BRAUER'S HEIGHT ZERO CONJECTURE FOR TWO PRIMES

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Abstract. Let p and q be two primes. We propose that Brauer's Height Zero Conjecture for the principal p-blocks of finite groups can naturally be extended from the perspective of q. We prove one direction of this new conjecture, and show the reverse direction assuming that the Inductive Alperin–McKay condition holds for the finite simple groups.

1. Introduction

Unlike some celebrated theorems in number theory, there are few results in the representation theory of finite groups (outside solvable groups) that take two different primes into account.

In this paper, we propose what might be an exception. In fact, Conjecture A below constitutes a generalisation of a famous conjecture of Richard Brauer. If G is a finite group and p is a prime, let us denote by $B_p(G)$ the set of irreducible complex characters in the principal p-block of G.

Conjecture A. *Let G be a finite group, and let p and q be primes. Then the elements of some Sylow p-subgroup of G commute with the elements of some Sylow q-subgroup of G if and only if the characters in $B_p(G)$ have degree not divisible by q and the characters in $B_q(G)$ have degree not divisible by p.*

Of course, if $p = q$, then Conjecture A is Brauer's Height Zero Conjecture for principal p-blocks.

In this paper we prove the "only if" direction of Conjecture A in Theorem 4.1 below for $p \neq q$. (If $p = q$, the "only if" direction of Conjecture A is known to hold by Kessar–Malle [12].) In Theorem 5.2 again for $p \neq q$, we show the "if" direction *under the assumption* of the Inductive Alperin–McKay condition for principal blocks (see Späth [20]). (For $p = q$ this is the (principal block case of the) main result of Navarro–Späth [17].)

Is there some version of Conjecture A for arbitrary blocks of finite groups? If $G = 6.\mathfrak{A}_7$ and $p = 5, q = 7$, then G has a p-block of maximal defect such that all of its irreducible characters have degree coprime to q, and the other way around with the roles of p and

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q reversed. But in G, the order of the normaliser of a Sylow p-subgroup is not divisible by q, and the order of the normaliser of a Sylow q-subgroup is not divisible by p. So the extension of Conjecture A to blocks of maximal defect is false in general (although it does hold for $\{p, q\}$-separable groups, by Navarro–Wolf [19]). It is perhaps worth remarking that this is the very same example found by C. Bessenrodt in [3] to the question of when the irreducible characters of p and q-blocks coincide.

It is an interesting question to characterise when the irreducible characters of the principal p-block have degree not divisible by q. For $q = 2$, this was studied by Giannelli–Malle–Vallejo [8].

2. Extending Characters and Principal Blocks

If $N \triangleleft G$ and $\theta \in \mathrm{Irr}(N)$ is in the principal block of N and extends to G, it is not necessarily true that θ has an extension in the principal block of G. (For instance, if G is the solvable group `SmallGroup`$([144, 187])$ of order 144, and $p = 3$, then G has two normal subgroups of type $(C_3 \times C_3) : C_2$. Let N be the one that has a complement, Q_8, in G. Also $Z = \mathbf{Z}(G) = \mathbf{O}_{3'}(G)$ has order 2. Now, the non-trivial linear character of N extends to G but does not have an extension containing Z in its kernel.)

Let p be a prime. We fix a maximal ideal M of the ring of algebraic integers $\mathbf{R}$ containing p, and we let R be the localisation of $\mathbf{R}$ at M. We denote by $^* : R \to \mathbf{R}/M = F$ the natural ring homomorphism. Notice that if χ is in the principal block of a group X, and $x \in X$ is such that $|X : \mathbf{C}_X(x)|$ is not divisible by p, then $\chi(x)/\chi(1) \in R$ and $(\chi(x)/\chi(1))^* = 1^*$. We denote by $\lambda_\chi : \mathbf{Z}(FG) \to F$ the algebra homomorphism

$$\lambda_\chi(\hat{K}) = \left(\frac{|K|\chi(x)}{\chi(1)}\right)^*$$

for conjugacy classes K of G, where $\hat{K} = \sum_{g \in K} g$ is the conjugacy class sum. In general, we follow the notation of [15] for blocks.

Theorem 2.1. *Suppose that $N \triangleleft G$, $P \in \mathrm{Syl}_p(N)$, $G = N\mathbf{C}_G(P)$, $\theta \in B_p(N)$ is in the principal p-block of N and extends to G. Then there is $\chi \in B_p(G)$ in the principal block of G extending θ such that χ_J is in the principal block of J for every subgroup $N \leq J \leq G$. Furthermore, if $\beta \in \mathrm{Irr}(G)$ has N in its kernel, then $\chi\beta$ lies in the principal block of G if and only if β lies in the principal block of G.*

