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# SPRINGER CORRESPONDENCE FOR DISCONNECTED EXCEPTIONAL GROUPS

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Abstract

We complete the determination of the generalized Springer correspondence for disconnected reductive groups by determining it in the case of disconnected groups of type $D_4$ in characteristic 3 and $E_6$ in characteristic 2. It is expected that this correspondence is an important ingredient for the computation of character values of finite groups of Lie type. Some evidence for this is provided.

## 1. *Introduction*

Let $G$ be a connected reductive algebraic group over an algebraically closed field, with Weyl group $W$. The Springer correspondence was originally introduced by Tonny Springer as an injective map from the set of irreducible characters of $W$ to the set of pairs $(C, E)$, where $C$ is a unipotent conjugacy class of $G$ and $E$ is an irreducible $G$-equivariant local system on $C$. This correspondence was later generalized by Lusztig (see [**4**] for references) to a bijection between the set of pairs $(C, E)$ as above and the union over sets of irreducible characters of certain relative Weyl groups. It turned out to be a crucial ingredient for the computation of character values of reductive groups over finite fields via Shoji's and Lusztig's algorithm (see [**3**, §24]). In the connected case, this generalized Springer correspondence is explicitly known in all cases, see [**5**] and [**11**], except for some small amibiguities for $E_8$ in characteristic 2.

Sorlin [**9**] showed that the original Springer correspondence can be extended to the case of disconnected groups $\tilde{G}$ obtained from a connected reductive algebraic group $G$ by adjoining an automorphism $\sigma$ of order a power of the characteristic. In [**8**] we determined the restricted Springer correspondence for disconnected groups of classical type, that is, the part of the Springer correspondence where the images are of the form $(C, 1)$. Recently, Lusztig [**4**] made a thorough study of the generalized Springer correspondence for not necessarily connected groups and determined it completely for disconnected groups of classical type in characteristic 2. In Section 3 of this paper, we complete the explicit description of the generalized Springer correspondence for disconnected groups by handling the two open cases: groups of type $D_4$ in characteristic 3, extended by the graph automorphism of order 3, and groups of type $E_6$ in characteristic 2 extended by the graph automorphism of order 2.

It is not yet known whether the correspondence plays a rôle in the computation of character values of such disconnected groups. In Section 4 we verify that a straightforward generalization of Lusztig's algorithm, with this new correspondence as input, yields the values of the Green functions computed in [**6**] and [**7**] by different methods.

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## 2. *Properties of the Springer correspondence*

We start by introducing some notation and recalling important properties of the generalized Springer correspondence from the work of Lusztig [**4**]. Let $\tilde{G}$ be a reductive algebraic group over the algebraic closure of $\mathbb{F}_p$, with connected component $G := \tilde{G}^\circ$. Let $\sigma \in \tilde{G} \setminus G$ be unipotent. Let $\mathcal{N}_{G\sigma}$, or $\mathcal{N}$ if the coset is clear from the context, be the set consisting of all pairs $(C, E)$ where $C$ is a unipotent $G$-conjugacy class in the coset $G\sigma$ and $E$ is an irreducible $G$-equivariant local system on $C$. We shall identify $\mathcal{N}$ with the set of pairs $(u, \psi)$ where $u$ runs over a set of representatives for the unipotent conjugacy classes in $G\sigma$ and $\psi$ over the irreducible characters of the group of components $A_G(u) := C_G(u)/C_G(u)^\circ$ of the centralizer of $u$ in $G$. We will write $[u]_G$ for the $G$-conjugacy class of $u$.

For such pairs Lusztig defines a notion of cuspidality. If $L$ is a $\sigma$-stable Levi subgroup of a $\sigma$-stable parabolic subgroup of $G$, then to each cuspidal pair $(v, \psi)$ for $L\sigma$ there is attached the relative Weyl group $N_{G^\sigma}(L)/L$ in $G$, a finite Coxeter group, where $G^\sigma$ denotes the centralizer in $G$ of $\sigma$. For example, if $T$ is a $\sigma$-stable maximal torus, then the pair $(1, 1)$ (trivial element and trivial character) is cuspidal. Its relative Weyl group in $G$ is the group of fixed points $W^\sigma$ under $\sigma$ of the Weyl group $W$ of $G$.

