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Financial Mathematics I

Exercise Sheet 5

Please submit your solutions by 17:00 on Monday, June 08.

Exercise 1 (Itô's Product Rule)
Let X and Y be one-dimensional Itô processes. Show that the product XY is an Itô process such that

$$X_tY_t = X_0Y_0 + \int_0^t X_s\,dY_s + \int_0^t Y_s\,dX_s + \langle X, Y\rangle_t, \qquad t \geq 0.$$

Use this result to prove the integration by parts formula

$$\int_0^t s\,dW_s = tW_t - \int_0^t W_s\,ds, \qquad t \geq 0.$$

Exercise 2 (The Ornstein-Uhlenbeck Process)
Let $\sigma : [0, t] \to \mathbb{R}$ be a deterministic Borel-measurable function such that $\int_0^t \sigma_s^2\,ds < \infty$.

(i) Show that

$$\int_0^t \sigma_s\,dW_s \sim \mathcal{N}\left(0, \int_0^t \sigma_s^2\,ds\right).$$

(ii) Determine the distribution of $\int_0^t W_s\,ds$.

(iii) For $\kappa, \theta, \sigma > 0$, define

$$X_t \triangleq xe^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \int_0^t e^{-\kappa(t-s)}\sigma\,dW_s.$$

Calculate $\lim_{t\to\infty} \mathbb{E}[X_t]$ and $\lim_{t\to\infty} Var[X_t]$.

Exercise 3 (Variation of Constants)
Let W be a one-dimensional Brownian motion, $\kappa, \theta, \sigma > 0$. Solve the SDE

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma\,dW_t, \qquad t \geq 0, \qquad X_0 = x.$$

Why is this process called mean-reverting?

Exercise 4 (The Cox-Ingersoll-Ross Process)
Let $\kappa, b > 0$ and $n \in \mathbb{N}$ with $n \geq 2$. Assume that $W = (W^1, \dots, W^n)$ is an n-dimensional Brownian motion. For each $i = 1, \dots, n$, let Y^i be the solution of

$$dY_t^i = -\frac{1}{2}\kappa Y_t^i \, dt + \frac{1}{2} b \, dW_t^i, \qquad Y_t^i = 0,$$

and set $Y = (Y^1, \dots, Y^n)$.

(i) Denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^n$. Prove that the process B defined as

$$B_t \triangleq \sum_{i=1}^{n} \int_0^t \frac{Y_t^i}{\|Y_t\|} \mathbb{1}_{\{\|Y_t\|>0\}} \, dW_t^i, \qquad t \geq 0,$$

is a one-dimensional Brownian motion.

(ii) Show that the process r defined as $r_t \triangleq \sum_{i=1}^{n} (Y_t^i)^2$ solves the SDE

$$dr_t = (\theta - \kappa r_t) \, dt + b\sqrt{r_t} \, dB_t, \qquad r_0 = 0.$$