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Coorbit spaces and Banach frames on homogeneous spaces with applications to the sphere

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This paper is concerned with the construction of generalized Banach frames on homogeneous spaces. The major tool is a unitary group representation which is square integrable modulo a certain subgroup. By means of this representation, generalized coorbit spaces can be defined. Moreover, we can construct a specific reproducing kernel which, after a judicious discretization, gives rise to atomic decompositions for these coorbit spaces. Furthermore, we show that under certain additional conditions our discretization method generates Banach frames. We also discuss nonlinear approximation schemes based on the atomic decomposition. As a classical example, we apply our construction to the problem of analyzing and approximating functions on the spheres.

Keywords: square integrable group representations, time-frequency analysis, atomic decompositions, frames, homogeneous spaces, coorbit spaces, modulation spaces, nonlinear approximation, spheres

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1. Introduction

A classical problem in applied mathematics is to analyze and to process a given set of signals. Usually, the first step is to decompose the signal into certain building blocks. A widespread strategy is to use Fourier transform, i.e., to analyze the signal with respect to its components corresponding to different frequencies. Although very successful in many applications, Fourier analysis has the serious disadvantage that the basis functions

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are not local so that small changes in the signal influence the whole Fourier spectrum. Therefore many attempts have been made to localize the Fourier transform in some natural way. In 1946, Gabor [20] introduced a time-frequency analysis which is often called the *short-time Fourier transform*. The idea is to use a window function g in order to localize the Fourier analysis. In the meantime, the short-time Fourier transform has indeed been established as a powerful tool in signal analysis. Another way to obtain some kind of local analysis would be to use the *wavelet transform*. Then the modulation term in the short-time Fourier transform is replaced by a dilation procedure, and it is possible to work with very localized basis functions. Starting with the pioneering work of Grossmann and Morlet [26], wavelet analysis has become a very important field in applied mathematics with many successful applications in image/signal analysis/compression, numerical analysis, geophysics and in many other fields. Although they may behave quite different in applications, there exists a common thread between Gabor and wavelet transform. Both can be derived from square integrable representations of a certain group, see, e.g., [27] and section 2 for details. Both transforms have their advantages and drawbacks, so that the decision which method to use depends on the specific application. For further information and a general overview on both transforms we refer to the excellent textbooks which have appeared quite recently [9,24,28,30–32,37].

In any case, when it comes to practical applications, only a *discrete* set of coefficients can be handled. It is therefore necessary to dicretize both transforms to obtain some stable basis for the function space under consideration. However, constructing some stable basis may be asking to much, nevertheless, it is usually possible to obtain at least a frame. In general, given a Hilbert space H, a system $\{h_m\}_{m\in\mathbb{Z}}$ is called a *frame* if there exist constants A and B, $0 < A \leqslant B < \infty$ such that

$$A\|F\|_H^2 \leqslant \sum_{m\in\mathbb{Z}} \left|\langle F, h_m\rangle\right|^2 \leqslant B\|F\|_H^2. \tag{1.1}$$

This setting can also be generalized to Banach spaces, see, e.g., [16,17,23] and section 4 for details. In our case, the frames are obtained by discretizing the underlying group representation in some clever way. A very general machinery for frame constructions has been developed in the pioneering work of Feichtinger and Gröchenig [15–18]. Once these frames are constructed, they usually also give rise to frames in certain smoothness spaces. These smoothness spaces are again defined by the underlying square integrable group representation, i.e., one collects all functions for which the associated (Gabor or wavelet) transform is contained in some (weighted) L_p-space on the group. These function spaces are usually called *coorbit spaces* and will be introduced more accurately in section 3. For the Gabor transform, the coorbit spaces are nothing else but the *modulation spaces*, whereas for the wavelet transform one obtains the *Besov spaces*. We refer to [10,11,15–18,24,32,36] for the definitions and the main properties of modulation and Besov spaces. At this point, the strong analytical properties of wavelets come into play. Indeed, it can be shown that moreover stable wavelet bases for a huge scale of Besov spaces involving those related with L_p-spaces for $p < 1$ can be established, see again [10,11,32] for details. These relationships have some very important consequences. In fact, it can be shown that the order of convergence of nonlinear approximation schemes such as best N-term approximation or adaptive wavelet Galerkin methods depends on the regularity of the approximated object in a very specific Besov scale, see, e.g., [6,8,10,11] for details. For the case of the Gabor transform, quite recently some results have been derived by Gröchenig and Samarah [25]. They have shown that the approximation order of nonlinear schemes based on local Fourier bases is determined by the regularity in some specific scale of modulation spaces. Nevertheless, these results are naturally weaker when compared with those for the wavelet case.

In any case, when it comes to practical applications, it is clearly desirable to generalize the theories developed so far to bounded domains and manifolds. This problem has been intensively studied in the last few years. Because of the strong analytical properties of wavelets, one might feel temptered to start with the wavelet transform. However, usually the dilation procedure involved in the wavelet transform does not fit together very well with the boundedness of the domain. Nevertheless, quite recently an almost complete solution to this problem has been given by Antoine and Vandergheynst [4,5]. Their approach makes heavy use of group theory and can also be employed to construct suitable wavelet frames [3]. However, the whole machinery is very complicated. It is fun for the specialists but terrible for the average consumer. In this context, Gabor analysis seems to have a serious advantage. It seems that the generalization of the Gabor transform to manifolds is much simpler than for the wavelet transform. Indeed, quite recently, a first approach for the case of the sphere in $\mathbb{R}^d$ has been presented by Torresani [35].

In summary, the current state of the art suggests the following questions:

- Is it possible to construct a generalized Gabor transform on manifolds and to properly define the associated coorbit spaces?
- Is it possible to generalize the machinery developed by Feichtinger and Gröchenig to this case and to obtain atomic decompositions and generalized Gabor frames in these coorbit spaces?
- What are the smoothness spaces which determine the order of convergence of the associated best N-term approximation schemes?
- Is it possible to come from abstract general nonsense to concrete applications, e.g., by combining these investigations with Torresani's results, in order to obtain Gabor frames on spheres?

In order to execute this program, we proceed in the following way. We start by discussing the group theoretical background in section 2. Given our manifold $\mathcal{N}$, the first step is clearly to find a locally compact group $\mathcal{G}$ which admits a unitary representation in the Hilbert space $L_2(\mathcal{N})$. To be on safe side, this representation has to be irreducible and square integrable. The first property is usually relatively easy to realize whereas the second one often causes trouble because the group is too 'large'. To obtain a 'smaller' group, *one* natural way would be to extract a closed subgroup $\mathcal{G}_{\mathcal{F}}$ and to restrict the representation to the quotient space $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$. However, since $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$ has no longer a group structure, one has to ensure that nevertheless all the nice properties of square integrable representations can be saved. Once these relationships are clarified, we are able to define associated coorbit spaces in section 3. Loosely speaking, these generalized coorbit spaces consist of all functions for which the associated Gabor transform is contained in some L_p-space on the quotient manifold $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$. According to our program, the next step is to construct atomic decompositions and Banach frames for these coorbit spaces in section 4. To this end, we investigate to what extent the general approach of Feichtinger and Gröchenig can be adapted to our setting. The first step is always to introduce approximation operators. These operators are usually defined by means of a convolution with the Gabor transform of the analyzing function itself. Since a group structure doesn't longer exist in our setting, a convolution is no longer well-defined. We therefore suggest to replace this convolution by a suitable defined integral transform involving a specific kernel defined by means of the analyzing function, see section 4.2 for details. The next step is to discretize these approximation operators to obtain the desired atomic decomposition and the Banach frames. In section 4.3, we show that under very natural assumptions the required norm equivalences can be established for both cases. As outlined above, we also intend to analyze nonlinear approximation schemes based on the new atomic decompositions. In section 5, we show that a part of the results of Gröchenig and Samarah [25] on Banach frames carry over to our case without any serious difficulty. Finally, in section 6, we discuss some applications of our theory, i.e., we treat the problem of analyzing functions on spheres. Our approach is based on the fundamental investigations of Torresani [35]. We show that in the setting of [35] all our assumptions are satisfied so that our theory yields generalized coorbit spaces on spheres and also provides us with suitable atomic decompositions and Banach frames for these spaces.

Remark 1.1. (i) We want to emphasize that we do not claim to rediscover the whole world of square integrable group representations. It is clear that some of the building blocks used in this paper have already been established before, at least partially. However, we intend to establish the relationships between all these building blocks and to show that they fit together quite nicely.

(ii) The basic idea of this paper has been developed while listening to a talk of Gröchenig on "New Results in Time-Frequency Analysis".

2. Group theoretical background

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{G}$ be a separable Lie group with (right) Haar measure ν. A *continuous representation* of $\mathcal{G}$ in $\mathcal{H}$ is defined as a mapping

$$U : \mathcal{G} \to \mathcal{L}(\mathcal{H}) \tag{2.1}$$

of $\mathcal{G}$ into the space $\mathcal{L}(\mathcal{H})$ of unitary operators on $\mathcal{H}$, such that $U(gg') = U(g)U(g')$ for all $g, g' \in \mathcal{G}$, $U(e) = Id$ and for any $\phi, \psi \in \mathcal{H}$, the function $g \in \mathcal{G} \to \langle \phi, U(g)\psi \rangle_{\mathcal{H}}$ is continuous. The representation U is said to be *square-integrable* if it is irreducible and there exists a nonzero $\psi \in \mathcal{H}$ such that

$$\int_{\mathcal{G}} \left|\langle \psi, U(g)\psi\rangle_{\mathcal{H}}\right|^2 \mathrm{d}\nu(g) < \infty. \tag{2.2}$$

Such a function ψ is called *admissible*. In the sequel, we shall always be concerned with the case that the Hilbert space $\mathcal{H}$ is given as some L_2-space on a manifold $\mathcal{N}$, i.e., $\mathcal{H} = L_2(\mathcal{N})$. As an example, let us consider the *reduced Weyl–Heisenberg group* $\mathcal{G}_{\mathrm{WH}}^{\mathrm{red}} \cong \mathbb{R}^2 \times S^1$, generated by time and frequency translations on the real line. The group operation is explicitly given by

$$(p, q, \phi)\big(p', q', \phi'\big) = \big(p + p', q + q', \phi + \phi' + p'q\big).$$

The Weyl–Heisenberg group $\mathcal{G}_{\mathrm{WH}}^{\mathrm{red}}$ admits unitary irreducible representations on $L_2(\mathbb{R})$ which act as follows:

$$U(p, q, \phi)f(x) = \exp\big(\mathrm{i}\big(\lambda\phi + q(x - \lambda p)\big)\big)f(x - \lambda p).$$

Because S^1 is compact it is easy to check that U is square integrable and any nonzero $\psi \in \mathcal{H}$ is admissible. This specific representation can be viewed as the basic building block for the classical Gabor transform, see, e.g., [24] for details. However, there are cases in which square-integrable representations are not available. A simple example is the full Weyl–Heisenberg group $\mathcal{G}_{\mathrm{WH}} \cong \mathbb{R}^2 \times \mathbb{R}$. Nevertheless, its coefficients $\langle f, U(q, p, 0)\psi\rangle$ form a square integrable function of $(q, p) \in \mathbb{R}^2$. This example suggests a general strategy. Indeed, the cases where no square-integrable representations are available can very often be handled by restricting U to a convenient quotient $\mathcal{G}/\mathcal{P}$, where $\mathcal{P}$ is a closed subgroup of $\mathcal{G}$. Unless otherwise stated, we shall always consider right coset spaces, i.e.,

$$g_1 \sim g_2 \quad \text{if and only if} \quad g_1 = h \circ g_2 \text{ for some } h \in \mathcal{P}. \tag{2.3}$$

Because U is not directly defined on $\mathcal{G}/\mathcal{P}$, it is necessary to embed $\mathcal{G}/\mathcal{P}$ in $\mathcal{G}$. This can be realized by using the canonical fiber bundle structure of $\mathcal{G}$ with projection $\Pi : \mathcal{G} \to \mathcal{G}/\mathcal{P}$. Let $\sigma : \mathcal{G}/\mathcal{P} \to \mathcal{G}$ be a Borel section of this fiber bundle, i.e. $\Pi \circ \sigma(h) = h$ for all $h \in \mathcal{G}/\mathcal{P}$. We introduce $U \circ \sigma$ and a quasi-invariant measure, necessarily unique up to equivalence, μ on $\mathcal{G}/\mathcal{P}$ by

$$\int_{\mathcal{G}/\mathcal{P}} \left(\int_{\mathcal{P}} f(h \circ g)\,\mathrm{d}\zeta(h)\right) \mathrm{d}\mu\big([g]\big) = \int_{\mathcal{G}} f(g)\,\mathrm{d}\nu(g) \quad \text{for all } f \in C_0(\mathcal{G}), \tag{2.4}$$

where ζ denotes the (right) Haar measure on $\mathcal{P}$, see [2,34,35] for details. An attractive notation of square integrability on a homogeneous space appears in [2]. An irreducible representation U is *square integrable* mod $(\mathcal{P}, \sigma)$, if there exists a nonzero function $\psi \in L_2(\mathcal{N})$, called *admissible* (with respect to σ), such that

$$\int_{\mathcal{G}/\mathcal{P}} \left|\big\langle f, U\big(\sigma(h)^{-1}\big)\psi\big\rangle_{L_2(\mathcal{N})}\right|^2 \mathrm{d}\mu(h) < \infty \quad \text{for all } f \in L_2(\mathcal{N}),$$

i.e., the operator V_ψ given by

$$V_\psi f(h) := \big\langle f, U\big(\sigma(h)^{-1}\big)\psi\big\rangle_{L_2(\mathcal{N})} \tag{2.5}$$

maps $L_2(\mathcal{N})$ into $L_2(\mathcal{G}/\mathcal{P})$. The admissibility condition can be rewritten as

$$0 < \int_{\mathcal{G}/\mathcal{P}} \big|\big\langle f, U\big(\sigma(h)^{-1}\big)\psi\big\rangle_{L_2(\mathcal{N})}\big|^2 \, \mathrm{d}\mu(h) = \langle f, A_\sigma f\rangle < \infty \quad \text{for all } f \in L_2(\mathcal{N}),$$

where A_σ is a positive, bounded, and invertible operator. If $A_\sigma = \lambda\mathcal{I}$ for some $\lambda > 0$, then U is called *strictly square integrable* mod $(\mathcal{P}, \sigma)$ and ψ *strictly admissible*. Moreover, we say that (ψ, σ) is a *strictly admissible pair* [35]. Note that in case of a strictly admissible function ψ, the set $\{U(\sigma(h)^{-1})\psi\colon h \in \mathcal{G}/\mathcal{P}\}$ is called a (continuous) *tight frame* in [2]. In order to keep the notation simple and since we have mainly applications on the sphere in mind, we focus our attention to strictly square integrable representations, where we normalize ψ so that $\lambda = 1$. Then $V_\psi : L_2(\mathcal{N}) \to L_2(\mathcal{G}/\mathcal{P})$ in (2.5) is an isometry. A particular case of this construction is that considered independently by Gilmore [21,22] and Perelomov [33], namely where the subgroup $\mathcal{P}$ is in the subgroup $\mathcal{P}_\psi$ of $\mathcal{G}$ that leaves ψ invariant up to a phase, i.e., $U(h)\psi = \mathrm{e}^{\mathrm{i}w(h)}\psi$ $(h \in \mathcal{P}_\psi)$, where w is a real-valued function on $\mathcal{P}_\psi$. Then the admissibility condition is independent of the choice of the section σ and $A_\sigma = \mathcal{I}$. Unfortunately, the Euclidian group related to our construction on the sphere does not fit into this setting.

