

# THE CONTINUOUS SHEARLET TRANSFORM IN ARBITRARY SPACE DIMENSIONS

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ABSTRACT. This paper is concerned with the generalization of the continuous shearlet transform to higher dimensions. Similar to the two-dimensional case, our approach is based on translations, anisotropic dilations and specific shear matrices. We show that the associated integral transform again originates from a square-integrable representation of a specific group, the full  $n$ -variate shearlet group. Moreover, we verify that by applying the coorbit theory, canonical scales of smoothness spaces and associated Banach frames can be derived. We also indicate how our transform can be used to characterize singularities in signals.

## 1. INTRODUCTION

Modern technology allows for easy creation, transmission and storage of huge amounts of data. Confronted with a flood of data, such as internet traffic, or audio and video applications, nowadays the key problem is to extract the relevant information from these sets. To this end, usually the first step is to decompose the signal with respect to suitable building blocks which are well-suited for the specific application and allow a fast and efficient extraction. In this context, one particular problem which is currently in the center of interest is the analysis of *directional* information. Due to the bias to the coordinate axes, classical approaches such as, e.g., wavelet or Gabor transforms are clearly not the best choices, and hence new building blocks have to be developed. In recent studies, several approaches have been suggested such as ridgelets [2], curvelets [3], contourlets [9], shearlets [19] and many others. For a general approach see also [18]. Among all these approaches, the shearlet transform stands out because it is related to group theory, i.e., this transform can be derived from a square-integrable representation  $\pi : \mathcal{S} \rightarrow \mathcal{U}(L_2(\mathbb{R}^2))$  of a certain group  $\mathcal{S}$ , the so-called *shearlet group*, see [7]. An admissible function with respect to this group is called a shearlet. Therefore, in the context of the shearlet transform, all the powerful tools of group representation theory can be exploited.

So far, the shearlet transform is well developed for problems in  $\mathbb{R}^2$ . Given a shearlet  $\psi \in L_2(\mathbb{R}^2)$ , a signal  $f \in L_2(\mathbb{R}^2)$  can be analyzed by its voice transform, the *shearlet transform*

$$\mathcal{SH}_\psi f(a, s, t) = \langle f, \pi(a, s, t) \psi(\cdot) \rangle = \langle f, |a|^{-\frac{3}{4}} \psi(A_a^{-1} S_s^{-1}(\cdot - t)) \rangle, \quad (1)$$

where  $a \in \mathbb{R}^*$ ,  $s \in \mathbb{R}$ ,  $t \in \mathbb{R}^2$  and

$$A_a := \begin{pmatrix} a & 0 \\ 0 & \operatorname{sgn}(a)\sqrt{|a|} \end{pmatrix} \quad \text{and} \quad S_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad (2)$$

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denote the *parabolic scaling matrix* and the *shear matrix*. By now, the continuous shearlet transform of two-dimensional functions is already well-established; however, for analyzing *higher-dimensional* data sets, there is clearly an urgent need for further generalizations. This is exactly the concern of this paper.

To our best knowledge, it seems that there exist only few results in this direction: general semidirect products of subgroups of  $GL_n(\mathbb{R})$  with the translation group were considered with respect to square integrability in [21], see also [18]. Admissible subgroups of the semidirect product of the Heisenberg group and the symplectic group were examined in [5]. Some important progress has been achieved for the curvelet case in [1] and for surfacelets in [22]. Our paper provides the first detailed higher-dimensional shearlet approach.

The first step towards a higher-dimensional shearlet transform is the identification of a suitable shear matrix. Given an  $n$ -dimensional vector space  $V$  and a  $k$ -dimensional subspace  $W$  of  $V$ , a reasonable model reads as follows: the shear should fix the space  $W$  and translate all vectors parallel to  $W$ . That is, for  $V = W \oplus W'$  and  $v = w + w'$ , the shear operation  $S$  can be described as  $S(v) = w + (w' + s(w'))$  where  $s$  is a linear mapping from  $W'$  to  $W$ . Then, with respect to an appropriate basis of  $V$ , the shear operation  $S$  corresponds to a block matrix of the form

$$S = \begin{pmatrix} I_k & s^T \\ 0 & I_{n-k} \end{pmatrix}, \quad s \in \mathbb{R}^{n-k,k}.$$

Then we are faced with the problem how to choose the block  $s$ . Since we want to end up with a square integrable group representation, one has to be careful. Usually, the number of parameters has to fit together with the space dimension, for otherwise the resulting group would be either too large or too small. Since we have  $n$  degrees of freedom related with the translates and one degree of freedom related with the dilation,  $n - 1$  degrees of freedom for the shear component would be optimal. Therefore one natural choice would be  $s \in \mathbb{R}^{n-1,1}$ , i.e.,  $k = 1$ . Indeed, in Section 2 we show that with this choice the associated multivariate shearlet transform can be interpreted as a square integrable group representation of a  $(2n)$ -parameter group, the full shearlet group. It is a remarkable fact that this choice is in some sense a canonical one, other  $(n - 1)$ -parameter choices might lead to nice group structures, but the representation will usually not be square integrable, see Remark 2.5 for details.