Lusztig [**4**, Th. 12.9] has determined the number of cuspidal pairs in all cases. Let $W^1, W^2, \ldots, W^k$ denote the sequence of relative Weyl groups for the various cuspidal pairs $(v, \psi)$ in $\sigma$-stable Levi subgroups of $\sigma$-stable parabolic subgroups of $G$ up to conjugation. By our previous remark, one of them is always just $W^\sigma$. Lusztig [**4**] (see also Sorlin [**9**] for a less general approach) constructs a canonical bijection

$$i_{G\sigma} : \mathcal{N} \xrightarrow{1-1} \dot{\bigcup_i} \operatorname{Irr} W^i,$$

generalizing a construction of Springer valid for connected groups and a restricted set of pairs. This map is called the generalized Springer correspondence for $G\sigma$. We will not review the rather involved construction of this map here, but just recall some of its important properties.

We need some further notation. Let $W$ be a finite reflection group on a complex vector space $V$. For $\chi \in \operatorname{Irr}(W)$ let $b_\chi$ denote the smallest positive integer $d$ such that $\chi$ occurs in the natural representation of $W$ on the space of homogeneous polynomials of degree $d$ on $V$. If $W'$ is a parabolic subgroup of $W$, i.e., the centralizer in $W$ of a subspace of $V$, and $\chi' \in \operatorname{Irr}(W')$, then

$$j_{W'}^W(\chi') := \sum_{\substack{\chi \in \operatorname{Irr} W \\ b_\chi = b_{\chi'}}} \langle \operatorname{Ind}_{W'}^W(\chi'), \chi \rangle \, \chi$$

denotes the Macdonald-Lusztig-Spaltenstein induction (see for example [**2**, 5.2]).

The Springer correspondence $i_{G\sigma}$ satisfies the following properties:

(1) If $i_{G\sigma}((u, \psi)) = \chi$, then $b_\chi \geq d_u$, where $d_u = \dim(\mathcal{B}_u)$ denotes the dimension of the variety of Borel subgroups of $\tilde{G}$ containing $u$. Moreover, $b_\chi = d_u$ if $\psi = 1$ (see [**9**, 6.7(ii)]).

Let $P$ be a $\sigma$-stable parabolic subgroup of $G$, $L$ a $\sigma$-stable Levi subgroup of $P$ and $U_P$ the ($\sigma$-stable) unipotent radical of $P$. Denote by $W_1$ the Weyl group of $L$. Assume that $v \in L\sigma$ is unipotent.

(2) If $[u]_G$ is induced from $[v]_L$, i.e., it contains a dense open subset of $vU_P$, then $i_{G\sigma}(u, 1)$ is a constituent of $j_{W_1^\sigma}^{W^\sigma}(i_{L\sigma}(v, 1))$ (see [**8**, Th. 2]).

Now assume that $(v, \psi)$ is cuspidal for $L\sigma$. We write $\epsilon$ for the sign representation of the relative Weyl group $N_{G^\sigma}(L)/L$. Assume that $i_{G\sigma}(u, \psi) = \chi \in \operatorname{Irr}(N_{G^\sigma}(L)/L)$.

(3) If $\chi = \epsilon$ then $v \sim_G u$. If $\chi = 1$ then $[u]_G$ is induced from $[v]_L$ (see [**4**, Lemma 11.8]).

