Prof. Dr. Ralf Korn
Dr. Christoph Belak

Financial Mathematics I

Exercise Sheet 11

Please submit your solutions by 17:00 on Monday, July 20.

Exercise 1 (Martingale Optimality Principle)
We consider the problem of maximizing expected utility of consumption and terminal wealth in a d-dimensional Black-Scholes market (that is r, b, σ are constant), i.e.

$$\max_{(\pi,c)\in\mathcal{A}'(x)} \mathbb{E}\left[\int_0^T U_1(t, c(t))\,\mathrm{d}t + U_2(X^{x,\pi,c}(T))\right].$$

Derive and prove the martingale optimality principle for this problem and use it to derive the Hamilton-Jacobi-Bellman equation.
Hint: If you have trouble identifying the correct martingale, look at the hint in Exercise 2, part (i).

Exercise 2 (Drift Control with Quadratic Cost)
We consider the optimization problem

$$\min_{u\in L^2([0,T])} \mathbb{E}\left[\int_0^T a\cdot u(t)^2\,\mathrm{d}t - b\cdot X^{x,u}(T)\right],$$

where $a, b > 0$ and where for each $u \in L^2([0,T])$ and each $x \in \mathbb{R}$ we let $X^{x,u}$ denote the solution of

$$\mathrm{d}X^{x,u}(t) = u(t)\,\mathrm{d}t + \mathrm{d}W(t), \qquad X^{x,u}(0) = x.$$

Solve the optimization problem as follows:

(i) Use the martingale optimality principle to derive the Hamilton-Jacobi-Bellman (HJB) equation.
Hint: The martingale M in the martingale optimality principle should be of the form

$$M_t = \int_0^t a\cdot u^*(s)\,\mathrm{d}s + G(t, X^{x,u^*}(t)).$$

(ii) Find a candidate optimal strategy u^* by formally minimizing the HJB equation with respect to u.

(iii) Plug u^* back into the HJB equation and solve it.
Hint: Try a solution of the form $G(t,x) = -b\cdot x + h(t)$.

(iv) Verify the optimality of u^* using the martingale optimality principle.

Exercise 3 (The End)
No exercise 3, you have done enough this semester. Good luck for the exams and all the best for your future!