Option Pricing under the Threat of Crashes

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Literature: Hua and Wilmott (1996) and Mönnig (2012).

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- We make no assumption about the distribution of (τ, β). Instead, the investor takes a **worst-case perspective** towards the impact of the jump.
- The investor can **observe the price jump** and adjust her trading strategy afterwards.
- We denote the **set of all such jump scenarios** by $\mathcal{B}$.

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Trading strategies and wealth in the presence of jumps

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In this setting, given a jump scenario $\iota = (\tau, \beta)$, the wealth evolves as

$$\begin{aligned} dX_t &= (rX_t + \theta\sigma\pi_t)\, dt + \sigma\pi_t\, dW_t - dC_t, && t \in [0, \tau), \\ X_\tau &= X_{\tau-} - \beta\pi_{\tau-}, \\ dX_t &= (rX_t + \theta\sigma\pi_t^\iota)\, dt + \sigma\pi_t^\iota\, dW_t - dC_t^\iota, && t \in [\tau, T], \end{aligned}$$

where $\theta = (\alpha - r)\sigma$.

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Trading strategies are called **admissible**, if they are square-integrable and lead to non-negative wealth in all jump scenarios $\iota \in \mathcal{B}$.

The worst-case price

Our aim is to price an **option** with **payoff** $\xi(S_T)$ (square-integrable, non-negative).

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Well-definedness of the worst-case price

Denote by $\check{V}(t,s)$ the arbitrage-free price of the option $\xi(S_T)$ in the corresponding **jump-free market** at time t with stock price $S_t = s$ (Black-Scholes price). The worst-case price of the option is **well-defined** if and only if there exists a pre-jump strategy (π, C) such that

$$X_t - \sup_{\beta \in [\underline{\beta}, \overline{\beta}]} \left[\beta \pi_t + \check{V}\big(t, (1-\beta) S_{t-}\big)\right] \geq 0, \qquad \text{for all } t, \mathbb{P}\text{-a.s.}$$

BSDE characterization of the worst-case price

We denote the time-t worst-case price of the option $\xi(S_T)$ with $S_t = s$ by $V(t, s)$.

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Assume that the worst-case price $V(t,s)$ is **well-defined**. Assume further that the fair price $\check{V}(t,s)$ in the corresponding jump-free market is **jointly continuous** in (t,s). Then the worst-case price $V(t,s)$ of $\xi(S_T)$ **exists** and is given as the **smallest solution** $(X_t, \sigma\pi_t, C_t)$ of

$$dX_u = (rX_u + \theta\sigma\pi_u)\, du + \sigma\pi_u\, dW_u - dC_t, \qquad X_T = \xi(S_T),$$

under the **constraint**

$$X_u - \sup_{\beta\in[\underline{\beta},\overline{\beta}]} \left[\beta\pi_u + \check{V}\big(u,(1-\beta)S_{u-}\big)\right] \geq 0 \qquad \text{for all } u, \mathbb{P}\text{-a.s.}$$

Literature: Peng (1999).

Idea of the proof

Step 1: Since the worst-case price is well-defined, there exists at least one solution of the constrained BSDE. Fix an arbitrary solution $(\hat{X}, \sigma\hat{\pi}, \hat{C})$.

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$$\Phi(u, x, \sigma\pi) := \left(x - \sup_{\beta \in [\underline{\beta}, \overline{\beta}]} \left[\beta\pi + \check{V}\big(u, (1-\beta)S_{u-}\big)\right]\right)^{-}.$$

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Step 3: Consider the family of unconstrained BSDEs:

$$X_t^i = \xi(S_T) - \int_t^T rX_u^i + \theta\sigma\pi_u^i - i\Phi(u, X_u^i, \sigma\pi_u^i)\, du - \int_t^T \sigma\pi_u^i\, dW_u.$$

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Step 4: X^i converges monotonically to a solution X of the constrained BSDE (Peng). Moreover, $X \leq \hat{X}$ for any other solution $\hat{X}$. Hence X is the smallest solution of the constrained BSDE!

