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Financial Mathematics I

Exercise Sheet 6

Please submit your solutions by 17:00 on Monday, June 15.

Exercise 1 (Martingale Representation)
Let W be a one-dimensional Brownian motion and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, $(t,x) \mapsto f(t,x)$ such that f is continuously differentiable in t and twice continuously differentiable in x. Suppose that the process M given by $M_t = f(t, W_t)$ is a square-integrable Brownian martingale. Show that

$$M_t = M_0 + \int_0^t \frac{\partial}{\partial x} f(u, W_u) dW_u, \qquad t \geq 0.$$

Exercise 2 (Growth-Optimal Portfolio)
Let H and θ be given as in the theorem on complete markets.

(i) Show that $1/H_t$ is the wealth process corresponding to

$$\pi_t \triangleq (\sigma_t^{-1})^\top \sigma_t^{-1}(b_t - r_t 1) \quad \text{and} \quad c_t = 0$$

with $1/H_0 = 0$.

(ii) Let $(\pi, c) \in \mathcal{A}(1)$ be such that

$$\mathbb{E}\left[\int_0^T \pi_t^\top \sigma_t \, dW_t\right] = 0, \quad \text{and} \quad c = 0.$$

Assume moreover that $\mathbb{E}[\log(X_T)]$ exists, where X denotes the wealth process corresponding to (π, c) with $X_0 = 1$. Show that

$$\mathbb{E}\left[\log(X_T)\right] \leq \mathbb{E}\left[\log(1/H_T)\right].$$

Exercise 3 (Option Pricing)
Determine the price of

$$B = P_1(T) 1_{\{P_1(T) \geq K\}}$$

in the one-dimensional Black-Scholes model.

Exercise 4 (The Greeks)
In the one-dimensional Black-Scholes model, determine the following derivatives of the price of a European call option (at time $t = 0$) with strike $K > 0$ and maturity $T > 0$.

(i) The first- and second-order derivative with respect to the price of the underlying asset.

(ii) The derivative with respect to the volatility.

(iii) The derivative with respect to the interest rate.

(iv) The derivative with respect to the maturity.