Risk Measures, Backward SDEs and g-Expectations

Christoph Belak

Department of Mathematics
Kaiserslautern University of Technology
Germany

Reading course on Nonlinear Expectations
TU Kaiserslautern, May 07, 2014

Suppose that X is some (risky) **financial position** due at time $T > 0$. How much **capital** should be set aside now to cover potential losses associated with X in the future?

Risk Measures

Suppose that X is some (risky) **financial position** due at time $T > 0$. How much **capital** should be set aside now to cover potential losses associated with X in the future?

A **risk measure** is simply a mapping $\rho : L^2(\mathcal{F}_T) \to \mathbb{R}$. The risk associated with $X \in L^2(\mathcal{F}_T)$ is said to be **acceptable**, if $\rho(X) \leq 0$.

Suppose that X is some (risky) **financial position** due at time $T > 0$. How much **capital** should be set aside now to cover potential losses associated with X in the future?

A **risk measure** is simply a mapping $\rho : L^2(\mathcal{F}_T) \to \mathbb{R}$. The risk associated with $X \in L^2(\mathcal{F}_T)$ is said to be **acceptable**, if $\rho(X) \leq 0$.

Example: Value at Risk

Denote by F_X the cumulative distribution function of $X \in L^2(\mathcal{F}_T)$. We call

$$\rho(X) \triangleq V@R_\alpha(X) \triangleq \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}$$

the **value at risk** at level $\alpha \in (0, 1)$.

Risk Measures

Suppose that X is some (risky) **financial position** due at time $T > 0$. How much **capital** should be set aside now to cover potential losses associated with X in the future?

A **risk measure** is simply a mapping $\rho : L^2(\mathcal{F}_T) \to \mathbb{R}$. The risk associated with $X \in L^2(\mathcal{F}_T)$ is said to be **acceptable**, if $\rho(X) \leq 0$.

Example: Value at Risk

Denote by F_X the cumulative distribution function of $X \in L^2(\mathcal{F}_T)$. We call

$$\rho(X) \triangleq V@R_\alpha(X) \triangleq \inf \{x \in \mathbb{R} : F_X(x) \geq \alpha\}$$

the **value at risk** at level $\alpha \in (0, 1)$.

Question: How can we judge the **quality** of a risk measure?

Coherent risk measures

Coherent risk measure

A risk measure $\rho : \mathcal{L}(\mathcal{F}_T) \to \mathbb{R}$ is called **coherent**, if it is

Coherent risk measures

Coherent risk measure

A risk measure $\rho : \mathcal{L}(\mathcal{F}_T) \to \mathbb{R}$ is called **coherent**, if it is

- **normalized**, i.e. $\rho(0) = 0$,

Coherent risk measures

Coherent risk measure

A risk measure $\rho : \mathcal{L}(\mathcal{F}_T) \to \mathbb{R}$ is called **coherent**, if it is

- **normalized**, i.e. $\rho(0) = 0$,
- **translation invariant**, i.e. $\rho(X + c) = \rho(X) - c$, $c \in \mathbb{R}$,

Coherent risk measures

Coherent risk measure

A risk measure $\rho : \mathcal{L}(\mathcal{F}_T) \to \mathbb{R}$ is called **coherent**, if it is

- **normalized**, i.e. $\rho(0) = 0$,
- **translation invariant**, i.e. $\rho(X + c) = \rho(X) - c$, $c \in \mathbb{R}$,
- **monotone**, i.e. if $X \leq Y$, then $\rho(X) \geq \rho(Y)$,

Coherent risk measures

Coherent risk measure

A risk measure $\rho : \mathcal{L}(\mathcal{F}_T) \to \mathbb{R}$ is called **coherent**, if it is

- **normalized**, i.e. $\rho(0) = 0$,
- **translation invariant**, i.e. $\rho(X + c) = \rho(X) - c$, $c \in \mathbb{R}$,
- **monotone**, i.e. if $X \leq Y$, then $\rho(X) \geq \rho(Y)$,
- **sub-additive**, i.e. $\rho(X + Y) \leq \rho(X) + \rho(Y)$,

