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TU Kaiserslautern
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Financial Mathematics I

Exercise Sheet 3

Please submit your solution by 17:00 on Tuesday, May 26.

Exercise 1 (Moments of Brownian Motion)
Let W be a standard $\mathbb{R}$-valued Brownian motion and define $\mu_k(t) := \mathbb{E}[(W_t)^k]$, $k \in \mathbb{N}$. Use Itô's formula to show that

$$\mu_k(t) = \frac{1}{2}k(k-1)\int_0^t \mu_{k-2}(s)\,ds, \qquad k \geq 2.$$

Use this to compute $\mathbb{E}[(W_t)^4]$.

Exercise 2 (Local Martingales and Martingales)
Let $M = \{M_t\}_{t\in[0,T]}$ be a continuous local martingale and suppose that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |M_t|^p\Big] < \infty \quad \text{for some } p \in [1,\infty).$$

Show that M is in fact a martingale.

Exercise 3 (Two-dimensional Brownian Motion)
Suppose that B^1 and B^2 are $\mathbb{R}$-valued Brownian motions with

$$\langle B^1, B^2\rangle_t = \int_0^t \rho_s\,ds, \qquad t \geq 0,$$

where ρ is a progressively measurable process with values in $(-1,1)$. Define a process $W = (W^1, W^2)^\top$ through

$$W_t^1 := B_t^1, \quad W_t^2 := -\int_0^t \frac{\rho_s}{\sqrt{1-\rho_s^2}}\,dB_s^1 + \int_0^t \frac{1}{\sqrt{1-\rho_s^2}}\,dB_s^2, \quad t \geq 0.$$

Show that W is a two-dimensional Brownian motion.
Hint: You may use Lévy's characterization of Brownian motion.

Exercise 4 (Application of Itô's Formula)
Let $f : [0,\infty) \to \mathbb{R}$ be continuously differentiable and let W be a Brownian motion.

(i) Prove that

$$\mathbb{E}\left[\left(\int_0^T f'(s)W_s\,ds\right)^2\right] = f(T)^2T - 2f(T)\int_0^T f(s)\,ds + \int_0^T f(s)^2\,ds.$$

(ii) Calculate $\mathbb{E}[(\int_0^T W_s\,ds)^2]$.