbf{H}_{\Phi_1}$ for a Sylow 1-torus of $(\mathbf{H}, F)$. For $w \in W_0$ we have

$$\dim(\mathbf{G}_{\Phi_1}) - \dim((\mathbf{T}_w)_{\Phi_1}) \geq \dim(\mathbf{L}_{\Phi_1}) - \dim(\mathrm{Z}(\mathbf{L})_{\Phi_1}) = r,$$

where r is the rank of a Sylow 1-torus of the semisimplification of $\mathbf{L}$, hence the semisimple $\mathbb{F}_q$-rank of $\mathbf{L}$ (see [DM, Def. 8.6]). So the degree polynomial f_χ is divisible by $(X-1)^r$. □

We also recall the following facts from ordinary Harish-Chandra theory (see e.g. [C, Thm. 10.11.5]). Let $L \leq G$ be a Levi subgroup with a cuspidal character $\lambda \in \mathrm{Irr}(L)$, and let $W(\lambda)$ denote the relative Weyl group of this cuspidal pair. Assume that $\chi \in \mathrm{Irr}(G)$ lies in the Harish-Chandra series above (L, λ). Let $\eta \in \mathrm{Irr}(W(\lambda))$ be the character associated to χ and $D_\chi \in \mathbb{Q}(X)$ the inverse of the Schur element of η of the corresponding generic Hecke algebra, so numerator and denominator of D_χ are prime to $X-1$. Then

$$\chi(1) = |G\colon L|_{p'}\, D_\chi(q)\, \lambda(1) \qquad \text{and} \quad D_\chi(1) = \eta(1)/|W(\lambda)|. \tag{7.2}$$

With the degree polynomial f_λ of λ we define a *degree function* $f'_\chi \in \mathbb{Q}(X)$ for χ as $f'_\chi = (|\mathbb{G}|/|\mathbb{L}|)_{X'} D_\chi f_\lambda$, where $\mathbb{L}$ denotes the complete root datum associated to the standard Levi subgroup $(\mathbf{L}, F)$ with $\mathbf{L}^F = L$. The following is shown in [BMM, Thm. 3.2]:

Lemma 7.2. *Let $G = \mathbf{G}^F$ be as above. If $\chi \in \mathrm{Irr}(G)$ is unipotent, then $f_\chi = f'_\chi$.*

7.B. **Unipotent characters of odd degree.** From now on and for the rest of this section assume that p and hence q *is odd.*

Proposition 7.3. *Let $G = \mathbf{G}^F$ be as above. Then every non-trivial cuspidal unipotent character χ of G has even degree. More precisely, if χ has degree polynomial $a(X-1)^m f$, with $a \in \mathbb{Q}$, $m \geq 0$ and $f \in \mathbb{Z}[X]$ monic and prime to $X-1$, then $a(q-1)^m$ is even.*

Proof. First note that unipotent characters of G restrict irreducibly to unipotent characters of $[\mathbf{G}, \mathbf{G}]^F$, so we may assume that $\mathbf{G}$ is semisimple. Furthermore, degrees of unipotent characters are insensitive to the isogeny type of $\mathbf{G}$, whence we may assume that $\mathbf{G}$ is of simply connected type and hence a direct product of simple algebraic groups. As unipotent characters of a direct product are the exterior products of the unipotent characters of the factors, we may reduce to the case that $\mathbf{G}$ is a direct product of r isomorphic simple groups $\mathbf{H}_i \cong \mathbf{H}$, $1 \leq i \leq r$, transitively permuted by F. But then $\mathbf{G}^F \cong \mathbf{H}^{F^r}$, and $f(X^r)$ is divisible by the same power of $X-1$ as $f(X)$, so that finally we may assume that $\mathbf{G}$ is simple.

We then use Lusztig's classification of cuspidal unipotent characters. In fact, when $q \equiv 1 \pmod 4$ then the first claim is already proved in [Ma07, Prop. 6.5]. But a quick check of that argument shows that it only relies on the fact that the degree of χ is divisible by a sufficiently high power of the even number $q-1$. It thus also works for $q \equiv 3 \pmod 4$ and does even yield the second assertion. □

Proposition 7.4. *Let $G = \mathbf{G}^F$ be as above. Then all unipotent characters of G of odd degree lie in the principal series of G.*

Proof. We distinguish two cases. If $q \equiv 1 \pmod 4$ then our claim is contained in [Ma07, Cor. 6.6]. So for the rest of the proof we may suppose that $q \equiv 3 \pmod 4$.

