Construction of low-discrepancy point sets and sequences

Josef Dick

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Local discrepancy function

Point set \( P_{N,s} = \{x_0, x_1, \ldots, x_{N-1}\} \subset [0, 1]^s \),
\( t = (t_1, \ldots, t_s) \in [0, 1]^s \).

- Local discrepancy:

\[
\Delta_{P_{N,s}}(t) = \frac{1}{N} \sum_{n=0}^{N-1} 1_{[0,t]}(x_n) - \prod_{i=1}^{s} t_i,
\]

where \([0, t] = \prod_{i=1}^{s} [0, t_i]\).
Local discrepancy function of point set $P_{N,s} = \{x_0, x_1, \ldots, x_{N-1}\} \subset [0, 1]^s$:

$$\Delta_{P_{N,s}}(t) = \frac{1}{N} \sum_{n=0}^{N-1} 1_{[0,t]}(x_n) - \text{Vol}([0,t]), \quad \text{for } [0, t) \subseteq [0, 1]^s.$$
Local discrepancy function of point set $P_{N,s} = \{x_0, x_1, \ldots, x_{N-1}\} \subset [0, 1]^s$:

$$\Delta_{P_{N,s}}(t) = \frac{1}{N} \sum_{n=0}^{N-1} 1_{[0,t]}(x_n) - \text{Vol}([0,t]), \quad \text{for } [0, t) \subseteq [0, 1]^s.$$

$L^q$ Discrepancy:

$$L^q(P_{N,s}) = \left( \int_{[0,1]^s} |\Delta_{P_{N,s}}(t)|^q \, dt \right)^{1/q} \quad \text{for } 1 \leq q \leq \infty.$$ 

(With obvious modifications for $q = \infty$.)
For fixed $s \in \mathbb{N}$, we are interested in

$$L^q_{N,s} = \inf_{P_{N,s} \subset [0,1]^s} L^q(P_{N,s}) \quad \text{as} \quad N \to \infty$$
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The case $s = 1$ is trivial, so we assume that $s > 1$. 
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We suppress the dependence of constants on dimension $s$. Friedrich Pillichshammer will discuss dimension dependence.
Roth’s lower bound (1954):

\[ L_{N,s}^2 \gg s \left( \frac{(\log N)^{s-1}}{N} \right). \]
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\[ L_{N,s}^2 \gg s \left( \frac{\log N}{2} \right)^{s-1}. \]

Schmidt’s lower bound (1977):

\[ L_{N,s}^q \gg s \left( \frac{\log N}{2} \right) \quad \text{for } 1 < q < 2. \]
Lower bounds

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For \( 1 < q < \infty \) these results yield optimal lower bounds

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Roth’s lower bound (1954):

\[ L_{N,s}^2 \gg s \left( \frac{\log N} {N} \right)^{\frac{s-1}{2}}. \]

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Lower bounds will be discussed in more detail by Dmitriy Bilyk.
Endpoint estimates: $q = 1$

For $q = 1$ and $s = 2$: Halász (1981):

$$L_{N,2}^1 \gg s \frac{(\log N)^{1/2}}{N}.$$ 

This result is best possible.
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For $q = 1$ and $s > 2$: Dichotomy result by Amirhkanyan, Bilyk, Lacey (2013):

Let $q \in (1, \infty)$. If $L^q(P_{N,s}) \ll s \frac{(\log N)^{s-1}}{N}$, then

$$L^1(P_{N,s}) \gg s \frac{(\log N)^{s-1}}{N}.$$
For dimension $s = 2$: lower bound by Schmidt (1972), upper bound by van der Corput (1934), Lerch (1904):

$$L_{N,2}^\infty \approx \frac{\log N}{N}.$$
For dimension $s = 2$: lower bound by Schmidt (1972), upper bound by van der Corput (1934), Lerch (1904):

$$L_{N,2}^\infty \gtrsim \frac{\log N}{N}.$$

Best lower bound for $q = \infty$ and $s > 2$: (Bilyk, Lacey (2008) for $s=3$; Bilyk, Lacey, Vagharshakyan (2008) for $s \geq 4$):

$$L_{N,s}^\infty \gg s \left(\frac{\log N}{N}\right)^{\frac{s-1}{2} + \eta},$$

where $\eta > c/s^2$ for some $c > 0$. 