Proof. Let $\eta \in \mathrm{Irr}(G)$ be an extension of θ. Write $C = \mathbf{C}_G(P)$, $D = \mathbf{C}_N(P)$. Let $x \in C$ be p-regular. Write $N_x = N\langle x \rangle$. Then θ has a unique extension $\theta_x \in \mathrm{Irr}(N_x)$ in the principal block, using [1]. Hence

$$\eta_{N_x} = \lambda_x \theta_x$$

for a unique linear character $\lambda_x \in \mathrm{Irr}(N_x/N)$. If $x \in C$, we define

$$\nu(x) = \lambda_{x_{p'}}(x_{p'})\,.$$

We claim that ν is a linear character of C. Since $N_x = N_{x^{-1}}$, then $\lambda_x = \lambda_{x^{-1}}$, and notice that $\nu(x^{-1}) = \nu(x)^{-1}$. Suppose that $H = R \times Q$ is a nilpotent subgroup of C, where R is a p-group and Q is a p'-group. We show that ν_H is a generalised character of H. By [18, Thm. 3.2], there exists a unique $\hat{\theta} \in \mathrm{Irr}(NQ)$ in the principal block of NQ, extending θ. Furthermore, $\hat{\theta}_J$ lies in the principal block of J for $N \subseteq J \subseteq NQ$. By Gallagher's theorem ([10, Corollary 6.17]), write $\eta_{NQ} = \lambda\hat{\theta}$, for some $\lambda \in \mathrm{Irr}(NQ/N)$. Thus $\theta_y = \hat{\theta}_{N_y}$ for $y \in Q$. By the uniqueness in Gallagher's theorem we conclude that

$$\lambda_y = \lambda_{N_y}\,.$$

Therefore, if $h \in H$, then $\nu(h) = \nu(h_{p'}) = \lambda(h_{p'})$, and we conclude that ν_H is a linear character of H. Thus ν is a generalised character of C, by Brauer's characterisation of characters. Now

$$|C|[\nu, \nu] = \sum_{c \in C} \nu(c)\overline{\nu(c)} = |C|\,,$$

and we conclude that ν is an irreducible (linear) character of C. Since $D \subseteq \ker \nu$, we may view ν as a character of G/N. We claim that $\chi = \eta\nu^{-1}$ is in the principal block of G. Let E be a defect group of the block of χ. By [15, Thm. 9.26], we have that $E \cap N \in \mathrm{Syl}_p(N)$. Since $\chi_N \in \mathrm{Irr}(N)$, it follows that p does not divide $|G : NE|$, by [17, Prop. 2.5(d)]. Hence $E \in \mathrm{Syl}_p(G)$, and χ lives in a block of maximal defect. By [15, Problem 4.5], it suffices to show that $\lambda_\chi(\hat{K}) = |K|^*$ for every conjugacy class $K = x^G$, where x is p-regular and $|K|$ is not divisible by p. Now,

$$\lambda_\chi(\hat{K}) = |K|^* \left(\frac{\chi(x)}{\chi(1)}\right)^* = |K|^* \left(\frac{\theta_x(x)}{\theta(1)}\right)^* = |K|^*\,,$$

as wanted. The same proof shows that χ_J is in the principal block of J for every $N \leq J \leq G$.

Let $\beta \in \mathrm{Irr}(G)$ with $N \subseteq \ker \beta$, and consider the corresponding character $\bar{\beta}$ of $\bar{G} = G/N$. Suppose that $\gamma \in \mathrm{Irr}(G)$ is such that $\gamma_N \in \mathrm{Irr}(N)$, $x \in G$, and $\mathbf{C}_{G/N}(xN) = L/N$. Let $K = x^G$, $X = x^L$, and $Y = (xN)^{\bar{G}}$, be the conjugacy classes of x in G, L and of xN in $\bar{G}$, respectively. Then, by [17, Lemma 2.2], we have that

$$\lambda_{\gamma\beta}(\hat{K}) = \lambda_{\gamma_L}(\hat{X})\lambda_{\bar{\beta}}(\hat{Y})\,.$$

Now, since $\chi_L \in \mathrm{Irr}(L)$ lies in the principal block of L, we have that

$$\lambda_\beta(\hat{K}) = \lambda_{1_G\beta}(\hat{K}) = \lambda_{1_L}(\hat{X})\lambda_{\bar{\beta}}(\hat{Y}) = \lambda_{\chi_L}(\hat{X})\lambda_{\bar{\beta}}(\hat{Y}) = \lambda_{\chi\beta}(\hat{K})\,.$$

Therefore, we conclude that β and $\chi\beta$ lie in the same p-block of G. □

3. Quasi-simple Groups

In this section we show the "only if" direction of Conjecture A for quasi-simple groups. The condition of possessing Sylow p- and Sylow q-subgroups for different primes p and q dividing the order and that commute elementwise is quite restrictive for finite simple groups. In fact, apart from two "accidents" in sporadic groups, it can only happen in groups of Lie type, and only if both Sylow subgroups are abelian, as was already shown in [2].

We need an easy lemma.