To exploit this concept, the first step is clearly to define an appropriate subgroup of $\mathcal{G}$. We begin with the *adjoint mapping* of $\mathcal{G}$ acting on itself by inner automorphism, i.e., $ad(h)g := hgh^{-1}$, where $g, h \in \mathcal{G}$. This action induces a corresponding action $Ad(h)$ on the Lie algebra $\mathcal{T}_e\mathcal{G}$ of $\mathcal{G}$, $Ad(h)X = hXh^{-1}$ with $X \in \mathcal{T}_e\mathcal{G}$. Finally, the *coadjoint* $Ad(h)^*$ on the dual Lie algebra $\mathcal{T}_e^*\mathcal{G}$ is defined by

$$\big\langle X, Ad(h)^* F\big\rangle := \big\langle Ad(h)X, F\big\rangle, \quad \text{for } F \in \mathcal{T}_e^*\mathcal{G}.$$

For $\mathcal{F} \in \mathcal{T}_e^*\mathcal{G}$, let

$$\mathcal{G}_\mathcal{F} := \big\{g \in \mathcal{G}\colon Ad(g)^*\mathcal{F} = \mathcal{F}\big\} \tag{2.6}$$

denote the stability subgroup of $\mathcal{F}$. Whenever the coadjoint orbit $\mathcal{O}_\mathcal{F} \cong \mathcal{G}/\mathcal{G}_\mathcal{F}$ can be associated with the representation under consideration, the quotient space $\mathcal{G}/\mathcal{G}_\mathcal{F}$ is a natural candidate to perform the previous construction.

Assume now that (ψ, σ) is a strictly admissible pair for our setting. Then the following facts are well-known [2]:

- The set $S_\sigma := \big\{U\big(\sigma(h)^{-1}\big)\psi\colon h \in \mathcal{G}/\mathcal{P}\big\}$ is total in $L_2(\mathcal{N})$, i.e., $(S_\sigma)^\perp = \{0\}$.
- The map V_ψ is an isometry from $L_2(\mathcal{N})$ onto the reproducing kernel Hilbert space

$$\mathcal{M}_2 := \big\{F \in L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})\colon \big\langle F(\cdot), R(h, \cdot)\big\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})} = F(h)\big\} \tag{2.7}$$

with reproducing kernel

$$R(h,l) = R_\psi(h,l) := \big\langle U\big(\sigma(h)^{-1}\big)\psi, U\big(\sigma(l)^{-1}\big)\psi\big\rangle_{L_2(\mathcal{N})} \tag{2.8}$$

$$= \big\langle \psi, U\big(\sigma(h)\sigma(l)^{-1}\big)\psi\big\rangle_{L_2(\mathcal{N})}$$

$$= V_\psi\big(U\big(\sigma(h)^{-1}\big)\psi\big)(l). \tag{2.9}$$

In other words, the spaces $L_2(\mathcal{N})$ and $\mathcal{M}_2$ are isometrically isomorphic. In particular, $\|f\|_{L_2(\mathcal{N})} = \|V_\psi f\|_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})}$.

Note that $R(h,l) = \overline{R(l,h)}$. Further, we see by (2.9) that $R(h,\cdot) \in L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})$ for any fixed $h \in \mathcal{G}/\mathcal{G}_\mathcal{F}$ and by applying Schwarz's inequality in (2.8) that $R \in L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F} \times \mathcal{G}/\mathcal{G}_\mathcal{F})$.

- The map V_ψ can be inverted on its image by its adjoint V_ψ^*, which is obviously given by

$$V_\psi^* F(s) := \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(h) U\big(\sigma(h)^{-1}\big)\psi(s)\,\mathrm{d}\mu(h).$$

This provides us with the reconstruction formula

$$f = V_\psi^* V_\psi f = \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big\langle f, U\big(\sigma(h)^{-1}\big)\psi\big\rangle_{L_2(\mathcal{N})} U\big(\sigma(h)^{-1}\big)\psi\,\mathrm{d}\mu(h) \tag{2.10}$$

for $f \in L_2(\mathcal{N})$.

3. Coorbit spaces on homogeneous spaces

We want to modify the concept of coorbit spaces [18] to functions defined on manifolds. In order to keep comparisons as simple as possible, we adapt the notations given in [14–18]. Furthermore, to keep the technical difficulties at a reasonable level, we only consider the 'simplest' case, e.g., the weight function w involved in the usual definition of coorbit spaces is assumed to be $w \equiv 1$. The general case will be studied in a forthcoming paper.

Let U be a strictly square integrable representation of $\mathcal{G}$ mod $(\mathcal{G}_\mathcal{F}, \sigma)$ with a strictly admissible function ψ. In order to handle other spaces than Hilbert spaces it is necessary to require further conditions. For the kernel R in (2.8), we shall need the basic assumption that

$$\int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|R(h,l)\big|\,\mathrm{d}\mu(l) \leqslant C_\psi \tag{3.1}$$

with a constant $C_\psi < \infty$ independent of $h \in \mathcal{G}/\mathcal{G}_\mathcal{F}$. This requirement replaces the usual integrability condition in the group case. In our setting, the general problem occurs that a group structure no longer exists and therefore we need a substitute for the usual convolution operation. It seems to us that a powerful approach is to use the generalized Young inequality, see, e.g., [19, p. 185, theorem 6.18]. However, the application of this inequality requires exactly integrability conditions of the form (3.1).

The first problem is to provide a suitable large set that may serve as a reservoir of selection for the objects of our coorbit spaces. By H_1' we denote the space of all continuous linear functionals on

$$H_1 := \{f \in L_2(\mathcal{N}):\ V_\psi f \in L_1(\mathcal{G}/\mathcal{G}_\mathcal{F})\}.$$

As usual, the norm $\|\cdot\|_{H_1}$ on H_1 is defined as

$$\|f\|_{H_1} := \|V_\psi f\|_{L_1(\mathcal{G}/\mathcal{G}_\mathcal{F})}.$$

For $f \in H_1$, we have by Schwarz's inequality and since $R \in L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F} \times \mathcal{G}/\mathcal{G}_\mathcal{F})$ that

$$\begin{aligned}\|V_\psi f\|^2_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})} &= \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \left|\left\langle f, U\left(\sigma(h)^{-1}\right)\psi\right\rangle_{L_2(\mathcal{N})}\right| \left|V_\psi f(h)\right| \mathrm{d}\mu(h) \\ &\leqslant \|f\|_{L_2(\mathcal{N})} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \|\psi\|_{L_2(\mathcal{N})} \left|V_\psi f(h)\right| \mathrm{d}\mu(h) \\ &\leqslant \|f\|_{L_2(\mathcal{N})} C \|V_\psi f\|_{L_1(\mathcal{G}/\mathcal{G}_\mathcal{F})}\end{aligned}$$

which implies the following continuous embeddings

$$H_1 \hookrightarrow L_2(\mathcal{N}) \hookrightarrow H_1'.$$

In this paper C always denotes a generic constant which is independent of all the other parameters under consideration, but whose concrete value may be different in each particular estimate. Further, we note by (3.1) that $U(\sigma(h)^{-1})\psi \in H_1$ for all $h \in \mathcal{G}/\mathcal{G}_\mathcal{F}$. Consequently, the following generalization of the operator V_ψ in (2.5) on H_1' is well defined:

$$V_\psi f(h) := \left\langle f, U\left(\sigma(h)^{-1}\right)\psi\right\rangle, \tag{3.2}$$

where $f \in H_1'$. For any $f \in H_1'$, we obtain by (3.1) that

$$\begin{aligned}\|V_\psi f\|_{L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})} &= \left\|\left\langle f, U\left(\sigma(h)^{-1}\right)\psi\right\rangle\right\|_{L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})} \\ &\leqslant \|f\|_{H_1'} \operatorname*{ess\,sup}_{h\in\mathcal{G}/\mathcal{G}_\mathcal{F}} \left\|U\left(\sigma(h)^{-1}\right)\psi\right\|_{H_1} \\ &= \|f\|_{H_1'} \operatorname*{ess\,sup}_{h\in\mathcal{G}/\mathcal{G}_\mathcal{F}} \|R\|_{L_1(\mathcal{G}/\mathcal{G}_\mathcal{F})} \\ &\leqslant C_\psi \|f\|_{H_1'}.\end{aligned} \tag{3.3}$$

Thus, $V_\psi : H_1' \to L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})$. For $F \in L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})$ and $g \in H_1$, we have further that

$$\begin{aligned}\langle F, V_\psi g\rangle &= \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\overline{V_\psi g(l)}\,\mathrm{d}\mu(l) \\ &= \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\overline{\left\langle g, U\left(\sigma(l)^{-1}\right)\psi\right\rangle}\,\mathrm{d}\mu(l) \\ &= \left\langle \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l) U\left(\sigma(l)^{-1}\right)\psi\,\mathrm{d}\mu(l), g\right\rangle.\end{aligned}$$

We define the operator $\widetilde{V}_\psi : L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F}) \to H_1'$ by

$$\widetilde{V}_\psi F := \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l) U\big(\sigma(l)^{-1}\big)\psi \,\mathrm{d}\mu(l),$$

where the integral is considered in the weak sense. Then we obtain for $F \in L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})$ that

$$\begin{aligned} V_\psi \widetilde{V}_\psi F &= \left\langle \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l) U\big(\sigma(l)^{-1}\big)\psi \,\mathrm{d}\mu(l),\, U\big(\sigma(h)^{-1}\big)\psi \right\rangle \\ &= \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l) \big\langle U\big(\sigma(l)^{-1}\big)\psi,\, U\big(\sigma(h)^{-1}\big)\psi \big\rangle \,\mathrm{d}\mu(l) \\ &= \big\langle F, R(h,\cdot)\big\rangle. \end{aligned} \tag{3.4}$$

Similar to the usual coorbit spaces we define

$$M_p := \big\{ f \in H_1' \colon V_\psi f \in L_p(\mathcal{G}/\mathcal{G}_\mathcal{F}) \big\}, \tag{3.5}$$

with $1 \leqslant p \leqslant \infty$ and norm

$$\|f\|_{M_p} := \|V_\psi f\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}.$$

It is straightforward to check that $\|\cdot\|_{M_p}$ defines a seminorm. The property that $\|f\|_{M_p} = 0$, i.e., $V_\psi f = 0$, implies $f = 0$ follows similarly as in [15] by proving that $\{U\big(\sigma(h)^{-1}\big)\psi \colon h \in \mathcal{G}/\mathcal{G}_\mathcal{F}\}$ is a dense subset of H_1.

The next natural question is to which extent the spaces M_p are independent of the choice of the analyzing function. In the following lemma, we classify analyzing functions which give rise to the same coorbit spaces.

Lemma 3.1. Let $\psi, \eta \in H_1$ be two analyzing functions such that the corresponding kernels R_ψ and R_η satisfy (3.1). Further, let

$$\int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|\big\langle U\big(\sigma(h)^{-1}\big)\eta,\, U\big(\sigma(l)^{-1}\big)\psi\big\rangle\big| \,\mathrm{d}\mu(l) \leqslant C_{\eta\psi},$$

$$\int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|\big\langle U\big(\sigma(h)^{-1}\big)\eta,\, U\big(\sigma(l)^{-1}\big)\psi\big\rangle\big| \,\mathrm{d}\mu(h) \leqslant C_{\eta\psi}$$

with a constant $C_{\eta\psi}$ independent of h and l, respectively. Then the norms

$$\|f\|_{M_p,\psi} := \|V_\psi f\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}, \qquad \|f\|_{M_p,\eta} := \|V_\eta f\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}$$

are equivalent.