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Endpoint estimates: $q = \infty$

For $s > 2$ the correct order is unknown. There are two conjectures:

$$L^{\infty}_{N,s} \approx \frac{(\log N)^{s-1}}{N}$$

and

$$L^{\infty}_{N,s} \approx \frac{(\log N)^{\frac{s}{2}}}{N}.$$
For $s > 2$ the correct order is unknown. There are two conjectures:

$$L_{N,s}^\infty \asymp \frac{(\log N)^{s-1}}{N}$$

and

$$L_{N,s}^\infty \asymp \frac{(\log N)^{\frac{s}{2}}}{N}.$$

Note: For $s = 2$ both are correct!

$$L_{N,2}^\infty \asymp \frac{\log N}{N}.$$
For $s \geq 2$ and $q = 2$: Roth (1980):

$$L^2_{N,s} \ll s \left( \frac{\log N}{N} \right)^{s-1}.$$
Upper bounds $1 \leq q < \infty$ (existence result)

For $s \geq 2$ and $q = 2$: Roth (1980):

$$L_{N,s}^2 \ll s \frac{(\log N)^{s-1}}{N}.$$ 

For dimension $s \geq 2$ and $1 \leq q < \infty$: Chen (1980/1983):

$$L_{N,s}^q \ll q,s \frac{(\log N)^{s-1}}{N}.$$ 

This result is based on shifted Halton sequence / shifted digital nets. The proof shows that bound holds for the average over all (digital) shifts. There is another construction by Frolov (1980).
Explicit constructions: $s = 2, q = 2$

**Theorem (Davenport (1956))**

Let $s = q = 2$. For a real number $x$ denote the fractional part by $\{x\} = x - \lfloor x \rfloor$. Let $\theta$ be any irrational number having a continued fraction expansion with bounded partial quotients. Then for the set of $2N$ points

$$P_{2N,2} = \{ (\{\pm n\theta\}, n/N) : 0 \leq n < N \},$$

we have

$$L^2(P_{2N,2}) \ll \frac{\sqrt{\log N}}{N}.$$

This is called *Davenport’s reflection principle*. 

Skriganov (Algebra i Analiz 1994): Explicit construction for $s = 2$ and $q > 2$. 

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**Theorem (Davenport (1956))**

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\[
P_{2N,2} = \left\{ \left( \{\pm n\theta\}, \frac{n}{N} \right) : 0 \leq n < N \right\},
\]

we have

\[
L^2(P_{2N,2}) \ll \frac{\sqrt{\log N}}{N}.
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This is called Davenport's reflection principle.

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Explicit constructions: $s = 2, q = 2$
Bilyk (2014): $s = q = 2$: Davenport construction without reflection principle;
Bilyk, Lacey, Ioannis, Vagharshakyan (2009): $s = 2$, $1 < q \leq \infty$: Scrambled van der Corput set;
Bilyk, Temlyakov, Rui (2012): $s = 2$, $1 < q < \infty$: Symmetrized Fibonacci point sets;
Faure, Pillichshammer, Pirsic: (2011): $s = q = 2$: generalized Hammersley point sets;
Faure, Pillichshammer, Pirsic, Schmid (2010): $s = q = 2$: generalized Hammersley point sets;
Kritzer, Larcher, Pillichshammer: (2007): $s = 2$, $q = \infty$: digitally shifted Hammersley point set;
Kritzer, Pillichshammer (2007): $s = 2$, $q > 1$: digitally shifted Hammersley point set;
Larcher, Pillichshammer (2001): $s = q = 2$: symmetrized digital nets;
Larcher, Pillichshammer (2002): $s = 3$, $q = 2$: almost
Explicit constructions: $q = \infty$

For $s = 2$: Lerch 1904, van der Corput (1934),...