The strongest property is a certain restriction result of Lusztig's [**4**, Prop. 9.4]. Assume that

$(u,\psi)\in\mathcal{N}_{G\sigma}$ and $(u',\psi')\in\mathcal{N}_{M\sigma}$ lie over the same cuspidal pair of a $\sigma$-stable Levi subgroup $L$ of $M$ (hence of $G$). Let

$$Y_{u,u'} := \{x\in G \mid x^{-1}ux\in u'U\}$$

where $U$ denotes the unipotent radical of a $\sigma$-stable parabolic subgroup with Levi subgroup $M$, and denote by $X_{u,u'}$ the set of irreducible components of $Y_{u,u'}$ of maximal dimension.

(4) $X_{u,u'}$ is an $A_G(u)\times A_M(u')$-permutation module with character $\epsilon_{u,u'}$ such that

$$\langle \psi\otimes\psi'^*, \epsilon_{u,u'}\rangle = \langle \mathrm{Res}\, i_{G\sigma}(u,\psi), i_{M\sigma}(u',\psi')\rangle$$

where Res denotes restriction from $N_{G^\sigma}(L)/L$ to $N_{M^\sigma}(L)/L$ and $\psi'^*$ is the contragredient representation of $\psi'$.

In the situation described above, the definition shows that $\epsilon_{u,u'}$ is zero unless $u'$ is contained in the closure $\overline{[u]_G}$ of the conjugacy class of $u$. Hence we have in particular:

(5) If $u'$ is not contained in $\overline{[u]_G}$, then

$$\langle \mathrm{Res}\, i_{G\sigma}(u,\psi), i_{M\sigma}(u',\psi')\rangle = 0.$$

By the reductions made in [**8**, Sec. 3] and [**4**, 12.1–12.6] for a complete knowledge of the Springer correspondence it is sufficient to determine it in the case that $G$ is of prime index $p$ in $\tilde{G}$, hence $\tilde{G}/G$ is generated by the image of a unipotent element $\sigma\in\tilde{G}$ of order $p$ inducing a graph automorphism on $G$. Moreover, we may assume that $G$ is simple. Thus we are in one of the following four cases:

(a) $G$ of type $A_n$, $n\geq 2$, $p=2$,
(b) $G$ of type $D_n$, $n\geq 4$, $p=2$,
(c) $G$ of type $D_4$, $p=3$,
(d) $G$ of type $E_6$, $p=2$.

The first two cases were solved completely by Lusztig [**4**], see also [**8**] for a partial result. We consider the remaining two possibilities.

## 3. *Springer correspondence for exceptional groups*

In this section we consider the Springer correspondence for the two remaining, exceptional cases of simple groups with graph automorphisms.

3.1. **The Springer correspondence for** $D_4$: **3.** Let $\tilde{G}$ be the split extension of a simple algebraic group $G$ of type $D_4$ (an 8-dimensional orthogonal group) over an algebraically closed field of characteristic 3 by the exceptional graph automorphism $\sigma$ of order 3. The unipotent classes of $\tilde{G}$ not contained in $G$ and the groups $A_G(u)$ were determined in [**6**, Prop. 11] whose notation we follow (see also Spaltenstein [**10**, I.3]). In loc. cit. we gave representatives for the conjugacy classes over a finite field; over the algebraic closure the classes of the three elements $u_4, u_5, u_6$ fuse. This is the only class with non-trivial group of components, equal to the symmetric group $\mathfrak{S}_3$. As usual the characters of $\mathfrak{S}_3$ are denoted by the corresponding partitions of 3.

THEOREM 3.2. *The Springer correspondence for outer unipotent classes of* $D_4$: 3 *is given by Table 1. The unique cuspidal pair is given by* $(u_4, 1^3)$.