Coherent risk measure

A risk measure $\rho : \mathcal{L}(\mathcal{F}_T) \to \mathbb{R}$ is called **coherent**, if it is

- **normalized**, i.e. $\rho(0) = 0$,
- **translation invariant**, i.e. $\rho(X + c) = \rho(X) - c$, $c \in \mathbb{R}$,
- **monotone**, i.e. if $X \leq Y$, then $\rho(X) \geq \rho(Y)$,
- **sub-additive**, i.e. $\rho(X + Y) \leq \rho(X) + \rho(Y)$,
- **positively homogeneous**, i.e. $\rho(\lambda X) = \lambda \rho(X)$, $\lambda \geq 0$.

Coherent risk measures

Coherent risk measure

A risk measure $\rho : \mathcal{L}(\mathcal{F}_T) \to \mathbb{R}$ is called **coherent**, if it is

- **normalized**, i.e. $\rho(0) = 0$,
- **translation invariant**, i.e. $\rho(X + c) = \rho(X) - c$, $c \in \mathbb{R}$,
- **monotone**, i.e. if $X \leq Y$, then $\rho(X) \geq \rho(Y)$,
- **sub-additive**, i.e. $\rho(X + Y) \leq \rho(X) + \rho(Y)$,
- **positively homogeneous**, i.e. $\rho(\lambda X) = \lambda \rho(X)$, $\lambda \geq 0$.

Note: Value at Risk is **not** a coherent risk measure!

Coherent risk measures

Coherent risk measure

A risk measure $\rho : \mathcal{L}(\mathcal{F}_T) \to \mathbb{R}$ is called **coherent**, if it is

- **normalized**, i.e. $\rho(0) = 0$,
- **translation invariant**, i.e. $\rho(X + c) = \rho(X) - c$, $c \in \mathbb{R}$,
- **monotone**, i.e. if $X \leq Y$, then $\rho(X) \geq \rho(Y)$,
- **sub-additive**, i.e. $\rho(X + Y) \leq \rho(X) + \rho(Y)$,
- **positively homogeneous**, i.e. $\rho(\lambda X) = \lambda \rho(X)$, $\lambda \geq 0$.

Note: Value at Risk is **not** a coherent risk measure!

Risk-Neutral Price

Let $X \in L^2(\mathcal{F}_T)$ be a contingent claim. Any **risk-neutral price**

$$\rho(X) := \mathbb{E}^{\mathbb{Q}}[-X]$$

is a coherent risk measure.

Dynamic Risk Measures

Dynamic Risk Measure

A **dynamic risk measure** is a family $(\rho_t)_{t\in[0,T]}$ of mappings

$$\rho_t : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t).$$

Dynamic Risk Measures

Dynamic Risk Measure

A **dynamic risk measure** is a family $(\rho_t)_{t \in [0,T]}$ of mappings

$$\rho_t : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t).$$

Coherent Dynamic Risk Measure

A dynamic risk measure $(\rho_t)_{t \in [0,T]}$ is said to be **coherent**, if ρ_t satisfies the **axioms of coherence** for every $t \in [0, T]$.

Dynamic Risk Measures

Dynamic Risk Measure

A **dynamic risk measure** is a family $(\rho_t)_{t\in[0,T]}$ of mappings

$$\rho_t : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t).$$

Coherent Dynamic Risk Measure

A dynamic risk measure $(\rho_t)_{t\in[0,T]}$ is said to be **coherent**, if ρ_t satisfies the **axioms of coherence** for every $t \in [0, T]$.

Time-Consistent Dynamic Risk Measure

A dynamic risk measure $(\rho_t)_{t\in[0,T]}$ is said to be **time-consistent**, if

$$\rho_t(-\rho_s(X)) = \rho_t(X), \qquad t \leq s.$$

How can we construct (time-consistent) dynamic risk measures? We use our new tool: **Nonlinear Expectations**!