$$L^\infty(P_{N,2}) \ll \frac{\log N}{N}.$$
Explicit constructions: \( q = \infty \)

For \( s = 2 \): Lerch 1904, van der Corput (1934),...

\[
L^\infty (P_{N,2}) \ll \frac{\log N}{N}.
\]

For \( s \geq 2 \): (Halton/Hammersley 1960, Faure 1982, Niederreiter 1987, ...)

\[
L^\infty (P_{N,s}) \ll_s \left( \frac{\log N}{N} \right)^{s-1}.
\]
Explicit constructions: \( q = \infty \)

For \( s = 2 \): Lerch 1904, van der Corput (1934), ...

\[
L^\infty(P_{N,2}) \ll \frac{\log N}{N}.
\]

For \( s \geq 2 \): (Halton/Hammersley 1960, Faure 1982, Niederreiter 1987, ...)

\[
L^\infty(P_{N,s}) \ll_s \frac{(\log N)^{s-1}}{N}.
\]

Note the gap between this bound and the best known lower bound of order \( \frac{(\log N)^{s-1}}{2} + \eta \).
There are two explicit constructions:

- Chen and Skriganov (2002): based on $\mathbb{F}_b$ with $b \geq 2s^2$;
- D. and Pillichshammer (2013) based on $\mathbb{F}_2$;

which achieve

$$L^2(P_{N,s}) \ll s \frac{(\log N)^{s-1}}{N}.$$
Choose prime number $b$ and finite field $\mathbb{Z}_b = \{0, 1, \ldots, b - 1\}$ of order $b$.
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Choose $C_1, \ldots, C_s \in \mathbb{Z}_b^{m \times m}$. 

Many explicit constructions, by Sobol', Faure, Niederreiter, Niederreiter-Xing, ...
Choose prime number $b$ and finite field $\mathbb{Z}_b = \{0, 1, \ldots, b - 1\}$ of order $b$.

Choose $C_1, \ldots, C_s \in \mathbb{Z}_b^{m \times m}$.

Let $n = n_0 + n_1 b + \cdots + n_{m-1} b^{m-1}$ and set $\vec{n} = (n_0, \ldots, n_{m-1})^\top \in \mathbb{Z}_b^m$. 
Digital nets

Choose prime number \( b \) and finite field \( \mathbb{Z}_b = \{0, 1, \ldots, b - 1\} \) of order \( b \).

Choose \( C_1, \ldots, C_s \in \mathbb{Z}_b^{m \times m} \).

Let \( n = n_0 + n_1 b + \cdots + n_{m-1} b^{m-1} \) and set \( \bar{n} = (n_0, \ldots, n_{m-1})^\top \in \mathbb{Z}_b^m \).

Let

\[
\tilde{y}_{n,i} = C_i \bar{n} \quad \text{for } 1 \leq i \leq s, \ 0 \leq n < b^m.
\]
Choose prime number $b$ and finite field $\mathbb{Z}_b = \{0, 1, \ldots, b - 1\}$ of order $b$.

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Let

$$\vec{y}_{n,i} = C_i \vec{n} \quad \text{for } 1 \leq i \leq s, 0 \leq n < b^m.$$

For $\vec{y}_{n,i} = (y_{n,i,1}, \ldots, y_{n,i,m})^\top \in \mathbb{Z}_b^m$ let

$$x_{n,i} = \frac{y_{n,i,1}}{b} + \cdots + \frac{y_{n,i,m}}{b^m}.$$
Choose prime number $b$ and finite field $\mathbb{Z}_b = \{0, 1, \ldots, b-1\}$ of order $b$.

Choose $C_1, \ldots, C_s \in \mathbb{Z}_b^{m \times m}$.

Let $n = n_0 + n_1 b + \cdots + n_{m-1} b^{m-1}$ and set $\tilde{n} = (n_0, \ldots, n_{m-1})^\top \in \mathbb{Z}_b^m$.