*Proof.* The Weyl group $W^\sigma$ is of type $G_2$. The $b$-values of the irreducible characters of $W(G_2)$ are all distinct, except for $\phi'_{1,3}, \phi''_{1,3}$, which both have $b$-value 3 (see [**2**, 5.3.4] for example). Thus the Springer correspondent of $(u_i,1)$, $i\neq 2$, is uniquely determined by property (1) above. The representatives of conjugacy classes in [**6**, Table VIII] show that a regular unipotent element $u'$ of a $\sigma$-stable Levi subgroup $L$ of $G$ of type $A_1$ fuses into the class of $u_2$. Thus $\epsilon_{u_2,u'}\neq 0$. On the other hand, the restriction of $\phi'_{1,3}$ to the Weyl group $W_L^\sigma$ of $L$ does not contain the Springer correspondent of $(u',1)$. Thus by property (3) it cannot be the Springer correspondent of $(u_2,1)$, leaving only the possibility $i_{G\sigma}(u_2,1)=\phi''_{1,3}$. Finally, direct computation shows that $X_{u_4,u'}$ has cardinality 3, thus $X_{u_4,u'}$ has permutation character $3+21$, and $(u_4,21)$ corresponds to $\phi'_{1,3}$. Thus $(u_4,1^3)$ is the cuspidal pair for $D_4\colon 3$. □

TABLE 1. Springer correspondence for $D_4\colon 3$

| $u$ | $d_u$ | $A_G(u)$ | $\psi$ | $i_{G\sigma}(u,\psi)$ |
|---|---|---|---|---|
| $u_7$ | 0 | 1 | 1 | $\phi_{1,0}$ |
| $u_4$ | 1 | $\mathfrak{S}_3$ | 3 | $\phi_{2,1}$ |
| | | | 21 | $\phi'_{1,3}$ |
| | | | $1^3$ | cuspidal |
| $u_3$ | 2 | 1 | 1 | $\phi_{2,2}$ |
| $u_2$ | 3 | 1 | 1 | $\phi''_{1,3}$ |
| $u_1$ | 6 | 1 | 1 | $\phi_{1,6}$ |

3.3. **The Springer correspondence for $E_6\colon 2$.** We now consider the Springer correspondence for the group $\tilde{G}$ obtained by extending the simply-connected exceptional group $G$ of type $E_6$ over an algebraically closed field of characteristic 2 by the non-trivial graph automorphism $\sigma$ of order 2. The unipotent classes of $\tilde{G}$ not contained in $G$ are given in Spaltenstein [**10**, II.10.14], the groups $A_G(u)$ were determined in [**7**]. We follow the notation for the classes in terms of suitable Dynkin diagrams used in [**10**], but for easier reference we also give the notation for class representatives as in [**7**, Table 10] in Table 2. In this case, according to Lusztig [**4**, Th. 12.9] there is a further cuspidal pair in a proper $\sigma$-stable Levi subgroup $L$ different from a maximal torus, namely for $L$ of type $A_5$. The relative Weyl group in this case is cyclic of order 2.

THEOREM 3.4. *The Springer correspondence for outer unipotent classes of $E_6\colon 2$ is given by Table 2. In particular, $(F_4(a_3),1^4)$ is the unique cuspidal pair.*

