ON THE NUMBER OF SIMPLE MODULES
IN A BLOCK OF A FINITE GROUP

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To the memory of Sandy Green

ABSTRACT. We prove that if B is a p-block with non-trivial defect group D of a finite p-solvable group G, then $\ell(B) < p^r$, where r is the sectional rank of D. We remark that there are infinitely many p-blocks B with non-Abelian defect groups and $\ell(B) = p^r - 1$.

We conjecture that the inequality $\ell(B) \leq p^r$ holds for an arbitrary p-block with defect group of sectional rank r. We show this to hold for a large class of p-blocks of various families of quasi-simple and nearly simple groups.

1. INTRODUCTION

Almost 60 years ago, Richard Brauer posed his fundamental problem on $k(B)$, the number of ordinary irreducible characters in a p-block B of defect d of a finite group: Is it always the case that $k(B) \leq p^d$? Despite partial progress, this question remains open to the present day. Here, we propose a conjecture of a similar flavour for the number of irreducible Brauer characters in a p-block. In order to formulate it, we need the following notion:

For a p-block B let $s(B)$ denote the sectional p-rank of a defect group D of B, that is, the largest rank of an elementary Abelian section of D. Further, let $\ell(B)$ denote the number of isomorphism classes of simple modules in B.

Conjecture 1. *Let B be a p-block of a finite group. Then $\ell(B) \leq p^{s(B)}$.*

Let us mention that in all of our results, the inequality in Conjecture 1 turns out to be strict for blocks of positive defect, unless possibly when $p = 3$ and B has extra-special defect group of order 27, see Remark 5.4.

The first main result of this paper is the proof of our conjecture (with strict inequality) in the case of p-solvable groups, which will be given in Section 2 using the positive solution of the $k(GV)$-problem:

Theorem 2. *Let B be a p-block of a finite p-solvable group with non-zero defect. Then $\ell(B) < p^{s(B)}$.*

This has the following consequence:

Corollary 3. *Let B be a block with an abelian defect group, and assume that B either satisfies Alperin's weight conjecture, or Broué's perfect isometry conjecture. Then Conjecture 1 holds for B, with strict inequality if B has non-zero defect.*

Our second main result concerns certain blocks of nearly simple groups. More precisely we show the following:

Theorem 4. *Conjecture 1 holds for the following p-blocks of nearly simple groups:*
  (a) *all p-blocks of all covering groups of symmetric, alternating, sporadic, special linear and special unitary groups;*
  (b) *all p-blocks of all quasi-simple groups of Lie type in characteristic p;*
  (c) *all unipotent p-blocks of quasi-simple groups G of Lie type in characteristic different from p when p is good for G; and*
  (d) *the principal p-blocks of all quasi-simple groups of Lie type.*

This is shown in Propositions 5.2, 5.3, 6.1, Theorems 6.6, 6.7 and Proposition 6.10 respectively. The proofs in those cases involve interesting combinatorial questions on numbers of unipotent characters and of unipotent classes in groups of Lie type.

In view of Conjecture 1 it would be interesting to know whether or under what conditions the sectional rank of a defect group is an invariant under Morita equivalences of blocks.

We also have some partial results on minimal counter examples to Conjecture 1 with a non-trivial normal p-subgroup, which we hope to pursue in future work.

2. Proof of the p-solvable case

We recall for the convenience of the reader that the *sectional rank* of a finite group G is the maximum over all sections of G of the minimum number of generators of the section. The trivial group is considered here to have sectional rank 0. Since every section of G is a homomorphic image of a subgroup of G, the sectional rank of G is also the maximum over all subgroups of G of the minimal number of generators of that subgroup. When G is a finite p-group for some prime p, the sectional rank of G is r, where p^r is the order of the largest elementary Abelian section of G. For a p-block B we let $s(B)$ denote the sectional p-rank of any defect group of B.

Proof of Theorem 2. Let B be a p-block of a finite p-solvable group G such that $s(B) = r \geq 1$. We wish to prove that $\ell(B) < p^r$, where $\ell(B)$ is the number of simple modules in B. By Fong's first reduction, we may suppose that B is a primitive block. Hence, whenever $N \lhd G$ has order prime to p, $\mathrm{Irr}(B)$ covers a single irreducible character of N. By Fong's second reduction, we may then suppose that $O_{p'}(G) \leq Z(G)$. Let $Z = Z(G)$ and let $\lambda \in \mathrm{Irr}(Z)$ lie under B. Note that we now have $D \in \mathrm{Syl}_p(G)$.

Let $U = O_p(G)$ and let H be a Hall p'-subgroup of G. Then there are $\ell(B)$ projective indecomposable characters of G lying over λ, each of which is induced from an irreducible character of H lying over λ. Hence $\ell(B) \leq k(H|\lambda) \leq k(H/Z)$.

The following lemma (stated in somewhat greater generality than required here) now completes the proof. □

Lemma 2.1. *Let X be a non-trivial finite p-solvable group with $O_{p'}(X) = 1$ such that $U = O_p(X)$ has sectional rank r. Then a Hall p'-subgroup of X has fewer than p^r conjugacy classes. In particular, the number of p-regular conjugacy classes of X is less than p^r.*

Proof. Let $V = U/\Phi(U)$. Then $Y = X/U$ acts faithfully as a group of linear transformations of V and $|V| \leq p^r$. Let T be a Hall p'-subgroup of Y. Then the number of p-regular conjugacy classes of X is the number of p-regular conjugacy classes of Y, which is at most the number of conjugacy classes of T. Now $k(T) < k(VT) \leq |V| \leq p^r$, using the solution of the $k(GV)$-problem [13]. $\square$

Remark 2.2. Note that the inequality $k(B) \leq p^{s(B)}$ need not hold (consider $\mathfrak{S}_4$ for $p = 2$, B its unique 2-block). Also, there are 2-blocks B of solvable groups with dihedral defect group of order 8 with $\ell(B) = 3 = 2^{s(B)} - 1$, so the given upper bound in Theorem 2 for the strict inequality for $\ell(B)$ can be attained.