Proof. By using the definitions of V_ψ and $\widetilde{V}_\eta$, we obtain

$$\begin{aligned} V_\psi\big(\widetilde{V}_\eta(F)\big) &= \big\langle \widetilde{V}_\eta(F),\, U\big(\sigma(h)^{-1}\big)\psi \big\rangle \\ &= \left\langle \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l) U\big(\sigma(l)^{-1}\big)\eta \,\mathrm{d}\mu(l),\, U\big(\sigma(h)^{-1}\big)\psi \right\rangle \end{aligned}$$

$$= \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} \big\langle F(l)U\big(\sigma(l)^{-1}\big)\eta, U\big(\sigma(h)^{-1}\big)\psi\big\rangle \,\mathrm{d}\mu(l)$$
$$= \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} F(l)\big\langle U\big(\sigma(l)^{-1}\big)\eta, U\big(\sigma(h)^{-1}\big)\psi\big\rangle \,\mathrm{d}\mu(l)$$

and further by applying the generalized Young inequality

$$\big\| V_\psi\big(\widetilde{V}_\eta(F)\big)\big\|_{L_p(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} \leqslant C_{\eta\psi}\|F\|_{L_p(\mathcal{G}/\mathcal{G}_{\mathcal{F}})}.$$

Employing this inequality with $F = V_\eta f$ we get

$$\begin{aligned}\|f\|_{M_p,\psi} = \|V_\psi f\|_{L_p(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} &= \big\| V_\psi\big(\widetilde{V}_\eta V_\eta f\big)\big\|_{L_p(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} \\ &= \big\| \big(V_\psi \widetilde{V}_\eta\big) V_\eta f\big\|_{L_p(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} \\ &\leqslant C_{\eta\psi}\|V_\eta f\|_{L_p(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} \leqslant C_{\eta\psi}\|f\|_{M_p,\eta}.\end{aligned}$$

By interchanging the roles of ψ and η, the assertion follows. □

The basic step for the investigations outlined below is a correspondence principle between these coorbit spaces and certain subspaces on the quotient group $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$ which are defined by means of the reproducing kernel R. To this end, we consider the subspaces

$$\mathcal{M}_p := \big\{F \in L_p(\mathcal{G}/\mathcal{G}_{\mathcal{F}}):\ \big\langle F, R(h,\cdot)\big\rangle = F\big\} \tag{3.6}$$

of $L_p(\mathcal{G}/\mathcal{G}_{\mathcal{F}})$ with $1 \leqslant p \leqslant \infty$. Then the desired correspondence principle can be formulated as follows:

Theorem 3.1. Let U be a strictly square integrable representation of $\mathcal{G}$ mod $(\mathcal{G}_{\mathcal{F}}, \sigma)$ and ψ a strictly admissible function. Let V_ψ be defined by (3.2) and let R in (2.8) fulfill (3.1).

(i) For every $f \in M_p$, the following equation is satisfied

$$\big\langle V_\psi f, R(h,\cdot)\big\rangle = V_\psi f,$$

i.e., $V_\psi f \in \mathcal{M}_p$.

(ii) For every $F \in \mathcal{M}_p$, $1 \leqslant p \leqslant \infty$, there exists a uniquely determined functional $f \in M_p$ such that $F = V_\psi f$.

Consequently, the spaces M_p and $\mathcal{M}_p$, $1 \leqslant p \leqslant \infty$, are isometrically isomorph.

Proof. (i) Since $U\big(\sigma(h)^{-1}\big)\psi \in L_2(\mathcal{N})$ we have by (2.10) that

$$\begin{aligned}V_\psi f(h) &= \big\langle f, U\big(\sigma(h)^{-1}\big)\psi\big\rangle \\ &= \Big\langle f, \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} R(h,l)U\big(\sigma(l)^{-1}\big)\psi \,\mathrm{d}\mu(l)\Big\rangle \\ &= \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} \overline{R(h,l)}\big\langle f, U\big(\sigma(l)^{-1}\big)\psi\big\rangle \,\mathrm{d}\mu(l) \\ &= \big\langle V_\psi f, R(h,\cdot)\big\rangle.\end{aligned}$$

(ii) For $F \in \mathcal{M}_p$, $1 \leqslant p \leqslant \infty$, we obtain

$$\|F\|_{L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})} = \left\| \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\overline{R(h,l)}\,\mathrm{d}\mu(l) \right\|_{L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})}$$
$$= \operatorname{ess}\sup_{h\in\mathcal{G}/\mathcal{G}_\mathcal{F}} \left| \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\overline{R(h,l)}\,\mathrm{d}\mu(l) \right|,$$

and further, by applying Hölder's inequality with $1/p + 1/q = 1$, the fact that $R \in L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F} \times \mathcal{G}/\mathcal{G}_\mathcal{F})$ and (3.1),

$$\left| \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\overline{R(h,l)}\,\mathrm{d}\mu(l) \right|$$
$$\leqslant \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \left|F(l)\right| \left|R(h,l)\right|^{1/p+1/q}\,\mathrm{d}\mu(l)$$
$$\leqslant \left(\int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \left|F(l)\right|^p \left|R(h,l)\right| \mathrm{d}\mu(l) \right)^{1/p} \left(\int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \left|R(h,l)\right| \mathrm{d}\mu(l) \right)^{1/q}$$
$$\leqslant C\|F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}.$$

Consequently, we have that

$$\|F\|_{L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})} \leqslant C\|F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}.$$

Thus, $F \in L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})$ and by (3.4) we obtain that $F = V_\psi(\widetilde{V}_\psi F)$, where $\widetilde{V}_\psi F \in H_1'$ and since $F \in L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})$ also $\widetilde{V}_\psi F \in M_p$. The uniqueness condition follows by definition of M_p. □

Applying theorems 3.1(i) and (3.4) we get for $f \in H_1'$ that

$$V_\psi \widetilde{V}_\psi(V_\psi f) = \langle V_\psi f, R(h,\cdot)\rangle = V_\psi f.$$

Hence, $\widetilde{V}_\psi V_\psi$ is the identity in H_1' and we have the reconstruction formula

$$f = \widetilde{V}_\psi V_\psi f = \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \langle f, U(\sigma(h)^{-1})\psi\rangle U(\sigma(h)^{-1})\psi\,\mathrm{d}\mu(h).$$

We finish this section by establishing the relationships of our generalized coorbit spaces to the fundamental spaces $L_2(\mathcal{N})$ and H_1'.

Lemma 3.2. Under the assumptions outlined above, the following relations are valid:

(i) $M_\infty = H_1'$.

(ii) $M_2 = L_2(\mathcal{N})$.

Proof. (i) For $f \in H_1'$ we have by (3.3) that $\|V_\psi f\|_{L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})} \leqslant C\|f\|_{H_1'}$. Conversely, we have for $f \in M_\infty$

$$\|f\|_{H_1'} = \sup_{\|g\|_{H_1}=1} \left|\langle f, g\rangle\right| = \sup_{\|g\|_{H_1}=1} \left|\langle \widetilde{V}_\psi V_\psi f, g\rangle\right|$$
$$= \sup_{\|g\|_{H_1}=1} \left|\langle V_\psi f, V_\psi g\rangle\right| \leqslant \|V_\psi f\|_{L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})},$$

which yields the first assertion.

(ii) Let $f \in L_2(\mathcal{N})$. Then we have $V_\psi(f) \in \mathcal{M}_2$. By theorem 3.1 there exists $g \in M_2$ such that $V_\psi(f) = V_\psi(g)$ which implies by definition of M_2 that $f = g$. Conversely, let $f \in M_2$. Then we have by theorem 3.1 that $V_\psi(f) \in \mathcal{M}_2$ and there exists $g \in L_2(\mathcal{N})$ such that $V_\psi(f) = V_\psi(g)$ which again implies that $f = g$. □

Note that one can prove results similar to [16, theorem 4.9] for the dual spaces M_p'. Using these results one can show for $1 \leqslant p \leqslant q$ that $M_p \subset M_q$ with continuous embeddings.

4. Atomic decomposition and Banach frames for coorbit spaces

Once our generalized coorbit spaces are established, the next step is to derive some atomic decomposition for these spaces and to construct suitable Banach frames. This program is performed in several steps. In the next subsection, we present some preparations and state our main results. The remaining two subsections are devoted to the building blocks which are necessary to prove these results. The major step is the construction of suitable approximation operators which are defined and analyzed in section 4.2.

The results in this section are again inspired by the pioneering work of Feichtinger and Gröchenig [15–18].

4.1. Setting and main results

Before we can state and prove our main results, some preparations are necessary. Given some neighborhood $\mathcal{U}$ of the identity in $\mathcal{G}$, a family $X = (x_i)_{i\in\mathcal{I}}$ in $\mathcal{G}$ is called *$\mathcal{U}$-dense* if $\bigcup_{i\in\mathcal{I}} \mathcal{U}x_i = \mathcal{G}$. A family $X = (x_i)_{i\in\mathcal{I}}$ in $\mathcal{G}$ is called *relatively separated,* if for any compact set $\mathcal{Q} \subseteq \mathcal{G}$ there exists a finite partition of the index set $\mathcal{I}$, i.e., $\mathcal{I} = \bigcup_{r=1}^{r_0} \mathcal{I}_r$, such that $\mathcal{Q}x_i \cap \mathcal{Q}x_j = \emptyset$ for all $i, j \in \mathcal{I}_r$ with $i \neq j$. Note that these technical conditions can be easily fulfilled by some families X in all the settings we are interested in.

Let $\mathcal{U}$ be an arbitrary compact neighborhood of the identity in $\mathcal{G}$. By [13], there exists a bounded uniform partition of unity (of size $\mathcal{U}$), i.e., a family of continuous functions $(\varphi_i)_{i\in\mathcal{I}}$ on $\mathcal{G}$ such that

- $0 \leqslant \varphi_i(g) \leqslant 1$ for all $g \in \mathcal{G}$;
- there is a $\mathcal{U}$-dense, relatively separated family $(x_i)_{i\in\mathcal{I}}$ in $\mathcal{G}$ such that supp $\varphi_i \subseteq \mathcal{U}x_i$;
- $\sum_{i\in\mathcal{I}} \varphi_i(g) \equiv 1$ for all $g \in \mathcal{G}$.

Furthermore, we define the (left and right) $\mathcal{U}$*-oscillation* with respect to the analyzing wavelet ψ as

$$\operatorname{osc}^l_{\mathcal{U}}(l,h) := \sup_{u\in\mathcal{U}} \bigl|\bigl\langle \psi, U\bigl(\sigma(l)\sigma(h)^{-1}\bigr)\psi - U\bigl(u^{-1}\sigma(l)\sigma(h)^{-1}\bigr)\psi\bigr\rangle\bigr|, \tag{4.1}$$

$$\operatorname{osc}^r_{\mathcal{U}}(l,h) := \sup_{u\in\mathcal{U}} \bigl|\bigl\langle \psi, U\bigl(\sigma(l)\sigma(h)^{-1}\bigr)\psi - U\bigl(\sigma(l)\sigma(h)^{-1}u\bigr)\psi\bigr\rangle\bigr|. \tag{4.2}$$

In the sequel, we shall always assume that $(x_i)_{i\in\mathcal{I}}$ can be chosen such that $\sigma(\mathcal{G}/\mathcal{G}_{\mathcal{F}}) \cap \mathcal{U}x_i \neq \emptyset$ implies $x_i \in \sigma(\mathcal{G}/\mathcal{G}_{\mathcal{F}})$. Let

$$\mathcal{I}_\sigma := \bigl\{i \in \mathcal{I}\colon\ \sigma(\mathcal{G}/\mathcal{G}_{\mathcal{F}}) \cap \mathcal{U}x_i \neq \emptyset\bigr\}.$$

Then there exist h_i such that $x_i = \sigma(h_i)$, where $i \in \mathcal{I}_\sigma$. Note that

$$\sum_{i\in\mathcal{I}_\sigma} \varphi_i\bigl(\sigma(h)\bigr) = 1,$$

where $h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}$.

In this setting, we can formulate our main theorems which we will prove in the following subsections. The first one is a decomposition theorem which says that discretizing the representation $U(\sigma(\cdot)^{-1})$ by means of a $\mathcal{U}$-dense set indeed produces an atomic decomposition of M_p.

Theorem 4.1. Let $\mathcal{G}$ be a separable Lie group with closed subgroup $\mathcal{G}_{\mathcal{F}}$ and let μ be a quasi-invariant measure on $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$. Further, let U be a strictly square integrable representation of $\mathcal{G}$ mod $(\mathcal{G}_{\mathcal{F}}, \sigma)$ in $L_2(\mathcal{N})$ with strictly admissible function ψ. Let a compact neighborhood $\mathcal{U}$ of the identity in $\mathcal{G}$ be chosen so small that

$$\int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} \operatorname{osc}^l_{\mathcal{U}}(l,h)\,\mathrm{d}\mu(l) < 1 \quad \text{and} \quad \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} \operatorname{osc}^l_{\mathcal{U}}(l,h)\,\mathrm{d}\mu(h) < 1. \tag{4.3}$$

Let $X = (x_i)_{i\in\mathcal{I}}$ be an $\mathcal{U}$-dense and relatively separated family. Furthermore, suppose that for any compact neighborhood $\mathcal{Q}$ of the identity in $\mathcal{G}$

$$\mu\bigl\{h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}\colon\ \sigma(h) \in \mathcal{Q}\sigma(h_i)\bigr\} \geqslant C_{\mathcal{Q}} > 0$$

holds for all $i \in \mathcal{I}_\sigma$. Finally, let us assume that for any compact neighborhood $\mathcal{Q}$ of the identity in $\mathcal{G}$ our analyzing function ψ fulfills the following inequality

$$\int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} \sup_{q\in\mathcal{Q}} \bigl|\bigl\langle U\bigl(\sigma(h)^{-1}\bigr)\psi, U\bigl(\sigma(l)^{-1}q\bigr)\psi\bigr\rangle_{L_2(\mathcal{N})}\bigr|\,\mathrm{d}\mu(l) \leqslant \widetilde{C}_{\mathcal{Q}} \tag{4.4}$$

with a constant $\widetilde{C}_{\mathcal{Q}} < \infty$ independent of $h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}$. Then M_p, $1 \leqslant p < \infty$, has the following atomic decomposition: if $f \in M_p$, $1 \leqslant p < \infty$, then

$$f = \sum_{i\in\mathcal{I}_\sigma} c_i U\bigl(\sigma(h_i)^{-1}\bigr)\psi, \tag{4.5}$$

where the sequence of coefficients $(c_i)_{i\in I_\sigma} = \big(c_i(f)\big)_{i\in I_\sigma} \in \ell_p$ depends linearly on f and satisfies

$$\big\|(c_i)_{i\in I_\sigma}\big\|_{\ell_p} \leqslant A\|f\|_{M_p}. \tag{4.6}$$

If $(c_i)_{i\in I_\sigma} \in \ell_p$, then $f = \sum_{i\in\mathcal{I}_\sigma} c_i U\big(\sigma(h_i)^{-1}\big)\psi$ is contained in M_p and

$$\|f\|_{M_p} \leqslant B\big\|(c_i)_{i\in I_\sigma}\big\|_{\ell_p}. \tag{4.7}$$

Given such an atomic decomposition, the problem arises under which conditions a function f is completely determined by the moments or coefficients $\langle f, U\big(\sigma(h_i)^{-1}\big)\psi\rangle$ and how f can be reconstructed from these coefficients. This question is answered by the following theorem which shows that our generalized coherent states indeed give rise to Banach frames.