TABLE 2. Springer correspondence for $E_6\colon 2$

| classes in [**10**] | representatives in [**7**] | $d_u$ | $A_G(u)$ | $\psi$ | $\emptyset$ $i_{G\sigma}(u,\psi)$ | $A_5\colon 2$ $i_{G\sigma}(u,\psi)$ |
|---|---|---|---|---|---|---|
| $F_4$ | $u_{26}, u_{27}$ | 0 | $Z_2$ | 1 | $\phi_{1,0}$ | |
| | | | | $\epsilon$ | | 1 |
| $F_4(a_1)$ | $u_{24}, u_{25}$ | 1 | $\mathfrak{S}_2$ | 1 | $\phi_{4,1}$ | |
| | | | | $\epsilon$ | $\phi'_{2,4}$ | |
| $F_4(a_2)$ | $u_{23}$ | 2 | 1 | 1 | $\phi_{9,2}$ | |
| $B_3$ | $u_{21}, u_{22}$ | 3 | $\mathfrak{S}_2$ | 1 | $\phi''_{8,3}$ | |
| | | | | $\epsilon$ | $\phi''_{2,4}$ | |
| $C_3$ | $u_{19}, u_{20}$ | 3 | $\mathfrak{S}_2$ | 1 | $\phi'_{8,3}$ | |
| | | | | $\epsilon$ | | $\epsilon$ |
| $F_4(a_3)$ | $u_{14} - u_{18}$ | 4 | $\mathfrak{S}_4$ | 4 | $\phi_{12,4}$ | |
| | | | | 31 | $\phi'_{9,6}$ | |
| | | | | $2^2$ | $\phi''_{6,6}$ | |
| | | | | $21^2$ | $\phi'_{1,12}$ | |
| | | | | $1^4$ | cuspidal | |
| $C_3(a_1)$ | $u_{12}, u_{13}$ | 5 | $\mathfrak{S}_2$ | 1 | $\phi_{16,5}$ | |
| | | | | $\epsilon$ | $\phi'_{4,7}$ | |
| $B_2$ | $u_{10}, u_{11}$ | 6 | $\mathfrak{S}_2$ | 1 | $\phi''_{9,6}$ | |
| | | | | $\epsilon$ | $\phi_{4,8}$ | |
| $\tilde{A}_2 + A_1$ | $u_9$ | 6 | 1 | 1 | $\phi'_{6,6}$ | |
| $\tilde{A}_1 + A_2$ | $u_7$ | 7 | 1 | 1 | $\phi''_{4,7}$ | |
| $\tilde{A}_2$ | $u_8$ | 9 | 1 | 1 | $\phi'_{8,9}$ | |
| $A_2$ | $u_6$ | 9 | 1 | 1 | $\phi''_{8,9}$ | |
| $A_1 + \tilde{A}_1$ | $u_4$ | 10 | 1 | 1 | $\phi_{9,10}$ | |
| $(A_2)_2$ | $u_5$ | 12 | 1 | 1 | $\phi''_{1,12}$ | |
| $\tilde{A}_1$ | $u_2, u_3$ | 13 | $\mathfrak{S}_2$ | 1 | $\phi_{4,13}$ | |
| | | | | $\epsilon$ | $\phi'_{2,16}$ | |
| $A_1$ | $u_1$ | 16 | 1 | 1 | $\phi''_{2,16}$ | |
| $\emptyset$ | $u_0$ | 24 | 1 | 1 | $\phi_{1,24}$ | |

*Proof.* The Weyl group $W^\sigma$ is of type $F_4$. There exist unique irreducible characters of $W(F_4)$ with $b$-values $0, 1, 2, 5, 10, 13, 24$ (see for example [**2**, Table C3]), so the Springer correspondents for the pairs $(u, 1)$ with $d_u \in \{0, 1, 2, 5, 10, 13, 24\}$ are determined by property (1).

By [**10**, II.10.14], the unipotent classes $B_3, C_3, F_4(a_3), B_2, \tilde{A}_2$ and $A_2$ in $G\sigma$ are induced from unipotent classes of proper $\sigma$-stable Levi subgroups according to Table 3. (Here $G(6)$ denotes the extension of $\mathrm{GL}_6$ by the graph automorphism of order 2, that is, the Levi subgroup of type $A_5\colon 2$; the notation for unipotent conjugacy classes in the Levi subgroups is as in [**10**].) So their Springer correspondents can be determined by property (2) from our knowledge of the Springer correspondence for classical groups (see [**8**] and [**4**]). Note that in all these cases, the $j$-induction is irreducible. Now the two pairs lying above the cuspidal pair in the Levi subgroup of type $A_5$ are uniquely determined by property (3).