Let

$$\tilde{y}_{n,i} = C_i \tilde{n} \quad \text{for } 1 \leq i \leq s, 0 \leq n < b^m.$$ 

For $\tilde{y}_{n,i} = (y_{n,i,1}, \ldots, y_{n,i,m})^\top \in \mathbb{Z}_b^m$ let

$$x_{n,i} = \frac{y_{n,i,1}}{b} + \cdots + \frac{y_{n,i,m}}{b^m}.$$

Set $x_n = (x_{n,1}, \ldots, x_{n,s})$ for $0 \leq n < b^m$. 
Choose prime number $b$ and finite field $\mathbb{Z}_b = \{0, 1, \ldots, b - 1\}$ of order $b$.

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Set $\mathbf{x}_n = (x_{n,1}, \ldots, x_{n,s})$ for $0 \leq n < b^m$.

Many explicit constructions, by Sobol’, Faure, Niederreiter, Niederreiter-Xing, ...
Let $m, s \geq 1$ and $b \geq 2$ be integers. A point set $P_{b^m,s} = \{x_0, \ldots, x_{b^m-1}\}$ is called a $(t, m, s)$-net in base $b$, if for all integers $d_1, \ldots, d_s \geq 0$ with

$$d_1 + \cdots + d_s = m - t$$
Let $m, s \geq 1$ and $b \geq 2$ be integers. A point set $P_{b^m,s} = \{x_0, \ldots, x_{b^m-1}\}$ is called a $(t, m, s)$-net in base $b$, if for all integers $d_1, \ldots, d_s \geq 0$ with

$$d_1 + \cdots + d_s = m - t$$

the number of points in the elementary intervals

$$\prod_{i=1}^{s} \left[ \frac{a_i}{b^{d_i}}, \frac{a_i + 1}{b^{d_i}} \right)$$

where $0 \leq a_i < b^{d_i}$, is $b^t$. 

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Hammersley point set with 2 points.
The shaded region contains exactly one point.
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Hammersley point set with 4 points.
The shaded region contains exactly one point.
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If $P_{b^m,s} = \{x_0, \ldots, x_{b^m-1}\} \subset [0, 1]^s$ is a $(t, m, s)$-net, then local discrepancy function

$$\Delta_{P_{b^m,s}} \left( \frac{a_1}{b^{d_1}}, \ldots, \frac{a_s}{b^{d_s}} \right) = 0$$

for all $0 \leq a_i < b^{d_i}$, $d_i \geq 0$, $d_1 + \cdots + d_s = m - t$. 
Walsh functions

i) $k \geq 0$ integer with $k = k_0 + k_1 b + \cdots + k_{a-1} b^{a-1}$

ii) $x \in [0, 1)$ with $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots$

Then

$$\text{wal}_k(x) = \omega_b^{k_0 x_1 + k_1 x_2 + \cdots + k_{a-1} x_a},$$

where $\omega_b = e^{2\pi i / b}$. 
Walsh functions

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Then

$$\text{wal}_k(x) = \omega_b^{k_0 x_1 + k_1 x_2 + \cdots + k_{a-1} x_a},$$

where $\omega_b = e^{2\pi i/b}$.

For vectors $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $x = (x_1, \ldots, x_s) \in [0, 1)^s$ we write

$$\text{wal}_k(x) = \prod_{j=1}^s \text{wal}_{k_j}(x_j).$$
Walsh coefficients: let $\mathbf{k} \in \mathbb{N}_0^s$, then

$$\hat{f}(\mathbf{k}) = \int_{[0,1]^s} f(\mathbf{x}) \text{wal}_k(\mathbf{x}) \, d\mathbf{x}.$$
Walsh coefficients: let $k \in \mathbb{N}_0^s$, then

$$\hat{f}(k) = \int_{[0,1]^s} f(x) \text{wal}_k(x) \, dx.$$ 

Walsh series:

$$f(x) \sim \sum_{k \in \mathbb{N}_0^s} \hat{f}(k) \text{wal}_k(x).$$
Digital net $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{b^m-1}$ with generating matrices $C_1, C_2, \ldots, C_s$:

$$
\frac{1}{b^m} \sum_{n=0}^{b^m-1} \text{wal}_k(\mathbf{x}_n) = \begin{cases} 
1 & \text{if } \mathbf{x} \in \mathcal{D}, \\
0 & \text{otherwise,}
\end{cases}
$$

where

$$
\mathcal{D} = \{ \mathbf{k} \in \mathbb{N}_0^s : C_1^\top \mathbf{k}_1 + \cdots + C_s^\top \mathbf{k}_s \equiv 0 \pmod{b} \},
$$

and $\mathbf{k} = (\kappa_0, \kappa_1, \ldots, \kappa_{m-1})^\top \in \mathbb{F}_b^m$ for

$$
k = \kappa_0 + \kappa_1 b + \cdots + \kappa_{a-1} b^{a-1}.
$$
Walsh series expansion of $L^2$ discrepancy

Let $P_{b^m,s}$ be a digital net. Then

$$[L^2(P_{b^m,s})]^2 = \int_{[0,1]^s} |\Delta P_{b^m,s}(t)|^2 \, dt = \sum_{k,\ell \in \mathcal{D} \setminus \{0\}} r(k, \ell),$$

where for $k = (k_1, \ldots, k_s)$ and $\ell = (\ell_1, \ldots, \ell_s)$

$$r(k, \ell) = \prod_{j=1}^{s} r(k_j, \ell_j).$$
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Known result: For digital $(t, m, s)$- net:

$$\sum_{k \in \mathcal{D}\backslash\{0\}} r(k, k) \ll_s \frac{m^{s-1}}{b^{2(m-t)}}.$$
Two approaches:

- Chen-Skriganov: quasi-orthogonality
  \[ b \geq 2 s^2 \implies r(k, \ell) = 0 \text{ for } k \neq \ell; \]

- D.-Pillichshammer: new metric
  \[ |r(k, \ell)| \text{ is small}; \]
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Two approaches:

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- D.-Pillichshammer: new metric:
  
  \[ |r(k, \ell)| \text{ is small}; \]
Sparsity of coefficients

For $k, \ell \in \mathbb{N}$:

$$|r(k, \ell)| \ll \begin{cases} b^{-2\mu_1(k)} & \text{if } k = \ell, \\ b^{-\max(\mu_2(k), \mu_2(\ell))} & \text{if } k \sim \ell, \\ 0 & \text{otherwise}. \end{cases}$$

Let

$$k = \kappa_1 b^{a_1-1} + \kappa_2 b^{a_2-1} + \cdots + \kappa_\nu b^{a_\nu-1} = \kappa_1 b^{a_1-1} + k',$$

$$\ell = \lambda_1 b^{c_1-1} + \kappa_2 b^{c_2-1} + \cdots + \kappa_\nu b^{c_\nu-1} = \lambda_1 b^{c_1-1} + \ell';$$

Then

$$\mu_\alpha(k) = \sum_{i=1}^{\min(\alpha, \nu)} a_i \quad (\text{where } a_1 > a_2 > \cdots > a_\nu;)$$

and

$$k \sim \ell \iff \text{either } k = \ell', k' = \ell' \text{ or } k' = \ell'.$$
Idea is to construct digital net such that $k \not\sim \ell$ for all $k, \ell \in \mathcal{D}$. 
Quasi-orthogonality

- Idea is to construct digital net such that $k \not\sim \ell$ for all $k, \ell \in \mathcal{D}$.
- Find $\mathcal{D}$ such that Hamming weight of $k \oplus \ell$ is large for all $k, \ell \in \mathcal{D}$.
Quasi-orthogonality