Lemma 3.1. *Let G be a finite group, and suppose that $Z \leq \mathbf{Z}(G)$. Let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, where p and q are different primes. If $[PZ/Z, QZ/Z] = 1$, then $[P, Q] = 1$.*

Proof. We have that $P \triangleleft PZ$. Hence P is characteristic in PZ. Since Q normalises PZ, it follows that Q normalises P. Also, $[P, Q] \subseteq Z$ by hypothesis. Hence $[P, Q, Q] = 1$, and by coprime action [10, Lemma 4.29], $[P, Q] = 1$. □

We will deal with the various cases according to the classification of finite simple groups.

Proposition 3.2. *Let G be a quasi-simple group such that $G/\mathbf{Z}(G)$ is sporadic. Suppose that $p \neq q$ are primes dividing $|G|$. If $[P, Q] = 1$ for some $P \in \mathrm{Syl}_p(G)$ and some $Q \in \mathrm{Syl}_q(G)$, then either $G = J_1$ for $\{p, q\} = \{3, 5\}$, or $G = J_4$ for $\{p, q\} = \{5, 7\}$.*

Proof. This is easily read off from the known character tables [6]. □

Proposition 3.3. *Let G be a covering group of an alternating group $\mathfrak{A}_n$, $n \geq 5$, and $p < q \leq n$ two primes. Then there are no $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$.*

Proof. First observe that the exceptional 3- and 6-fold coverings of $\mathfrak{A}_6$ and $\mathfrak{A}_7$ satisfy the conclusion, so we may assume that G is at most a 2-fold cover. Now Sylow 2-subgroups of $\mathfrak{A}_n$, $n \geq 5$, are self-normalising and thus certainly do not centralise other Sylow subgroups. Clearly this also holds for Sylow 2-subgroups of the covering groups $2.\mathfrak{A}_n$. Thus we may assume that $p, q \neq 2$. But then, our statement will follow if we prove it for the symmetric groups $\mathfrak{S}_n$ instead. Let P be a Sylow p-subgroup of $G = \mathfrak{S}_n$. Then $\mathbf{C}_G(P)/\mathbf{Z}(P) \cong \mathfrak{S}_m$ where m is the residue of n modulo p. Clearly, this has order prime to any prime $q > p$, so indeed there are no examples. □

Thus we are left to consider the quasi-simple groups of Lie type.

Proposition 3.4. *Let G be quasi-simple of Lie type in characteristic p. Then the centraliser of a Sylow p-subgroup P of G is $\mathbf{Z}(P)\mathbf{Z}(G)$. In particular, P does not centralise a Sylow q-subgroup for $q \neq p$.*

Proof. This follows immediately from the fact that the centraliser of a regular unipotent element of G is an extension of a p-group with the centre of G, see [5, Prop. 5.1.5]. □

In order to study the non-defining primes we introduce the following setup. Let $\mathbf{G}$ be a simple algebraic group of simply connected type over an algebraically closed field of positive characteristic and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism, with finite group of fixed points $G = \mathbf{G}^F$. Then, as is well-known, $S := G/\mathbf{Z}(G)$ is almost always simple, and moreover the universal covering groups of all but finitely many simple groups of Lie type are among the groups thus constructed (see, e.g., [9, §6.1]).

We start off by characterising the situations in which there exist commuting Sylow subgroups; this can also be extracted from the proof of [2, Thm. 26], but we prefer to give a shorter, more conceptual proof. Note that by Lemma 3.1 we can pass freely between the various perfect central extensions of a simple group and in particular solve the problem for just one such extension, for example for the group G as constructed above.

If F is a Frobenius endomorphism it defines an $\mathbb{F}_r$-rational structure on $\mathbf{G}$ for some power r of the characteristic. If F is not a Frobenius endomorphism we let r be the absolute value of all eigenvalues of F^2 on the character group of an F-stable maximal torus of $\mathbf{G}$; it is an integral power of the characteristic as well.

For a prime p not dividing r we denote by $d_p(r)$ the order of r modulo p when p is odd, respectively the order of r modulo 4 when $p = 2$, and we set $e_p(r) := d_p(r)/\gcd(2, d_p(r))$.

Proposition 3.5. *Let $G = \mathbf{G}^F$ be quasi-simple of Lie type and $p \neq q$ two prime divisors of $|G|$ different from the defining characteristic of $\mathbf{G}$. Assume that $[P, Q] = 1$ for some $P \in \mathrm{Syl}_p(G)$ and some $Q \in \mathrm{Syl}_q(G)$. Then p and q are odd, $d := d_p(r) = d_q(r)$, P and Q are abelian and PQ lies in a Sylow d-torus of $\mathbf{G}$.*

