Financial Mathematics I

Exercise Sheet 7

Please submit your solutions by 17:00 on Monday, June 22.

Throughout this exercise sheet, we fix a one-dimensional Brownian motion $\{(W_t, \mathcal{F}_t)\}_{t\geq 0}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ denotes the Brownian filtration. We fix moreover progressively measurable processes $\{a_t\}_{t\in[0,T]}$, $\{b_t\}_{t\in[0,T]}$, $\{c_t\}_{t\in[0,T]}$. We assume that a and b are bounded processes and that $c \in L^2([0,T])$. Finally, we fix some $\mathcal{F}_T$-measurable random variable ξ with $\mathbb{E}[|\xi|^2] < \infty$ and we define a process $\{\Gamma_t\}_{t\in[0,T]}$ as the solution of the SDE

$$\mathrm{d}\Gamma_t = a_t\Gamma_t\,\mathrm{d}t + b_t\Gamma_t\,\mathrm{d}W_t, \qquad \Gamma_0 = 1.$$

The aim of this exercise sheet is to study solutions of SDEs for which we specify a terminal condition at time $t = T$ instead of an initial condition at time $t = 0$. That is, we consider equations of the form

$$Y_t = \xi + \int_t^T a_sY_s + b_sZ_s + c_s\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s, \qquad t \in [0,T], \tag{1}$$

or, equivalently, in differential form

$$-\mathrm{d}Y_t = a_tY_t + b_tZ_t + c_t\,\mathrm{d}t - Z_t\,\mathrm{d}W_t, \qquad Y_T = \xi.$$

We say that a pair of progressively measurable processes (Y_t, Z_t) is a solution of such a backward SDE if it satisfies Equation (1) as well as the integrability condition

$$\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|^2 + \int_0^T |Z_s|^2\,\mathrm{d}s\right] < \infty.$$

Exercise 1 (Explicit Solutions of Linear Backward SDEs; 8 points)
We proceed step-by-step to construct the unique solution of (1).

(i) Assume that $a \equiv b \equiv c \equiv 0$. Show that the backward SDE (1) admits a unique solution and provide an explicit formula for the process Y in terms of ξ.
Hint: Apply conditional expectation on both sides of (1).

(ii) Assume that $a \equiv b \equiv 0$, but possibly $c \neq 0$. Show that the backward SDE (1) admits a unique solution and provide an explicit formula for the process Y in terms of ξ and c.

(iii) Assume that $b \equiv 0$, but possibly $a, c \neq 0$. Using the results of part (ii) of this exercise, show that the backward SDE (1) admits a unique solution and provide an explicit formula for the process Y in terms of ξ, c and Γ.
Hint: Apply Itô's formula to reduce to the case of (ii).

(iv) Assume that $a, b, c \neq 0$. Using the results of part (iii) of this exercise, show that the backward SDE (1) admits a unique solution and show that the process Y is given via the equation

$$\Gamma_t Y_t = \mathbb{E}\left[\Gamma_T \xi + \int_t^T \Gamma_s c_s \, \mathrm{d}s \middle| \mathcal{F}_t\right], \qquad t \in [0, T].$$

Hint: Try to solve the BSDE (1) under a measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$.

Exercise 2 (Comparison Principle for Linear Backward SDEs; 4 points)
Let $\{(Y_t, Z_t)\}_{t\in[0,T]}$ be the unique solution of (1). Let ξ' be an $\mathcal{F}_T$-measurable random variable with $\mathbb{E}[|\xi'|^2] < \infty$, and let $\{a'_t\}_{t\in[0,T]}$, $\{b'_t\}_{t\in[0,T]}$, $\{c'_t\}_{t\in[0,T]}$ be progressively measurable with a' and b' being bounded and $c' \in L^2([0,T])$. Let $\{(Y'_t, Z'_t)\}_{t\in[0,T]}$ satisfy the equation

$$Y'_t = \xi' + \int_t^T a'_s Y'_s + b'_s Z'_s + c'_s \, \mathrm{d}s - \int_t^T Z'_s \, \mathrm{d}W_s, \qquad t \in [0, T].$$

Suppose that

(a) $\xi \geq \xi'$ $\mathbb{P}$-almost surely and

(b) $a_s Y_s + b_s Z_s + c_s \geq a'_s Y_s + b'_s Z_s + c'_s$ $\mathrm{d}t \otimes \mathbb{P}$-almost everywhere.

Show that this implies that

$$\mathbb{P}\Big[Y_t \geq Y'_t \text{ for all } t \in [0, T]\Big] = 1.$$

Moreover, if any one of the inequalities in (a) or (b) is strict on a set of positive measure, then $Y_0 > Y'_0$.
Hint: You may use part (iv) of Exercise 1.

Exercise 3 (Pricing Contingent Claims via Backward SDEs; 4 points)
Consider the standard market model from the lecture with $d = m = 1$. Let (g, B) be a contingent claim such that

$$\mathbb{E}\left[|B|^2 + \int_0^T |g(s)|^2 \, \mathrm{d}s\right] < \infty.$$

Denote the fair price of the claim (g, B) at time t by $\hat{p}(t)$. Find a backward SDE such that the solution $\{(Y_t, Z_t)\}_{t\in[0,T]}$ of this backward SDE satisfies $Y_t = \hat{p}(t)$. In other words, show that the fair price $\hat{p}$ can be expressed as the unique solution of a suitable backward SDE.
Hint: It may be useful to express any wealth process in terms of trading strategies φ instead of the portfolio process π.