Prof. Dr. Ralf Korn
Dr. Christoph Belak

Financial Mathematics I

Exercise Sheet 9

Please submit your solutions by 17:00 on Monday, July 06.

Exercise 1 (Option Pricing in a Two-Dimensional Black-Scholes Market; 4 points)
Determine the price of a European option with payoff $B \triangleq 1_{\{P_1(T) \geq P_2(T)\}}$ in a two-dimensional Black-Scholes model.

Exercise 2 (Black-Scholes Formula with Dividend Rates; 4 points)
If the stock P_1 in the one-dimensional Black-Scholes model pays a dividend rate $\delta P_1(t)$ per unit of time, then its price is modeled as the solution of

$$\mathrm{d}P_1(t) = P_1(t)\left[(b-\delta)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\right], \qquad P_1(0) = p.$$

Imitate the arguments from the lecture to derive an option pricing PDE in this market. Use the pricing PDE to show that the price $C(t, P_1(t))$ of a European call on the stock P_1 with strike K is given by

$$C(t, P_1(t)) = e^{-\delta(T-t)} P_1(t)\Phi(\delta_1(t)) - e^{-r(T-t)} K\Phi(\delta_2(t))$$

with

$$\delta_1(t) \triangleq \frac{\log(P_1(t)/K) + (r-\delta+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad \delta_2(t) \triangleq \delta_1(t) - \sigma\sqrt{T-t}.$$

Remark: You only have to verify that the function $C(t, P_1(t))$ given above satisfies your pricing PDE. You do not have to verify that $C(t, P_1(t))$ indeed coincides with the price of the option.

Exercise 3 (Reflection Principle, Running Maximum and Hitting Time Distribution; 8 points)
Let $(W_t, \mathcal{F}_t)_{t\geq 0}$ be a Brownian motion and $\mu, b > 0$. We define two stopping times T_b and $\tilde{T}_b$ through

$$T_b(\omega) = \inf\{t > 0 : W_t(\omega) = a\} \quad \text{and} \quad \tilde{T}_b(\omega) = \inf\{t > 0 : W_t(\omega) + \mu t = a\}.$$

(i) Show that the process $\bar{W}_t \triangleq W_t 1_{\{t<T_b\}} + (2b - W_t)1_{\{t\geq T_b\}}$ is a Brownian motion.

(ii) Define the running maximum M of W through $M_t \triangleq \sup_{u\in[0,t]} W_u$. Show that, for any $a, y, t \geq 0$,
$$\mathbb{P}\left[M_t \geq a, W_t \leq a - y\right] = \mathbb{P}\left[W_t \geq a + y\right].$$

(iii) Determine the distribution of T_b as well as the joint distribution of (M_t, W_t).

(iv) Use Girsanov's theorem to derive the distribution of $\tilde{T}_b$ and $(\tilde{M}_t, W_t + \mu t)$, where $\tilde{M}_t$ denotes the running maximum of $W_t + \mu t$.

Hint: In (i) you are allowed to use the strong Markov property of W, i.e. you can use that the process $Z_t \triangleq (W_{t+T_b} - b)_{t\geq 0}$ is a Brownian motion with respect to $(\mathcal{F}_{t+T_b})_{t\geq 0}$ and Z_0 is independent of $\mathcal{F}_{T_b}$.