TABLE 3. Induced classes for $E_6\colon 2$

| $L$ | $C$ | $\mathrm{Ind}_{L\sigma}^{G\sigma}(C)$ |
|---|---|---|
| $\mathrm{GO}_6 \setminus \mathrm{SO}_6$ | $21^4$ | $F_4(a_3)$ |
| $\mathrm{GO}_8 \setminus \mathrm{SO}_8$ | $3^2 2$ | $B_3$ |
| $\mathrm{GO}_8 \setminus \mathrm{SO}_8$ | $21^6$ | $\tilde{A}_2$ |
| $G(6) \setminus \mathrm{GL}_6$ | $1^6$ | $B_2$ |
| $G(6) \setminus \mathrm{GL}_6$ | $1^6_0$ | $A_2$ |
| $\mathrm{GL}_3 \wr 2 \setminus \mathrm{GL}_3^2$ | $1^3$ | $C_3$ |

According to property (1), we have the following candidates for Springer correspondents for the remaining pairs $(u,1)$:

$$\begin{array}{rl} (\tilde A_2+A_1,1): & \phi'_{6,6},\phi''_{6,6},\phi'_{9,6}\\ (\tilde A_1+A_2,1): & \phi'_{4,7},\phi''_{4,7}\\ ((A_2)_2,1): & \phi'_{1,12},\phi''_{1,12}\\ (A_1,1): & \phi'_{2,16},\phi''_{2,16} \end{array}$$

By choosing $L$ of type $A_5$ and $u'$ in class $C_3(a_1)$ or class $\tilde A_2$ correct Springer correspondents for $\tilde A_2+A_1$ and $\tilde A_1+A_2$ are determined with property (5). Similarly with $L$ of type $D_4$ this determines the other two Springer correspondents. Induction from $D_4$ also shows that $(B_3,\epsilon)$ corresponds to $\phi''_{2,4}$, and then by (1) the only possibility for $\phi'_{2,4}$ is $(F_4(a_1),\epsilon)$. Again by (5) we find $i_{G\sigma}(\tilde A_1,\epsilon)=\phi'_{2,16}$ and $i_{G\sigma}(B_2,\epsilon)=\phi_{4,8}$.

We are left with the five pairs

$$(F_4(a_3),31),\ (F_4(a_3),2^2),\ (F_4(a_3),21^2),\ (F_4(a_3),1^4),\ (C_3(a_1),\epsilon)$$

in $\mathcal{N}_{G\sigma}$ and the four irreducible characters $\phi'_{1,12},\phi'_{4,7},\phi''_{6,6},\phi'_{9,6}$ of $W(F_4)$. Here, the Springer correspondence is finally determined by property (4). It turns out that $(F_4(a_3),1^4)$ is the cuspidal pair. □

## 4. *Green functions*

4.1. **Character values.** Let $G$ be a connected reductive group over the algebraic closure of a finite field $\mathbb{F}_q$, $\sigma$ an automorphism of $G$ of prime order and $\tilde G := G.\langle\sigma\rangle$ the semidirect product of $G$ with the cyclic group generated by $\sigma$. We are interested in the irreducible characters of the group of $\mathbb{F}_q$-points of $\tilde G$. The values of these characters on elements lying in the connected component $G$ are now quite well understood, thanks to work of Lusztig, Shoji and others. If the order of $\sigma$ is prime to the underlying characteristic, then the only new information needed for the computation of character values on the coset $G\sigma$ is the decomposition of certain generalized Deligne–Lusztig characters. For generalized Deligne–Lusztig characters corresponding to maximal tori this decomposition has been completely determined by Digne–Michel [**1**] for classical groups and by the author [**6**] for exceptional groups. On the other hand, if the order of $\sigma$ is divisible by the characteristic, then the coset $G\sigma$ contains unipotent classes. The determination of character values on outer elements then in addition requires the knowledge of so-called Green functions on these classes. We computed the ordinary Green functions, corresponding to tori, for the exceptional groups and for classical groups of low rank [**6**], [**7**], but the case of classical groups of arbitrary rank and the generalized Green functions corresponding to arbitrary Levi subgroups has not yet been settled.

In the case of connected groups, the Green functions can be computed by an algorithm of Lusztig. The input to this algorithm consists of the generalized Springer correspondence. The validity of the obvious generalization of this algorithm to disconnected groups has not yet been shown. Here, we apply this algorithm for groups of low rank and check that the results in these cases agree with the known Green functions.