Construction of Dynamic Risk Measures

How can we construct (time-consistent) dynamic risk measures? We use our new tool: **Nonlinear Expectations**!

Dynamic Risk Measures via Nonlinear Expectations

Let $(\mathcal{E}_{s,t}[\cdot])_{0\leq s\leq t<\infty}$ be an $\mathcal{F}_t$-consistent nonlinear expectation. Define

$$\rho_t(X) \triangleq \mathcal{E}_{t,T}\left[-X\right].$$

Then $(\rho_t)_{t\in[0,T]}$ is a time-consistent dynamic risk measure.

How can we construct (time-consistent) dynamic risk measures? We use our new tool: **Nonlinear Expectations**!

Dynamic Risk Measures via Nonlinear Expectations

Let $(\mathcal{E}_{s,t}[\cdot])_{0 \leq s \leq t < \infty}$ be an $\mathcal{F}_t$-consistent nonlinear expectation. Define

$$\rho_t(X) \triangleq \mathcal{E}_{t,T}[-X].$$

Then $(\rho_t)_{t \in [0,T]}$ is a time-consistent dynamic risk measure.

In what follows, we will take a closer look at this construction in a concrete example: The **Black-Scholes price** of a contingent claim.

Option Pricing from a Different Perspective

Suppose we are in a **Black-Scholes market**, i.e.

$$dS_u = \mu S_u \, du + \sigma S_u \, dW_u, \qquad dB_u = 0, \qquad u \in [t, T].$$

Suppose we are in a **Black-Scholes market**, i.e.

$$dS_u = \mu S_u \, du + \sigma S_u \, dW_u, \qquad dB_u = 0, \qquad u \in [t, T].$$

We denote by Ψ_u the investor's wealth invested in the stock at time u. The investor's **total wealth** Y_u then follows

$$dY_u = \mu \Psi_u \, du + \sigma \Psi_u \, dW_u$$

Option Pricing from a Different Perspective

Suppose we are in a **Black-Scholes market**, i.e.

$$dS_u = \mu S_u \, du + \sigma S_u \, dW_u, \qquad dB_u = 0, \qquad u \in [t, T].$$

We denote by Ψ_u the investor's wealth invested in the stock at time u. The investor's **total wealth** Y_u then follows

$$dY_u = \mu \Psi_u \, du + \sigma \Psi_u \, dW_u = \theta Z_u \, du + Z_u \, dW_u,$$

where $\theta \triangleq \mu/\sigma$ and $Z_u \triangleq \sigma \Psi_u$.

Option Pricing from a Different Perspective

Suppose we are in a **Black-Scholes market**, i.e.

$$dS_u = \mu S_u \, du + \sigma S_u \, dW_u, \qquad dB_u = 0, \qquad u \in [t, T].$$

We denote by Ψ_u the investor's wealth invested in the stock at time u. The investor's **total wealth** Y_u then follows

$$dY_u = \mu \Psi_u \, du + \sigma \Psi_u \, dW_u = \theta Z_u \, du + Z_u \, dW_u,$$

where $\theta \triangleq \mu/\sigma$ and $Z_u \triangleq \sigma \Psi_u$.

Aim: Calculate the time-t **Black-Scholes price** of a claim $X \in L^2(\mathcal{F}_T)$. Use this to define a risk measure.

Option Pricing from a Different Perspective

In order to find the fair price of X, we need to **replicate** it.

In order to find the fair price of X, we need to **replicate** it. That is, we have to solve the **backward SDE**

$$dY_u = \theta Z_u \, du + Z_u \, dW_u, \qquad Y_T = X, \qquad u \in [t, T],$$

for the unknowns (Y_u, Z_u).

Option Pricing from a Different Perspective

In order to find the fair price of X, we need to **replicate** it. That is, we have to solve the **backward SDE**

$$dY_u = \theta Z_u\,du + Z_u\,dW_u, \qquad Y_T = X, \qquad u \in [t, T],$$

for the unknowns (Y_u, Z_u).