Assume if possible that χ is a unipotent character of G of odd degree and not lying in the principal series. So χ lies above a cuspidal unipotent character $\lambda \neq 1_L$ of a Levi subgroup $L \leq G$. Let $f_\chi, f_\lambda \in \mathbb{Q}[X]$ denote the degree polynomials of χ, λ respectively. As $\chi(1)$ is odd and $4|(q+1)$ we have that χ must lie in the principal 2-series of G by [Ma07, Cor. 6.6] applied with $d = 2$. Thus f_χ is prime to $X+1$ according to [BMM, Prop. 2.4] (an analogue of our Lemma 7.1).

Now by what we recalled before Lemma 7.2 there exists a rational function $g \in \mathbb{Q}(X)$ with numerator and denominator products of cyclotomic polynomials times an integer, both prime to $X-1$, such that $f_\chi = g \cdot f_\lambda$, and such that $g(1)$ is the degree of an irreducible character of the relative Weyl group $W(\lambda)$ of (L, λ) in G. Write $f_\lambda = (X+1)^k f_1$ with a non-negative integer k such that $f_1 \in \mathbb{Q}[X]$ is prime to $X+1$. Then by our observations on f_χ and f_λ there exists a rational function g_1 such that $g = g_1/(X+1)^k$ and both numerator and denominator of g_1 are prime to X^2-1. Then $f_\chi = g_1 \cdot f_1$.

Now let Φ_i be a cyclotomic polynomial dividing g_1. Then $\Phi_i(q)$ is odd unless $i = 2^{j+1}$ for some $j \geq 1$, in which case we have $\Phi_i(q)_2 = (q^{2^j}+1)_2 = 2 = \Phi_i(1)_2$. Thus, $g_1(q)$ is divisible by the same 2-power as $g_1(1)$, which is an integer. Since $f_1(q)$ is even by Proposition 7.3 we conclude that

$$\chi(1) = g(q) \cdot f_\lambda(q) = g_1(q) \cdot f_1(q) \equiv g_1(1) \cdot f_1(q) \pmod 2$$

is even as well, a contradiction. □

7.C. **Characters of odd degree and the principal series.**

Lemma 7.5. *Let* $\mathbf{H}$ *be simple of adjoint type* B_l *(*$l \geq 1$*),* C_l *(*$l \geq 2$*),* D_{2l} *(*$l \geq 2$*) or* E_7*, and* $F : \mathbf{H} \to \mathbf{H}$ *a Steinberg endomorphism. Let* $s \in \mathbf{H}^F$ *be semisimple centralising a Sylow 2-subgroup of* $\mathbf{H}^F$*. Then* $s^2 = 1$*.*

This was observed in [Ma08b, Lemma 4.1] for type B_l; the proof given there carries over word by word, since in all listed cases the longest element of the Weyl group acts by inversion on a maximal torus.

Recall that an element of a connected reductive algebraic group $\mathbf{H}$ is called *quasi-isolated* if its centraliser is not contained in any proper Levi subgroup of $\mathbf{H}$.

Lemma 7.6. *Let* $\mathbf{H}$ *be simple of adjoint type* B_l *(*$l \geq 2$*),* C_l *(*$l \geq 3$*) or* D_l *(*$l \geq 4$*), and* $F : \mathbf{H} \to \mathbf{H}$ *a Frobenius endomorphism defining an* $\mathbb{F}_q$*-rational structure such that* $q \equiv 3 \pmod 4$*. Let* $s \in \mathbf{H}$ *be semisimple quasi-isolated with disconnected centraliser* $\mathbf{C} = \mathrm{C}_{\mathbf{H}}(s)$ *such that* $\mathbf{C}^F$ *contains a Sylow 2-subgroup of* $\mathbf{H}^F$*. Then* $\mathbf{C}^F$ *is as in Table 1.*