Once we have established a square integrable group representation, there is a very natural link to another useful concept, namely the coorbit space theory introduced by Feichtinger and Gröchenig in a series of papers [10, 11, 12, 13, 15]. By means of the coorbit space theory, it is possible to derive in a very natural way scales of smoothness spaces associated with the group representation. In this setting, the smoothness of functions is measured by the decay of the associated voice transform. Moreover, by a tricky discretization of the representation, it is possible to obtain (Banach) frames for these smoothness spaces. Fortunately, it turns out that for our multivariate continuous shearlet transform, all the necessary conditions for the application of the coorbit space theory can be established, so that we end up with new canonical smoothness spaces, the multivariate shearlet coorbit spaces, together with their discretizations.

One of the most important advantages of the two-dimensional continuous shearlet transform is the fact that it can be used to analyze singularities. Indeed, as outlined in [20], see also [4] for curvelets, it turns out that the decay of the continuous shearlet transform exactly describes the location and orientation of the singularities. By our approach these characterizations carry over to higher-dimensions.

This paper is organized as follows: in Section 2, we introduce the multivariate continuous shearlet transform and investigate its properties. We establish the full shearlet group and show that its

representation in  $L_2(\mathbb{R}^n)$  is indeed square integrable. Then, in Section 3, we discuss the relations with coorbit space theory. We show that all the necessary building block for the application of this theory can be provided. Consequently, the shearlet coorbit spaces and the associated families of Banach frames can be established. Finally, in Section 4, we investigate how the  $n$ -variate shearlet transform can be used to detect singularities in signals. It turn out that, similar to the two-dimensional case, the shape of the singularity is reflected by the decay of the shearlet transform.

## 2. MULTIVARIATE CONTINUOUS SHEARLET TRANSFORM

In this section, we introduce the shearlet transform on  $L_2(\mathbb{R}^n)$ . This requires the generalization of the parabolic dilation matrix and of the shear matrix. We will start with a rather general definition of shearlet groups in Subsection 2.1 and then restrict ourselves to those groups having square integrable representations in Subsection 2.1. Let  $I_n$  denote the  $(n, n)$ -identity matrix and  $0_n$ , resp.  $1_n$  the vectors with  $n$  entries 0, resp. 1.

**2.1. Unitary Representations of the Shearlet Group.** We define *dilation matrices* depending on one parameter  $a \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$  by

$$A_a := \text{diag}(a_1(a), \dots, a_n(a)),$$

where  $a_1(a) := a$  and  $a_j(a) = \text{sgn}(a)|a|^{\alpha_j}$  with  $\alpha_j \in (0, 1)$ ,  $j = 2, \dots, n$ . In order to have directional selectivity, the dilation factors at the diagonal of  $A_a$  should be chosen in an anisotropic way, i.e.,  $|a_k(a)|$ ,  $k = 2, \dots, n$  should increase less than linearly in  $a$  as  $a \rightarrow \infty$ . Our favorite choice will be

$$A_a := \begin{pmatrix} a & 0_{n-1}^T \\ 0_{n-1} & \text{sgn}(a)|a|^{\frac{1}{n}} I_{n-1} \end{pmatrix}. \quad (3)$$

In Section 4, we will see that this choice leads to an increase of the shearlet transform at hyperplane singularities as  $|a| \rightarrow 0$ . For fixed  $k \in \{1, \dots, n\}$ , we define our *shear matrices* by

$$S = \begin{pmatrix} I_k & s^T \\ 0_{n-k,k} & I_{n-k} \end{pmatrix}, \quad s \in \mathbb{R}^{n-k,k}. \quad (4)$$

The shear matrices form a subgroup of  $GL_n(\mathbb{R})$ .

**Remark 2.1.** Shear matrices on  $\mathbb{R}^n$  were also considered in [18]. We want to show the relation of those matrices to our setting (4). The authors in [18] call  $S \in \mathbb{R}^{n,n}$  a *general shear matrix* if

$$(I_n - S)^2 = 0_{n,n}. \quad (5)$$

Of course, our matrices in (4) fulfill this condition. Condition (5) is equivalent to the fact that  $S$  decomposes as

$$S = P^{-1} \text{diag}(J_1, \dots, J_r, 1_{n-2r}) P, \quad J_j := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad r \leq n/2.$$

With  $P := (p_1, \dots, p_n)$  and  $P^{-1} = (q_1, \dots, q_n)^T$  this can be written as

$$S = I_n + \sum_{j=1}^r q_{2j-1} p_{2j}^T, \quad \text{with } p_{2j}^T q_{2i-1} = 0, \quad i, j = 1, \dots, r.$$