3. Remarks on Conjecture 1

We remark that it is not possible to replace "sectional rank" by "the maximal rank of an Abelian subgroup" in Conjecture 1 (consider the principal 2-block of $\mathrm{SL}_2(3)$, with quaternion defect group, for example).

We note that Conjecture 1 gives a somewhat less precise statement than that predicted by Alperin's Weight Conjecture. It is not clear to us whether an affirmative answer to Conjecture 1 would follow from an affirmative answer to Alperin's Weight Conjecture.

In what follows, we will say that Conjecture 1 holds *in strong form* for B if we have $\ell(B) < p^{s(B)}$. We have seen that in the case that B is a p-block of positive defect of a p-solvable group then Conjecture 1 holds in strong form for B.

Proposition 3.1. *Conjecture 1 holds in strong form for blocks with non-trivial cyclic defect, as well as for 2-blocks with dihedral, (generalised) quaternion or semi-dihedral defect groups.*

Proof. This is clear for p-blocks with cyclic defect groups, since any such block has at most $p-1$ simple modules. If B is a 2-block with dihedral defect group D then $\ell(B) \leq 3 < 4 = 2^{s(B)}$. For 2-blocks B with (generalised) quaternion defect group D, we have $\ell(B) = \ell(b)$, with b the unique Brauer correspondent of B for $C_G(t)$, where t is the central involution of D. Furthermore, we have $\ell(b) = \ell(\tilde{b})$, where $\tilde{b}$ is the unique block of $C_G(t)/\langle t \rangle$ dominated by b, and $\tilde{b}$ has a dihedral (possibly including Klein 4) defect group. Finally, for any 2-block B with semi-dihedral defect groups we again always have $\ell(B) \leq 3$ (see e.g. [25, Thm. 8.1]). $\square$

As stated in Corollary 3 there is a strong relation to other standard conjectures in block theory.

Proof of Corollary 3. If either of Broué's perfect isometry conjecture or Alperin's Weight Conjecture holds for a block B with abelian defect groups, then we have $\ell(B) = \ell(b)$ where b is the unique Brauer correspondent of B for the normaliser $N_G(D)$ of a defect group D of B. By theorems of Külshammer and Reynolds (see [25, Thms. 1.18, 1.19], for example), b is Morita equivalent to a block $\tilde{b}$ of a finite group H with normal Sylow p-subgroup D, so it satisfies Conjecture 1 in strong form. $\square$

As with Brauer's $k(B) \leq |D|$ question, Clifford theoretic reductions for Conjecture 1 are not entirely straightforward (in fact, they are opaque at present). We concentrate on the minimal case.

Lemma 3.2. *If B is a p-block with $\ell(B) > p^{s(B)}$ of a finite group G of minimal order subject to this occurring, then we have $O_{p'}(G) = Z(G)$, as well as $G = O^p(G)$ and $O_p(G)$ elementary Abelian.*

Proof. The fact that $G = O^p(G)$ follows since B is necessarily primitive and B covers a unique block b of $O^p(G)$. For every projective indecomposable of B is induced from a projective indecomposable of b. Hence $\ell(B) \leq \ell(b)$, while the defect group of B contains the defect group of b. The fact that $U = O_p(G)$ is elementary Abelian follows, since by [24, Cor. 7], B dominates a unique block $\tilde{B}$ of $G/\Phi(U)$ with defect group $D/\Phi(U)$, and $\ell(B) = \ell(\tilde{B})$. The fact that $O_p(Z(G)) = 1$ follows for the same reason. By Fong's second reduction we have that $O_{p'}(G) \leq Z(G)$. $\qquad\square$

Proposition 3.3. *Let b be a p-block of G of positive defect satisfying Conjecture 1 and C a cyclic group of prime order $q \neq p$. Then any p-block B of $G \wr C$ which covers $b \otimes \ldots \otimes b$ (q factors) satisfies the strong form of Conjecture 1.*

Proof. As b has positive defect, B has at most $\frac{\ell^q - \ell}{q} + q\ell$ simple modules when b has ℓ simple modules. This is less than ℓ^q when $\ell > 2$, so we may suppose that $\ell \leq 2$. For note that the sectional rank of a defect group of B is qr, where r is the sectional rank of a defect group of b. If we have $\frac{2^q - 2}{q} + 2q \geq p^{rq}$ then $2^q + 2q^2 - 2 \geq qp^{rq}$. In particular, $2^q \leq 2(q+1)$ which forces $q \leq 3$. If $q = 2$ then $p^{2r} \leq 5$ so $p = 2$, contrary to the fact that $p \neq q$. If $q = 3$, we have $p^{3r} \leq 8$, so $p = 2, r = 1$. But then b has cyclic defect (2-)group, so we have $\ell = 1$, and there are just 3 simple modules in B, while $2^{3r} = 8$. $\qquad\square$

Similarly, our conjecture is compatible with direct products. For this note that the sectional p-rank of a direct product of finite p-groups is the sum of the sectional p-ranks of the factors.

4. Implications of the conjecture for Brauer's question

Recall Brauer's question whether it is always true that $k(B) \leq p^d$ for a p-block of defect d. The best known general bounds for $k(B)$ are given by Brauer and Feit in [2]. They show that $k(B) \leq p^{2d-2}$.

Here we note that a proof of our conjecture on $\ell(B)$ would imply bounds of a different nature for a block B with defect group D of order p^d and sectional rank r. For it is a consequence of Brauer's Second Main Theorem that $k(B) = \sum_x \sum_b \ell(b)$, where x runs through a set of representatives for the G-conjugacy classes of elements of D and b runs through blocks of $C_G(x)$ with $b^G = B$.

Using the Alperin–Broué theory of B-subpairs, we may write this expression as $k(B) = \sum_{(x,b)} \ell(b)$, where (x, b) is a Brauer element contained in the maximal B-subpair (D, b), and these are taken up to G-conjugacy.

If Conjecture 1 is correct, then we would certainly have $k(B) \leq \sum_x \sum_b p^r$ where x and b are as previously. So $k(B) \leq p^r k_G(D, b)$, where $k_G(D, b)$ is the number of G-conjugacy classes of Brauer elements contained in (D, b). In any case, there will be many blocks

for which this can be sharpened, for it is quite likely that $s(b) < s(B)$ for many of the blocks b.