Proof. We first assume that F is a Frobenius endomorphism. Let $d := d_p(r)$, $d' := d_q(r)$. We may and will assume that $d \leq d'$. Now by [14, Thm. 5.9] any q-element g centralising a Sylow p-subgroup P of G lies in a torus of $\mathbf{G}$ centralising a Sylow d-torus $\mathbf{S}$ of $\mathbf{G}$. The centraliser $\mathbf{C}_{\mathbf{G}}(\mathbf{S})$ is an F-stable Levi subgroup of $\mathbf{G}$. If $G = \mathrm{SL}_n(r)$ is a special linear group then it has the structure

$$\mathbf{C}_G(\mathbf{S}) \cong (r^d - 1)^a/(r-1)\,.\,\mathrm{GL}_s(r),$$

where $n = ad + s$ with $0 \leq s < d$. Since $s < d \leq d'$ the order of $\mathrm{GL}_s(r)$ is prime to q, so in fact g lies in the torus of order $(r^d - 1)^a/(r-1)$. But then necessarily $d' \leq d$, whence the two are equal. The situation for $\mathrm{SU}_n(r)$ is entirely similar, with r formally replaced by $-r$.

If $\mathbf{G}$ is of classical type then set $e = e_p(r)$, $e' = e_q(r)$, and now assume that $e \leq e'$. Here we have

$$\mathbf{C}_G(\mathbf{S}) \cong (r^e + (-1)^d)^a\, G_s(r),$$

where $\mathbf{G}$ has rank n, $n = ae + s$ with $0 \leq s < e$, and $G_s(r)$ is a group of the same classical type as G but of rank s. As before we see that q must divide $r^e + (-1)^d$ and so $d' = d$. Finally, for groups of exceptional type an easy case-by-case check shows that the same conclusion holds. So $d = d'$ in all cases.

Now first assume that $p, q \neq 2$. Then by [14, Thm. 5.14], apart from very few exceptions when $p = 3$, the centraliser of our Sylow p-subgroup P of G is contained in the normaliser of a Sylow d-torus $\mathbf{S}$ of $\mathbf{G}$. So then Q lies in $\mathbf{C}_{\mathbf{G}}(\mathbf{S})$ and thus, by what we showed before, in a torus containing $\mathbf{S}$. In particular, it is abelian. Interchanging the roles of p, q shows the same for P. In the three cases with $p = 3$ and $\mathbf{G}$ of rank 2, the claim is easily checked directly.

If $q = 2$ then the preceding argument shows that a Sylow 2-subgroup is abelian, but this never happens in groups $\mathbf{G}^F$ for $\mathbf{G}$ simple.

If G is a covering group of a Suzuki or Ree group then a slight modification of the previous argument still applies, or alternatively the claim is readily checked from the well-known Sylow structure of these groups. □

Theorem 3.6. *Let G be a finite quasi-simple group, and let p, q be primes. Suppose that there exist $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$. Then q does not divide the degrees of the irreducible characters in the principal p-block of G.*

Proof. Let G be quasi-simple and $p \neq q$ two primes dividing $|G|$ such that there exists a Sylow p-subgroup P and a Sylow q-subgroup Q of G with $[P, Q] = 1$. By Propositions 3.2, 3.3, 3.4 and 3.5 either $G = J_1$ or $G = J_4$, or G is of Lie type and p, q are non-defining primes. In the first case, the claim is immediate from the known character tables [6].

Now assume that G is of Lie type. By Lemma 3.1 the universal covering group of $S = G/\mathbf{Z}(G)$ satisfies the same assumptions, and clearly it is enough to prove the claim for it. So assume that G is the universal covering group of S. For the finitely many groups S with an exceptional Schur multiplier we can again refer to the known character tables [6]. Thus we have that $G = \mathbf{G}^F$ for some simple algebraic group $\mathbf{G}$ of simply connected type, as above.

First observe that if p is a bad prime for $\mathbf{G}$ then it divides at least two distinct cyclotomic polynomials dividing the order polynomial of $\mathbf{G}$ and so the Sylow p-subgroups of G are non-abelian. So p, q are both good for $\mathbf{G}$, and furthermore they do not divide $|\mathbf{Z}(\mathbf{G})|$. Moreover, by Proposition 3.5 both primes p and q are odd. In this case the principal p-block of G is described by Cabanes and Enguehard [4]: a character $\chi \in \mathrm{Irr}(G)$ lying in the principal p-block of G must lie in a Lusztig series $\mathcal{E}(G, s)$ where $s \in G^*$ is a p-element and moreover the Jordan correspondent of χ in $\mathcal{E}(\mathbf{C}_{G^*}(s), 1)$ lies in the principal d-Harish-Chandra series of $\mathbf{C}_{G^*}(s)$, where $d = d_p(r)$. Observe that $\mathbf{C}_{\mathbf{G}^*}(s)$ is connected as p does not divide $|\mathbf{Z}(\mathbf{G})|$. Thus, Φ_d is the unique cyclotomic polynomial dividing the order polynomial of $\mathbf{G}$ such that $p|\Phi_d(r)$. Then $\chi(1)$ is prime to p if and only if Φ_d does not divide the degree polynomial of χ. In particular, these properties of χ only depend on combinatorial data: the rational structure of $\mathbf{C}_{\mathbf{G}^*}(s)$ and the label of the Jordan corresponding unipotent character of $\mathbf{C}_{G^*}(s)$.