Table 1. Disconnected centralisers of 2-central elements

$\mathbf{H}^F$	$\mathbf{C}^F$	conditions
$\mathsf{B}_l(q)$	$\mathsf{B}_{l-2k}(q) \cdot \mathsf{D}_{2k}(q).2$	$1 \leq k \leq l/2$
$\mathsf{B}_{2l+1}(q)$	$\mathsf{B}_{2k}(q) \cdot {}^2\mathsf{D}_{2(l-k)+1}(q).2$	$0 \leq k \leq l$
$\mathsf{C}_{2l}(q)$	$(\mathsf{C}_l(q) \cdot \mathsf{C}_l(q)).2$	
$\mathsf{D}_l(q)$	$(\mathsf{D}_k(q) \cdot \mathsf{D}_{l-k}(q)).2$	$1 \leq k < l/2$
$\mathsf{D}_{4l}(q)$	$(\mathsf{D}_{2l}(q) \cdot \mathsf{D}_{2l}(q)).4$	
${}^2\mathsf{D}_l(q)$	$(\mathsf{D}_k(q) \cdot {}^2\mathsf{D}_{l-k}(q)).2$	$2 \leq k \leq l-1,\ k \neq l/2$

Here $\mathsf{D}_1(q)$, ${}^2\mathsf{D}_1(q)$ are to be interpreted as tori of order $q-1$, $q+1$ respectively.

Proof. The conjugacy classes of quasi-isolated elements s in classical groups of adjoint type were classified in [Bo05, Tab. 2]. From that list, we may exclude those s with connected centraliser. It then remains to determine the various rational forms of $\mathbf{H}$ and $\mathbf{C}$ and to decide when s is 2-central. We treat the cases individually. For $\mathbf{H}$ of adjoint type B_l, the table contains all examples from loc. cit. For $\mathbf{H}$ of type C_l, an easy calculation shows that only the listed case occurs. Similarly, it can be checked in type D_l from the order formulas that only the listed types of disconnected centralisers can possibly contain a Sylow 2-subgroup. □

We thus obtain the following classification of characters of odd degree:

Theorem 7.7. *Let* $\mathbf{G}$ *be simple, of simply connected type, not of type* A_l*, with* F*,* $\mathbf{G}^*$ *as introduced in Section 2.B. Let* $\chi \in \mathrm{Irr}_{2'}(G)$*. Then either* χ *lies in the principal series of* G*, or* $q \equiv 3 \pmod 4$*,* $G = \mathrm{Sp}_{2l}(q)$ *with* $l \geq 1$ *odd,* $\chi \in \mathcal{E}(G,s)$ *with* $\mathrm{C}_{G^*}(s) = \mathsf{B}_{2k}(q) \cdot {}^2\mathsf{D}_{l-2k}(q).2$ *where* $0 \leq k \leq (l-3)/2$*, and* χ *lies in the Harish-Chandra series of a cuspidal character of degree* $\frac{1}{2}(q-1)$ *of a Levi subgroup* $\mathrm{Sp}_2(q) \times (q-1)^{l-1}$*.*

Proof. We follow the line of arguments in [Ma07, §7]. Let $\chi \in \mathrm{Irr}(G)$ be a character of odd degree and not lying in the principal series of G. Then the degree polynomial of χ is divisible by $X-1$, by Lemma 7.1. Let $s \in G^*$ be semisimple such that $\chi \in \mathcal{E}(G,s)$ and set $\mathbf{C} := \mathrm{C}_{\mathbf{G}^*}(s)$, $C := \mathbf{C}^F$ and $C^\circ := \mathbf{C}^{\circ F}$. Let $\Psi(\chi) \in \mathcal{E}(C,1)$ denote the unipotent Jordan correspondent of χ. Then by Lusztig's Jordan decomposition $\chi(1) = |G^* : C|_{p'}\, \Psi(\chi)(1)$ (see (7.1)), so both $|G^* : C|$ and $\Psi(\chi)(1)$ have to be odd. Thus $\Psi(\chi)$ lies above a unipotent character of $\mathbf{C}^{\circ F}$ of odd degree, and hence in the principal series of C by Proposition 7.4. So its degree polynomial is prime to $X-1$ by Lemma 7.2 and (7.2). Hence the order polynomial of $|G^* : C|$ must be divisible by $X-1$ by our assumption. On the other hand, as $|G^* : C|$ is odd, C contains a Sylow 2-subgroup of G^*.

