Hybrid Galerkin-Collocation Methods for Modeling of Nonlinear Problems

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Introduction

Hybrid Galerkin-Collocation methods for the Surface-Oriented Modeling of Nonlinear Problems in Solid Mechanics, is a DFG project with members from TU Kaiserslautern (NA group) and RWTH Aachen (SAD group).

Boundary element method (BEM)

For complex geometries with an arbitrary number of boundary surfaces, boundary oriented-methods may be considered, in order to avoid the finite element meshing of the interior domain. While the BEM has the advantage of completely avoiding any discretization of the interior domain and is also applicable to arbitrary geometries, it has its limitations for nonlinear problems.

Scaled boundary FEM (SB-FEM)

Here, the structure is parametrized with the radial scaling factor with respect to a scaling center and a parameter in circumferential direction along the boundary. The domain occupied by the solid must be star-shaped. The weak form is enforced in cicumferential direction and is approximated by standard finite element method using Lagrange polynomials. The strong form is enforced in scaling directions. Assuming linear behaviour, it results in an ODE in terms of the scaling parameter.

Goals

First objective is to develop a computational method that combines the features of isogeometric analysis with the scaled-boundary approach. A further objective is the numerical analysis of the method class in terms of accuracy, stability, error estimation.

Tasks for NA group:

- Efficient and stable discretization of the ODE in scaling direction: while a Galerkin projection in accordance with the isogeometric paradigm will be used to treat surface integrals, the remaining ODE problem with its singularity demands for novel discretization methods.
- Error and convergence analysis of the overall SB-IGA method: the two discretizations on the boundary surface and along the scaling direction determine the overall properties that can't be subsumed under standard techniques from numerical PDEs.

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http://gepris.dfg.de/gepris/projekt/285973342

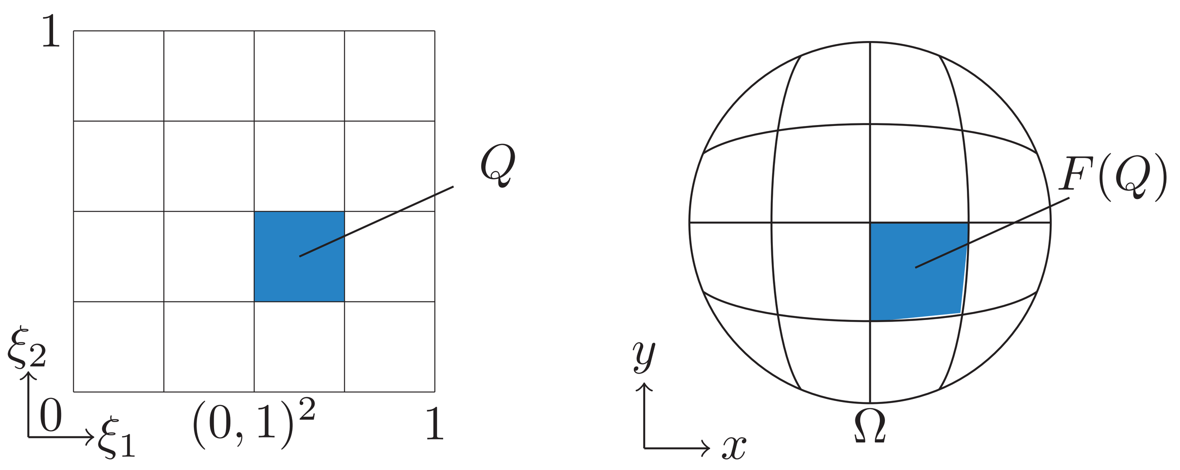
FEM vs IGA

The finite element method (FEM) is a numerical technique to solve partial differential equations (PDEs). A finite element method is characterized by a variational formulation and a discretization strategy. It is based on Lagrange polynomials to model the geometry of the structure and the displacement response. Furthermore, the approach of FEM is isoparametric, hence one of its drawbacks is the lack of an exact representation of the geometry.

