Prof. Dr. Ralf Korn
Dr. Christoph Belak
TU Kaiserslautern
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Financial Mathematics I

Exercise Sheet 2

Please submit your solution by 17:00 on Monday, May 18.

Exercise 1 (Geometric Brownian Motion)
Let $\{W_t\}_{t\geq 0}$ be a Brownian motion and fix $b \in \mathbb{R}$ and $\sigma > 0$. Define a new process $\{X_t\}_{t\geq 0}$ through

$$X_t := e^{bt+\sigma W_t}, \quad t \geq 0.$$

(i) Under which conditions on b and σ is the process X a supermartingale, a submartingale or a martingale, respectively?

(ii) Calculate the variance of X_t for t fixed.

Exercise 2 (Modifications and Indistinguishability)
Suppose that $\{X_t\}_{t\in[0,T]}$ and $\{Y_t\}_{t\in[0,T]}$ are processes such that X is a modification of Y.

(i) Provide an example which shows that X and Y need not be indistinguishable.

(ii) Show that if both X and Y are right-continuous, then X and Y are indistinguishable.

Exercise 3 (Measurability of Stopped Processes)
Let τ be a stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Show that if $\{(X_t, \mathcal{F}_t)\}_{t\geq 0}$ is progressively measurable, then so is the stopped process $\{(X_{t\wedge\tau}, \mathcal{F}_t)\}_{t\geq 0}$. Moreover, $X_{t\wedge\tau}$ is both $\mathcal{F}_t$- and $\mathcal{F}_{t\wedge\tau}$-measurable.

Exercise 4 (Stopping Time Characterization of Martingales)
Let $\{M_t\}_{t\in[0,T]}$ be a right-continuous stochastic process adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ and assume that $\mathcal{F}_0$ is trivial.

(i) Show that M is a supermartingale if and only if $\mathbb{E}[M_\tau] \geq \mathbb{E}[M_\sigma]$ for all $[0,T]$-valued stopping times τ, σ with $\tau \leq \sigma$ almost surely.

(ii) Show that M is a martingale if and only if $\mathbb{E}[M_\tau] = M_0$ for every $[0,T]$-valued stopping time τ.