Proposition 4.1. *Let G be a finite group with p dividing $|G|$. Assume that Conjecture 1 holds for the principal p-block of the centraliser $C_G(x)$ for each p-element $x \in G$. Then for the principal block B of G we have $k(B) \leq p^r(k(S)-1)$, where S is a Sylow p-subgroup of G.*

Proof. As B is the principal block, using Brauer's Third Main Theorem the above considerations show that our conjecture implies $k(B) \leq p^r k_G(S)$, where $k_G(S)$ is the number of G-conjugacy classes of S, and r is the sectional rank of S. Clearly this yields $k(B) \leq p^r k(S)$, with $k(S)$ the number of conjugacy classes of S. In fact if $k_G(S) = k(S)$, then we have $S \cap G' = S'$, and by a theorem of Tate, G has a normal p-complement (that is, the principal p-block of G is nilpotent). Then we have $\ell(B) = 1$ and $k(B) = k(S)$. Hence we always have $k(B) \leq p^r(k(S) - 1)$ when p divides $|G|$. $\square$

If any two Brauer elements contained in (D, b) are already conjugate via an element of D, then we may use the Broué–Puig star construction and follow the procedure used in [5]: we start with an irreducible character χ of height zero in B, and for each irreducible character μ of D we form the irreducible character $\chi_*\mu$ in B. Then it follows that $k(B) = k(D)$ and $\ell(B) = 1$. It follows that in all cases we should have $k(B) \leq p^r(k(D)-1)$ when B has positive defect.

Remark 4.2. We note that $p^r k_G(D, b)$ can sometimes be less than $|D|$. For example, if B has extra-special defect groups of order 27 and we have $k_G(D, b) = 2$ (which does happen in genuine examples), then $p^r k_G(D, b) = 18$. In any event, the weaker inequality $k(B) \leq p^r(k(D)-1)$ is likely to improve on $k(B) \leq p^{2d-2}$ for most defect groups (though not when D is extra-special or elementary Abelian).

5. ALTERNATING AND SPORADIC GROUPS

We now turn to verifying Conjecture 1 for certain blocks of nearly simple groups. Since the conjecture holds for blocks with cyclic defect, we may and will restrict ourselves to considering blocks whose defect groups are not cyclic. Moreover, by Brauer–Feit, we have $k(B) \leq p^2$ for blocks B with defect group of order p^2, so certainly $\ell(B) < p^2$ and Conjecture 1 holds in strong form for B. Hence we may assume that B has defect at least 3.

Secondly, assume $Z \lhd G$ is a normal subgroup of order prime to p. Then any p-block of G/Z can be considered as a p-block of G with isomorphic defect groups and the same number of irreducible Brauer characters. Thus, when investigating the validity of Conjecture 1 for blocks of a quasi-simple group G we may pass to the universal p'-covering group of G (as by Lemma 3.2 we need not consider p-covers).

For integers $s, t \geq 1$ let us denote by $k(s, t)$ the number of s-tuples of partitions of t. In particular, $k(1, t)$ is the number of partitions of t. We will use the following estimates:

Lemma 5.1 (Olsson). *Let $s, t \geq 1$. Then $k(s, t) < (s+1)^t$. If moreover $s \geq 2$ then $k(s, t) \leq s^t$ unless $s = 2$ and $t \leq 6$.*

Proof. By [21, Prop. 5] for all $s \geq 2$ we have $k(s, t) \leq s^t$ unless $s = 2$ and $t \leq 6$. As $k(s, t) < k(s+1, t)$ and $k(1, t) < 2^t$ for $t \leq 6$, all claims follow. $\square$

Proposition 5.2. *Conjecture 1 in strong form holds for the blocks of alternating and symmetric groups and their covering groups for all primes.*

Proof. The 2-blocks of the 3-fold covering groups of $\mathfrak{A}_6$ and $\mathfrak{A}_7$ can be dealt with directly. There is nothing to check for the 6-fold covers since Sylow p-subgroups for $p > 3$ are cyclic. In all other cases the numbers $\ell(B)$ have been computed by G. de B. Robinson (see [23]). First let B be a p-block of $\mathfrak{S}_n$ of weight w. Then any defect group of B has an elementary Abelian subgroup of rank w, so $s(B) \geq w$. On the other hand, by [23, Prop. 11.14] we have $\ell(B) = k(p-1, w)$. By Lemma 5.1 we find $\ell(B) = k(p-1, w) < p^w \leq p^{s(B)}$ as claimed.

Next let B be a p-block of $\mathfrak{A}_n$, covered by a block $\hat{B}$ of $\mathfrak{S}_n$ of weight w. First assume that $p > 2$. Then the defect groups of B and $\hat{B}$ agree. If $1 \leq w < p$, then $\hat{B}$ has elementary Abelian defect groups, whence $s(B) = w$, while $\ell(B) < k(B) \leq k(\hat{B})$ by [22, Prop. 4.10]. So the claim follows from the already proven result for $\mathfrak{S}_n$. So we have $p \leq w$. Then clearly still $\ell(B) \leq 2\ell(\hat{B}) = 2k(p-1, w)$. Now again $2k(p-1, w) \leq 2(p-1)^w$ unless $p = 3$ and $w \leq 6$, and an easy estimate shows that $2(p-1)^w < p^w$ whenever $3 \leq p \leq w$. The finitely many cases when $p = 3$ and $w \leq 6$ can be checked easily. If $p = 2$ then by [23, Prop. 12.9] we have that $\ell(B) = \ell(\hat{B})$ if w is odd and $\ell(B) = \ell(\hat{B}) + k(1, w/2)$ if w is even. Moreover, B satisfies $s(B) \geq s(\hat{B}) - 1$. When $w = 2$ the defect groups are elementary Abelian of order 4, while $\ell(B) = 3$. When $w = 3$ then $\ell(B) = w(1, 3) = 3$ and $|D| = 2^3$, so $s(B) \geq 2$. Finally, if $w \geq 4$ then

$$\ell(B) \leq \ell(\hat{B}) + k(1, w/2) = k(1, w) + k(1, w/2) \leq 2^w \leq 2^{s(B)}.$$