Isogeometric Analysis (IGA) was introduced in 2005 [1] to replace traditional finite elements by Non-Uniform Rational B-Splines (NURBS). The isogeometric paradigm is based on employing the same set of basis functions for both description of the geometry in the design process and structural analysis, thus ensuring that geometries in analysis processes are always exactly represented. In such a way we provide a formulation of the problem on a reference domain.

Isogeometric Analysis

We first consider a bounded physical domain $\Omega \in \mathbb{R}^d$ parametrized by a global geometry function $F\colon (0,1)^d \to \Omega$



F can be defined via

$$F(\xi) = \sum_{i=1}^{n} N_i(\xi) P_i$$

with n control points $P_i \in \mathbb{R}^d$ and NURBS functions $N_i(\xi)\colon (0,1)^d \to \mathbb{R}$.

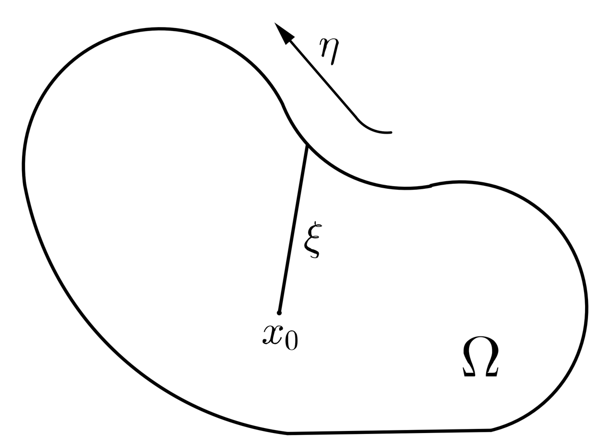
Let $Lu = f$, for $u \in \mathcal{V}, u\colon \mathbb{R}^d \to \mathbb{R}^m, m \leq d$, a PDE with differential operator L. To solve it, we use Galerkin's Projection from $\mathcal{V} \to \mathcal{V}_h$, where $\mathcal{V}_h \subset \mathcal{V} \subset (H^1)^m$.

Using the isogeometric paradigm the space of basis functions is given by

$$\mathcal{V}_h = \{\phi_i := e_{k_i} N_i \circ F^{-1}, i = 1, \dots, n,\ k_i = 1, \dots, m\}.$$

SB-IGA method

Model problem: $-\Delta u = f$ in Ω with some boundary conditions.



Assume there is a center point $C = x_0$ such that

- $\bar{x} = N(\eta)X = \sum_i N_i(\eta)x_i$ on $\partial\Omega$, with splines/NURBS N and control points X
- $x = x_0 + \xi(N(\eta)X - x_0)$ in Ω, $\xi \in [0,1]$

N.B.: Ω is required to be a star-shaped domain.

After transforming the PDE, we work with the main idea for a scaled boundary method: taken a test function $\delta u = N(\eta)v(\xi)$, we consider the weak form along the boundary while the strong form in scaling direction. In this way the structure of boundary value problem in scaling direction is given by a ODE BVP problem:

$$A_1 U'' + \frac{1}{\xi} A_2 U' - \frac{1}{\xi^2} A_3 U + c = 0, \quad \xi \in [0,1] \quad \text{with BC at } \xi = 1.$$

From [2] an exact solution exists.
Relationship with IGA: define

$$F(\xi, \eta) := \sum_{i=1}^{n} \sum_{j=1}^{2} d_{i,j} N_i(\eta) Q_j(\xi)$$

where $Q_1(\xi) = 1 - \xi$, $Q_2(\xi) = \xi$, $d_{i,1} = x_0$, $d_{i,2} = x_i$ for $i = 1, \dots, n$.

$\Rightarrow F(\xi, \eta) = x_0 + \xi(N(\eta) - X - x_0)$.

References

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