If $q \equiv 1 \pmod 4$ then we may argue as follows. By [Ma07, Thm. 5.9], $\mathbf{C}$ must contain a Sylow 1-torus of $\mathbf{G}^*$. But then the order polynomial of $|G^* : C|$ cannot be divisible by $X-1$, a contradiction.

So now assume that $q \equiv 3 \pmod 4$. Then again by [Ma07, Thm. 5.9], $\mathbf{C}$ must contain a Sylow 2-torus of $\mathbf{G}^*$. The order $|C|$ is given by a polynomial in q of the form $cf(q)$, where $c = |C : C^\circ|$ and $f \in \mathbb{Z}[X]$ is monic. Note that $\mathbf{C}/\mathbf{C}^\circ$ is isomorphic to a subgroup of the fundamental group of $\mathbf{G}$, hence of the center of $\mathbf{G}$ (see [MT, Prop. 14.20]). In particular, as $\mathbf{G}$ is simple and not of type A_l we have $|C : C^\circ|_2 \le 4$, and in fact $|C : C^\circ|_2 \le 2$ unless $\mathbf{G}$ is of type D_l. As $X-1$ divides the order polynomial of G^* divided by f, we are done if either G has odd order center, or if $\mathbf{C}$ is connected.

So $\mathbf{G}$ is of type $\mathsf{B}_l, \mathsf{C}_l, \mathsf{D}_l$ or E_7. For $\mathbf{G}$ not of type D_l with l odd we know by Lemma 7.5 applied to $\mathbf{H} := \mathbf{G}^*$ that s must be an involution. For $\mathbf{G}$ of type E_7, the 2-central involutions of G^* have centraliser of type $\mathsf{D}_6(q)\mathsf{A}_1(q)$, whose order polynomial is divisible by the full power $(X-1)^7$ of $X-1$ occurring in the polynomial order of G^*, contrary to what we showed. For the groups of classical types B_l, C_l, D_l and ${}^2\mathsf{D}_l$, let us first observe that $\mathbf{C}$ cannot be contained inside a proper F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}^*$, because $L = \mathbf{L}^F$ has even index in G^*. Indeed, a Sylow 2-subgroup of G^*, or L, is contained in the normaliser of a Sylow 2-torus of $\mathbf{G}^*$ (see [MT, Cor. 25.17]), respectively of $\mathbf{L}$. But this normaliser is an extension of that Sylow 2-torus by the Weyl group of G^*, L respectively. The claim then follows since any proper parabolic subgroup of a Weyl group W of type B_l or D_l has even index in W.

Thus, s is quasi-isolated in $\mathbf{G}^*$, and hence occurs in Table 1. For $\mathbf{G}$ of type B_l the dual group is of adjoint type C_l, and there the listed centraliser does contain a Sylow 1-torus. For $\mathbf{G}$ of type C_l and so $\mathbf{G}^*$ of type B_l the listed centralisers either contain a Sylow 1-torus or are given in the statement with l odd. As the order polynomial of $\mathbf{C}^\circ$ is divisible by $(X-1)^{l-1}$ in these cases, the degree polynomial of χ is divisible by $X-1$ just once, so by Lemma 7.1, χ lies in the Harish-Chandra series of a cuspidal character of a Levi subgroup L of G of rank 1, hence a Levi subgroup of type A_1. Now G has two conjugacy classes of such Levi subgroups, one with connected center lying in the stabiliser $\mathrm{GL}_l(q)$ of a maximally isotropic subspace, the other isomorphic to $\mathrm{Sp}_2(q) \times (q-1)^{l-1}$. The degrees of cuspidal characters of these two types of subgroups are $q-1$, and also $\frac{1}{2}(q-1)$ for the second type. The latter ones are thus the only ones of odd degree. An easy variation of the proof of

Proposition 7.4, using that $X-1$ is prime to $X+1$ now shows that if χ has odd degree, it must lie above the cuspidal characters of $\mathrm{Sp}_2(q)\times(q-1)^{l-1}$ of degree $\frac{1}{2}(q-1)$.

