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Financial Mathematics I

Exercise Sheet 10

Please submit your solutions by 17:00 on Monday, July 13.

The objective of this exercise sheet is to solve the so-called *constrained portfolio optimization problem* in which the investor is only allowed to choose portfolio processes π which take values in a given convex cone $\mathcal{K} \subset \mathbb{R}^d$. The unconstrained portfolio optimization problem is given by

$$\mathcal{V}_0(x) \triangleq \max_{(\pi,c)\in\mathcal{A}'(x)} \mathbb{E}\left[\int_0^T U(c(t))\,\mathrm{d}t + U\left(X^{x,\pi,c}(T)\right)\right],$$

where $x > 0$ denotes the initial wealth and $U(x) = \log(x)$. The constrained portfolio problem is given by

$$\mathcal{V}_{\mathcal{K}}(x) \triangleq \max_{(\pi,c)\in\mathcal{A}'_{\mathcal{K}}(x)} \mathbb{E}\left[\int_0^T U(c(t))\,\mathrm{d}t + U\left(X^{x,\pi,c}(T)\right)\right],$$

where

$$\mathcal{A}'_{\mathcal{K}}(x) \triangleq \{(\pi,c) \in \mathcal{A}'(x) : \pi(t,\omega) \in \mathcal{K} \text{ for } \mathrm{d}t \otimes \mathbb{P}\text{-a.e. } (t,\omega)\}.$$

We denote by δ the support function of $-\mathcal{K}$ given by

$$\delta(x) \triangleq \sup_{\pi\in\mathcal{K}}[-\pi^\top x], \qquad x \in \mathbb{R}^d,$$

and we let $\tilde{\mathcal{K}}$ denote the effective domain of δ, i.e.

$$\tilde{\mathcal{K}} \triangleq \left\{x \in \mathbb{R}^d : \delta(x) < \infty\right\}.$$

We denote by $\mathcal{D}$ the set of all $\tilde{\mathcal{K}}$-valued, bounded and progressively measurable processes ν. For every $\nu = (\nu_1, \dots, \nu_d) \in \mathcal{D}$, we introduce a new market with prices $(P_0^\nu, P_1^\nu, \dots, P_d^\nu)$ given by

$$\mathrm{d}P_0^\nu(t) = P_0^\nu(t)[r(t) + \delta(\nu(t))]\,\mathrm{d}t, \qquad P_0^\nu(0) = p_0,$$

$$\mathrm{d}P_i^\nu(t) = P_i^\nu(t)[b_i(t) + \nu_i(t) + \delta(\nu(t))]\,\mathrm{d}t + P_i^\nu(t)\sum_{j=1}^d \sigma_{i,j}(t)\,\mathrm{d}W_j(t), \qquad P_i^\nu(0) = p_i.$$

The unconstrained portfolio optimization problem in this auxiliary market is given by

$$\mathcal{V}_\nu(x) \triangleq \max_{(\pi,c)\in\mathcal{A}'_\nu(x)} \mathbb{E}\left[\int_0^T U(c(t))\,\mathrm{d}t + U\left(X_\nu^{x,\pi,c}(T)\right)\right],$$

where $\mathcal{A}'_\nu(x)$ denotes the set of all admissible strategies with respect to $(P_0^\nu, P_1^\nu, \dots, P_d^\nu)$ and where $X_\nu^{x,\pi,c}$ denotes the wealth process started in x corresponding to $(\pi, c) \in \mathcal{A}'_\nu(x)$ and $(P_0^\nu, P_1^\nu, \dots, P_d^\nu)$. In what follows, we restrict ourselves to the following three cases:

1. The unconstrained case: $\mathcal{K} = \mathbb{R}^d$.
2. No borrowing, no short-selling: $d = 1$ and $\mathcal{K} = [0, 1]$.
3. Incomplete market: $d = 2$, $\mathcal{K} = \mathbb{R} \times \{0\}$ and r, b, σ constant, σ symmetric.

Exercise (Solution of the Constrained Portfolio Problem)
Solve the constrained portfolio problem as follows:

(i) In all three cases, calculate the function δ and the effective domain $\tilde{K}$.

(ii) Show that $\mathcal{A}_{\mathcal{K}}(x) \subset \mathcal{A}_\nu(x)$ for every $\nu \in \mathcal{D}$ and $\mathcal{V}_{\mathcal{K}}(x) \leq \inf_{\nu \in \mathcal{D}} \mathcal{V}_\nu(x)$.

(iii) Use the martingale method to solve the auxiliary problem $\mathcal{V}_\nu$. Show that the optimal strategy (π_ν, c_ν) and the value function $\mathcal{V}_\nu(x)$ are given by

$$\begin{aligned} \pi_\nu(t) &= (\sigma(t)^\top)^{-1}\theta_\nu(t), \\ c_\nu(t) &= \frac{x}{(T+1)H_\nu(t)}, \\ \mathcal{V}_\nu(x) &= \log(x) + \mathbb{E}\left[\int_0^T r(s) + \delta(\nu(s)) + \frac{1}{2}\|\theta_\nu(s)\|^2 \, \mathrm{d}s\right], \end{aligned}$$

where $\theta_\nu(t) \triangleq \theta(t) + \sigma^{-1}(t)\nu(t)$ and

$$H_\nu(t) \triangleq \exp\left\{-\int_0^t r(s) + \delta(\nu(s)) + \frac{1}{2}\|\theta_\nu(t)\| \, \mathrm{d}s - \int_0^t \theta_\nu(s)^\top \, \mathrm{d}W(s)\right\}.$$

(iv) Define a process λ through

$$\lambda(t,\omega) = \arg\min_{\lambda \in \tilde{\mathcal{K}}} \left[\delta(\lambda) + \frac{1}{2}\|\theta(t,\omega) + \sigma^{-1}(t,\omega)\lambda\|^2\right].$$

Compute λ in all three cases and show that $\lambda \in \mathcal{D}$.

(v) Show that $\mathcal{V}_\lambda(x) \leq \inf_{\nu \in \mathcal{D}} \mathcal{V}_\nu(x)$.

(vi) In all three cases, compute the optimal strategy (π_λ, c_λ) in the auxiliary market corresponding to λ and show that $\pi_\lambda \in \mathcal{K}$ almost everywhere.

(vii) In all three cases, show that

$$\delta(\lambda(t)) + \pi_\lambda(t)^\top \lambda(t) = 0, \qquad \mathrm{d}t \otimes \mathbb{P}\text{-a.e.}$$

(viii) Use (vii) to prove that the processes $X^{x,\pi_\lambda,c_\lambda}$ and $X_\nu^{x,\pi_\lambda,c_\lambda}$ are indistinguishable and argue that $(\pi_\lambda, c_\lambda) \in \mathcal{A}'_{\mathcal{K}}(x)$. Use this to prove that (π_λ, c_λ) is optimal for the constrained portfolio problem.

Remark: You can obtain up to 2 points for each step. You are always allowed to use the results from the previous steps, even if you do not manage to prove them. So if you get stuck somewhere in between, just proceed with the next step. Step (viii) can be proved without using the explicit form of λ and (π_λ, c_λ).