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Financial Mathematics I

Exercise Sheet 1

Please submit your solution until 17:00 on Monday, May 04.

Exercise 1 (Put-Call Parity)
Consider an arbitrage-free, frictionless (no transaction costs, tax influence, etc.; buying and selling at the same price, ...) market in which two assets B and S are traded continuously in time. Asset B is assumed to be risk-less with known time-t price B_t (e.g. a bank account). Asset S is assumed to be risky (e.g. a stock). We make no assumptions about the distribution of the time-t price S_t of this asset.
Denote by C_t^{eu} the time-t price of a European call on S with maturity T and strike price K and denote by P_t^{eu} the time-t price of a European put on S with the same maturity and strike. Prove the so-called put-call parity:

$$C_t^{eu} + \frac{B_t}{B_T} K = P_t^{eu} + S_t.$$

Hint: Use the replication principle: Construct two portfolios with the same value at maturity and use the absence of arbitrage to conclude that the time-t value of the two portfolios must coincide as well.

Exercise 2 (Arbitrage Bounds and the American Call Option)
Consider the same market as in Exercise 1 and assume in addition that $B_t < B_T$ for all $t < T$.

(i) Prove that the following arbitrage bounds hold:

$$\max\left\{S_t - \frac{B_t}{B_T}K, 0\right\} \leq C_t^{eu} \leq S_t, \quad \text{for all } t \leq T.$$

(ii) Denote by C_t^{am} the time-t price of an American call with strike price K and maturity T. Prove that it is never optimal to exercise the American call before time T and

$$C_t^{am} = C_t^{eu}, \quad \text{for all } t \leq T.$$

Hint: What is the value of C_t^{am} if exercising at time t is optimal?

Exercise 3 (The Trinomial Market Model)
We consider a one-period model with one bond B and one stock S. For the deterministic bond we have

$$B_0 = 1, \qquad\qquad B_1 = 1 + r,$$

where $r > 0$. For the stock we assume

$$S_0 = s > 0, \qquad\qquad S_1 = \begin{cases} us, & \text{with probability } p_u, \\ ms, & \text{with probability } p_m, \\ ds, & \text{with probability } p_d, \end{cases}$$

where $0 < d < m < u$ as well as $p_u, p_m, p_d > 0$ and $p_u + p_m + p_d = 1$.

(i) Show that the model does not admit arbitrage if and only if $d < 1 + r < u$.

(ii) Construct a contingent claim which is not attainable.

(iii) Find all risk-neutral measures, i.e. all probability measures $\mathbb{Q}$ which are equivalent to $\mathbb{P}$ such that

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S_1}{S_0}\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{B_1}{B_0}\right].$$

Exercise 4 (A Market with Transaction Costs)
Consider a one-period model with bond $B_0 = B_1 = 1$. We suppose there are bid and ask prices for the underlying risky asset S, i.e. the stock is sold at the bid price S^{bid} and bought at the ask price S^{ask}. At terminal time $T = 1$ there are two possible states g and b, each with bid and ask prices. At T, the investor is assumed to liquidate her stock position. Consider the following situation:

$$(S_0^{bid}, S_0^{ask}) = (3, 5), \qquad (S_1^{bid}(g), S_1^{ask}(g)) = (4, 6), \qquad (S_1^{bid}(b), S_1^{ask}(b)) = (2, 3).$$

(i) Replicate a claim C with payoff $C_1(g) = 2$ and $C_1(b) = 0$.

(ii) Determine a strategy with higher payoff than C (a superhedging strategy) which is less expensive than the replication strategy in (i).