- Idea is to construct digital net such that $k \not\sim \ell$ for all $k, \ell \in D$.
- Find $D$ such that Hamming weight of $k \ominus \ell$ is large for all $k, \ell \in D$.
- Then $r(k, \ell) = 0$ for all $k, \ell \in D$. 
Define the *digit interlacing function* with interlacing factor \( \alpha \) by

\[
D_\alpha : [0, 1)^\alpha \rightarrow [0, 1) \\
(x_1, \ldots, x_\alpha) \mapsto \sum_{a=1}^{\infty} \sum_{j=1}^{\alpha} \xi_{j,a} b^{-j-(a-1)\alpha},
\]

where \( x_j = \xi_{j,1} b^{-1} + \xi_{j,2} b^{-2} + \cdots \) for \( 1 \leq j \leq \alpha \).
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We also define such a function for vectors by setting

\[
D_\alpha : [0, 1)^{\alpha s} \rightarrow [0, 1)^s
\]

\[
(x_1, \ldots, x_{\alpha s}) \mapsto (D_\alpha(x_1, \ldots, x_\alpha), \ldots, D_\alpha(x_{(s-1)\alpha+1}, \ldots, x_{s\alpha})).
\]
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\]

**Interlaced digital net of order \( \alpha \):**

\[
D_\alpha([y_0, y_1, \ldots, y_{b^m-1}])
\]
Define the *digit interlacing function* with interlacing factor \( \alpha \) by

\[
D_{\alpha} : [0, 1)^{\alpha} \rightarrow [0, 1)
\]

\[
(x_1, \ldots, x_\alpha) \mapsto \sum_{a=1}^{\infty} \sum_{j=1}^{\alpha} \xi_{j,a} b^{-j-(a-1)\alpha},
\]

where \( x_j = \xi_{j,1} b^{-1} + \xi_{j,2} b^{-2} + \cdots \) for \( 1 \leq j \leq \alpha \).

We also define such a function for vectors by setting

\[
D_{\alpha} : [0, 1)^{\alpha s} \rightarrow [0, 1)^s
\]

\[
(x_1, \ldots, x_{\alpha s}) \mapsto (D_{\alpha}(x_1, \ldots, x_\alpha), \ldots, D_{\alpha}(x_{(s-1)\alpha + 1}, \ldots, x_{s\alpha})).
\]

**Interlaced digital net of order \( \alpha \):**

\[
D_{\alpha}([y_0, y_1, \ldots, y_{b^m-1}])
\]

Order \( \alpha = 5 \) is sufficient.
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Construction of low-discrepancy point sets and sequences
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Construction of low-discrepancy point sets and sequences
Choose $C_1, \ldots, C_{\alpha s} \in \mathbb{Z}_b^{N \times N}$.
Choose $C_1, \ldots, C_{\alpha s} \in \mathbb{Z}_b^{\mathbb{N} \times \mathbb{N}}$.

Let $n = n_0 + n_1 b + \cdots + n_{m-1} b^{m-1}$ and set

$\mathbf{n} = (n_0, \ldots, n_{m-1}, 0, 0, \ldots)^\top \in \mathbb{Z}_b^{\mathbb{N}}$. 

Apply interlacing of order $\alpha \geq 5$: $D_{\alpha}(x_0), D_{\alpha}(x_1), \ldots$. 

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Construction of low-discrepancy point sets and sequences
Choose $C_1, \ldots, C_{\alpha s} \in \mathbb{Z}^N_b$.

Let $n = n_0 + n_1 b + \cdots + n_{m-1} b^{m-1}$ and set $\vec{n} = (n_0, \ldots, n_{m-1}, 0, 0, \ldots) \top \in \mathbb{Z}^N_b$.

Let

$$\vec{y}_{n,i} = C_i \vec{n} \quad \text{for } 1 \leq i \leq \alpha s, n \geq 0.$$
Choose \( C_1, \ldots, C_{\alpha s} \in \mathbb{Z}_b^{N \times N} \).

Let \( n = n_0 + n_1 b + \cdots + n_{m-1} b^{m-1} \) and set \( \vec{n} = (n_0, \ldots, n_{m-1}, 0, 0, \ldots)^T \in \mathbb{Z}^N_b \).

Let \( \vec{y}_{n,i} = C_i \vec{n} \) for \( 1 \leq i \leq \alpha s, n \geq 0 \).