Now choose $f \equiv 1 \pmod d$ to be an integer big enough such that

- there exists a Zsigmondy primitive prime divisor ℓ of $\Phi_d(r^f)$ (that is, $d_\ell(r^f) = d$); and
- for all p- and q-elements $s \in G^*$ there exists an ℓ-element $s' \in G_1^* := \mathbf{G}^{*F^f}$ having the same centraliser $\mathbf{C}_{\mathbf{G}^*}(s') = \mathbf{C}_{\mathbf{G}^*}(s)$ in $\mathbf{G}$.

Then for $\chi \in \mathcal{E}(G, s)$ lying in the principal p-block of G, for some p-element $s \in G^*$, there is an ℓ-element $s' \in G_1^*$ and a corresponding character χ' in $\mathcal{E}(G_1^*, s')$ lying in the principal ℓ-block having the same degree polynomial as χ. Since $d_\ell(r^f) = d$ by assumption, the Sylow ℓ-subgroups of G' are also abelian, and then by the proved direction of Brauer's height zero conjecture [12, Thm. 1.1], the degree of χ' is not divisible by ℓ, thus its degree polynomial is not divisible by Φ_d and so the degree polynomial of χ is not divisible by

Φ_d. But then $\chi(1)$ is prime to q as well. Now reversing the roles of p and q we reach our conclusion. □

4. The "only if" direction

We can now present the proof of the "only if" direction of Conjecture A .

Theorem 4.1. *Let G be a finite group, and let p, q be primes. Suppose that there exists $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$. Then q does not divide the degrees of the irreducible characters in the principal p-block of G.*

Proof. Let $\chi \in \mathrm{Irr}(B_p(G))$. We prove that q does not divide $\chi(1)$ by induction first on $\chi(1)$, and then on $|G|$. By the proved direction of Brauer's height zero conjecture (see [12]), we may assume that p and q are different.

Suppose that $N \triangleleft G$ is a maximal normal subgroup of G, and let $\theta \in \mathrm{Irr}(B_p(N))$ be under χ. Since $P \cap N \in \mathrm{Syl}_p(N)$, and $Q \cap N \in \mathrm{Syl}_q(N)$, we have that $[P \cap N, Q \cap N] = 1$, and by induction we have that q does not divide $\theta(1)$. If G/N is a q'-group, then $\chi(1)/\theta(1)$ divides $|G/N|$, and we are done. So we may assume that q divides $|G/N|$. Let $P_0 = P \cap N$. Since $[Q, P_0] = 1$, it follows then that $N < N\mathbf{C}_G(P_0)$. Since $G = N\mathbf{N}_G(P_0)$ by the Frattini argument, then we have that $G = N\mathbf{C}_G(P_0)$, using that G/N is simple. Now, if r is a prime different from p, we have that θ extends to RN, where $R \in \mathrm{Syl}_r(G)$ by the theory of isomorphic blocks [1]. Hence, if $T = G_\theta$ is the stabiliser of θ in G, we have that $|G : T|$ is a power of p. In particular, $T \cap P \in \mathrm{Syl}_p(T)$ and $Q \subseteq T$. Let $\psi \in \mathrm{Irr}(T)$ be the Clifford correspondent of χ over θ. By [15, Cor. 6.2] and the Third Main Theorem [15, Thm. 6.7], we have that $\psi \in \mathrm{Irr}(B_p(T))$. Now, $[T \cap P, Q] = 1$. If $T < G$, by induction q does not divide $\psi(1)$, and therefore q does not divide $|G : T|\psi(1) = \chi(1)$. Hence, we may assume that θ is G-invariant. If G/N is cyclic, then $\chi_N = \theta$, and we are done. So we may assume that G/N is a non-abelian simple group. Since N was an arbitrary maximal subgroup, we conclude that G is a perfect group.

Now we use the representation group $\widehat{G}$ of G with respect to θ (see [16, Sect. 5.3]). Let $\pi : \widehat{G} \to G$ be the canonical epimorphism, with kernel $Z \subseteq \mathbf{Z}(\widehat{G})$. Since $\hat{G}/Z$ has a commuting Sylow p and q-subgroup, so does $\hat{G}$ by Lemma 3.1. Now $N \triangleleft \hat{G}$ and θ extends to $\hat{G}$ (by [16, Thm. 5.6]). Using that in $\hat{G}$, we have that $(n, 1)(g, z) = (ng, z)$ for $n \in N$, $g \in G$ and $z \in Z$ (by [16, Lemma 5.3(a)]), we readily check that $\mathbf{C}_{\widehat{G}}(P_0) = \mathbf{C}_G(P_0) \times Z$, and that $\widehat{G} = N\mathbf{C}_{\widehat{G}}(P_0)$. By Theorem 2.1, there is an extension $\tau \in \mathrm{Irr}(\widehat{G})$ in the principal block of $\widehat{G}$. View now χ as an irreducible character of $\widehat{G}$ with Z in its kernel. Then χ lies over θ, and lies in the principal block of $\widehat{G}$ (because it lies in the principal block of $\widehat{G}/Z$). By Gallagher, we can write