Theorem 4.2. Impose the same assumptions as in theorem 4.1 with

$$\int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \operatorname{osc}^r_{\mathcal{U}}(l,h)\,\mathrm{d}\mu(l) < \frac{1}{C_\psi} \quad \text{and} \quad \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \operatorname{osc}^r_{\mathcal{U}}(l,h)\,\mathrm{d}\mu(h) < \frac{1}{C_\psi} \tag{4.8}$$

instead of (4.3), where C_ψ is defined by (3.1). Then the set

$$\big\{\psi_i := U\big(\sigma(h_i)^{-1}\big)\psi\colon\, i\in\mathcal{I}_\sigma\big\} \tag{4.9}$$

is a Banach frame for M_p. This means that

(i) $f\in M_p$ if and only if $\big(\langle f,\psi_i\rangle\big)_{i\in\mathcal{I}_\sigma} \in \ell_p$;

(ii) there exist two constants $0 < A' \leqslant B' < \infty$ such that

$$A'\|f\|_{M_p} \leqslant \big\|\big(\langle f,\psi_i\rangle\big)_{i\in\mathcal{I}_\sigma}\big\|_{\ell_p} \leqslant B'\|f\|_{M_p}; \tag{4.10}$$

(iii) there exists a bounded, linear reconstruction operator S from ℓ_p to M_p such that $S\big(\big(\langle f,\psi_i\rangle\big)_{i\in\mathcal{I}_\sigma}\big) = f$.

For further information concerning Banach frames see [23]. We finish this section with some illuminating remarks.

Remark 4.1.

(i) In our definition of U-density, we made use of the right translation for the following reason: the proofs in the following subsections show that the setting of U-density has to fit together with the definition of the quotient space $\mathcal{G}/\mathcal{G}_\mathcal{F}$. As explained in section 2, we always use right coset spaces in this paper.

(ii) One might conjecture that only one of the two integrability conditions in (4.3) and (4.8) is sufficient. However, the proofs of both, theorems 4.1 and 4.2, make use of the generalized Young inequality, and the application of this inequality requires both integrability conditions.

(iii) It is a remarkable fact that in Hilbert spaces the norm equivalence (4.6), (4.7) alone guarantees an efficient method for the reconstruction of f and (iii) is redundant in this case. For Banach spaces, however, conditions (ii) and (iii) are independent, and to find the reconstruction operator S poses additional difficulties.

(iv) The conditions on the family $(x_i)_{i\in\mathcal{I}_\sigma}$ may sound very technical and one might wonder if they are ever satisfied in conjunction with all other assumptions. However, as we shall explain in detail in section 6 below all these conditions can be satisfied, e.g., for the sphere S^1.

4.2. Approximation operators

In this section, we examine two different approximation operators on $\mathcal{M}_p$. We use the results to construct expansions for the spaces $\mathcal{M}_p$, which then, by the correspondence principle in theorem 3.1, lead to expansions for the coorbit spaces M_p.

We consider the following approximation operators on $\mathcal{M}_p$:

$$T_\varphi F(h) := \sum_{i\in\mathcal{I}_\sigma} \langle F, \varphi_i \circ \sigma\rangle R(h_i, h) \tag{4.11}$$

$$= \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\varphi_i\bigl(\sigma(l)\bigr)\, \mathrm{d}\mu(l) R(h_i, h), \tag{4.12}$$

$$S_\varphi F(h) := \sum_{i\in\mathcal{I}_\sigma} F(h_i)\bigl\langle\varphi_i \circ \sigma, R(h, \cdot)\bigr\rangle \tag{4.13}$$

$$= \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(h_i)\varphi_i\bigl(\sigma(l)\bigr) R(l, h)\, \mathrm{d}\mu(l). \tag{4.14}$$

The first step is to establish the invertibility of the operators T_φ and S_φ.

Theorem 4.3. (i) If the conditions (4.3) are fulfilled, then the operator T_φ is invertible. (ii) If the conditions (4.8) are fulfilled, then the operator S_φ is invertible.

Proof. By definition of $\mathcal{M}_p$, we have that

$$F(h) = \bigl\langle F, R(h, \cdot)\bigr\rangle = \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\overline{R(h, l)}\, \mathrm{d}\mu(l)$$
$$= \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\varphi_i\bigl(\sigma(l)\bigr) R(l, h)\, \mathrm{d}\mu(l)$$

and consequently

$$F(h) - T_\varphi F(h) = \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\varphi_i\bigl(\sigma(l)\bigr)\bigl[R(l, h) - R(h_i, h)\bigr]\, \mathrm{d}\mu(l),$$

$$F(h) - S_\varphi F(h) = \sum_{i \in \mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big[F(l) - F(h_i)\big]\varphi_i\big(\sigma(l)\big)R(l,h)\,\mathrm{d}\mu(l). \tag{4.15}$$

Let us first consider $F - T_\varphi F$. By the definition of R we obtain

$$\begin{aligned}\big|F(h) - T_\varphi F(h)\big| &\leqslant \sum_{i \in \mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|F(l)\big|\varphi_i\big(\sigma(l)\big)\big|R(l,h) - R(h_i,h)\big|\,\mathrm{d}\mu(l) \\ &= \sum_{i \in \mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|F(l)\big|\varphi_i\big(\sigma(l)\big)\big|\big\langle\psi, U\big(\sigma(l)\sigma(h)^{-1}\big)\psi \\ &\qquad - U\big(\sigma(h_i)\sigma(h)^{-1}\big)\psi\big\rangle_{L_2(\mathcal{N})}\big|\,\mathrm{d}\mu(l).\end{aligned}$$

Now $\sigma(l) \in \mathcal{U}x_i$ implies that there exists $u \in \mathcal{U}$ such that $\sigma(l) = ux_i = u\sigma(h_i)$. Thus $\sigma(h_i) = u^{-1}\sigma(l)$ and we get

$$\begin{aligned}\big|F(h) - T_\varphi F(h)\big| &\leqslant \sum_{i \in \mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|F(l)\big|\varphi_i\big(\sigma(l)\big)\mathrm{osc}^l_\mathcal{U}(l,h)\,\mathrm{d}\mu(l) \\ &= \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|F(l)\big|\mathrm{osc}^l_\mathcal{U}(l,h)\,\mathrm{d}\mu(l),\end{aligned}$$

where $\mathrm{osc}^l_\mathcal{U}(l,h)$ is defined by (4.1). By applying the generalized Young inequality and recalling the assumptions (4.3), we obtain

$$\|F - T_\varphi F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} = \big\|(I - T_\varphi)F\big\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} \leqslant \gamma\|F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})},$$

where $\gamma < 1$. Consequently $|||(I - T_\varphi)||| < 1$, i.e., $I - T_\varphi$ is a contraction on $\mathcal{M}_p$ and T_φ is invertible on $\mathcal{M}_p$.

Next we consider $F - S_\varphi F$. Since $F \in \mathcal{M}_p$ and by the definition of R we obtain

$$\begin{aligned}\big|F(l) - F(h_i)\big| &\leqslant \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|F(g)\big|\,\big|R(g,l) - R(g,h_i)\big|\,\mathrm{d}\mu(g) \\ &= \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|F(g)\big|\,\big|\big\langle\psi, U\big(\sigma(g)\sigma(l)^{-1}\big)\psi \\ &\qquad - U\big(\sigma(g)\sigma(h_i)^{-1}\big)\psi\big\rangle_{L_2(\mathcal{N})}\big|\,\mathrm{d}\mu(g).\end{aligned}$$

By (4.15) we are only interested in $l \in \mathcal{G}/\mathcal{G}_\mathcal{F}$ with $\sigma(l) \in \mathcal{U}x_i$, i.e., $\sigma(l) = u\sigma(h_i)$ for some $u \in \mathcal{U}$ and $\sigma(h_i)^{-1} = \sigma(l)^{-1}u$. Thus

$$\big|F(l) - F(h_i)\big| \leqslant \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|F(g)\big|\mathrm{osc}^r_\mathcal{U}(g,l)\,\mathrm{d}\mu(g)$$

and since (φ_i) is a partition of unity

$$\sum_{i \in \mathcal{I}_\sigma} \big|F(l) - F(h_i)\big|\varphi_i\big(\sigma(l)\big) \leqslant \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|F(g)\big|\mathrm{osc}^r_\mathcal{U}(g,l)\,\mathrm{d}\mu(g).$$

By the generalized Young inequality and (4.8) this implies

$$\Big\| \sum_{i\in\mathcal{I}_\sigma} \big|F(l) - F(h_i)\big| \varphi_i\big(\sigma(l)\big) \Big\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} < \frac{1}{C_\psi} \|F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}.$$

Now we obtain by (4.15) and (3.1)

$$\|F - S_\varphi F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} \leqslant C_\psi \Big\| \sum_{i\in\mathcal{I}_\sigma} \big|F(l) - F(h_i)\big| \varphi_i\big(\sigma(l)\big) \Big\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} \leqslant \gamma \|F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})},$$

where $\gamma < 1$. Consequently, $I - S_\varphi$ is a contraction on $\mathcal{M}_p$ and S_φ is invertible on $\mathcal{M}_p$. □

Using the correspondence principle we can derive the following representations of functions from our coorbit spaces.

Corollary 4.1. Any function $f \in M_p$ can be decomposed as

$$f = \sum_{i\in\mathcal{I}_\sigma} c_i U\big(\sigma(h_i)^{-1}\big)\psi, \tag{4.16}$$

where

$$c_i = c_i(f) := \big\langle T_\varphi^{-1} F, \varphi_i \circ \sigma \big\rangle.$$

Proof. By theorems 3.1(ii) and 4.3 we have that

$$V_\psi f(h) = F(h) = T_\varphi T_\varphi^{-1} F(h) = \sum_{i\in\mathcal{I}_\sigma} \big\langle T_\varphi^{-1} F, \varphi_i \circ \sigma \big\rangle R(h_i, h)$$

and further by the definition of V_ψ that

$$\begin{aligned} V_\psi f &= \big\langle f, U\big(\sigma(h)^{-1}\big)\psi \big\rangle = \sum_{i\in\mathcal{I}_\sigma} \big\langle T_\varphi^{-1} F, \varphi_i \circ \sigma \big\rangle R(h_i, h) \\ &= \sum_{i\in\mathcal{I}_\sigma} \big\langle T_\varphi^{-1} F, \varphi_i \circ \sigma \big\rangle \big\langle U\big(\sigma(h_i)^{-1}\big)\psi, U\big(\sigma(h)^{-1}\big)\psi \big\rangle_{L_2(\mathcal{N})} \\ &= \Big\langle \sum_{i\in\mathcal{I}_\sigma} \big\langle T_\varphi^{-1} F, \varphi_i \circ \sigma \big\rangle U\big(\sigma(h_i)^{-1}\big)\psi, U\big(\sigma(h)^{-1}\big)\psi \Big\rangle_{L_2(\mathcal{N})}. \end{aligned}$$

This yields the assertion. □

Moreover, the operator S_φ induces the reconstruction operator as stated in theorem 4.2(iii).

Corollary 4.2. Any function $f \in M_p$ can be reconstructed as

$$f = \sum_{i\in\mathcal{I}_\sigma} \big\langle f, U\big(\sigma(h_i)^{-1}\big)\psi \big\rangle e_i,$$

where

$$e_i = \widetilde{V}_\psi(E_i), \quad E_i := S_\varphi^{-1}\big(\langle \varphi_i \circ \sigma, R(h, \cdot)\rangle\big).$$

Proof. Since S_φ is invertible, we obtain

$$\begin{aligned} F(h) &= S_\varphi^{-1}(S_\varphi)(F)(h) \\ &= \sum_{i\in\mathcal{I}_\sigma} F(h_i) S_\varphi^{-1}\langle \varphi_i \circ \sigma, R(h, \cdot)\rangle = \sum_{i\in\mathcal{I}_\sigma} F(h_i) E_i. \end{aligned}$$

Therefore the correspondence principle implies

$$\begin{aligned} f = \widetilde{V}_\psi V_\psi f &= \widetilde{V}_\psi\Bigg(\sum_{i\in\mathcal{I}_\sigma} V_\psi(f)(h_i) E_i\Bigg) \\ &= \sum_{i\in\mathcal{I}_\sigma} \big\langle f, U\big(\sigma(h_i)^{-1}\big)\psi\big\rangle \widetilde{V}_\psi(E_i) = \sum_{i\in\mathcal{I}_\sigma} \big\langle f, U\big(\sigma(h_i)^{-1}\big)\psi\big\rangle e_i. \end{aligned}$$

□

4.3. Frame bounds

In this section, we want to prove the norm equivalences in theorems 4.1 and 4.2. For the verification that the infinite sums appearing in the following lemmatas converge (unconditionally) in $\mathcal{M}_p$, respectively M_p, it suffices to obtain for $p < \infty$ the estimates for finite sequences. Then all the estimates can be extended in the usual way, see again [15–17] for details. Only the case $p = \infty$ requires some additional effort. The necessary modifications are left to the reader.

We start with theorem 4.1.