Matrices of the type  $S_{qp} := I_n + qp^T$  with  $p^T q = 0$  are called *elementary shear matrices*. The general shear matrices do not form a group. In particular, the product of two elementary shear matrices  $S_{q_1 p_1}$  and  $S_{q_2 p_2}$  is again a shear matrix if and only if the matrices commute which is the case if and only if  $p_1^T q_2 = p_2^T q_1 = 0$ . Then  $S_{q_1 p_1} S_{q_2 p_2} = I_n + \sum_{j=1}^2 q_j p_j^T$  holds true. Hence we see

that any general shear matrix is the product of elementary shear matrices. In [18] any subgroup of  $GL_n(\mathbb{R})$  generated by *finitely* many pairwise commuting elementary matrices is called a shear group. A shear group is maximal if it is not a proper subgroup of any other shear group. It is not difficult to show that maximal shear groups are those of the form

$$G := \left\{ I_n + \begin{pmatrix} k \\ \sum_{i=1}^k c_i q_i \end{pmatrix} \begin{pmatrix} n-k \\ \sum_{j=1}^{n-k} d_j p_j^T \end{pmatrix} : c_i, d_j \in \mathbb{R} \right\}, \quad p_j^T q_i = 0,$$

with linearly independent vectors  $q_i$ ,  $i = 1, \dots, k$ , resp.,  $p_j$ ,  $j = 1, \dots, k$ . Let  $\{\tilde{q}_i : i = 1, \dots, k\}$  be the dual basis of  $\{q_i : i = 1, \dots, k\}$  in the linear space  $V$  spanned by these vectors and let  $\{\tilde{p}_j : j = 1, \dots, n-k\}$  be the dual basis of  $\{p_j : j = 1, \dots, n-k\}$  in  $V^\perp$ . Set  $P := (q_1, \dots, q_k, \tilde{p}_1, \dots, \tilde{p}_{n-k})$  so that  $P^{-1} = (\tilde{q}_1, \dots, \tilde{q}_k, p_1, \dots, p_{n-k})^T$ . Then we see that for all  $S \in G$

$$P^{-1} S P = \begin{pmatrix} I_k & c d^T \\ 0_{n-k,k} & I_{n-k} \end{pmatrix}, \quad c = (c_1, \dots, c_k)^T, d = (d_1, \dots, d_{n-k})^T.$$

In other words, up to a basis exchange, the maximal shear groups  $G$  coincide with our block matrix groups in (4).

For our shearlet transform we have to combine dilation matrices and shear matrices. Let  $A_{a,1} := \text{diag}(a_1, \dots, a_k)$  and  $A_{a,2} := \text{diag}(a_{k+1}, \dots, a_n)$ . We will use the relations

$$S_s^{-1} = \begin{pmatrix} I_k & -s^T \\ 0_{n-k,k} & I_{n-k} \end{pmatrix} \quad \text{and} \quad S_s A_a S_{s'} A_{a'} = S_{s+A_{a,2}^{-1} s' A_{a,1}} A_{aa'}. \quad (6)$$

For the special setting in (3), the last relation simplifies to

$$S_s A_a S_{s'} A_{a'} = S_{s+|a|^{1-\frac{1}{n}} s'} A_{aa'}.$$

**Lemma 2.2.** *The set  $\mathbb{R}^* \times \mathbb{R}^{k(n-k)} \times \mathbb{R}^n$  endowed with the operation*

$$(a, s, t) \circ (a', s', t') = (aa', s + A_{a,2}^{-1} s' A_{a,1}, t + S_s A_a t')$$

*is a locally compact group  $\mathbb{S}$ . The left and right Haar measures on  $\mathbb{S}$  are given by*

$$d\mu_l(a, s, t) = \frac{|\det A_{a,2}|^{k-1}}{|a| |\det A_{a,1}|^{n-k+1}} da ds dt \quad \text{and} \quad d\mu_r(a, s, t) = \frac{1}{|a|} da ds dt.$$

*Proof.* By the left relation in (6) it follows that  $e := (1, 0_{n-1}, 0_n)$  is the neutral element in  $\mathbb{S}$  and that the inverse of  $(a, s, t) \in \mathbb{R}^* \times \mathbb{R}^{k(n-k)} \times \mathbb{R}^n$  is given by

$$(a, s, t)^{-1} = (a^{-1}, -A_{a,2} s A_{a,1}^{-1}, -A_a^{-1} S_s^{-1} t).$$

By straightforward computation it can be checked that the multiplication is associative.

Further, we have for a function  $F$  on  $\mathbb{S}$  that

$$\int_{\mathbb{S}} F((a', s', t') \circ (a, s, t)) d\mu_l(a, s, t) = \int_{\mathbb{R}} \int_{\mathbb{R}^{k(n-k)}} \int_{\mathbb{R}^n} F(a'a, s' + A_{a',2}^{-1} s A_{a',1}, t' + S_{s'} A_{a'} t) d\mu_l(a, s, t).$$

By substituting  $\tilde{t} := t' + S_{s'} A_{a'} t$ , i.e.,  $d\tilde{t} = |\det A_{a'}| dt$  and  $\tilde{s} := s' + A_{a',2}^{-1} s A_{a',1}$ , i.e.,

$d\tilde{s} = |\det A_{a',1}|^{n-k} / |\det A_{a',2}|^k ds$  and  $\tilde{a} := a'a$  this can be rewritten as