Arbitrage arguments dictate that Y_t must be the time-t price of X. Hence, we can define

$$\mathcal{E}_{t,T}[X] \triangleq Y_t$$

In order to find the fair price of X, we need to **replicate** it. That is, we have to solve the **backward SDE**

$$dY_u = \theta Z_u \, du + Z_u \, dW_u, \qquad Y_T = X, \qquad u \in [t, T],$$

for the unknowns (Y_u, Z_u).

Arbitrage arguments dictate that Y_t must be the time-t price of X. Hence, we can define

$$\mathcal{E}_{t,T}[X] \triangleq Y_t \qquad (= \mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}_t]).$$

(Don't worry: It can be shown that Y_t is indeed the usual Black-Scholes price.)

In order to find the fair price of X, we need to **replicate** it. That is, we have to solve the **backward SDE**

$$dY_u = \theta Z_u \, du + Z_u \, dW_u, \qquad Y_T = X, \qquad u \in [t, T],$$

for the unknowns (Y_u, Z_u).

Arbitrage arguments dictate that Y_t must be the time-t price of X. Hence, we can define

$$\mathcal{E}_{t,T}[X] \triangleq Y_t \qquad (= \mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}_t]).$$

(Don't worry: It can be shown that Y_t is indeed the usual Black-Scholes price.)

With this, we can define

$$\rho_t(X) \triangleq \mathcal{E}_{t,T}(-X),$$

and obtain a time-consistent **dynamic risk measure**.

Idea: Generalize this!

Idea: Generalize this!

1. Solve more general **backward SDEs**

$$dY_u = g(u, Y_u, Z_u)\, du + Z_u\, dW_u, \qquad Y_T = X, \qquad u \in [t, T],$$

for $(Y_u, Z_u)_{u \in [t,T]}$ to find Y_t.

Idea: Generalize this!

1. Solve more general **backward SDEs**

$$dY_u = g(u, Y_u, Z_u)\,du + Z_u\,dW_u, \qquad Y_T = X, \qquad u \in [t, T],$$

   for $(Y_u, Z_u)_{u\in[t,T]}$ to find Y_t.

2. Define a **nonlinear expectation** via

$$\mathcal{E}^g_{t,T}[X] \triangleq Y_t.$$

   $\mathcal{E}^g$ constructed like this is usually called a **g-expectation**.

Idea: Generalize this!

1. Solve more general **backward SDEs**

$$dY_u = g(u, Y_u, Z_u)\,du + Z_u\,dW_u, \qquad Y_T = X, \qquad u \in [t, T],$$

   for $(Y_u, Z_u)_{u\in[t,T]}$ to find Y_t.

2. Define a **nonlinear expectation** via

$$\mathcal{E}^g_{t,T}[X] \triangleq Y_t.$$

   $\mathcal{E}^g$ constructed like this is usually called a **g-expectation**.

3. Use the g-expectation to construct a (time-consistent) **dynamic risk measure** via

$$\rho_t(X) \triangleq \mathcal{E}_{t,T}[-X].$$

Idea: Generalize this!

1. Solve more general **backward SDEs**

$$dY_u = g(u, Y_u, Z_u)\,du + Z_u\,dW_u, \qquad Y_T = X, \qquad u \in [t, T],$$

   for $(Y_u, Z_u)_{u\in[t,T]}$ to find Y_t.

2. Define a **nonlinear expectation** via

$$\mathcal{E}^g_{t,T}[X] \triangleq Y_t.$$

   $\mathcal{E}^g$ constructed like this is usually called a **g-expectation**.