Lemma 4.1. Suppose that the conditions in theorem 4.1 are satisfied. For any $f \in M_p$ let the sequence

$$(c_i)_{i\in\mathcal{I}_\sigma} = \Big(\big\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma\big\rangle\Big)_{i\in\mathcal{I}_\sigma}$$

be given by (4.16). Then there exists a constant $A < \infty$ such that the following inequality holds:

$$\big\|(c_i)_{i\in\mathcal{I}_\sigma}\big\|_{\ell_p} \leqslant A\|f\|_{M_p}.$$

In particular, we have that $(c_i)_{i\in\mathcal{I}_\sigma} \in \ell_p$.

Proof. 1. First we show that for any sequence $(\eta_i)_{i\in\mathcal{I}_\sigma} \in \ell_p$ the inequality

$$\big\|(\eta_i)_{i\in\mathcal{I}_\sigma}\big\|_{\ell_p} \leqslant C_{\mathcal{U}}^{1/p} \Bigg\|\sum_{i\in\mathcal{I}_\sigma} |\eta_i| \mathbb{1}_{\mathcal{U}x_i} \circ \sigma\Bigg\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} \tag{4.17}$$

holds, where again $x_i = \sigma(h_i)$ and where $\mathbb{1}_{\mathcal{U}x_i}$ denotes the characteristic function of $\mathcal{U}x_i$.

Since $(x_i)_{i\in\mathcal{I}}$ is a relatively separated family, there exists a splitting $\mathcal{I} = \bigcup_{r=1}^{r_0} \mathcal{I}_r$ such that $\mathcal{U}x_i \cap \mathcal{U}x_j = \emptyset$ for $i, j \in \mathcal{I}_r$ and $i \neq j$. This results in a decomposition $\mathcal{I}_\sigma = \bigcup_{r=1}^{r_0} \mathcal{I}_{\sigma_r}$, where

$$\mathcal{I}_{\sigma_r} = \bigl\{i \in \mathcal{I}_r\colon\ \mathcal{U}x_i \cap \sigma(\mathcal{G}/\mathcal{G}_\mathcal{F}) \neq \emptyset\bigr\}.$$

Then we obtain (4.17) by

$$\begin{aligned}\Bigl\|\sum_{i\in\mathcal{I}_\sigma} |\eta_i|\mathbb{1}_{\mathcal{U}x_i}\circ\sigma\Bigr\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}^p &= \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \Biggl(\sum_{r=1}^{r_0}\sum_{i\in\mathcal{I}_{\sigma_r}} |\eta_i|\mathbb{1}_{\mathcal{U}x_i}\bigl(\sigma(h)\bigr)\Biggr)^p \,\mathrm{d}\mu(h)\\ &\geqslant \sum_{r=1}^{r_0}\int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \Biggl(\sum_{i\in\mathcal{I}_{\sigma_r}} |\eta_i|\mathbb{1}_{\mathcal{U}x_i}\bigl(\sigma(h)\bigr)\Biggr)^p \,\mathrm{d}\mu(h)\\ &= \sum_{r=1}^{r_0}\int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \sum_{i\in\mathcal{I}_{\sigma_r}} |\eta_i|^p\mathbb{1}_{\mathcal{U}x_i}\bigl(\sigma(h)\bigr)\,\mathrm{d}\mu(h)\\ &\geqslant C_\mathcal{U}\sum_{i\in\mathcal{I}_\sigma}|\eta_i|^p.\end{aligned}$$

2. Let $F \in L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})$. Then the application of (4.17) yields

$$\begin{aligned}\bigl\|\bigl(\langle F,\varphi_i\circ\sigma\rangle\bigr)_{i\in\mathcal{I}_\sigma}\bigr\|_{\ell_p} &\leqslant \bigl\|\bigl(\langle |F|,\varphi_i\circ\sigma\rangle\bigr)_{i\in\mathcal{I}_\sigma}\bigr\|_{\ell_p}\\ &\leqslant C_\mathcal{U}^{-1/p}\Bigl\|\sum_{i\in\mathcal{I}_\sigma}\bigl\langle |F|,\varphi_i\circ\sigma\bigr\rangle\mathbb{1}_{\mathcal{U}x_i}\circ\sigma\Bigr\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}.\end{aligned}$$

Further, we see for an arbitrary fixed $h \in \mathcal{G}/\mathcal{G}_\mathcal{F}$ that

$$\sum_{i\in\mathcal{I}_\sigma}\bigl\langle |F|,\varphi_i\circ\sigma\bigr\rangle\mathbb{1}_{\mathcal{U}x_i}\bigl(\sigma(h)\bigr) = \sum_{i\in\mathcal{I}_h}\bigl\langle |F|,\varphi_i\circ\sigma\bigr\rangle,$$

where $\mathcal{I}_h := \{i \in \mathcal{I}_\sigma\colon\ x_i \in \mathcal{U}^{-1}\sigma(h)\}$ and further that

$$\begin{aligned}\sum_{i\in\mathcal{I}_h}\bigl\langle |F|,\varphi_i\circ\sigma\bigr\rangle &= \sum_{i\in\mathcal{I}_h}\bigl\langle |F|,\varphi_i\bigl(\sigma(\cdot)\bigr)\bigr\rangle\\ &\leqslant \bigl\langle |F|,\mathbb{1}_{\mathcal{U}\mathcal{U}^{-1}}\bigl(\sigma(\cdot)\sigma(h)^{-1}\bigr)\bigr\rangle.\end{aligned}$$

Since

$$\int_{\mathcal{G}/\mathcal{G}_\mathcal{F}}\mathbb{1}_{\mathcal{U}\mathcal{U}^{-1}}\bigl(\sigma(l)\sigma(h)^{-1}\bigr)\,\mathrm{d}\mu(l) = \mu\bigl\{l\in\mathcal{G}/\mathcal{G}_\mathcal{F}\colon\ \sigma(l)\in\mathcal{U}\mathcal{U}^{-1}\sigma(h)\bigr\} \leqslant C,$$

for all $h \in \mathcal{G}/\mathcal{G}_\mathcal{F}$ we obtain by the generalized Young inequality

$$\begin{aligned}\bigl\|\bigl(\langle F,\varphi_i\circ\sigma\rangle\bigr)_{i\in I_\sigma}\bigr\|_{\ell_p} &\leqslant C_\mathcal{U}^{-1/p}\bigl\|\bigl\langle |F|,\mathbb{1}_{\mathcal{U}\mathcal{U}^{-1}}\bigl(\sigma(\cdot)\sigma(h)^{-1}\bigr)\bigr\rangle\bigr\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}\\ &\leqslant C_\mathcal{U}^{-1/p}C\|F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}.\end{aligned}$$

Finally, we conclude by using $F = T_\varphi^{-1} V_\psi f \in \mathcal{M}_p$ in the above inequality that

$$\begin{aligned}\big\| \big(\big\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma\big\rangle\big)_{i\in\mathcal{I}_\sigma} \big\|_{\ell_p} &\leqslant C \big\| T_\varphi^{-1} V_\psi f \big\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} \\ &\leqslant C \;|||\; T_\varphi^{-1} \;|||\; \|V_\psi f\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} \\ &\leqslant C \;|||\; T_\varphi^{-1} \;|||\; \|f\|_{M_p}. \qquad \square\end{aligned}$$

The next step is to establish (4.7).

Lemma 4.2. Suppose that the conditions in theorem 4.1 are satisfied. Then there exists a constant $B < \infty$ such that for any sequence $(c_i)_{i\in\mathcal{I}_\sigma} \in \ell_p$, $1 \leqslant p \leqslant \infty$, the following inequality holds:

$$\Big\| \sum_{i\in\mathcal{I}_\sigma} c_i U\big(\sigma(h_i)^{-1}\big)\psi \Big\|_{M_p} \leqslant B \big\| (c_i)_{i\in\mathcal{I}_\sigma} \big\|_{\ell_p}. \tag{4.18}$$

In particular, we have by (4.16) that

$$\|f\|_{M_p} \leqslant B \big\| \big(\big\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma\big\rangle\big)_{i\in\mathcal{I}_\sigma} \big\|_{\ell_p}.$$

Proof. By definition of the norm in M_p and (2.9) we have

$$\Big\| \sum_{i\in\mathcal{I}_\sigma} c_i U\big(\sigma(h_i)^{-1}\big)\psi \Big\|_{M_p} = \Big\| \sum_{i\in\mathcal{I}_\sigma} c_i R(h_i, h) \Big\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}.$$

By the Riesz–Thorin Interpolation Theorem, see, e.g., [19, chapter 6] for details, it suffices to prove the inequality (4.18) for $p = 1$ and $p = \infty$. For $p = 1$, we obtain by (3.1)

$$\begin{aligned}\Big\| \sum_{i\in\mathcal{I}_\sigma} c_i U\big(\sigma(h_i)^{-1}\big)\psi \Big\|_{M_1} &= \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \Big| \sum_{i\in\mathcal{I}_\sigma} c_i R(h_i, h) \Big| \,\mathrm{d}\mu(h) \\ &\leqslant \sum_{i\in\mathcal{I}_\sigma} |c_i| \sup_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big| R(h_i, h) \big| \,\mathrm{d}\mu(h) \\ &\leqslant C_\psi \big\| (c_i)_{i\in\mathcal{I}_\sigma} \big\|_{\ell_1}.\end{aligned}$$

For $p = \infty$ it follows that

$$\begin{aligned}\Big\| \sum_{i\in\mathcal{I}_\sigma} c_i U\big(\sigma(h_i)^{-1}\big)\psi \Big\|_{M_\infty} &= \sup_{h\in\mathcal{G}/\mathcal{G}_\mathcal{F}} \Big| \sum_{i\in\mathcal{I}_\sigma} c_i R(h_i, h) \Big| \\ &\leqslant \sup_{i\in\mathcal{I}_\sigma} |c_i| \sup_{h\in\mathcal{G}/\mathcal{G}_\mathcal{F}} \sum_{i\in\mathcal{I}_\sigma} \big| R(h_i, h) \big|. \end{aligned} \tag{4.19}$$

Since $(x_i)_{i\in\mathcal{I}}$ is a relatively separated family, we have for any compact neighborhood $\mathcal{Q}$ of the identity in $\mathcal{G}$ that $\mathcal{I}_\sigma = \bigcup_{r=1}^{r_0} \mathcal{I}_{\sigma_r}$ and $\mathcal{Q}x_i \cap \mathcal{Q}x_j = \emptyset$ for $i, j \in \mathcal{I}_{\sigma_r}$ and $i \neq j$. Hence we obtain

$$\sum_{i\in\mathcal{I}_\sigma} \bigl|R(h_i, h)\bigr| = \sum_{r=1}^{r_0} \sum_{i\in\mathcal{I}_{\sigma_r}} \bigl|R(h_i, h)\bigr|.$$

For all $l \in \mathcal{G}/\mathcal{G}_\mathcal{F}$ with the property that $\sigma(l) \in \mathcal{Q}\sigma(h_i)$, we have that $\sigma(h_i)^{-1} \in \sigma(l)^{-1}\mathcal{Q}$ and hence

$$\begin{aligned}\sup_{q\in\mathcal{Q}} \bigl|\bigl\langle U\bigl(\sigma(h)^{-1}\bigr)\psi, U\bigl(\sigma(l)^{-1}q\bigr)\psi\bigr\rangle_{L_2(\mathcal{N})}\bigr| &\geqslant \bigl|\bigl\langle U\bigl(\sigma(h)^{-1}\bigr)\psi, U\bigl(\sigma(h_i)^{-1}\bigr)\psi\bigr\rangle_{L_2(\mathcal{N})}\bigr| \\ &= \bigl|R(h, h_i)\bigr| = \bigl|R(h_i, h)\bigr|.\end{aligned}$$

Let $\mathcal{B}_i := \{l \in \mathcal{G}/\mathcal{G}_\mathcal{F}\colon\ \sigma(l) \in \mathcal{Q}\sigma(h_i)\}$. Then the above inequality implies

$$\int_{\mathcal{B}_i} \sup_{q\in\mathcal{Q}} \bigl|\bigl\langle U\bigl(\sigma(h)^{-1}\bigr)\psi, U\bigl(\sigma(l)^{-1}q\bigr)\psi\bigr\rangle_{L_2(\mathcal{N})}\bigr|\,\mathrm{d}\mu(l) \geqslant \bigl|R(h_i, h)\bigr|\mu(\mathcal{B}_i).$$

Now we have that for $i, j \in \mathcal{I}_{\sigma_r}$ and $i \neq j$ the sets $\mathcal{B}_i$ and $\mathcal{B}_j$ are disjoint. Consequently, we obtain

$$\begin{aligned}&\int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \sup_{q\in\mathcal{Q}} \bigl|\bigl\langle U\bigl(\sigma(h)^{-1}\bigr)\psi, U\bigl(\sigma(l)^{-1}q\bigr)\psi\bigr\rangle_{L_2(\mathcal{N})}\bigr|\,\mathrm{d}\mu(l) \\ &\quad\geqslant \sum_{i\in\mathcal{I}_{\sigma_r}} \int_{\mathcal{B}_i} \sup_{q\in\mathcal{Q}} \bigl|\bigl\langle U\bigl(\sigma(h)^{-1}\bigr)\psi, U\bigl(\sigma(l)^{-1}q\bigr)\psi\bigr\rangle_{L_2(\mathcal{N})}\bigr|\,\mathrm{d}\mu(l) \\ &\quad\geqslant \sum_{i\in\mathcal{I}_{\sigma_r}} \bigl|R(h_i, h)\bigr|\mu(\mathcal{B}_i) \geqslant C_\mathcal{Q} \sum_{i\in\mathcal{I}_{\sigma_r}} \bigl|R(h_i, h)\bigr|\end{aligned}$$

and further by (4.4) for all $h \in \mathcal{G}/\mathcal{G}_\mathcal{F}$

$$\sum_{i\in\mathcal{I}_{\sigma_r}} \bigl|R(h_i, h)\bigr| \leqslant \frac{\widetilde{C}_\mathcal{Q}}{C_\mathcal{Q}}, \qquad \sum_{i\in\mathcal{I}_\sigma} \bigl|R(h_i, h)\bigr| \leqslant \frac{r_0\widetilde{C}_\mathcal{Q}}{C_\mathcal{Q}}. \tag{4.20}$$

Together with (4.19) this yields

$$\Bigl\|\sum_{i\in\mathcal{I}_\sigma} c_i U\bigl(\sigma(h_i)^{-1}\bigr)\psi\Bigr\|_{M_\infty} \leqslant \bigl\|(c_i)_{i\in\mathcal{I}_\sigma}\bigr\|_{\ell_\infty} \frac{r_0\widetilde{C}_\mathcal{Q}}{C_\mathcal{Q}}.$$

□

Next let us turn to the estimates (4.10) in theorem 4.2.