For \( \vec{y}_{n,i} = (y_{n,i,1}, \ldots, y_{n,i,m_n}, 0, 0, \ldots)^T \in \mathbb{Z}_b^N \) let
\[
x_{n,i} = \frac{y_{n,i,1}}{b} + \cdots + \frac{y_{n,i,m_n}}{b^{m_n}}.
\]
Choose $C_1, \ldots, C_{\alpha s} \in \mathbb{Z}_{b}^{N \times N}$.

Let $n = n_0 + n_1 b + \cdots + n_{m-1} b^{m-1}$ and set
\[
\vec{n} = (n_0, \ldots, n_{m-1}, 0, 0, \ldots) \top \in \mathbb{Z}_b^N.
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\[
\vec{y}_{n, i} = C_i \vec{n} \quad \text{for } 1 \leq i \leq \alpha s, n \geq 0.
\]

For $\vec{y}_{n, i} = (y_{n, i, 1}, \ldots, y_{n, i, m_n}, 0, 0, \ldots) \top \in \mathbb{Z}_b^N$ let
\[
x_{n, i} = \frac{y_{n, i, 1}}{b} + \cdots + \frac{y_{n, i, m_n}}{b^{m_n}}.
\]

Set $\mathbf{x}_n = (x_{n, 1}, \ldots, x_{n, \alpha s})$ for $n \geq 0$. 

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Construction of low-discrepancy point sets and sequences
Choose $C_1, \ldots, C_{\alpha s} \in \mathbb{Z}_b^{N \times N}$.

Let $n = n_0 + n_1 b + \cdots + n_{m-1} b^{m-1}$ and set
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Let
\[ \tilde{y}_{n,i} = C_i \tilde{n} \quad \text{for } 1 \leq i \leq \alpha s, n \geq 0. \]

For $\tilde{y}_{n,i} = (y_{n,i,1}, \ldots, y_{n,i,mn}, 0, 0, \ldots)^\top \in \mathbb{Z}_b^N$ let
\[ x_{n,i} = \frac{y_{n,i,1}}{b} + \cdots + \frac{y_{n,i,mn}}{b^{mn}}. \]

Set $x_n = (x_{n,1}, \ldots, x_{n,\alpha s})$ for $n \geq 0$.

Apply interlacing of order $\alpha \geq 5$: $\mathcal{D}_\alpha(x_0), \mathcal{D}_\alpha(x_1), \ldots$;
Theorem (Proinov 1985)

For any sequence of points in $[0, 1]^s$ the $L^2$ discrepancy of the first $N$ points satisfies

$$L^2(P_{N,s}) \gg s \frac{(\log N)^s}{N}$$

for infinitely many $N$. 

Theorem (D.-Pillichshammer 2013)

A sequence $x_0, x_1, \ldots \in [0, 1]^s$ can be constructed such that for the first $N$ points $P_N$, we have

$$L^2(P_{N,s}) \ll s \sqrt{r} \frac{(\log N)^s}{s^{1/2} N}$$

for all $N \geq 2$, with $N = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_r}$. 

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Construction of low-discrepancy point sets and sequences
Theorem (Proinov 1985)

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Construction of low-discrepancy point sets and sequences
Explicit constructions of point sets with small $L^q$ discrepancy for $1 \leq q < \infty$

There are two explicit constructions:

- Skriganov (2002): constructed $P_{N,s}^{(b)}$ based on $\mathbb{F}_b$ with $b \geq qs^2$ such that for any $1 \leq q < \infty$:

  $$L^q(P_{N,s}^{(b)}) \ll_{q,s} \frac{(\log N)^{s-1}}{N^{s/2}}.$$
Explicit constructions of point sets with small $L^q$ discrepancy for $1 \leq q < \infty$

There are two explicit constructions:

- Skriganov (2002): constructed $P_{N,s}^{(b)}$ based on $\mathbb{F}_b$ with $b \geq qs^2$ such that for any $1 \leq q < \infty$:

\[
L^q(P_{N,s}^{(b)}) \ll_{q,s} \frac{(\log N)^{\frac{s-1}{2}}}{N}.
\]