So finally assume that $\mathbf{G}$ is of type D_l. Recall that $\mathbf{C}^\circ$ cannot contain a Sylow 1-torus of $\mathbf{G}^*$. The only centralisers in Table 1 not containing a Sylow 1-torus of the ambient group are $\mathsf{D}_k(q)\cdot{}^2\mathsf{D}_{l-k}(q).2$ with l even and k odd inside ${}^2\mathsf{D}_l(q)$. But these do not contain a Sylow 2-torus, contrary to what we know has to happen. So we get no example in type D_l. □

Remark 7.8. The precise conditions on k for $\mathsf{B}_{2k}(q)\cdot{}^2\mathsf{D}_{l-2k}(q)$ in Theorem 7.7 to contain a Sylow 2-subgroup of $G^*=\mathrm{SO}_{2l+1}(q)$ are worked out in [Ma08b, Prop. 4.2]. In fact, in [Ma08b, Thms. 4.10 and 4.11] it is shown that $G=\mathrm{Sp}_{2l}(q)$ satisfies the inductive McKay condition for the prime 2 if q is an odd power of p.

The following consequence will be used in the proof of Proposition 7.10:

Lemma 7.9. *Let* $\mathbf{G}$ *be simple, of simply connected type, not of type* A_l *or* C_l. *Let* $\chi\in\mathrm{Irr}_{2'}(G)$. *Then* $\chi=\mathrm{R}_T^G(\lambda)_\eta$, *where* T *is a maximally split torus of* G, $\lambda\in\mathrm{Irr}(T)$ *is such that* $2\nmid|W:W(\lambda)|$ *and* $\eta\in\mathrm{Irr}(W(\lambda))$ *is of odd degree.*

Proof. By Theorem 7.7 every $\chi\in\mathrm{Irr}_{2'}(G)$ occurs in the principal series, that is, it lies in the Harish-Chandra series of a linear character $\lambda\in\mathrm{Irr}(T)$. According to our remarks before Lemma 7.2 we have that $\chi(1)=|G:T|_{p'}\,D_\chi(q)\,\lambda(1)$, where $\lambda(1)=1$. Write $f'_\chi\in\mathbb{Q}[X]$ for the degree polynomial of χ. Since $\Phi_i(q)_2\geq\Phi_i(1)_2$ for all $i\geq 2$ it follows that $f_\chi(1)$ is odd. Now replacing $|G:T|_{p'}$ by the order polynomial and then specialising at $q=1$ we obtain that $|W|\,D_\chi(1)=|W|\,\eta(1)/|W(\lambda)|$ is odd, for $\eta\in\mathrm{Irr}(W(\lambda))$ the label of χ, whence the two integers $\eta(1)$ and $|W:W(\lambda)|$ are odd. □

This completes the proof of Theorem 2. Indeed, if $d=1$ then Theorem 2 follows from Theorem 6.4(a) and Lemma 6.6. Suppose that $\ell|(q-1)$, but $d\neq 1$. Then $\ell=d=2$, and one may assume by Proposition 6.1 that $\mathbf{G}^F$ is not of type $\mathrm{Sp}_{2n}(q)$ with $4|(q+1)$. In this case, Theorem 2 follows by combining Lemma 6.6, Corollary 5.3, Theorem 6.4(c) and Theorem 7.7.

7.D. **Proof of Theorem 1.** We now combine the above results on Harish-Chandra induction and on characters of odd degree to complete the proof that every simple group satisfies the inductive McKay condition for $\ell=2$, and thus Theorem 1 holds.

We have seen in Theorem 6.4 that our results are sufficient to prove that the inductive McKay condition holds for the prime 2, whenever $4|(q-1)$ and Φ is of type B_l, C_l or E_7. The result even applies to the simple groups which are the quotients of ${}^2\mathsf{D}_l(q)$ or ${}^2\mathsf{E}_6(q)$. Taking into account earlier results summarised in Proposition 6.1 the only cases that are left to consider when $4|(q-1)$ are the simple groups associated with $\mathsf{D}_{l,\mathrm{sc}}(q)$ and $\mathsf{E}_{6,\mathrm{sc}}(q)$.

The following specific considerations are tailored to the case where $\ell=2$.