By Lemma 3.2 the claim then also holds for the 2-fold covering groups of $\mathfrak{A}_n$ and $\mathfrak{S}_n$. Now assume that $p > 2$ and let B be a faithful p-block of a 2-fold covering group of $\mathfrak{S}_n$, of weight w. Then by [23, Prop. 13.17] we have that

$$\ell(B) = \begin{cases} k(t, w) & \text{if } w \text{ is even,} \\ 2k(t, w) & \text{if } w \text{ is odd,} \end{cases}$$

with $t = (p-1)/2$, while on the other hand $s(B) \geq w$. Thus, we may conclude as before. Finally, any faithful block of the 2-fold cover of an alternating group has exactly the same invariants as some faithful block of a 2-fold covering of some symmetric group of the same weight (see [23, Rem. 13.18]), so the claim here follows from the previous considerations. $\square$

Proposition 5.3. *Let G be such that $[G, G]$ is quasi-simple and $S \leq G/Z(G) \leq \mathrm{Aut}(S)$ for some sporadic simple group S or $S = {}^2F_4(2)'$. Then Conjecture 1 holds for all p-blocks B of G. It holds in strong form unless possibly when $p = 3$ and defect groups of B are extraspecial of order 27.*

Proof. This statement can be essentially verified by computer on the known ordinary character tables of the groups in question together with the known (lower bounds on) the p-ranks as given in [14, Tab. 5.6.1]. The only blocks B with defect at least 3 whose defect groups are not Sylow p-subgroups are a 2-block of the third Conway group Co_3 of defect 3 with $\ell(B) = 5$, and a 2-block of the Lyons group Ly of defect 7 with $\ell(B) = 8$. The former block is known to have elementary Abelian defect groups. In the latter case,

the defect groups have index 2 in a Sylow 2-subgroup, which is of rank 4, so the inequality in Conjecture 1 holds as well. We claim that in fact a defect group D of B has rank 4 in this case as well. Indeed, $G = Ly$ has a unique class of involutions with representative t, say, with centraliser $C = C_G(t)$ a double cover of $\mathfrak{A}_{11}$. There is a Brauer correspondent of B for C which has the same defect group D of order 2^7. Let $D_1 = D/\langle t \rangle$. Then D_1 is a defect group of a 2-block b of $\mathfrak{A}_{11}$ which is covered by a block of weight 4 of $\mathfrak{S}_{11}$ whose defect group intersected with $\mathfrak{A}_{11}$ has rank 4. $\qquad\square$

Remark 5.4. The principal 3-blocks of the Tits group ${}^2F_4(2)'$, of the Rudvalis group Ru and of the fourth Janko group J_4, as well as a further non-principal 3-block of J_4, have extra-special defect group $D \cong 3_+^{1+2}$ of order 27 and each possesses 9 irreducible Brauer characters, so they provide examples of blocks B with $\ell(B) = p^{s(B)}$. These (and related examples in automorphism groups as well as examples in groups ${}^2F_4(q^2)$ with $q^2 \geq 8$ also for $p = 3$ and with sectional rank equal to 2, see Proposition 6.5) are the only cases of positive defect known to us where equality in Conjecture 1 occurs.

6. Groups of Lie type

6.1. **Defining characteristic.** We first deal with the defining characteristic case.

Proposition 6.1. *Let G be a finite quasi-simple group of Lie type in characteristic p. Then Conjecture 1 holds in strong form for all p-blocks of G.*

Proof. By Lemma 3.2 we need not consider covering groups with p dividing the order of the centre. But then we may assume that G is the universal p'-covering group of its simple factor. In that case, G is obtained as $\mathbf{G}^F$ where $\mathbf{G}$ is a simple, simply connected linear algebraic group over an algebraic closure of $\mathbb{F}_p$ and $F : \mathbf{G} \to \mathbf{G}$ is a Steinberg endomorphism. By a result of Humphreys G has exactly one p-block of defect zero, containing the Steinberg character, and all other p-blocks of G have full defect. First assume that G is not of twisted type. Then Sylow p-subgroups of G have p-rank at least rf, where $q = p^f$ is the size of the underlying field of G and r denotes the rank of $\mathbf{G}$, see [14, Tab. 3.3.1]. On the other hand, the simple modules of G in characteristic p are parametrised by q-restricted weights, of which there exist exactly q^r. One of them belongs to the Steinberg character, so $\ell(B) \leq q^r - 1$ for all blocks B of positive defect. Thus

$$\ell(B) \leq q^r - 1 < q^r \leq p^{s(B)}.$$

Next assume that G is twisted, but not of Ree or Suzuki type. Then again by loc. cit., the p-rank of G is at least rf, where $q = p^f$ is the absolute value of the eigenvalues of F on the character group of a maximal torus of $\mathbf{G}$, unless $p = 2$ and $G = \mathrm{SU}_3(q)$. But note that the quotient of a Sylow p-subgroup of $\mathrm{SU}_3(q)$ by the highest root subgroup is elementary Abelian, so has p-rank $2f$. The number of simple modules of G again equals q^r. So we may conclude as before. Finally, the groups ${}^2B_2(q^2)$, ${}^2G_2(q^2)$ and ${}^2F_4(q^2)$ $(q^2 \geq 8)$ have p-ranks f, $2f$, $5f$ respectively, where $q^2 = p^f$, and the same argument applies. $\qquad\square$

Remark 6.2. The proof shows that for all primes $p \geq 5$ there exist p-blocks B of simple groups with extra-special defect group p^{1+2} in which $\ell(B) = p^2 - 1 = p^{s(B)} - 1$, so the bound in the strong form of Conjecture 1 cannot be improved to $p^{s(B)} - 2$ for any prime. Indeed, for $p \equiv 2 \pmod 3$ the principal p-block of $\mathrm{SL}_3(p)$ is as claimed, while for $p \equiv 1 \pmod 3$ we may take the principal p-block of $\mathrm{SU}_3(p)$.