$$\chi = \beta\tau\,,$$

where β lies in the principal block of $\hat{G}$ by Theorem 2.1. If $\beta(1) < \chi(1)$, then q does not divide $\beta(1)$, and therefore we are done (since $\tau(1) = \theta(1)$ has q'-degree). Therefore, we may assume that $\chi(1) = \beta(1)$. Hence $\tau(1) = \theta(1) = 1$. Now, since $\theta_{\mathbf{C}_N(P_0)} \in \mathrm{Irr}(\mathbf{C}_N(P_0))$, by [15, Lemma 6.8(d)] we have that $\chi_{\mathbf{C}_G(P_0)} \in \mathrm{Irr}(\mathbf{C}_G(P_0))$. Since $P \subseteq \mathbf{N}_G(P_0)$, we have that $|G : \mathbf{N}_G(P_0)|$ is not divisible by p. Then, by [11, Thm. A], $\chi_{\mathbf{N}_G(P_0)}$ is in the principal block of $\mathbf{N}_G(P_0)$. Since $\mathbf{C}_G(P_0) \triangleleft \mathbf{N}_G(P_0)$, we have that $\chi_{\mathbf{C}_G(P_0)}$ is in the principal block. Since $PQ \subseteq \mathbf{N}_G(P_0)$ and $\mathbf{C}_G(P_0) \triangleleft \mathbf{N}_G(P_0)$, we have that $P \cap \mathbf{C}_G(P_0) \in \mathrm{Syl}_p(\mathbf{C}_G(P_0))$, and the hypotheses of the theorem are satisfied in $\mathbf{C}_G(P_0)$. By induction, we deduce that $P_0 \subseteq \mathbf{Z}(G)$. Hence $N = P_0 \times K$, where $K = \mathbf{O}_{p'}(N)$. Since χ is in the principal p-block of G, we have that $K \subseteq \ker \chi$. By [15, Thm. 9.9(c)], $\chi \in \mathrm{Irr}(G/K)$ lies in the principal p-block of G/K, and since PK/K and QK/K commute, we may assume that $K = 1$, by induction. Thus $N \subseteq \mathbf{Z}(G)$, G is perfect, and G/N is a simple group, whence the assertion follows from Theorem 3.6. □

5. The "if" direction

In this section we show that the "if" direction of Conjecture A is true if we assume the inductive Alperin–McKay condition.

We start off with the case of simple groups:

Theorem 5.1. *Let G be non-abelian simple and assume that $[P, Q] \neq 1$ for every $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. Then, up to interchanging p and q, the principal p-block of G contains an irreducible character of degree divisible by q.*

Proof. If $p = q$ then the claim is a special case of the main result of Kessar–Malle [13]. So we may and will assume for the rest of the proof that the two primes p, q are different.

The sporadic groups and the Tits group can easily be checked from the known character tables [6]. The alternating groups, which probably constitute the most involved case, had already been handled by Giannelli–Malle–Vallejo [8, Thm. 3.5].

Now assume that G is simple of Lie type. Let p be the defining prime for G and let $q \neq p$. The principal p-block of G contains all irreducible characters of G except the Steinberg character. By the Ito–Michler theorem [16, Thm. 7.1] there is $\chi \in \mathrm{Irr}(G)$ of positive q-height, and this cannot be the Steinberg character, as the latter has degree a power of p. Thus χ lies in the principal p-block of G and we are done.

Finally assume that G is of Lie type and neither p nor q are equal to the defining characteristic of G, and such that a Sylow p-subgroup of G centralises no Sylow q-subgroup. We claim that the principal p-block of G contains an irreducible character χ of degree divisible by q. Observe that the hypotheses are also satisfied for any covering group of G. Thus we will assume that $G = \mathbf{G}^F$ for a simple algebraic group $\mathbf{G}$ of simply connected type and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism. We let $\mathbf{G}^*$ denote a group dual to $\mathbf{G}$ with corresponding Steinberg endomorphism also denoted F, and set $G^* = \mathbf{G}^{*F}$.