- D. (2013): constructed $P_{N,s}$ based on $\mathbb{F}_2$ such that for any $1 \leq q < \infty$:

\[
L^q(P_{N,s}) \ll_{q,s} \frac{(\log N)^{\frac{s-1}{2}}}{N}.
\]
For $b = (b_1, \ldots, b_s) \in \mathbb{N}_0^s$ we set

$$B(b) = \{ \ell \in \mathbb{N}_0^s : \lfloor b^j \ell^{-1} \rfloor \leq \ell_j < b^j \text{ for } 1 \leq j \leq s \}. $$

**Proposition (Skriganov (2006))**

Let $2 \leq q < \infty$. For functions $f \in L_q([0, 1]^s)$ and $b \in \mathbb{N}_0^s$ let

$$\sigma_b f(\theta) = \sum_{\ell \in B(b)} \hat{f}(\ell) \text{wal}_\ell(\theta)$$

where $\hat{f}(\ell) = \int_{[0,1]^s} f(x) \text{wal}_\ell(x) \, dx$ is the $\ell$th Walsh coefficient of $f$. Then for any $f \in L_q([0,1]^s)$ we have

$$\left( \int_{[0,1]^s} |f(\theta)|^q \, d\theta \right)^{1/q} \ll_{q,s} \left( \sum_{b \in \mathbb{N}_0^s} \left( \int_{[0,1]^s} |\sigma_b f(\theta)|^q \, d\theta \right)^{2/q} \right)^{1/2}. $$
An infinite-dimensional sequence $x_0, x_1, x_2, \ldots \in [0, 1]^\mathbb{N}$ can be constructed explicitly such that for any $s \in \mathbb{N}$, the projection onto the first $s$ coordinates satisfies for any $N = 2^m$:

$$L^q(P_{2^m}, s) \ll q, s \frac{m^{s-1}}{2^m} \quad \text{for all } 1 \leq q < \infty.$$
Beck-Chen (1987): Great open problem:

\[ L_{N,s}^\infty \sim_s \frac{(\log N)^{s-1}}{N} \quad \text{or} \quad L_{N,s}^\infty \sim_s \frac{(\log N)^s}{N} \quad \text{or?} \]
Beck-Chen (1987): Great open problem:

\[ L_{N,s}^\infty \simeq s \frac{(\log N)^{s-1}}{N} \quad \text{or} \quad L_{N,s}^\infty \simeq s \frac{(\log N)^s}{N} \quad \text{or?} \]

Say

\[ L^q(P_{2^m,s}) \ll_s q^{e(s)} \frac{m^{s-1}}{2^m}. \]
Beck-Chen (1987): Great open problem:

\[ L_{\infty,N,s} \asymp s \frac{(\log N)^{s-1}}{N} \quad \text{or} \quad L_{\infty,N,s} \asymp s \frac{(\log N)^{s}}{N} \quad \text{or?} \]

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For \( q = m \to \infty \) we obtain

\[ L^\infty(P_{2^m,s}) \ll_s \left( \frac{q}{m} \right)^{e(s)} \frac{m^{s-1}+e(s)}{2^m} = \frac{m^{s-1}+e(s)}{2^m}. \]
Beck-Chen (1987): Great open problem:

\[ L_{N,s}^\infty \lesssim s \frac{(\log N)^{s-1}}{N} \quad \text{or} \quad L_{N,s}^\infty \lesssim s \frac{(\log N)^s}{N} \quad \text{or?} \]

Say

\[ L^q(P_{2m,s}) \ll s q^{e(s)} \frac{m^{s-1}}{2^m}. \]

For \( q = m \to \infty \) we obtain

\[ L^\infty(P_{2m,s}) \ll s \left( \frac{q}{m} \right)^{e(s)} \frac{m^{s-1} + e(s)}{2^m} = \frac{m^{s-1} + e(s)}{2^m}. \]

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Using Littlewood-Paley theory yields \( e(s) \geq \frac{s-1}{2} \).
Conjecture (D. 2013)

The point sets constructed in [D., 2013] (and [D.-Pillichshammer 2013]) achieve the optimal rate of convergence of the $L^\infty$ discrepancy.
Thank You!