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The terminal condition

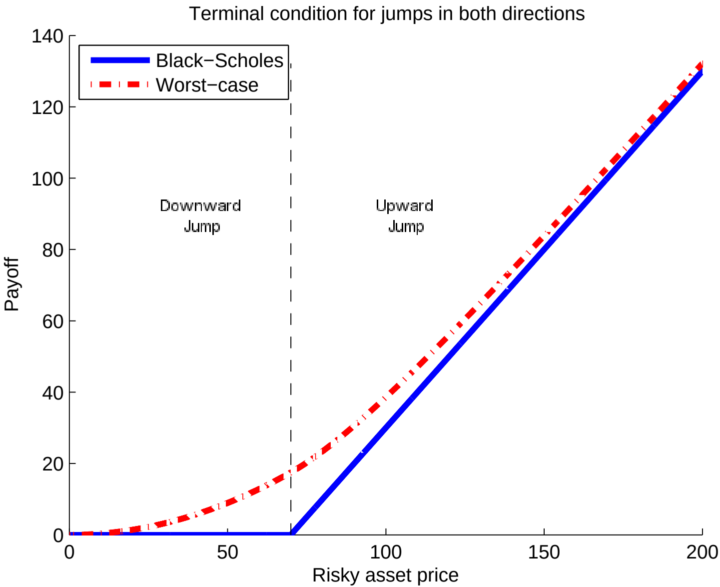
Define

$$T^*(s) = \lim \sup_{t \to T, s' \to s} V(t, s'), \qquad T_*(s) = \lim \inf_{t \to T, s' \to s} V(t, s').$$

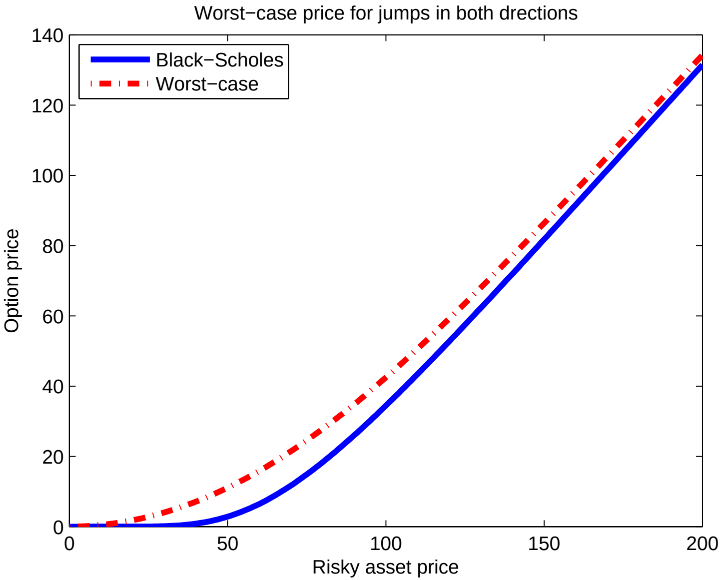
If $\xi(s)$ is Lipschitz-continuous, then $T(s) := T^*(s) = T_*(s)$ and T is the unique viscosity solution of

$$0 = \min\left\{ T(s) - \xi(s),\, T(s) - \sup_{\beta \in [\underline{\beta}, \overline{\beta}]} [\beta s T_s(s) + \xi((1-\beta)s)] \right\}.$$

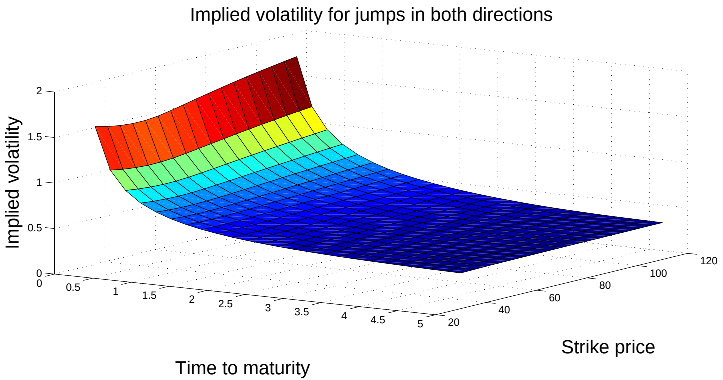
Numerical results: Terminal condition



Numerical results: Worst-case price



Numerical results: Implied volatility surface



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Thank you for your attention!