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^{k(n-k)}} \int_{\mathbb{R}} F(\tilde{a}, \tilde{s}, \tilde{t}) \frac{1}{|\det A_{a'}|} \frac{|\det A_{a',2}|^k}{|\det A_{a',1}|^{n-k}} \frac{1}{|a'|} \frac{|a'| |\det A_{a',1}|^{n-k+1}}{|\det A_{a',2}|^{k-1}} d\mu_l(\tilde{a}, \tilde{s}, \tilde{t})$$

so that  $d\mu_l$  is indeed the left Haar measure on  $\mathbb{S}$ . Similarly we can verify that  $d\mu_r$  is the right Haar measure on  $\mathbb{S}$ .  $\square$

In the following, we use only the left Haar measure and the abbreviation  $d\mu = d\mu_l$ . For  $f \in L_2(\mathbb{R}^n)$  we define

$$\pi(a, s, t)f(x) = f_{a,s,t}(x) := |\det A_a|^{-\frac{1}{2}} f(A_a^{-1}S_s^{-1}(x-t)). \quad (7)$$

It is easy to check that  $\pi : \mathbb{S} \rightarrow \mathcal{U}(L_2(\mathbb{R}^n))$  is a mapping from  $\mathbb{S}$  into the group  $\mathcal{U}(L_2(\mathbb{R}^n))$  of unitary operators on  $L_2(\mathbb{R}^n)$ . The *Fourier transform* of  $f_{a,s,t}$  is given by

$$\begin{aligned} \hat{f}_{a,s,t}(\omega) &= \int_{\mathbb{R}^n} f_{a,s,t}(x) e^{-2\pi i \langle x, \omega \rangle} dx \\ &= |\det A_a|^{\frac{1}{2}} e^{-2\pi i \langle t, \omega \rangle} \hat{f}(A_a^T S_s^T \omega) \\ &= |\det A_a|^{\frac{1}{2}} e^{-2\pi i \langle t, \omega \rangle} \hat{f} \left( \begin{array}{c} a\tilde{\omega}_1 \\ A_{a,2}(s\tilde{\omega}_1 + \tilde{\omega}_2) \end{array} \right) \end{aligned} \quad (8)$$

where  $\tilde{\omega}_1 := (\omega_1, \dots, \omega_k)^T$  and  $\tilde{\omega}_2 := (\omega_{k+1}, \dots, \omega_n)^T$ .

Recall that a *unitary representation* of a locally compact group  $G$  with the left Haar measure  $\mu$  on a Hilbert space  $\mathcal{H}$  is a homomorphism  $\pi$  from  $G$  into the group of unitary operators  $\mathcal{U}(\mathcal{H})$  on  $\mathcal{H}$  which is continuous with respect to the strong operator topology.

**Lemma 2.3.** *The mapping  $\pi$  defined by (7) is a unitary representation of  $\mathbb{S}$ .*

*Proof.* Let  $\psi \in L_2(\mathbb{R}^n)$ ,  $x \in \mathbb{R}^n$ , and  $(a, s, t), (a', s', t') \in \mathbb{S}$ . Using (6) we obtain

$$\begin{aligned} \pi(a, s, t)(\pi(a', s', t')\psi)(x) &= |\det A_a|^{-\frac{1}{2}} \pi(a', s', t')\psi(A_a^{-1}S_s^{-1}(x-t)) \\ &= |\det A_{aa'}|^{-\frac{1}{2}} \psi(A_{a'}^{-1}S_{s'}^{-1}(A_a^{-1}S_s^{-1}(x-t) - t')) \\ &= |\det A_{aa'}|^{-\frac{1}{2}} \psi(A_{a'}^{-1}S_{s'}^{-1}A_a^{-1}S_s^{-1}(x - (t + S_s A_a t'))) \\ &= |\det A_{aa'}|^{-\frac{1}{2}} \psi(A_{aa'}^{-1}S_{s+A_{a,2}^{-1}s' A_{a,1}}^{-1}(x - (t + S_s A_a t'))) \\ &= \pi((a, s, t) \circ (a', s', t'))\psi(x). \end{aligned}$$

$\square$

**2.2. Square Integrable Representations of the Shearlet Group.** A nontrivial function  $\psi \in L_2(\mathbb{R}^n)$  is called *admissible*, if

$$\int_{\mathbb{S}} |\langle \psi, \pi(a, s, t)\psi \rangle|^2 d\mu(a, s, t) < \infty.$$

If  $\pi$  is irreducible and there exists at least one admissible function  $\psi \in L_2(\mathbb{R}^n)$ , then  $\pi$  is called *square integrable*.

In the rest of this paper, we deal with shear matrices (4) with  $k = 1$ , i.e.,

$$S_s := \begin{pmatrix} 1 & s^T \\ 0_{n-1} & I_{n-1} \end{pmatrix}, \quad s \in \mathbb{R}^{n-1}. \quad (9)$$

Then we have that

$$d\mu_l(a, s, t) = \frac{1}{|a|^{n+1}} da ds dt.$$

We will see in Remark 2.5 that this choice is canonical. The following result shows that in this case the unitary representation  $\pi$  defined in (7) is square integrable.