Proposition 7.10. *Let* $G=\mathbf{G}^F$ *be as above. Let* $\chi\in\mathrm{Irr}_{2'}(G)$. *Then* χ *extends to its inertia group in* GD. *Thus the assumption 2.1(ii.2) holds.*

Proof. The statement is trivial whenever D_χ is cyclic. If $G\not\cong\mathsf{D}_{4,\mathrm{sc}}(q)$ the Sylow r-subgroups of D are cyclic for any odd prime r. Note that $\det\chi$ is trivial since G is perfect. Then

[I, Thm. 6.25] shows that χ extends to its inertia group in GD_2, where D_2 is a Sylow 2-subgroup of D. According to [I, (11.31)] this implies that χ extends to GD_χ.

It remains to consider the case where $G \cong \mathsf{D}_{4,\mathrm{sc}}(q)$. Following the considerations above we have to show that χ extends to its inertia group in GD_3 for any Sylow 3-subgroup D_3 of D. Assume that D_χ has a non-cyclic Sylow 3-subgroup.

According to Theorem 7.7 there exists a character $\lambda \in \mathrm{Irr}(T)$, where T is a maximally split maximal torus of G, such that χ is a constituent of $\mathrm{R}_T^G(\lambda)$. According to Lemma 7.9, χ corresponds to some character $\eta \in \mathrm{Irr}(W(\lambda))$ of odd degree such that χ has multiplicity $\eta(1)$ in $\mathrm{R}_T^G(\lambda)$. Moreover $2 \nmid |W : W(\lambda)|$.

Let $\gamma \in D$, $F' \in \langle F_0 \rangle$ be such that $\langle \gamma, F' \rangle$ is the Sylow 3-subgroup of D_χ. Direct computations show that any Sylow 2-subgroup of W is self-normalising in W and can be chosen to be γ-stable. Let P be such a γ-stable Sylow 2-subgroup of W. Then after some N-conjugation of λ we can assume that $W(\lambda)$ contains P.

Then there exist elements $n, n' \in N$ such that $n\gamma$ and $n'F'$ stabilise λ and P. Since P is $\langle \gamma, F' \rangle$-invariant the elements $\pi(n), \pi(n')$ are contained in $\mathrm{N}_W(P) = P$. As linear character λ has an extension $\widetilde{\lambda}$ to $\langle T, F', \gamma \rangle$. Since γ and F' stabilise the unipotent radical U of the Borel subgroup B of G, $\widetilde{\lambda}$ lifts to a character $\hat{\lambda} \in \mathrm{Irr}(\langle B, F', \gamma \rangle)$. The induced character $\Gamma = \hat{\lambda}^{\langle G, F', \gamma \rangle}$ is then an extension of the character $\mathrm{R}_T^G(\lambda)$. Hence $\Gamma\rceil_G$ has χ as constituent with odd multiplicity $\eta(1)$.

Since $|W : W(\lambda)|$ is odd and P has index 3 in W, we either have $W(\lambda) = W$ or $W(\lambda) = P$. In the first case, $\lambda = 1$ since W acts fixed point freely on $\mathrm{Irr}(T)$ and so χ is unipotent, in which case the statement is an easy consequence of [Ma08b, Thm. 2.4]. Else, the character η of $W(\lambda)$ of odd degree is linear since $W(\lambda)$ is a 2-group, and so χ has multiplicity one in $\mathrm{R}_T^G(\lambda)$. Hence Γ has a unique constituent $\widetilde{\chi}$ that is an extension of χ. This completes the proof. □

Proof of Theorem 1. By [IMN07, Thm. B] it is sufficient to show that all non-abelian simple groups S satisfy the inductive McKay condition. For the simple groups not of Lie type this is known by [Ma08a, Thm. 1.1]. So now assume that S is of Lie type, and not as in Proposition 6.1. Observe that $\ell = 2$ implies that $d_\ell(q) \in \{1, 2\}$. By Theorem 2 it suffices to verify the assumptions in Theorem 6.4(b) and (d). Condition 2.1(ii.1) holds since all characters of odd degree only lie in very specific Harish-Chandra series thanks to Theorem 7.7 and characters in those series, more precisely the structure of their stabilisers, have been studied in Corollary 5.3. The requirement 2.1(ii.2) is satisfied for characters in $\mathrm{Irr}_{2'}(G)$ thanks to Proposition 7.10. □

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