Concerning nearly simple groups, we make the following observations:

Corollary 6.3. *Let G be a finite quasi-simple group of Lie type in characteristic p and σ a field, graph or graph-field automorphism of G. Then Conjecture 1 holds in strong form for all p-blocks of the extension $\tilde{G} := G\langle\sigma\rangle$ of G.*

Proof. Let B be a p-block of G of positive defect and $\tilde{B}$ a p-block of $\tilde{G}$ covering B. As explained in the proof of Proposition 6.1 the simple modules in B are labelled by (a subset of the) q-restricted weights for G, which naturally can be given the structure of an elementary Abelian p-group P. Now the action of σ on the simple modules is induced by an action of σ on the associated weights, which may be considered as a faithful action of σ on P, where r is the rank of G. Note that we may assume that σ has order prime to p. Then by Lemma 2.1, the number of characters of $\tilde{G}$ above those in B is at most q^r minus the number of characters lying above the Steinberg weight (which belongs to a different block). Since $s(\tilde{B}) \geq s(B)$ the claim now follows from the arguments in the proof of Proposition 6.1. $\qquad\square$

6.2. **Exceptional coverings and small rank.** We first consider the finitely many exceptional covering groups.

Proposition 6.4. *Conjecture 1 in strong form holds for all blocks of exceptional covering groups of finite simple groups of Lie type and all primes.*

Proof. Again, the statement can be essentially verified by inspection on the known ordinary character tables of the groups in question together with obvious lower bounds on the p-ranks. All blocks are either of maximal defect, or of defect at most 2. $\qquad\square$

We next consider exceptional groups of small rank.

Proposition 6.5. *Let B be a p-block of a finite quasi-simple group G of Lie type ${}^2B_2, {}^2G_2$, $G_2, {}^3D_4$ or ${}^2F_4$. Then Conjecture 1 holds for B. It holds in strong form unless B is the principal 3-block of ${}^2F_4(q^2)^{(\prime)}$.*

Proof. The exceptional covering groups were dealt with in Proposition 6.4. Next, if p is the defining prime for G the result was already proved in Proposition 6.1. Furthermore, we may assume that the defect groups of B are non-cyclic. Thus G is of type $G_2, {}^3D_4$ or ${}^2F_4$, or $p = 2$ and $G = {}^2G_2(q^2)$. In the latter case the Sylow 2-subgroups are elementary Abelian of order 8, while $\ell(B) = 3$ for all 2-blocks of defect at least 2 by [26, p. 74].

For $G_2(q)$ the numbers $\ell(B)$ in the non-cyclic defect case were determined by Hiss and Shamash [17, 18, 19]: The principal block B has $\ell(B) \leq 7$ in all cases, while the p-rank is equal to 2 when $p \geq 3$, and 3 for $p = 2$. All other blocks B have $\ell(B) \leq 3$. For ${}^3D_4(q)$, then [11, Thm. A] shows that $\ell(B) \leq 7$ for the principal p-block when p is good (so when $p \neq 2$), while non-cyclic Sylow p-subgroups obviously have p-rank at least 2. When $p = 2$ then again $\ell(B) = 7$ by [15, Thm. 3.1] and Sylow 2-subgroups have rank 3. For all other blocks the validity of Conjecture 1 is immediate. Finally, for ${}^2F_4(q^2)$, $q^2 \geq 8$, the numbers $\ell(B)$ for primes $p \geq 3$ have been determined in [16]. From this the inequality can be checked, and as for the case of ${}^2F_4(2)'$ in Proposition 5.3 we obtain equality (only) when B is the principal 3-block. $\qquad\square$

Theorem 6.6. *Let B be a p-block of a quasi-simple group $\mathrm{SL}_n(q)$ ($n \geq 2$) or $\mathrm{SU}_n(q)$ ($n \geq 3$) in characteristic $r \neq p$. Then Conjecture 1 holds for B.*

Proof. Our argument crucially relies on the description of p-blocks of $\mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q)$ by Fong and Srinivasan [10], as well as on the result of Geck [11, Thm. A] that for any finite group of Lie type G and any semisimple p'-element $s \in G^*$ the set $\mathcal{E}(G, s)$ forms a basic set for $\mathcal{E}_p(G, s)$ whenever p is a good prime not dividing the order of the group of components of the center of the underlying algebraic group.

We first show our claim for unipotent p-blocks of $G = \mathrm{GL}_n(q)$. Note that the centre of GL_n is connected. Thus the unipotent characters form a basic set in any unipotent block B, and the number $\ell(B)$ is just the number of unipotent characters in B. Let d denote the multiplicative order of q modulo p; in particular we then have $d \leq p - 1$. Let B be a unipotent p-block of G. Then the unipotent characters in B are those labelled by partitions μ of n with a fixed d-core $\lambda \vdash n - wd$ for a suitable weight w, see [10], and [7, Thm. 21.14] for $p = 2$. The number of such partitions equals $k(d, w)$, so $\ell(B) = k(d, w)$. On the other hand, by [10] a defect group D of B contains an elementary Abelian subgroup of order p^w, so $s(B) \geq w$ (respectively for $p = 2$ we have $d = 1$ and B is the principal block). Now we have $k(d, w) < (d+1)^w$ by Lemma 5.1 and as $d \leq p - 1$, our claim follows for unipotent blocks.

Next let B be an arbitrary p-block of $\mathrm{GL}_n(q)$. By the theorem of Bonnafé–Rouquier [7, Thm. 10.1], B is Morita equivalent to a unipotent block of $C_{G^*}(s)$, with isomorphic defect groups. (Note that here $G^* \cong G = \mathrm{GL}_n(q)$.) But

$$C_{G^*}(s) = G_1 \times \cdots \times G_r \quad \text{with} \quad G_i \cong \mathrm{GL}_{n_i}(q^{f_i}) \text{ for certain } n_1 f_1 + \ldots + n_r f_r = n,$$

so B is the product of unipotent blocks B_i of G_i, and similarly for the defect groups. Thus our previous considerations show that B satisfies the assertion of Conjecture 1.