**Theorem 2.4.** *A function  $\psi \in L_2(\mathbb{R}^n)$  is admissible if and only if it fulfills the admissibility condition*

$$C_\psi := \int_{\mathbb{R}^n} \frac{|\hat{\psi}(\omega)|^2}{|\omega_1|^n} d\omega < \infty. \quad (10)$$

Then, for any  $f \in L_2(\mathbb{R}^n)$ , the following equality holds true:

$$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t} \rangle|^2 d\mu(a, s, t) = C_\psi \|f\|_{L_2(\mathbb{R}^n)}^2. \quad (11)$$

In particular, the unitary representation  $\pi$  is irreducible and hence square integrable.

*Proof.* Employing the Plancherel theorem and (8), we obtain with  $\psi_{a,s,0}^*(x) := \overline{\psi_{a,s,0}(-x)}$  that

$$\begin{aligned} \int_{\mathbb{S}} |\langle f, \psi_{a,s,t} \rangle|^2 d\mu(a, s, t) &= \int_{\mathbb{S}} |f * \psi_{a,s,0}^*(t)|^2 dt ds \frac{da}{|a|^{n+1}} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} |\hat{f}(\omega)|^2 |\hat{\psi}_{a,s,0}^*(\omega)|^2 d\omega ds \frac{da}{|a|^{n+1}} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} |\hat{f}(\omega)|^2 |\det A_a| |\hat{\psi}(A_a^T S_s^T \omega)|^2 d\omega ds \frac{da}{|a|^{n+1}} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} |\hat{f}(\omega)|^2 \frac{|\det A_{a,2}|}{|a|^n} |\hat{\psi}\left(\begin{smallmatrix} a\omega_1 \\ A_{a,2}(\tilde{\omega} + \omega_1 s) \end{smallmatrix}\right)|^2 ds d\omega da. \end{aligned} \quad (12)$$

Substituting  $\tilde{\xi} := A_{a,2}(\tilde{\omega} + \omega_1 s)$ , i.e.,  $|\det A_{a,2}| |\omega_1|^{n-1} ds = d\tilde{\xi}$ , we obtain

$$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t} \rangle|^2 d\mu(a, s, t) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} |\hat{f}(\omega)|^2 |a|^{-n} |\omega_1|^{-(n-1)} |\hat{\psi}\left(\begin{smallmatrix} a\omega_1 \\ \tilde{\xi} \end{smallmatrix}\right)|^2 d\tilde{\xi} d\omega da.$$

Next, we substitute  $\xi_1 := a\omega_1$ , i.e.,  $\omega_1 da = d\xi_1$  which results in

$$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t} \rangle|^2 d\mu(a, s, t) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} |\hat{f}(\omega)|^2 \frac{|\omega_1|^n}{|\xi_1|^n |\omega_1|^n} |\hat{\psi}\left(\begin{smallmatrix} \xi_1 \\ \tilde{\xi} \end{smallmatrix}\right)|^2 d\tilde{\xi} d\omega d\xi_1 = C_\psi \|f\|_{L_2(\mathbb{R}^n)}^2.$$

Setting  $f := \psi$ , we see that  $\psi$  is admissible if and only if  $C_\psi$  is finite.

The irreducibility of  $\pi$  follows from (11) in the same way as in [6].  $\square$

By the following remark, the choice of the shear matrix  $S_s$  with  $k = 1$  is canonical to ensure that  $\pi$  is a square integrable representation.

**Remark 2.5.** Assume that our shear matrix has the form (4) with  $s^T = (s_{ij})_{i,j=1}^{p,n-p} \in \mathbb{R}^{k,n-k}$ . Let  $s$  contain  $N$  different entries (variables). We assume that  $N \geq n - 1$  since we have one dilation parameter and otherwise the group becomes too small. Then we obtain instead of (12)

$$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t} \rangle|^2 d\mu(a, s, t) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} |\hat{f}(\omega)|^2 |\det A_a| |\hat{\psi}(A_a \left(\begin{smallmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 + s\tilde{\omega}_1 \end{smallmatrix}\right))|^2 d\mu(a, s, t) \quad (13)$$

where  $\tilde{\omega}_1 := (\omega_1, \dots, \omega_k)^T$  and  $\tilde{\omega}_2 := (\omega_{k+1}, \dots, \omega_n)^T$ . Now we can use the following substitution procedure:

$$\xi_{k+1} := (\omega_{k+1} + s_{11}\omega_1 + \dots + s_{1k}\omega_k), \quad (14)$$

i.e.,  $d\xi_{k+1} = |\omega_1| ds_{11}$  and with corresponding modifications if some of the  $s_{1j}$ ,  $j > 1$  are the same as  $s_{11}$ . Then we replace  $s_{11}$  in the other rows of  $\tilde{\omega}_2 + s\tilde{\omega}_1$  where it appears by (14). Next we continue to substitute the second row if it contains an integration variable from  $s$  ( $\neq s_{11}$ ). Continuing this substitution process up to the final row we have at the end replaced the lower  $n - k$  values in  $\hat{\psi}$  by  $n - r$ ,  $r \leq k$  variables  $\xi_1 = \xi_{j_1}, \dots, \xi_{j_{n-r}}$  and some functions depending only on  $a, \omega, \xi_{j_1}, \dots, \xi_{j_{n-r}}$ .