3. Use the g-expectation to construct a (time-consistent) **dynamic risk measure** via

$$\rho_t(X) \triangleq \mathcal{E}_{t,T}[-X].$$

It turns out that for **almost all** time-consistent coherent dynamic risk measures $(\rho_t)_{t\in[0,T]}$, we can find a function g, such that

$$\rho_t(X) = \mathcal{E}^g_{t,T}(-X).$$

Summary

We have seen that:

Summary

We have seen that:

1. Risk measures are an important concept to compute **capital requirements**.

Summary

We have seen that:

1. Risk measures are an important concept to compute **capital requirements**.
2. In order to judge the quality of a risk measure, we need properties like **coherence** and **time-consistency**.

Summary

We have seen that:

1. Risk measures are an important concept to compute **capital requirements**.
2. In order to judge the quality of a risk measure, we need properties like **coherence** and **time-consistency**.
3. Time-consistent dynamic risk measures can easily be constructed from **nonlinear expectations**.

Summary

We have seen that:

1. Risk measures are an important concept to compute **capital requirements**.
2. In order to judge the quality of a risk measure, we need properties like **coherence** and **time-consistency**.
3. Time-consistent dynamic risk measures can easily be constructed from **nonlinear expectations**.
4. A big class of nonlinear expectations is given by the class of **g-expectations**, defined through solutions of **backward SDEs**.

Summary

We have seen that:

1. Risk measures are an important concept to compute **capital requirements**.
2. In order to judge the quality of a risk measure, we need properties like **coherence** and **time-consistency**.
3. Time-consistent dynamic risk measures can easily be constructed from **nonlinear expectations**.
4. A big class of nonlinear expectations is given by the class of **g-expectations**, defined through solutions of **backward SDEs**.
5. **A big class** of time-consistent coherent dynamic risk measures can be **represented** by a g-expectation.

Summary

We have seen that:

1. Risk measures are an important concept to compute **capital requirements**.
2. In order to judge the quality of a risk measure, we need properties like **coherence** and **time-consistency**.
3. Time-consistent dynamic risk measures can easily be constructed from **nonlinear expectations**.
4. A big class of nonlinear expectations is given by the class of **g-expectations**, defined through solutions of **backward SDEs**.
5. **A big class** of time-consistent coherent dynamic risk measures can be **represented** by a g-expectation.

Hence, in this reading course, we will study

Summary

We have seen that:

1. Risk measures are an important concept to compute **capital requirements**.
2. In order to judge the quality of a risk measure, we need properties like **coherence** and **time-consistency**.
3. Time-consistent dynamic risk measures can easily be constructed from **nonlinear expectations**.
4. A big class of nonlinear expectations is given by the class of **g-expectations**, defined through solutions of **backward SDEs**.
5. **A big class** of time-consistent coherent dynamic risk measures can be **represented** by a g-expectation.

Hence, in this reading course, we will study

- **nonlinear expectations** and their properties,

Summary

We have seen that:

1. Risk measures are an important concept to compute **capital requirements**.
2. In order to judge the quality of a risk measure, we need properties like **coherence** and **time-consistency**.
3. Time-consistent dynamic risk measures can easily be constructed from **nonlinear expectations**.
4. A big class of nonlinear expectations is given by the class of **g-expectations**, defined through solutions of **backward SDEs**.
5. **A big class** of time-consistent coherent dynamic risk measures can be **represented** by a g-expectation.

Hence, in this reading course, we will study

- **nonlinear expectations** and their properties,
- **backward SDEs**, the related **g-expectations**, and some **g-stochastic calculus**,

Summary

We have seen that:

1. Risk measures are an important concept to compute **capital requirements**.
2. In order to judge the quality of a risk measure, we need properties like **coherence** and **time-consistency**.
3. Time-consistent dynamic risk measures can easily be constructed from **nonlinear expectations**.
4. A big class of nonlinear expectations is given by the class of **g-expectations**, defined through solutions of **backward SDEs**.
5. **A big class** of time-consistent coherent dynamic risk measures can be **represented** by a g-expectation.

Hence, in this reading course, we will study

- **nonlinear expectations** and their properties,
- **backward SDEs**, the related **g-expectations**, and some **g-stochastic calculus**,
- the **representation** of nonlinear expectations as g-expectations.

Thank you for your attention!