Now note that our condition for commuting Sylow subgroups is purely combinatorial, only in terms of the order of the underlying field size r modulo p and modulo q. Let $\mathbf{G}_{\mathrm{ad}}$ be a group of adjoint type but same root system as $\mathbf{G}$, with dual $\mathbf{G}_{\mathrm{ad}}^*$. Since $\mathbf{G}$, $\mathbf{G}_{\mathrm{ad}}$ and $\mathbf{G}_{\mathrm{ad}}^*$ have identical order polynomials, it follows that this condition is satisfied for G if and only if it is satisfied for $G_{\mathrm{ad}}^* := \mathbf{G}_{\mathrm{ad}}^{*F}$. Thus, no Sylow p-subgroup of G_{ad}^*

centralises a Sylow q-subgroup of G^*_{ad}. Now note that there is a morphism with central kernel $G^*_{\mathrm{ad}} \to [G^*, G^*]$, and so by Lemma 3.1 the same holds for $[G^*, G^*]$. Then by the main result of [2], there is a p-element $s \in [G^*, G^*]$ whose centraliser $\mathbf{C}_{G^*}(s)$ does not contain a Sylow q-subgroup of G^*. But then by Lusztig's Jordan decomposition any semisimple character χ in $\mathcal{E}(G, s)$ has degree divisible by q, and as $s \in [G^*, G^*]$, it has $\mathbf{Z}(G^*)$ in its kernel. Since s is a p-element, χ lies in a unipotent p-block of G. Moreover, by the description of unipotent blocks in [4, 7], χ being semisimple implies that it is even contained in the principal p-block. The proof is complete. □

Theorem 5.2. *Let p and q be different primes. Assume that the Inductive Alperin–McKay condition holds for the principal blocks of non-abelian simple groups. Suppose that p does not divide the degrees of the irreducible characters in $B_q(G)$, and q does not divide the degrees of the irreducible characters in $B_p(G)$. Then there are $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$.*

Proof. We argue by induction on $|G|$. The case of non-abelian simple groups holds by Theorem 5.1. We notice that the hypotheses are clearly inherited by factor groups and normal subgroups. Indeed, if $N \triangleleft G$, we know that $B_p(G/N) \subseteq B_p(G)$ (see the remark before Theorem 7.6 of [15]). Also, since every $\theta \in B_p(N)$ lies over some $\chi \in B_p(G)$ by [15, Thm. 9.4], it easily follows that normal subgroups also satisfy the hypotheses.

Let N be a maximal proper normal subgroup of G. By induction, there are $P_0 \in \mathrm{Syl}_p(N)$ and $Q_0 \in \mathrm{Syl}_q(N)$ such that $[P_0, Q_0] = 1$. We have that $G = N\mathbf{N}_G(P_0) = N\mathbf{N}_G(Q_0)$, by the Frattini argument.

Suppose first that G/N is a q-group. In this case, $P = P_0 \in \mathrm{Syl}_p(G)$. Since $[Q_0, P] = 1$, we have that $|N : \mathbf{N}_N(P)|$ is not divisible by q. Since $G = N\mathbf{N}_G(P)$, we have that $|G : \mathbf{N}_G(P)|$ is not divisible by q. Hence, there is $Q \in \mathrm{Syl}_q(G)$ that normalises P. Let $K = \mathbf{O}_{p'}(\mathbf{N}_N(P))$. Since $[Q_0, P] = 1$, we have that $Q_0 \subseteq \mathbf{C}_N(P) = \mathbf{Z}(P) \times K$ (using the Schur–Zassenhaus theorem). Hence $Q_0 \subseteq K$ and $\mathbf{N}_N(P)/K$ is a q'-group. Now, by using the hypothesis, we show that the irreducible characters of p'-degree of the principal p-block of N are Q-invariant. Indeed, if $\theta \in \mathrm{Irr}(B_p(N))$, then θ lies under some $\chi \in \mathrm{Irr}(B_p(G))$. Since q does not divide $\chi(1)$, it follows that $\chi_N = \theta$ by [16, Thm. 5.12]. By [17, Thm. B] (which assumes that the Inductive Alperin–McKay condition holds), we have that all of the irreducible characters of p'-degree in the principal p-block of $\mathbf{N}_N(P)$ are Q-invariant. Using [15, Thm. 10.20], we conclude that all the irreducible characters of $\mathbf{N}_N(P)/P'K$ are Q-invariant. Since this is a q'-group, by [16, Thm. 2.4], the group Q acts trivially on $\mathbf{N}_N(P)/P'K$. In particular, Q acts trivially on P/P'. Again by coprime action (use for instance [10, Lemma 4.28] and the fact that $P' \subseteq \Phi(P)$), it follows that Q acts trivially on P, and we are done in this case. The same happens if G/N is a p-group.

Let $C = \mathbf{C}_G(P_0)$. We claim that $G = NC$. Let $M = NC$. Notice that $M \triangleleft G$ since $G = N\mathbf{N}_G(P_0)$. By [18, Lemma 3.1], the principal p-block of G is the unique p-block covering the principal p-block of M. Hence, if $\theta \in \mathrm{Irr}(G/M)$, then θ belongs to the principal p-block of G. By hypothesis, q does not divide $\theta(1)$. Thus, if $Q \in \mathrm{Syl}_q(G)$, then it follows that $QM \triangleleft G$ by the Ito–Michler theorem [16, Thm. 7.1]. Assume that $M < G$.