Consequently, the integrand depends only on these variables. However, we have to integrate over  $a, \omega, \xi_{j_1}, \dots, \xi_{j_{n-r}}$  and over the remaining  $N - (n - r)$  variables from  $s$ . But then the integral in (13) becomes infinity unless  $N = n - r$ . Since  $n - 1 \leq N$  this implies  $r = k = 1$ , i.e., our choice of  $S_s$  with (9).

A function  $\psi \in L_2(\mathbb{R}^n)$  fulfilling the admissibility condition (10) is called a *continuous shearlet*, the transform  $\mathcal{SH}_\psi : L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{S})$ ,

$$\mathcal{SH}_\psi f(a, s, t) := \langle f, \psi_{a,s,t} \rangle = (f * \psi_{a,s,0}^*)(t), \quad (15)$$

*continuous shearlet transform* and  $\mathbb{S}$  defined in Lemma 2.2 with (9) a *shearlet group*.

**Remark 2.6.** An example of a continuous shearlet can be constructed as follows: Let  $\psi_1$  be a continuous wavelet with  $\hat{\psi}_1 \in C^\infty(\mathbb{R})$  and  $\text{supp } \hat{\psi}_1 \subseteq [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$ , and let  $\psi_2$  be such that  $\hat{\psi}_2 \in C^\infty(\mathbb{R}^{n-1})$  and  $\text{supp } \hat{\psi}_2 \subseteq [-1, 1]^{n-1}$ . Then the function  $\psi \in L^2(\mathbb{R}^n)$  defined by

$$\hat{\psi}(\omega) = \hat{\psi}(\omega_1, \tilde{\omega}) = \hat{\psi}_1(\omega_1) \hat{\psi}_2\left(\frac{1}{\omega_1} \tilde{\omega}\right)$$

is a continuous shearlet. The support of  $\hat{\psi}$  is depicted for  $\omega_1 \geq 0$  in Fig. 1.

**Remark 2.7.** In [21] the authors consider admissible subgroups  $G$  of  $GL_n(\mathbb{R})$ , i.e., those subgroups for which the semidirect product with the translation group gives rise to a square integrable representation  $\pi(g, t)f(x) = |\det g|^{-\frac{1}{2}}f(g^{-1}(x - t))$ . Let  $\Delta$  denotes the modular function on  $G$ , i.e.,  $d\mu(g) = \Delta(g)d\mu_r(g)$  and write  $\Delta \equiv |\det|$  to mean that  $\Delta(g) = |\det g|$  for all  $g \in G$ . Then [21] contains the following result:

- i) If  $G$  is admissible, then  $\Delta \not\equiv |\det|$  and  $G_x^0 := \{g \in G : gx = x\}$  is compact for a.e.  $x \in \mathbb{R}^n$ .
- ii) If  $\Delta \not\equiv |\det|$  and for a.e.  $x \in \mathbb{R}^n$  there exists  $\varepsilon(x) > 0$  such that  $G_x^\varepsilon := \{g \in G : |gx - x| \leq \varepsilon(x)\}$  is compact, then  $G$  is admissible.

Unfortunately, the above conditions "just fail" to be a characterization of admissibility by the " $\varepsilon$ -gap" in the compactness condition. In our case we have that  $\Delta \not\equiv |\det|$  since  $|a|^{-n} \neq |a| |a|^{\alpha_2 + \dots + \alpha_n}$  for  $|a| \neq 1$ . Further,  $G_x^0 = (1, 0_{n-1})$  a.e. and  $G_x^\varepsilon = \{(a, s) : |a| \in [1 - \varepsilon_1, 1 + \varepsilon_1], s_j \in [-\varepsilon_j, \varepsilon_j], j = 2, \dots, n\}$  a.e. for some small  $\varepsilon_j$ , so that the necessary condition i) and the sufficient condition ii) are fulfilled.

### 3. MULTIVARIATE SHEARLET COORBIT THEORY

In this section we want to establish a coorbit theory based on the square integrable representation (7) of the shearlet group defined with (3) and (9). We mainly follow the lines of [6]. For further information on coorbit space theory, the reader is referred to [10, 11, 12, 13, 15].