Now consider the case when $G = \mathrm{SL}_n(q)$. Embed G as a subgroup of $\tilde{G} = \mathrm{GL}_n(q)$ in the natural way. Let B be a p-block of G in series $\mathcal{E}_p(G, s)$. Let $\tilde{s} \in \tilde{G}^*$ be a preimage of s under the induced epimorphism $\tilde{G}^* \to G^*$ and $\tilde{B}$ be a p-block of $\tilde{G}$ in $\mathcal{E}_p(\tilde{G}, \tilde{s})$ covering B. By the previous case, the claim holds for $\tilde{B}$. But then it also holds for B whenever $p \nmid (q - 1)$, since on the one hand side $|Z(G)|$ divides $q - 1$, hence is prime to p and so $\mathrm{Irr}(B) \cap \mathcal{E}(G, s)$ is a basic set for B, and on the other hand $|\tilde{G} : G| = q - 1$ is prime to p in this case, so the defect groups of B and $\tilde{B}$ agree.

So now assume that $p | (q - 1)$, whence $d = 1$. As $\tilde{G}/G$ is cyclic, B satisfies $s(B) \geq s(\tilde{B}) - 1$, and furthermore $\ell(B) \leq n\ell(\tilde{B})$. As above write

$$C_{\tilde{G}^*}(\tilde{s}) = G_1 \times \cdots \times G_r$$

and let B_i denote the unipotent p-block of G_i such that $\tilde{B}$ is Morita equivalent to $B_1 \otimes \cdots \otimes B_r$. Since $p | (q - 1)$, each B_i is the principal block of G_i, and thus contains $k(1, n_i)$ unipotent characters, while $s(B_i) \geq n_i$. It is straightforward to check that the required inequality holds unless possibly when $n_i = 2$, $p \leq 3$. For $n_i = 2$ and $p = 3$ the Sylow p-subgroups of G_i are cyclic, while for $n_i = p = 2$ we have $\ell(B_i) = 3$ and $|D_i| = 8$.

The preceding arguments carry over almost word by word to $\mathrm{SU}_n(q)$. Again, we first deal with $G = \mathrm{GU}_n(q)$. Note that centralisers of semisimple elements $s \in G^* \cong \mathrm{GU}_n(q)$

have the form

$$C_{G^*}(s) = G_1 \times \cdots \times G_r \quad \text{with} \quad G_i \cong \mathrm{GL}_{n_i}((-q)^{f_i})$$

for suitable $n_1 f_1 + \ldots + n_r f_r = n$, where $\mathrm{GL}_m(-u)$ with $-u < 0$ is to be interpreted as $\mathrm{GU}_m(u)$. Then, setting d now to be the order of $-q$ modulo p we can argue as before. For $p = 2$ we again use the fact that all unipotent characters lie in the principal 2-block, see [7, Thm. 21.14]. We then deal with $\mathrm{SU}_n(q)$ via the regular embedding $\mathrm{SU}_n(q) \hookrightarrow \mathrm{GU}_n(q)$. Here, the critical case is when $p|(q+1)$, but then again all unipotent characters of $G_i \cong \mathrm{GL}_{n_i}((-q)^{f_i})$ lie in the principal block and the same estimates as above yield the claim. $\qquad\square$

6.3. **Unipotent blocks.** While our results for the classes of quasi-simple groups considered so far are complete, it seems that at present the knowledge on blocks of other groups of Lie type in non-defining characteristic is not yet strong enough to check Conjecture 1 in all cases; we only obtain partial results:

Theorem 6.7. *Let B be a unipotent p-block of a finite quasi-simple group G of classical Lie type in characteristic $r \neq p$. Then Conjecture 1 holds for B.*

Proof. We will use throughout the result of Geck [11, Thm. A] that the unipotent characters form a basic set for the unipotent p-blocks of any finite group of Lie type for which p is a good prime and such that p does not divide the order of the group of components of the center of the underlying algebraic group. Then in particular in any unipotent block B, the number $\ell(B)$ is just the number of unipotent characters in B.

First consider $G = \mathrm{Sp}_{2n}(q)$ and $G = \mathrm{SO}_{2n+1}(q)$. If $p > 2$ the unipotent characters form a basic set for the unipotent p-blocks of G. Let d denote the order of q modulo p. We need to distinguish two cases. First assume that d is odd. Then the unipotent blocks of G are indexed by Lusztig symbols $\mathcal{S}$ without d-hooks, called d-cores, and a unipotent character χ lies in the block B corresponding to $\mathcal{S}$ if and only if it has d-core $\mathcal{S}$. The number of such characters equals the number of irreducible characters of the corresponding relative Weyl group (see [4, Thm. 3.2]), of type $C_{2d} \wr \mathfrak{S}_w$, where w is the weight of B. Thus $\ell(B) = k(2d, w)$. Defect groups of B then contain an elementary Abelian subgroup of order p^w, so we have $s(B) \geq w$. Now note that d divides $p-1$, and as d is odd this forces $d \leq (p-1)/2$. By Lemma 5.1 we have $\ell(B) = k(2d, w) \leq k(p-1, w) < p^w \leq p^{s(B)}$. On the other hand, if d is even, then again using [4, Thm. 3.2] we have that $\ell(B) = k(C_d \wr \mathfrak{S}_w) = k(d, w)$ with $d \leq p-1$, while still $s(B) \geq w$, so we can conclude as before.

All of the above assertions remain valid for unipotent blocks of groups of type D_n or ${}^2D_n$, unless B contains the unipotent characters labelled by symbols whose d-core (when d is odd) or e-cocore (when $d = 2e$ is even) is a so-called degenerate symbol. First assume that d is odd. Then the relative Weyl group is the reflection subgroup $G(2d, 2, w)$ of $C_{2d} \wr \mathfrak{S}_w$ of index 2, defect groups of B still have rank at least w, and $\ell(B) < (2d+1)^w \leq p^w$ by Lemma 6.8. We may argue entirely similar when d is even.

If G is of type B_n, C_n or D_n and $p = 2$, then let $G \hookrightarrow \tilde{G}$ denote a regular embedding. By [6, Thm. 13] all unipotent characters of $\tilde{G}$ lie in the principal 2-block $\tilde{B}$ of $\tilde{G}$. Furthermore, by [12, Prop. 2.4] the number $\ell(\tilde{B})$ equals the number of unipotent classes of $\tilde{G}$. Upper bounds for these are given in Lemma 6.9. Now first assume that we are in type B_n or C_n. Then $\tilde{G}$ induces on G an outer automorphism of order 2. Thus if B is the principal

2-block of G, then $\ell(B) \leq 2\ell(\tilde{B})$. Lower bounds for the sectional 2-rank of G are given in Table 1.