Then $M = N$ and G/N has a normal Sylow q-subgroup. Since G/N cannot be a q-group by the claim in the third paragraph, we have that G/N is a q'-group. Thus $Q_0 \in \mathrm{Syl}_q(G)$. Suppose that $\mathbf{C}_G(Q_0) \subseteq N$. Again by [18, Lemma 3.1], the principal q-block of G is the unique q-block covering the principal q-block of N. If $\theta \in \mathrm{Irr}(G/N)$, then θ belongs to the principal q-block of G. By hypothesis, p does not divide $\theta(1)$. Thus, if $P \in \mathrm{Syl}_p(G)$, then it follows that $PN \triangleleft G$ by the Ito–Michler theorem [16, Thm. 7.1]. Since G/N cannot be a p-group, then it follows that G/N is a p'-group. In this case $P_0 \in \mathrm{Syl}_p(G)$, and we are done since $[P_0, Q_0] = 1$. Therefore we have that $G = N\mathbf{C}_G(Q_0)$. Since $[P_0, Q_0] = 1$, we have that $|N : \mathbf{C}_N(Q_0)|$ is not divisible by p. Therefore $|G : \mathbf{C}_G(Q_0)|$ is not divisible by p. This means that there is some Sylow p-subgroup of G that centralizes Q_0, and this proves the theorem. Hence, we conclude that $M = G$. That is, $G = N\mathbf{C}_G(P_0)$. By the same argument, we have that $G = N\mathbf{C}_G(Q_0)$.

By Wielandt's theorem on nilpotent Hall π-subgroups of finite groups, and the Frattini argument, we have that $G = N\mathbf{N}_G(P_0Q_0)$. Also, notice that $\mathbf{N}_G(P_0Q_0) = \mathbf{N}_G(P_0) \cap \mathbf{N}_G(Q_0)$. This is because if $x \in \mathbf{N}_G(P_0Q_0)$, then $P_0^xQ_0^x = P_0Q_0$, and therefore $P_0^x = P_0$ and $Q_0^x = Q_0$. Now, $|G : \mathbf{C}_G(P_0)| = |N : \mathbf{C}_N(P_0)|$ is not divisible by q. Therefore $\mathbf{C}_G(P_0)$ contains a Sylow q-subgroup of G. Since $Q_0 \subseteq \mathbf{C}_G(P_0)$, there is $Q_0 \subseteq Q \subseteq \mathbf{C}_G(P_0)$ a Sylow q-subgroup of G. Furthermore, since Q_0 is a q-Sylow of the normal subgroup $\mathbf{C}_N(P_0)$ of $\mathbf{C}_G(P_0)$, we have that $Q \cap \mathbf{C}_N(P_0) = Q_0$. In particular, $Q_0 \triangleleft Q$, and Q normalises Q_0. Hence $Q \subseteq \mathbf{N}_G(P_0Q_0)$. In the same way, we choose $P \in \mathrm{Syl}_p(G)$ such that $P \subseteq \mathbf{C}_G(Q_0)$, $P_0 \triangleleft P$. Again $P \subseteq \mathbf{N}_G(P_0Q_0)$.

Now, since $G/N \cong \mathbf{N}_G(P_0Q_0)/\mathbf{N}_N(P_0Q_0)$, we have that $\mathbf{N}_G(P_0Q_0)/\mathbf{N}_N(P_0Q_0)$ has a nilpotent $\{p, q\}$-Hall subgroup $R/\mathbf{N}_N(P_0Q_0)$. By the Schur–Zassenhaus theorem applied in the group R/P_0Q_0, we deduce that R has a Hall $\{p, q\}$-subgroup S, which therefore is a Hall $\{p, q\}$-subgroup of $\mathbf{N}_G(P_0Q_0)$. Hence there are $P_1 \in \mathrm{Syl}_p(\mathbf{N}_G(P_0Q_0))$ and $Q_1 \in \mathrm{Syl}_q(\mathbf{N}_G(P_0Q_0))$ such that $S = P_1Q_1$. Choose $x, y \in \mathbf{N}_G(P_0Q_0)$ such that $P_1 = P^x$ and $Q_1 = Q^y$. Since $x, y \in \mathbf{N}_G(P_0) \cap \mathbf{N}_G(Q_0)$, we have

$$P_0 \triangleleft P_1, \quad [P_1, Q_0] = 1, \quad Q_0 \triangleleft Q_1 \quad \text{and} \quad [Q_1, P_0] = 1.$$

Now Q_1 normalises P_1Q_0/P_0Q_0. Then Q_1 normalises $P_1Q_0 = P_1 \times Q_0$. Thus Q_1 normalises P_1. Since Q_1 centralises P_1Q_0/P_0Q_0 it follows that Q_1 centralises P_1/P_0. Therefore $[P_1, Q_1] \subseteq P_0$ and $[P_1, Q_1, Q_1] = 1$. Hence $[P_1, Q_1] = 1$, as desired. □

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