**3.1. Shearlet Coorbit Space.** We consider weight functions  $w(a, s, t) = w(a, s)$  that are locally integrable with respect to  $a$  and  $s$ , i.e.,  $w \in L_1^{loc}(\mathbb{R}^n)$  and fulfill  $w((a, s, t) \circ (a', s', t')) \leq w(a, s, t)w(a', s', t')$  and  $w(a, s, t) \geq 1$  for all  $(a, s, t), (a', s', t') \in \mathbb{S}$ . For  $1 \leq p < \infty$ , let

$$L_{p,w}(\mathbb{S}) := \{F \text{ measurable on } \mathbb{S} : \|F\|_{L_{p,w}(\mathbb{S})} := \left( \int_{\mathbb{S}} |F(g)|^p w(a, s, t)^p d\mu(a, s, t) \right)^{1/p} < \infty\},$$

and let  $L_{\infty,w}$  be defined with the usual modifications. In order to construct the coorbit spaces related to the shearlet group we have to ensure that there exists a function  $\psi \in L_2(\mathbb{R}^n)$  such that

$$\mathcal{SH}_\psi(\psi) = \langle \psi, \pi(a, s, t)\psi \rangle \in L_{1,w}(\mathbb{S}). \quad (16)$$

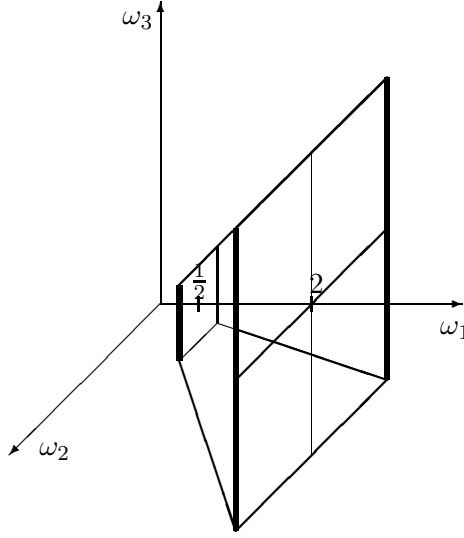


FIGURE 1. Support of the shearlet  $\hat{\psi}$  in Remark 2.6 for  $\omega_1 \geq 0$ .

Concerning the integrability of group extensions we also mention [17]. To this end, we need a preliminary lemma on the support of  $\psi$ .

**Lemma 3.1.** *Let  $a_1 > a_0 \geq \alpha > 0$  and  $b = (b_1, \dots, b_{n-1})^T$  be a vector with positive components. Suppose that  $\text{supp } \hat{\psi} \subseteq ([-a_1, -a_0] \cup [a_0, a_1]) \times Q_b$ , where  $Q_b := [-b_1, b_1] \times \dots \times [-b_{n-1}, b_{n-1}]$ . Then  $\hat{\psi}_{a,s,0} \neq 0$  implies  $a \in [-\frac{a_1}{a_0}, -\frac{a_0}{a_1}] \cup [\frac{a_0}{a_1}, \frac{a_1}{a_0}]$  and  $s \in Q_c$ , where  $c := \frac{1+(a_1/a_0)^{1/n}}{a_0} b$ .*

*Proof.* Let us first discuss the case  $a > 0$ . By (8) we see that the following conditions are necessary for  $\hat{\psi}(\omega)\hat{\psi}_{a,s,0}(\omega) \neq 0$ :

- i)  $a_0 \leq \omega_1 \leq a_1$  and  $\frac{a_0}{a} \leq \omega_1 \leq \frac{a_1}{a}$  or
- ii)  $-a_1 \leq \omega_1 \leq -a_0$  and  $-\frac{a_1}{a} \leq \omega_1 \leq -\frac{a_0}{a}$ ,
- ii)  $-b \leq \tilde{\omega} \leq b$  and  $-a^{-1/n}b - \omega_1 s \leq \tilde{\omega} \leq a^{-1/n}b - \omega_1 s$

where  $\tilde{\omega} := (\omega_2, \dots, \omega_{n-1})^T$  and ii) is meant componentwise. Condition i) implies that

$$a \in \left[ \frac{a_0}{a_1}, \frac{a_1}{a_0} \right]. \quad (17)$$

For  $s_i \geq 0$  and  $a_0 \leq \omega_1 \leq a_1$  the second condition in ii) becomes

$$-a^{-1/n}b_i - s_i a_1 \leq \omega_{i+1} \leq a^{-1/n}b_i - s_i a_0$$

and with (17) further

$$-\left(\frac{a_0}{a_1}\right)^{-1/n} b_i - s_i a_1 \leq \omega_{i+1} \leq \left(\frac{a_0}{a_1}\right)^{-1/n} b_i - s_i a_0.$$

Together with the first condition in ii) this results in  $s_i \leq \frac{b_i}{a_0} (1 + (\frac{a_1}{a_0})^{1/n})$ . The same condition can be deduced for  $s_i \geq 0$  and  $-a_1 \leq \omega_1 \leq -a_0$ .

For  $s_i < 0$  and  $a_0 \leq \omega_1 \leq a_1$  or  $-a_1 \leq \omega_1 \leq -a_0$ , we obtain that  $s_i \geq -\frac{b_i}{a_0} (1 + (\frac{a_1}{a_0})^{1/n})$  is necessary for  $\hat{\psi}(\omega)\hat{\psi}_{a,s,0}(\omega) \neq 0$ .



Finally, the case  $a < 0$  can be treated similarly which results in  $a \in [-\frac{a_1}{a_0}, -\frac{a_0}{a_1}]$ . This completes the proof.  $\square$

Now we can prove the required property (16) of  $\mathcal{SH}_\psi(\psi)$ .