Lemma 4.3. Suppose that the conditions in theorem 4.2 are satisfied. Let ψ_i be defined by (4.9). Then there exists a constant $B' < \infty$ such that

$$\bigl\|\bigl(\langle f, \psi_i\rangle\bigr)_{i\in I_\sigma}\bigr\|_{\ell_p} \leqslant B'\|f\|_{M_p}.$$

Proof. Let $F := V_\psi f$. By the correspondence principle the assertion is equivalent to

$$\big\| \big(F(h_i)\big)_{i\in I_\sigma} \big\|_{\ell_p} \leqslant B' \|F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}. \tag{4.21}$$

We prove (4.21) for $p = 1$ and $p = \infty$ and apply again the Riesz–Thorin Interpolation Theorem to obtain the inequality for all $1 \leqslant p \leqslant \infty$.

For $p = 1$, we conclude as follows

$$\begin{aligned}\sum_{i\in I_\sigma} \big|F(h_i)\big| &= \sum_{i\in I_\sigma} \big|\big\langle F, R(h_i,\cdot)\big\rangle\big| \leqslant \sum_{i\in I_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|F(l)\big| \, \big|R(h_i,l)\big| \, \mathrm{d}\mu(l) \\ &= \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|F(l)\big| \sum_{i\in I_\sigma} \big|R(h_i,l)\big| \, \mathrm{d}\mu(l) \leqslant \frac{r_0 \widetilde{C}_\mathcal{Q}}{C_\mathcal{Q}} \|F\|_{L_1(\mathcal{G}/\mathcal{G}_\mathcal{F})},\end{aligned}$$

where the last estimate involves (4.20).

For $p = \infty$, we get

$$\begin{aligned}\sup_{i\in I_\sigma} \big|F(h_i)\big| &= \sup_{i\in I_\sigma} \big|\big\langle F, R(h_i,\cdot)\big\rangle\big| \leqslant \sup_{i\in I_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|F(l)\big| \big|R(h_i,l)\big| \, \mathrm{d}\mu(l) \\ &\leqslant \sup_{l\in\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|F(l)\big| \sup_{i\in I_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \big|R(h_i,l)\big| \, \mathrm{d}\mu(l) \leqslant C_\psi \|F\|_{L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})},\end{aligned}$$

where we have used (3.1) for the last estimate. This finishes the proof. □

Lemma 4.4. Suppose that the conditions in theorem 4.2 are satisfied. Let ψ_i be defined by (4.9). Then there exists a constant $A' > 0$ such that

$$\|f\|_{M_p} \leqslant \frac{1}{A'} \big\| \big(\langle f, \psi_i\rangle\big)_{i\in I_\sigma} \big\|_{\ell_p}.$$

Proof. 1. First we show that

$$\mathcal{T} : (c_i)_{i\in I_\sigma} \mapsto \sum_{i\in I_\sigma} c_i S_\varphi^{-1} \big\langle \varphi_i \circ \sigma, R(h,\cdot)\big\rangle$$

is a bounded operator from ℓ_1 to $\mathcal{M}_1$ and from ℓ_∞ to $\mathcal{M}_\infty$. Then, by the Riesz–Thorin Interpolation Theorem, $\mathcal{T}$ is also a bounded operator from ℓ_p to $\mathcal{M}_p$ for all $1 \leqslant p \leqslant \infty$.

For $p = 1$, we get

$$\begin{aligned}\bigg\| \sum_{i\in I_\sigma} c_i S_\varphi^{-1} \big\langle \varphi_i \circ \sigma, R(h,\cdot)\big\rangle \bigg\|_{L_1(\mathcal{G}/\mathcal{G}_\mathcal{F})} &= \bigg\| S_\varphi^{-1} \bigg(\sum_{i\in I_\sigma} c_i \big\langle \varphi_i \circ \sigma, R(h,\cdot)\big\rangle \bigg) \bigg\|_{L_1(\mathcal{G}/\mathcal{G}_\mathcal{F})} \\ &\leqslant |||S_\varphi^{-1}||| \, \bigg\| \bigg\langle \sum_{i\in I_\sigma} c_i \varphi_i \circ \sigma, R(h,\cdot) \bigg\rangle \bigg\|_{L_1(\mathcal{G}/\mathcal{G}_\mathcal{F})}\end{aligned}$$

and further by (3.1) and the generalized Young inequality

$$\begin{aligned}\Big\|\sum_{i\in I_\sigma} c_i S_\varphi^{-1}\langle\varphi_i\circ\sigma, R(h,\cdot)\rangle\Big\|_{L_1(\mathcal{G}/\mathcal{G}_\mathcal{F})} &\leqslant |||S_\varphi^{-1}|||\, C_\psi \Big\|\sum_{i\in I_\sigma} c_i\varphi_i\circ\sigma\Big\|_{L_1(\mathcal{G}/\mathcal{G}_\mathcal{F})}\\ &\leqslant |||S_\varphi^{-1}|||\, C_\psi \sum_{i\in I_\sigma}|c_i|\|\varphi_i\circ\sigma\|_{L_1(\mathcal{G}/\mathcal{G}_\mathcal{F})}\\ &\leqslant |||S_\varphi^{-1}|||\, C_\psi C\big\|(c_i)_{i\in I_\sigma}\big\|_{\ell_1}.\end{aligned}$$

For $p=\infty$, we obtain in a similar way

$$\begin{aligned}\Big\|\sum_{i\in I_\sigma} c_i S_\varphi^{-1}\langle\varphi_i\circ\sigma, R(h,\cdot)\rangle\Big\|_{L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})} &\leqslant |||S_\varphi^{-1}|||\, C_\psi \sup_h\Big|\sum_{i\in I_\sigma} c_i\varphi_i\big(\sigma(h)\big)\Big|\\ &\leqslant |||S_\varphi^{-1}|||\, C_\psi \sup_{i\in I_\sigma}|c_i|\sup_h\sum_{i\in I_\sigma}\varphi_i\big(\sigma(h)\big)\\ &\leqslant |||S_\varphi^{-1}|||\, C_\psi \big\|(c_i)_{i\in I_\sigma}\big\|_{\ell_\infty}.\end{aligned}$$

2. Next, we observe for $F := V_\psi f \in \mathcal{M}_p$ by the correspondence principle and corollary 4.2 that

$$\|f\|_{M_p} = \|F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} = \Big\|\sum_{i\in I_\sigma} F(h_i)S_\varphi^{-1}\langle\varphi_i\circ\sigma, R(h,\cdot)\rangle\Big\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}$$

and by part 1 of the proof

$$\|f\|_{M_p} \leqslant |||S_\varphi^{-1}|||\, C_\psi\big\|\big(F(h_i)\big)_{i\in I_\sigma}\big\|_{\ell_p} = |||S_\varphi^{-1}|||\, C_\psi\big\|\big(\langle f, U\big(\sigma(h_i)^{-1}\big)\psi\rangle\big)_{i\in I_\sigma}\big\|_{\ell_p}$$

and we are done. □

5. Nonlinear approximation

The established atomic decomposition can now be used to decompose, to approximate and to analyze certain functions on $\mathcal{N}$. Then it is clearly desirable to determine the quality of certain approximation schemes based on our atomic decomposition, i.e., the approximation order comes into play. In this section, we are interested in the quality of the *best N-term approximation*. The setting can be described as follows.

Let $\{\psi_i = U(\sigma(h_i)^{-1})\psi\colon\ i\in\mathcal{I}_\sigma\}$ denote the set of atomic functions constructed in the previous section, i.e., we have for any $f\in M_p$ that

$$f = \sum_{i\in\mathcal{I}_\sigma} c_i\psi_i, \quad c_i := \big\langle T_\varphi^{-1}V_\psi f, \varphi_i\circ\sigma\big\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})}, \tag{5.1}$$

and

$$\big\|(c_i)_{i\in\mathcal{I}_\sigma}\big\|_{\ell_p} \sim \|f\|_{M_p}. \tag{5.2}$$

We want to approximate our functions $f \in M_p$ by elements from the nonlinear manifolds Σ_n, $n \in \mathbb{N}$, which consist of all functions $S \in M_p$ whose expansions with respect to our discrete coherent states have at most n nonzero coefficients, i.e.,

$$\Sigma_n := \left\{ S \in M_p\colon\ S = \sum_{i \in J} a_i \psi_i,\ J \subseteq \mathcal{I}_\sigma,\ \text{card}\, J \leqslant n \right\}.$$

Then we are interested in the asymptotic behavior of the error

$$E_n(f)_{M_p} := \inf_{S \in \Sigma_n} \|f - S\|_{M_p}.$$

Usually, the order of approximation which can be achieved depends on the regularity of the approximated function as measured in some associated smoothness space. For instance, for nonlinear wavelet approximation, the order of convergence is determined by the regularity as measured in a specific scale of Besov spaces. For nonlinear approximation based on Gabor frames, it has been shown in [25] that the 'right' smoothness spaces are given by a specific scale of modulation spaces. It turns out that at least a partial result from [25], i.e., an estimate in one direction, carries over to our case without any difficulty. The basic ingredient in the proof of the theorem is the following lemma which has been shown in [25], see also [12].

Lemma 5.1. Let $a = (a_i)_{i=1}^{\infty}$ be a decreasing sequence of positive numbers. For $p, q > 0$ set $\alpha := 1/p - 1/q$ and $E_{n,q}(a) := (\sum_{i=n}^{\infty} a_i^q)^{1/q}$. Then for $0 < p < q \leqslant \infty$ we have

$$2^{-1/p} \|a\|_{\ell_p} \leqslant \left(\sum_{n=1}^{\infty} \bigl(n^\alpha E_{n,q}(a)\bigr)^p \frac{1}{n} \right)^{1/p} \leqslant C \|a\|_{\ell_p}$$

with a constant $C > 0$ depending only on p.

Now one can prove the following theorem, see also [25].

Theorem 5.1. Let $\{\psi_i\colon\ i \in \mathcal{I}_\sigma\}$ be a set of atomic functions for M_p, $1 \leqslant p \leqslant \infty$, as constructed by theorem 4.1. If $1 \leqslant p < q$, $\alpha := 1/p - 1/q$ and $f \in M_p$, then

$$\left(\sum_{n=1}^{\infty} \frac{1}{n} \bigl(n^\alpha E_n(f)_{M_q}\bigr)^p \right)^{1/p} \leqslant C \|f\|_{M_p}$$

for a constant $C < \infty$.

Proof. Let π permutate the sequence $(c_i)_{i \in \mathcal{I}_\sigma}$ in (5.1) in a decreasing order, i.e., $|c_{\pi(1)}| \geqslant |c_{\pi(2)}| \geqslant \cdots$. Then we obtain

$$E_n(f)_{M_q} \leqslant \left\| \sum_{i=n+1}^{\infty} c_{\pi(i)} \psi_{\pi(i)} \right\|_{M_q}$$

and by (5.2) further that

$$E_n(f)_{M_q} \leqslant C\left(\sum_{i=n+1}^{\infty} |c_{\pi(i)}|^q\right)^{1/q} = C E_{n+1,q}\big(|c_{\pi(i)}|\big) \leqslant C E_{n,q}\big(|c_{\pi(i)}|\big).$$

Now we finish by applying lemma 5.1 and (5.2)

$$\left(\sum_{n=1}^{\infty} \frac{1}{n}\big(n^\alpha E_n(f)_{M_q}\big)^p\right)^{1/p} \leqslant \left(\sum_{n=1}^{\infty} \frac{1}{n}\big(n^\alpha C E_{n,q}\big)^p\right)^{1/p}$$
$$\leqslant C\big\|\big(|c_{\pi(i)}|\big)\big\|_{\ell_p} \leqslant C\|f\|_{M_p}. \qquad \square$$

6. Application to the sphere

In this section, we want to explain how the machinery developed in the previous sections can be applied to very specific manifolds, namely to the spheres S^{n-1} contained in $\mathbb{R}^n$. The aim is to derive a generalized windowed Fourier transform on the spheres and to construct the associated atomic Gabor functions. We therefore explain how the basic steps outlined above can be realized for this specific setting. First of all, in section 6.1, we construct a suitable group acting on the Hilbert space $L_2(S^{n-1})$. Here we follow the lines of Torresani [35]. Then, in section 6.2, we introduce and discuss the associated coorbit spaces. In case of the windowed Fourier transform these spaces can be interpreted as generalized modulation spaces. The basic technical step is to establish a generalized Young inequality, i.e., we have to verify (3.1). Section 6.3 is devoted to the frame construction. We therefore have to verify that all the assumptions in theorems 4.1 and 4.2, respectively, can be established.

Although some parts of the theory are presented for the general setting, we shall mainly confine the discussion to the simplest case, that is, to the sphere S^1 contained in $\mathbb{R}^2$. The reason for proceeding this way is to keep the technical difficulties at a reasonable level. The general case S^{n-1} will be discussed in a forthcoming paper.