TABLE 1. Lower bounds for sectional 2-ranks, q odd

G	$B_n(q)$	$C_n(q)$	$D_n(q)$	$^2D_n(q)$
	$2n$	$2n$	$2n-1$	$2n-1$

Indeed, according to [3, Table 2.5], the simple orthogonal group $\mathrm{O}_n^{(\pm)}(q)$, with q odd, contains a subgroup $2^{n-1}.\mathfrak{A}_n$ (inside the natural subgroup $\mathrm{O}_1(q) \wr \mathfrak{A}_n$), so has 2-rank at least $n-1$. Similarly, $\mathrm{Sp}_{2n}(q)$ contains a direct product $\mathrm{Sp}_2(q)^n$, each factor of which has sectional 2-rank 2. Then for $\mathbf{G}$ of type C_n, as $2^{2n} > 2 \cdot 2^{n+\lfloor\sqrt{n}\rfloor}$ for $n \geq 3$, the desired result follows for B. When $n = 2$ then $\tilde{G}$ has 5 unipotent classes and again our inequality is satisfied. For $\mathbf{G}$ of type B_n ($n \geq 3$) the sectional 2-rank of G is at least $2n$ by Table 1, and again we are done unless $n \in \{3, 4\}$. In the latter two cases, the actual number of unipotent classes of $\tilde{G}$ is $10, 21$ respectively, smaller than $2^{2n}/2$.

For $\mathbf{G}$ of type D_n ($n \geq 4$), $\tilde{G}$ induces on G a group of automorphisms of order dividing 4. The principal 2-block $\tilde{B}$ of $\tilde{G}$ has $\ell(\tilde{B}) \leq 2^{n+\lfloor\sqrt{2n}\rfloor}$ by Lemma 6.9. Thus the claim follows by Table 1 unless $4 \leq n \leq 6$. In the latter cases, $\tilde{G}$ has $13, 18, 37$ unipotent classes, respectively (and even fewer in the twisted case), which is smaller than $2^{2n-1}/4$. $\qquad\square$

The following estimates were used in the previous proof:

Lemma 6.8. *Let $w, d \geq 1$. Then $|\mathrm{Irr}(G(2d, 2, w))| < (2d + 1)^w$.*

Proof. The reflection group $G(2d, 1, w)$ is the wreath product $C_{2d} \wr \mathfrak{S}_w$, hence its irreducible characters are parametrised by $2d$-tuples of partitions of w, of which there are $k(2d, w)$. Such a character splits upon restriction to the normal reflection subgroup $G(2d, 2, w)$ if the parametrising $2d$-tuple λ has a symmetry of order 2, that is, if w is even and λ is the concatenation of twice a d-tuple μ of partitions of $w/2$, and it restricts irreducibly else. So by Lemma 5.1 $|\mathrm{Irr}(G(2d, 2, w))| = \frac{1}{2}k(2d, 1, w) < (2d + 1)^w$ when w is odd, and

$$|\mathrm{Irr}(G(2d, 2, w))| = \frac{1}{2}(k(2d, 1, w) - k(d, 1, v)) + 2k(d, 1, v)$$
$$= \frac{1}{2}k(2d, 1, w) + \frac{3}{2}k(d, 1, v) \leq \frac{1}{2}(2d + 1)^w + \frac{3}{2}(d + 1)^v < (2d + 1)^w$$

if $w = 2v$ is even. $\qquad\square$

Lemma 6.9. *Let $\mathbf{G}$ be a simple algebraic group in odd characteristic of adjoint type B_n, C_n or D_n respectively, and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism. Then $G = \mathbf{G}^F$ has at most $2^{n+\lfloor\sqrt{2n+1}\rfloor}$, respectively $2^{n+\lfloor\sqrt{n}\rfloor}$, $2^{n+\lfloor\sqrt{2n}\rfloor}$ unipotent conjugacy classes.*

Proof. The unipotent classes of $\mathbf{G}$ and G are described for example in [8, §13.1]. First assume that $\mathbf{G}$ is of type C_n. Its unipotent classes are parametrised by pairs of partitions (α, β) of n, where β has distinct parts. Thus, using Lemma 5.1 and direct computation for $n \leq 6$ there are at most 2^n unipotent classes in $\mathbf{G}$. Each class C of $\mathbf{G}$ splits into $a(C)$ classes in G, where $a(C)$ is the order of the component group of the centraliser of any

element $u \in C$. This component group has order $2^{n(C)}$, where $n(C)$ is at most the number of even i such that (α, β) has a part of length i. This becomes maximal if (α, β) has parts of lengths $2, 4, 6, \ldots$. Since $\sum_{i=1}^{k} 2i = k(k+1)$ we see that $n(C) \leq \lfloor \sqrt{n} \rfloor$, whence the result.

For **G** of type B_n, the unipotent classes are parametrised by pairs of partitions (α, β) such that $2|\alpha| + |\beta| = 2n+1$, where β has distinct odd parts. To any such pair we associate a pair of partitions (α, β') of n as follows: if $\beta = (\beta_1 \leq \beta_2 \leq \ldots)$ then $\beta' = ((\beta_1 - 1)/2 \leq (\beta_2 + 1)/2 \leq (\beta_3 - 1)/2 \leq \ldots)$ (note that β necessarily has an odd number of parts, so $|\beta'| = (|\beta| - 1)/2$). Clearly this map is injective, so again the number of unipotent classes of **G** is at most 2^n. The component group of the centraliser of a unipotent element in the class C parametrised by (α, β) has order $2^{n(C)}$, where $n(C)$ is at most the number of distinct odd parts of (α, β). As $\sum_{i=1}^{k}(2i-1) = k^2$ we find $n(C) \leq \lfloor \sqrt{2n+1} \rfloor$ and hence the claim.

