Prof. Dr. Ralf Korn
Dr. Christoph Belak

Financial Mathematics I

Exercise Sheet 8

Please submit your solutions by 17:00 on Monday, June 29.

Exercise 1 (Binary Options and Gap Options)
We consider a one-dimensional Black-Scholes market.

(i) The payoff of a *European binary put* option or *digital put* with strike $K > 0$ is given by

$$B^{\text{Bin}}_{\text{Put}} = 1_{\{P_1(T)<K\}}.$$

How are the prices of the binary call (see lecture notes) and the binary put related? Use this relation to compute the fair price of the binary put.

(ii) The payoff of a *European gap call* with strike $K > 0$ and gap $G < K$ is given by

$$B^{\text{Gap}}_{\text{Call}} = (P_1(T) - G)1_{\{P_1(T)\geq K\}}.$$

Compute the fair price of this option.

(iii) The payoff of a *European gap put* with strike $K > 0$ and gap $G < K$ is given by

$$B^{\text{Gap}}_{\text{Put}} = (G - P_1(T))1_{\{P_1(T)<K\}}.$$

Compute the fair price of this option.

(iv) Derive the following put-call parity for gap options:

$$X_{B^{\text{Gap}}_{\text{Put}}}(t) + P_1(t) = X_{B^{\text{Gap}}_{\text{Call}}}(t) + Ge^{-r(T-t)}, \qquad t \in [0, T].$$

Exercise 2 (Paylater Call)
A *paylater call* is a standard European call option which costs nothing to initiate, but the holder pays the premium at maturity and then only if the the option is in the money. More precisely, in the Black-Scholes setting, the payoff of a paylater call is given by

$$B^{\text{PL}} = (P_1(T) - K - L)1_{\{P_1(T)\geq K\}},$$

where at time $t = 0$ the parameter L is chosen such that $X_{B^{\text{PL}}}(0) = 0$, i.e. the initial price of the option is zero.

(i) Determine L at time $t = 0$.

(ii) Compute the price of this option at time $t \in (0, T)$. Assume that L was already fixed at time $t = 0$.

Exercise 3 (Feynman-Kac Representation)
Let $\mu, \sigma, \nu \in \mathbb{R}$. Using the Feynman-Kac representation theorem, solve the PDE

$$\frac{\partial}{\partial t}u(t,x) + \mu x \frac{\partial}{\partial x}u(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}u(t,x) = 0, \qquad (t,x) \in [0,T) \times \mathbb{R}_+,$$
$$u(T,x) = \log(x^2), \qquad x \in (0,\infty),$$

as well as the PDE

$$\frac{\partial}{\partial t}u(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_1^2}u(t,x) + \frac{1}{2}\nu^2 \frac{\partial^2}{\partial x_2^2}u(t,x) = 0, \qquad (t,x) \in [0,T) \times \mathbb{R}^2,$$
$$u(T,x) = x_1 x_2, \qquad x \in \mathbb{R}^2,$$

where $x = (x_1, x_2)$.

Exercise 4 (Equivalent Martingale Measures)
Let $S = \{S_t\}_{t \in [0,T]}$ be the solution of

$$\mathrm{d}S_t = 2\,\mathrm{d}t + \mathrm{d}W_t^1 + 2\,\mathrm{d}W_t^2, \qquad S_0 = s,$$

where $W = (W^1, W^2)$ is a two-dimensional Brownian motion under $\mathbb{P}$. Show that there exist infinitely many measures $\tilde{\mathbb{P}} \sim \mathbb{P}$ such that S is a martingale under $\tilde{\mathbb{P}}$.