6.1. Basic setting

In this subsection, we want to establish a suitable group representation for the Hilbert space $\mathcal{H} = L_2(S^{n-1})$. To this end, we shall mainly follow the lines of fundamental approach derived by Torresani [35]. We are interested in building a version of the windowed Fourier transform on the sphere. Since the usual windowed Fourier transform is generated with translations and modulations, we need similar transformations on the sphere. A good candidate to start with is the Euclidean group $E(n)$. Let $SO(n)$ denote the special orthogonal group of rotations in $\mathbb{R}^n$, then

$$\mathcal{G} := E(n) = SO(n) \ltimes \mathbb{R}^n$$

with group operation

$$(R,p)\circ\big(\widetilde{R},\tilde{p}\big)=\big(R\widetilde{R},R\tilde{p}+p\big),\quad (R,p)^{-1}=\big(R^{-1},-R^{-1}p\big). \tag{6.1}$$

The group $\mathcal{G}$ is a separable Lie group with Haar measure ν. As a natural analogue to the Schrödinger representation of the Weyl–Heisenberg group on $L_2(\mathbb{R}^n)$, we consider the continuous unitary representation U of $\mathcal{G}$ on $L_2(S^{n-1})$ defined by

$$\big(U(R,p)\big)f(s):=\mathrm{e}^{\mathrm{i}\langle s,p\rangle}f\big(R^{-1}s\big), \tag{6.2}$$

where $s\in S^{n-1}$. Note that U can be derived in a more sophisticated way by Mackey's induction from some subgroup $\mathcal{P}$ of $\mathcal{G}$ with $\mathcal{G}/\mathcal{P}\cong S^{n-1}$, see, e.g., [35] for details. Unfortunately, there does not exist any function $\psi\in L_2(S^{n-1})$ satisfying

$$\int_{\mathcal{G}}\big|\langle\psi,U\big(g^{-1}\big)\psi\rangle_{L_2(S^{n-1})}\big|^2\,\mathrm{d}\nu(g)<\infty,$$

so that the representation U in $L_2(S^{n-1})$ is not square integrable. However, the way out clearly consists in considering representations modulo a subgroup of $\mathcal{G}$ as explained in section 2.

As already stated above, we shall mainly restrict ourselves to the case $\mathcal{H}=L_2(S^1)$ in the sequel. In this setting, $R\in SO(2)$ and $s\in S^1$ are given explicitly by

$$R=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix},\qquad s=\begin{pmatrix}\sin\gamma\\ \cos\gamma\end{pmatrix}.$$

Hence, we have by this parametrization $L_2(S^1)\cong L_2([-\pi,\pi])$. This leads to

$$U(\theta,p_1,p_2)\psi(\gamma)=\mathrm{e}^{\mathrm{i}(p_1\sin\gamma+p_2\cos\gamma)}\psi(\gamma-\theta). \tag{6.3}$$

To overcome the integrability problem we have to choose an appropriate subgroup. A natural candidate is given by the stability group $\mathcal{G}_{\mathcal{F}}\cong\{(0,0,p_2)\in\mathcal{G}\}$. As explained in the previous sections, the whole construction depends on the choice of the section σ of the principal bundle $\Pi:\mathcal{G}\to\mathcal{G}/\mathcal{G}_{\mathcal{F}}$. In the following, we will primarily consider the flat section defined by $\sigma(\theta,p_1)=(\theta,p_1,0)$. We have to verify that U is strictly square integrable mod $(\mathcal{G}_{\mathcal{F}},\sigma)$. To this end, we have to show that there exists a function $\psi\in L_2(S^1)$ such that the associated wavelet transform

$$\begin{aligned}V_\psi g(h)&=\langle g,U\big(\sigma(h)^{-1}\big)\psi\rangle_{L_2(S^1)}\\&=\langle g,U\big((\theta,p_1,0)^{-1}\big)\psi\rangle_{L_2([-\pi,\pi])}\\&=\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}p_1\sin\gamma}\bar{\psi}(\gamma)g(\gamma-\theta)\,\mathrm{d}\gamma\end{aligned} \tag{6.4}$$

is an isometry. The next lemma can also be found in [35].

Lemma 6.1. Assume that the function $\psi \in L_2([-\pi, \pi])$ is such that $\operatorname{supp} \psi \subset [-\pi/2, \pi/2]$ and

$$2\pi \int_{-\pi/2}^{\pi/2} \frac{|\psi(\gamma)|^2}{\cos\gamma} \, \mathrm{d}\gamma = 1. \tag{6.5}$$

Then the map

$$L_2(S^1) \ni g \quad \mapsto \quad V_\psi g \in L_2(\mathcal{G}/\mathcal{G}_F),$$

where $V_\psi g$ is defined by (6.4) is an isometry.

Proof. Assume that $g \in L_2([-\pi, \pi])$ and $\psi \in L_2([-\pi, \pi])$. Then we can write

$$\begin{aligned} V_\psi g(\theta, p) &= \langle g, U(\sigma(\theta, p)^{-1})\psi \rangle_{L_2(S^1)} = \langle U(\sigma(\theta, p))g, \psi \rangle_{L_2(S^1)} \\ &= \int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{i}p\sin\gamma} g(\gamma - \theta)\bar{\psi}(\gamma) \, \mathrm{d}\gamma. \end{aligned}$$

By using the substitution $\sin\gamma = t$ we obtain

$$\begin{aligned} \int_{\mathcal{G}/\mathcal{G}_F} |V_\psi g(\theta, p)|^2 \, \mathrm{d}\mu(\theta, p) &= \int_{\mathbb{R}} \int_{-\pi}^{\pi} \left| \int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{i}p\sin\gamma} g(\gamma - \theta)\bar{\psi}(\gamma) \, \mathrm{d}\gamma \right|^2 \mathrm{d}\theta \, \mathrm{d}p \\ &= \int_{\mathbb{R}} \int_{-\pi}^{\pi} \left| \int_{-1}^{1} \mathrm{e}^{\mathrm{i}pt} \frac{g(\arcsin t - \theta)\bar{\psi}(\arcsin t)}{\sqrt{1 - t^2}} \, \mathrm{d}t \right|^2 \mathrm{d}\theta \, \mathrm{d}p \end{aligned}$$

and further by Parseval's equality

$$\begin{aligned} \int_{\mathcal{G}/\mathcal{G}_F} |V_\psi g(\theta, p)|^2 \, \mathrm{d}\mu(\theta, p) &= 2\pi \int_{-\pi}^{\pi} \int_{-1}^{1} \left| \frac{g(\arcsin t - \theta)\bar{\psi}(\arcsin t)}{\sqrt{1 - t^2}} \right|^2 \mathrm{d}t \, \mathrm{d}\theta \\ &= 2\pi \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \frac{|g(\gamma - \theta)|^2 |\psi(\gamma)|^2}{\cos\gamma} \, \mathrm{d}\gamma \, \mathrm{d}\theta \\ &= \|g\|_{L_2(S^1)}^2 2\pi \int_{-\pi/2}^{\pi/2} \frac{|\psi(\gamma)|^2}{\cos\gamma} \, \mathrm{d}\gamma. \end{aligned}$$

□

As a consequence, the wavelet transform can be inverted by using the adjoint V_ψ^*. Of course the approach works also if

$$0 < c_\psi := 2\pi \int_{-\pi/2}^{\pi/2} \frac{|\psi(\gamma)|^2}{\cos\gamma} \, \mathrm{d}\gamma < \infty.$$

Then the inverse of the wavelet transform is given by $V_\psi^* / \sqrt{c_\psi}$.

6.2. *Modulation spaces on the sphere S^1*

To construct properly defined modulation spaces, it is clearly necessary to ensure the correspondence principle in theorem 3.1. Therefore we have to establish the basic property (3.1). Hence, we have to verify that $R(l,\cdot) \in L_1(\mathcal{G}/\mathcal{G}_{\mathcal{F}})$ for every $l \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}$ with a norm that can be bounded independently of l. We shall always work with an admissible wavelet ψ in the sense of lemma 6.1, i.e., we assume that supp$\psi \subset [-\pi/2, \pi/2]$ and that condition (6.5) is satisfied. The group law (6.1) combined with the Euler angle parameterization yields for $h = (\theta_h, p_h, 0), l = (\theta_l, p_l, 0) \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}$

$$\sigma(h)\sigma(l)^{-1} = \bigl(\theta_h - \theta_l,\, p_h - p_l\cos(\theta_h - \theta_l),\, p_l \sin(\theta_h - \theta_l)\bigr).$$

We therefore obtain

$$\begin{aligned} R(l,h) &= \int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{i}(\sin\gamma(-p_l\cos\theta + p_h) + \cos\gamma(p_l\sin\theta))}\psi(\gamma-\theta)\bar{\psi}(\gamma)\,\mathrm{d}\gamma \\ &= \int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{i}(p_h\sin\gamma - p_l\sin(\gamma-\theta))}\psi(\gamma-\theta)\bar{\psi}(\gamma)\,\mathrm{d}\gamma, \end{aligned}$$

where $\theta := \theta_h - \theta_l$. By substituting $t = \sin\gamma$ one has

$$R(l,h) = \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}p_l\sin(\arcsin t - \theta)}\mathrm{e}^{\mathrm{i}p_h t}\psi(\arcsin t - \theta)\bar{\psi}(\arcsin t)\frac{\mathrm{d}t}{\sqrt{1-t^2}}.$$

Furthermore, by defining

$$F_{\theta,p_l}(t) := \mathrm{e}^{-\mathrm{i}p_l\sin(\arcsin t - \theta)}\psi(\arcsin t - \theta)\frac{\bar{\psi}(\arcsin t)}{\sqrt{1-t^2}}$$

and recalling the fact that supp$\psi \subset [-\pi/2, \pi/2]$ we may write

$$R(l,h) = \widehat{F}_{\theta,p_l}(-p_h). \tag{6.6}$$

The quasi-invariant measure $\mathrm{d}\mu(h)$ of the quotient space $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$ is given by $\mathrm{d}p_h\,\mathrm{d}\theta_h$, hence we have

$$\int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} \bigl|R(l,h)\bigr|\,\mathrm{d}\mu(h) = \int_{-\pi}^{\pi}\int_{\mathbb{R}} \bigl|\widehat{F}_{\theta,p_l}(p_h)\bigr|\,\mathrm{d}p_h\,\mathrm{d}\theta_h.$$

Interpreting $\int |\widehat{F}_{\theta,p_l}(p_h)|\,\mathrm{d}p_h$ as the inverse Fourier transform at point 0 and regarding that the outer integration is over a finite interval, we see that property (3.1) is equivalent to

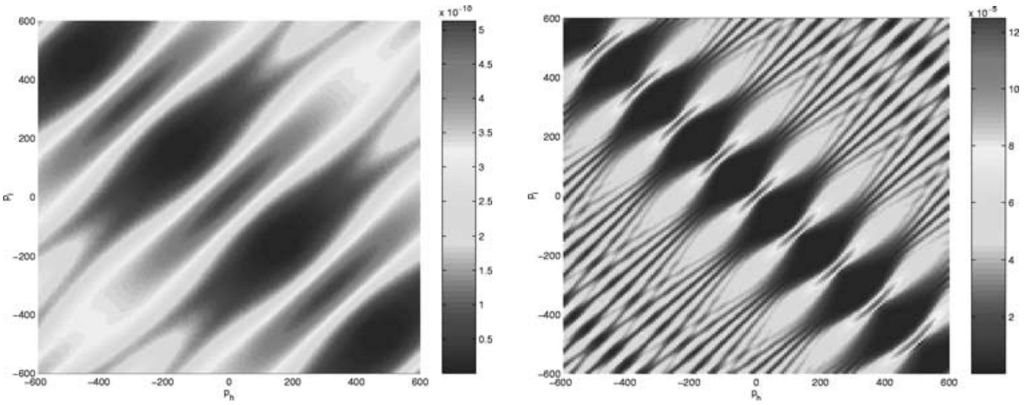
$$\bigl|\widehat{F}_{\theta,p_l}(\cdot)\bigr|^{\vee}(0) < C, \tag{6.7}$$

with some constant C independent of p_l and θ_l.

We have checked numerically that for one of the typical admissible functions suggested by Torresani [35] condition (6.7) is satisfied. We have chosen the function ψ by

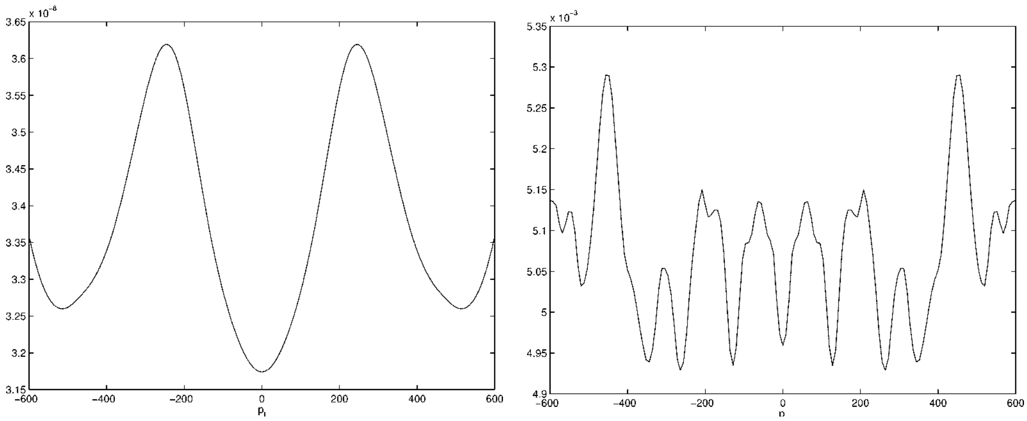
$$\psi(x) = \cos^6 x \cdot \chi_{[-\pi/2,\pi/2]}(x),$$

which is admissible in the sense of lemma 6.1. In figure 1 we have displayed two typical plots of $\widehat{F}_{\theta,p_l}(-p_h)$ for $\theta = -2.7416$ and $\theta = 2.0584$. Numerical experiments were



(a) (b)

Figure 1. (a) $|\widehat{F}_{\theta,p_l}(-p_h)|$ for $\theta = -2.7416$, (b) $|\widehat{F}_{\theta,p_l}(-p_h)|$ for $\theta = 2.0584$.