Finally, for **G** of type D_n the unipotent classes are parametrised by pairs of partitions (α, β) with $2|\alpha| + |\beta| = 2n$, where β has distinct odd parts. If β is empty and all parts of α are even, then (α, β) parametrises two unipotent classes. Arguing as in the previous case we see that there are at most $2^n - 2^{n/2}$ non-degenerate cases, and $2^{n/2+1}$ degenerate ones. The component group of the corresponding centraliser has order $2^{n(C)}$, where $n(C)$ is at most the number of distinct odd parts. Hence we get a factor of at most $2^{\lfloor \sqrt{2n} \rfloor}$ for non-degenerate classes, while degenerate classes have connected centralisers. This yields at most

$$(2^n - 2^{n/2})2^{\lfloor \sqrt{2n} \rfloor} + 2^{n/2+1} = 2^{n+\lfloor \sqrt{2n} \rfloor} - 2^{n/2+\lfloor \sqrt{2n} \rfloor} + 2^{n/2+1} \leq 2^{n+\lfloor \sqrt{2n} \rfloor}$$

unipotent classes for G. $\square$

We now turn to the groups of exceptional type. The small rank groups have already been considered in Proposition 6.5.

Proposition 6.10. *Let B be a unipotent p-block of a finite quasi-simple group G of exceptional Lie type in characteristic $r \neq p$ of rank at least 4. If p is bad for G, then assume that B is the principal block. Then Conjecture 1 holds for B.*

Proof. For the remaining types $F_4, E_6, {}^2E_6, E_7$ and E_8 first assume that p is a good prime (so $p \geq 5$, and $p \geq 7$ for type E_8). Let d denote the order of q modulo p. The numbers $\ell(B)$ for the principal p-block and the p-ranks of Sylow p-subgroups are then as given in Table 2 by [4, Thm. 3.2 and Table 3]. (The entries "−" signify that the Sylow p-subgroups are cyclic.) It ensues that the required inequality is satisfied in all cases, even in the strict form. More generally, the non-principal unipotent blocks are described in [4, Tab. 1]. An easy check, similar to that for the principal blocks, shows that the inequality in its strict form also holds for those blocks.

If p is bad for G, then the total number of simple modules in unipotent p-blocks of G is as given in Table 3, see [9, §4.1]. Note that the assumption in loc. cit. on the underlying field size of G is now unnecessary since by results of Lusztig cuspidal character sheaves are always 'clean', see [20]. Table 3 also gives obvious lower bounds for the p-ranks in the respective cases (obtained by looking at suitable maximal tori of the groups in question). Furthermore, for $p = 2$ in $F_4(q)$ note that the sectional 2-rank is at least 8, since $F_4(q)$ contains a central product A of 4 commuting A_1-type subgroups, see [14, Tab. 4.10.6].

Then $A/Z(A)$ contains a direct product $\prod_{i=1}^{4} \mathrm{PSL}_2(q)$; since $\mathrm{PSL}_2(q)$ has 2-rank 2, the claim follows. $\qquad\square$

TABLE 2. $\ell(B)$ for principal blocks and p-ranks

$(\ell(B):s)$		F_4	E_6	$^2E_6$	E_7	E_8
	1	$25:4$	$25:6$	$25:4$	$60:7$	$112:8$
	2	$25:4$	$25:4$	$25:6$	$60:7$	$112:8$
$d=$	3	$21:2$	$24:3$	$21:2$	$48:3$	$102:4$
	4	$-$	$16:2$	$16:2$	$16:2$	$59:4$
	6	$21:2$	$21:2$	$24:3$	$48:3$	$102:4$

TABLE 3. $\ell(B)$ and lower bounds on p-ranks for bad primes

$(\ell(B):s)$		F_4	$^{(2)}E_6$	E_7	E_8
	2	$28:4$	$27:6$	$64:7$	$131:8$
$p=$	3	$35:4$	$28:4$	$72:7$	$150:8$
	5	$-$	$-$	$-$	$162:4$

6.4. **Towards general blocks.** We give some further partial results for general blocks of finite quasi-simple groups of Lie type.

Theorem 6.11. *Let* $\mathbf{G}$ *be simple with Frobenius endomorphism* F, *and let* $p \geq 3$ *be a good prime for* $\mathbf{G}$. *Then Conjecture 1 holds for all* p-*blocks of* $\mathbf{G}^F$ *if it holds for all non-unipotent quasi-isolated* p-*blocks of all* F-*stable Levi subgroups* $\mathbf{L} \leq \mathbf{G}$.

Proof. Let B be a p-block of $\mathbf{G}^F$, in the Lusztig series of the semisimple p'-element $s \in \mathbf{G}^{*F}$. Let $\mathbf{L}^* \leq \mathbf{G}^*$ be the minimal F-stable Levi subgroup containing $C_{\mathbf{G}^*}(s)$ (this is uniquely determined since the intersection of two Levi subgroups containing a common maximal torus is again a Levi subgroup). Then s is quasi-isolated in $\mathbf{L}^*$. Let $\mathbf{L} \leq \mathbf{G}$ be dual to $\mathbf{L}^*$. According to the theorem of Bonnafé, Dat and Rouquier [1, Thm. 7.7] Lusztig induction $R_{\mathbf{L}}^{\mathbf{G}}$ induces a Morita equivalence between the p-blocks in $\mathcal{E}_p(\mathbf{L}^F, s)$ and those in $\mathcal{E}_p(\mathbf{G}^F, s)$, which sends $\mathrm{Irr}(B_1)$ bijectively to $\mathrm{Irr}(B)$ for some p-block B_1 contained in $\mathcal{E}_p(\mathbf{L}^F, s)$ and that preserves defect groups. In particular $\ell(B) = \ell(B_1)$ and $s(B) = s(B_1)$, and so Conjecture 1 holds for B if it holds for the quasi-isolated block B_1 of the Levi subgroup $\mathbf{L}^F$.

If B_1 is a unipotent block, then since unipotent characters are insensitive to isogeny types, our conjecture follows from our proof for unipotent blocks given in Theorems 6.6, 6.7 and Proposition 6.10. $\qquad\square$

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