4.2. **Lusztig's algorithm.** We recall Lusztig's algorithm for the computation of Green functions in the case of connected groups. Let $G$ be a connected reductive algebraic group with Weyl group $W$. The Springer correspondence $i_G$ gives an injection

$$\mathrm{Irr}\,(W) \longrightarrow \mathcal{N}_G$$

of the set of irreducible characters into the set of pairs $\mathcal{N}_G$ as above. For $\chi \in \mathrm{Irr}(W)$ define $d(\chi) := \dim \mathcal{B}_u = d_u$ if $\chi = i_G(u,\psi)$ under the Springer correspondence.

For $f$ a class function on $W$ define its *fake degree* to be the polynomial

$$R(f) := (x-1)^n P_W \frac{1}{|W|} \sum_{w \in W} \frac{\epsilon(w) f(w)}{\det_V(x\,\mathrm{id}_V - w)},$$

where $P_W$ denotes the Poincaré polynomial of $W$, $V$ is the module of the natural reflection representation of $W$, $n = \dim V$, and $\epsilon$ is the sign-character of $W$.

Let $\omega_{\chi,\chi'} := x^N R(\chi \otimes \chi' \otimes \epsilon) \in \mathbb{Z}[x]$, where $N$ is the number of reflections in $W$, and set $\Omega := (\omega_{\chi,\chi'})_{\chi,\chi'}$ with respect to some total ordering $\leq$ on $\mathrm{Irr}(W)$ satisfying $\chi \geq \chi'$ whenever $d(\chi) \leq d(\chi')$. Then there exist unique matrices $P = (p_{\chi,\chi'}), \Lambda = (\lambda_{\chi,\chi'})$ such that
(1) $\lambda_{\chi,\chi'} = 0$ unless $d(\chi) = d(\chi')$,
(2) $p_{\chi,\chi'} = 0$ unless $d(\chi) < d(\chi')$ or $\chi = \chi'$,
(3) $p_{\chi,\chi} = x^{d(\chi)}$,
(4) $P \Lambda P^{\mathrm{tr}} = \Omega$.
Moreover $\lambda_{\chi,\chi'}, p_{\chi,\chi'} \in \mathbb{Q}[x]$, and the coefficients of $p_{\chi,\chi'}$ are non-negative.

Lusztig and Shoji have shown that the $p_{\chi,\chi'}$, evaluated at $q$, allow to compute the values of the almost characters (a twisted form of the Green functions) for the $\mathbb{F}_q$-points of $G$ [**3**, §24].

4.3. **Green functions for $D_4 : 3$ and $E_6 : 2$.** The above algorithm has an obvious formal generalization to disconnected groups, taking as input the Springer correspondence for unipotent classes in a non-trivial coset of the connected component.

An implementation of Lusztig's algorithm now allows us to verify the following:

PROPOSITION 4.4.
(a) *For disconnected groups of type $D_4 : 3$, Lusztig's algorithm, with the Springer correspondence in Theorem 3.2 as input, yields the values of the almost characters given in [**6**, Table IX].*
(b) *For disconnected groups of type $E_6 : 2$, Lusztig's algorithm, with the Springer correspondence in Theorem 3.4 as input, yields the values of the almost characters given in [**7**, Table 11].*

The almost characters for the disconnected groups of type $G(n)$, $3 \leq n \leq 6$, and for the disconnected groups $\mathrm{GO}_8$ over a field of characteristic 2 were computed in [**6**, §9]. The generalized Springer correspondence for these groups was determined by Lusztig [**4**, 13.10]. Again we have:

PROPOSITION 4.5. *For $G(n)$, $n = 3, 4, 5$, and for* $\mathrm{GO}_8$ *Lusztig's algorithm, with the generalized Springer correspondence as input, yields the values of the almost characters given in [**6**, Tables VI and VII], respectively in [**6**, Table XI]*

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