(a) (b)

Figure 2. $\int|\widehat{F}_{\theta,p_l}(-p_h)|\,\mathrm{d}p_h$ for (a) $\theta = -2.7416$, (b) $\theta = 2.0584$.

done for θ on the whole grid $-\pi/2 : \pi/16 : \pi/2$. These figures indicate that for fixed θ the expression

$$\int \left|\widehat{F}_{\theta,p_l}(-p_h)\right| \mathrm{d}p_h$$

is bounded independently of p_l. This is confined by figure 2 which shows the approximated values of $\int|\widehat{F}_{\theta,p_l}(-p_h)|\,\mathrm{d}p_h$ as functions of p_l. Finally, in figure 3 we have displayed the maximal value of $\int|\widehat{F}_{\theta,p_l}(-p_h)|\,\mathrm{d}p_h$ with respect to p_l as a function of θ. From this figure, we observe that condition (6.7) is satisfied.

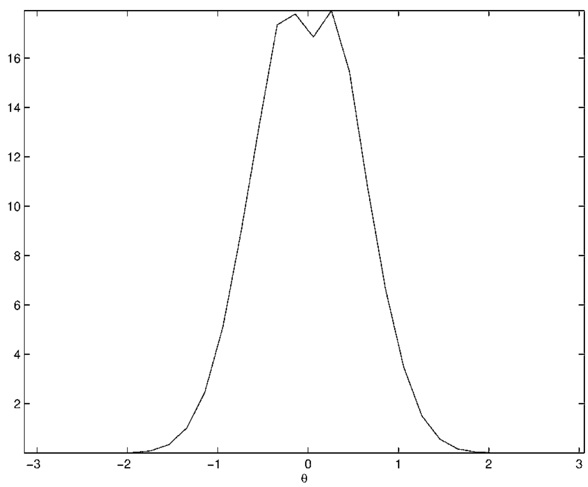


Figure 3. $\max_{p_l} \int |\widehat{F}_{\theta,p_l}(-p_h)|\,\mathrm{d}p_h$ as a function of θ.

6.3. *Banach frames on the sphere S^1*

In this subsection, we want to derive some atomic decompositions and associated Banach frames for the new modulation spaces. To this end, we have to check that all assumptions in theorems 4.1 and 4.2 can be satisfied. Therefore we have to define some neighborhood $\mathcal{U}$ and a related $\mathcal{U}$-dense family X which is relatively separated.

Let $\mathcal{U}$ be given by $\mathcal{U} := (-\pi/N, \pi/N) \times (-\pi/M, \pi/M) \times (-\pi/M, \pi/M)$ and $X := (x_{n,m})_{(n,m)\in\mathcal{I}}$ by $x_{n,m} = (\theta_n, p_m, q_m)$. One basic premise we have to verify is that the right $\mathcal{U}$-oscillation (4.2) fulfills (4.8). For $u = (\theta_u, p_u, q_u) \in \mathcal{U}$ we start by evaluating

$$\sigma(h)\sigma(l)^{-1}u = (\theta + \theta_u,\ p_h - p_l\cos\theta + p_u\cos\theta + q_u\sin\theta,\ p_l\sin\theta - p_u\sin\theta + q_u\cos\theta),$$

where $\theta := \theta_h - \theta_l$. By (6.3) and since $\operatorname{supp}\psi \in [-\pi/2, \pi/2]$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \big(U\big(\sigma(h)\sigma(l)^{-1}\big)\psi(\gamma)\bar{\psi}(\gamma) - U\big(\sigma(h)\sigma(l)^{-1}u\big)\psi(\gamma)\bar{\psi}(\gamma)\big)\,\mathrm{d}\gamma \\
&\quad= \int_{-\pi/2}^{\pi/2} \big(U(\theta, p_h - p_l\cos\theta, p_l\sin\theta)\psi(\gamma)\bar{\psi}(\gamma) \\
&\qquad - U(\theta+\theta_u, p_h - p_l\cos\theta + p_u\cos\theta + q_u\sin\theta, p_l\sin\theta - p_u\sin\theta \\
&\qquad + q_u\cos\theta)\psi(\gamma)\bar{\psi}(\gamma)\big)\,\mathrm{d}\gamma \\
&\quad= \int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{i}(p_h\sin\gamma + p_l\sin(\theta-\gamma))} \\
&\qquad \times \big[\psi(\gamma-\theta) - \mathrm{e}^{\mathrm{i}(p_u\sin(\gamma-\theta) + q_u\cos(\gamma-\theta))}\psi(\gamma-\theta-\theta_u)\big]\bar{\psi}(\gamma)\,\mathrm{d}\gamma
\end{aligned}$$

$$= \int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{i}(p_h \sin\gamma + p_l \sin(\theta-\gamma))} \bigl\{\bigl[\psi(\gamma-\theta) - \psi(\gamma-\theta-\theta_u)\bigr]\bar{\psi}(\gamma) + \bigl[1 - \mathrm{e}^{\mathrm{i}(p_u \sin(\gamma-\theta)+q_u\cos(\gamma-\theta))}\bigr]\psi(\gamma-\theta-\theta_u)\bar{\psi}(\gamma)\bigr\}\,\mathrm{d}\gamma.$$

Now we can estimate $\mathrm{osc}^r_{\mathcal{U}}(l,h)$ by

$$\mathrm{osc}^r_{\mathcal{U}}(l,h) \leqslant \sup_{u\in\mathcal{U}} \left| \int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{i}(p_h \sin\gamma + p_l \sin(\theta-\gamma))} \bigl[\psi(\gamma-\theta) - \psi(\gamma-\theta-\theta_u)\bigr]\bar{\psi}(\gamma)\,\mathrm{d}\gamma \right| + \sup_{u\in\mathcal{U}} \left| \int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{i}(p_h \sin\gamma + p_l \sin(\theta-\gamma))} \bigl[1 - \mathrm{e}^{\mathrm{i}(p_u \sin(\gamma-\theta)+q_u\cos(\gamma-\theta))}\bigr] \times \psi(\gamma-\theta-\theta_u)\bar{\psi}(\gamma)\,\mathrm{d}\gamma \right|.$$

We have to verify that $\mathrm{osc}^r_{\mathcal{U}}(l,h)$ fulfills the conditions (4.8). We restrict our attention to the condition

$$I := \int_{\mathcal{G}/\mathcal{G}_F} \mathrm{osc}^r_{\mathcal{U}}(l,h)\,\mathrm{d}\mu(h) < \frac{1}{C_\psi}.$$

The other condition follows in a similar way. By our estimate of $\mathrm{osc}^r_{\mathcal{U}}(l,h)$, we have that

$$I \leqslant \int_{-\pi}^{\pi} (I_1 + I_2)\,\mathrm{d}\theta_h, \tag{6.8}$$

where

$$I_1 := \int_{\mathbb{R}} \sup_{u\in\mathcal{U}} \left| \int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{i}(p_h \sin\gamma + p_l \sin(\theta-\gamma))} \bigl[\psi(\gamma-\theta) - \psi(\gamma-\theta-\theta_u)\bigr]\bar{\psi}(\gamma)\,\mathrm{d}\gamma \right| \mathrm{d}p_h$$

and

$$I_2 := \int_{\mathbb{R}} \sup_{u\in\mathcal{U}} \left| \int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{i}(p_h \sin\gamma + p_l \sin(\theta-\gamma))} \bigl[1 - \mathrm{e}^{\mathrm{i}(p_u \sin(\gamma-\theta)+q_u\cos(\gamma-\theta))}\bigr] \times \psi(\gamma-\theta-\theta_u)\bar{\psi}(\gamma)\,\mathrm{d}\gamma \right| \mathrm{d}p_h.$$

Substituting $t = \sin\gamma$ in I_1, we get

$$I_1 = \int_{\mathbb{R}} \sup_{u\in\mathcal{U}} \left| \int_{-1}^{1} \mathrm{e}^{\mathrm{i}p_h t} e^{\mathrm{i}p_l \sin(\theta - \arcsin t)} \bigl[\psi(\arcsin t - \theta) - \psi(\arcsin t - \theta - \theta_u)\bigr] \times \frac{\bar{\psi}(\arcsin t)}{\sqrt{1-t^2}}\,\mathrm{d}t \right| \mathrm{d}p_h.$$

By introducing the functions

$$g(t) := \begin{cases} \dfrac{e^{ip_l \sin(\theta - \arcsin t)} \bar{\psi}(\arcsin t)^{1/2}}{\sqrt{1-t^2}} & \text{for } t \in [-1,1], \\ 0 & \text{otherwise,} \end{cases}$$

and

$$w_{\theta_u}(t) := \begin{cases} \big[\psi(\arcsin t - \theta) - \psi(\arcsin t - \theta - \theta_u)\big]\bar{\psi}(\arcsin t)^{1/2} & \text{for } t \in [-1,1], \\ 0 & \text{otherwise,} \end{cases}$$

the above expression can be written as

$$\begin{aligned} I_1 &= \int_{\mathbb{R}} \sup_{u \in \mathcal{U}} \left| \int_{\mathbb{R}} w_{\theta_u}(t) g(t)\, e^{ip_h t}\, dt \right| dp_h \\ &= \int_{\mathbb{R}} \sup_{u \in \mathcal{U}} \big| (\widehat{w}_{\theta_u} * \hat{g})(-p_h) \big|\, dp_h \\ &\leqslant \int_{\mathbb{R}} \sup_{u \in \mathcal{U}} \int_{\mathbb{R}} \big|\widehat{w}_{\theta_u}(v)\big| \big|\hat{g}(p_h - v)\big|\, dv\, dp_h. \end{aligned} \tag{6.9}$$

We choose ψ sufficiently smooth, e.g., $\psi(t) = \cos^6(t)$, so that $w_{\theta_u}^{(r)}(t)$ is a continuous function for some $r \geqslant 2$ and $\hat{g} \in L_1$. Note that $w_{\theta_u}^{(r)}(t)$ has compact support. Then $\lim_{\theta_u \to 0} w_{\theta_u}^{(r)}(t) = 0$ and we obtain by dominated convergence that

$$\lim_{\theta_u \to 0} \big\| w_{\theta_u}^{(r)} \big\|_{L_1} = 0.$$

The Fourier transform maps L_1 continuously onto a dense subalgebra of C_0. Here C_0 denotes the Banach space of continuous functions which tend to zero at $\pm\infty$ with norm

$$\|f\|_\infty := \max\big\{ \big|f(t)\big| : t \in \mathbb{R} \big\}.$$

Thus

$$\lim_{\theta_u \to 0} \big\| \big(w_u^{(r)}\big)\hat{}\, \big\|_\infty = 0. \tag{6.10}$$

Further, we have that

$$\widehat{w}_{\theta_u}(v) = (-iv)^{-r} \big(w_{\theta_u}^{(r)}\big)\hat{}\,(v),$$

which by (6.10) implies

$$\big|\widehat{w}_{\theta_u}(v)\big| \leqslant \big(1 + |v|\big)^{-r} C(\theta_u), \tag{6.11}$$

where $C(\theta_u)$ is a continuous function with $\lim_{\theta_u \to 0} C(\theta_u) = 0$. Inserting (6.11) into (6.9), we get

$$\begin{aligned} I_1 &\leqslant \int_{\mathbb{R}} \sup_{u \in \mathcal{U}} C(\theta_u) \int_{\mathbb{R}} \big(1 + |v|\big)^{-r} \big|\hat{g}(p_h - v)\big|\, dv\, dp_h \\ &= \|\hat{g}\|_{L_1} \sup_{|\theta_u| \leqslant \pi/N} C(\theta_u) \int_{\mathbb{R}} \big(1 + |v|\big)^{-r}\, dv \\ &\leqslant C \sup_{|\theta_u| \leqslant \pi/N} C(\theta_u). \end{aligned}$$

This expression becomes arbitrary small for sufficiently large N. The term I_2 can be treated in a similar way. Now (4.8) follows by (6.8).

Similarly one can prove that $\operatorname{osc}^l_{\mathcal{U}}(l, h)$ fulfills (4.3) for sufficiently small $\mathcal{U}$. Finally, it is easy to check that

$$\mu\{h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}\colon \sigma(h) \in \mathcal{Q}\sigma(h_i)\} \geqslant C_{\mathcal{Q}}$$

for all $i \in I_\sigma$ as follows: Let $\mathcal{Q}$ be of the standard form $\mathcal{Q} = [-\pi/N, \pi/N] \times [-\pi/M, \pi/M] \times [-\pi/M, \pi/M]$ and let $\sigma(h_i) = (\theta_i, p_i, 0)$. For $l = (\gamma, q_1, q_2) \in \mathcal{Q}$ we obtain

$$\begin{aligned}(\gamma, q_1, q_2) \circ \sigma(h_i) &= (\gamma, q_1, q_2) \circ (\theta_i, p_i, 0) \\ &= \big(\gamma + \theta_i, q_1 + \cos(\gamma) p_i, q_2 - \sin(\gamma) p_i\big).\end{aligned}$$

The term on the right-hand side can be interpreted as some $\sigma(h)$, $h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}$ if $q_2 - \sin(\gamma) p_i = 0$, i.e.,

$$\sin\gamma = \frac{q_2}{p_i} \quad \text{if } p_i \neq 0, \qquad q_2 = 0 \quad \text{if } p_i = 0.$$

For fixed $p_i \neq 0$, the above equation can be satisfied if $q_2 \in [-\varepsilon, \varepsilon]$ and $\gamma \in [-\delta, \delta]$ for some sufficiently small parameters ε and δ. Then we obtain

$$(\gamma, q_1, q_2) \circ \sigma(h_i) = \big(\gamma + \theta_i, q_1 + \big(p_i^2 - q_2^2\big)^{1/2}, 0\big).$$

For $\gamma \in [-\delta, \delta]$, $q_2 \in [-\varepsilon, \varepsilon]$ and $q_1 \in [-\pi/M, \pi/M]$ this set has obviously a positive measure.

The remaining condition (4.4) can be checked numerically by performing similar calculations as in section 6.2.

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