**Theorem 3.2.** *Let  $\psi$  be a Schwartz function such that  $\text{supp } \hat{\psi} \subseteq ([-a_1, -a_0] \cup [a_0, a_1]) \times Q_b$ . Then we have that  $\mathcal{SH}_\psi(\psi) \in L_{1,w}(\mathbb{S})$ , i.e.,*

$$\|\langle \psi, \pi(\cdot)\psi \rangle\|_{L_{1,w}(\mathbb{S})} = \int_{\mathbb{S}} |\mathcal{SH}_\psi(\psi)(a, s, t)| w(a, s, t) d\mu(a, s, t) < \infty.$$

*Proof.* Straightforward computation gives

$$\begin{aligned} \|\langle \psi, \pi(\cdot)\psi \rangle\|_{L_{1,w}(\mathbb{S})} &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} |\langle \psi, \psi_{a,s,t} \rangle| w(a, s) dt ds \frac{da}{|a|^{n+1}} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} |\psi * \psi_{a,s,0}^*(t)| w(a, s) dt ds \frac{da}{|a|^{n+1}} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} |\mathcal{F}^{-1} \mathcal{F}(\psi * \psi_{a,s,0}^*)(t)| dt w(a, s) ds \frac{da}{|a|^{n+1}} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \|\mathcal{F}(\psi * \psi_{a,s,0}^*)\|_{\mathcal{F}^{-1}L_1} w(a, s) ds \frac{da}{|a|^{n+1}} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \|\hat{\psi} \hat{\psi}_{a,s,0}^*\|_{\mathcal{F}^{-1}L_1} w(a, s) ds \frac{da}{|a|^{n+1}}, \end{aligned}$$

where  $\|f\|_{\mathcal{F}^{-1}L_1(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\mathcal{F}^{-1}f(x)| dx$  for  $f \in L_1(\mathbb{R}^n)$ . By Lemma 3.1 this can be rewritten as

$$\|\langle \psi, \pi(\cdot)\psi \rangle\|_{L_{1,w}(\mathbb{S})} = \left( \int_{-a_1/a_0}^{-a_0/a_1} + \int_{a_0/a_1}^{a_1/a_0} \right) \int_{Q_c} \|\hat{\psi} \hat{\psi}_{a,s,0}^*\|_{\mathcal{F}^{-1}L_1(\mathbb{R}^n)} w(a, s) ds \frac{da}{|a|^{n+1}},$$

which is obviously finite.  $\square$

For  $\psi$  satisfying (16) we can consider the space

$$\mathcal{H}_{1,w} := \{f \in L_2(\mathbb{R}^n) : \mathcal{SH}_\psi(f) = \langle f, \pi(\cdot)\psi \rangle \in L_{1,w}(\mathbb{S})\}, \quad (18)$$

with norm  $\|f\|_{\mathcal{H}_{1,w}} := \|\mathcal{SH}_\psi f\|_{L_{1,w}(\mathbb{S})}$  and its anti-dual  $\mathcal{H}_{1,w}^\sim$ , the space of all continuous conjugate-linear functionals on  $\mathcal{H}_{1,w}$ . The spaces  $\mathcal{H}_{1,w}$  and  $\mathcal{H}_{1,w}^\sim$  are  $\pi$ -invariant Banach spaces with continuous embeddings  $\mathcal{H}_{1,w} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{1,w}^\sim$ , and their definition is independent of the shearlet  $\psi$ . Then the inner product on  $L_2(\mathbb{R}^n) \times L_2(\mathbb{R}^n)$  extends to a sesquilinear form on  $\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}$ , therefore for  $\psi \in \mathcal{H}_{1,w}$  and  $f \in \mathcal{H}_{1,w}^\sim$  the *extended representation coefficients*

$$\mathcal{SH}_\psi(f)(a, s, t) := \langle f, \pi(a, s, t)\psi \rangle_{\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}}$$

are well-defined. Now, for  $1 \leq p \leq \infty$ , we define the *shearlet coorbit spaces*

$$\mathcal{SC}_{p,w} := \{f \in \mathcal{H}_{1,w}^\sim : \mathcal{SH}_\psi(f) \in L_{p,w}(\mathbb{S})\} \quad (19)$$

with norms  $\|f\|_{\mathcal{SC}_{p,w}} := \|\mathcal{SH}_\psi f\|_{L_{p,w}(\mathbb{S})}$ . It holds that  $\mathcal{SC}_{1,w} = \mathcal{H}_{1,w}$  and  $\mathcal{SC}_{1,1} = L_2(\mathbb{R}^n)$ .

**3.2. Shearlet Banach Frames.** The Feichtinger-Gröchenig theory provides us with a machinery to construct atomic decompositions and Banach frames for our shearlet coorbit spaces  $\mathcal{SC}_{p,w}$ . In a first step, we have to determine, for a compact neighborhood  $U$  of  $e \in \mathbb{S}$  with non-void interior, so-called  $U$ -dense sets. A (countable) family  $X = ((a, s, t)_\lambda)_{\lambda \in \Lambda}$  in  $\mathbb{S}$  is said to be  $U$ -dense if  $\cup_{\lambda \in \Lambda} (a, s, t)_\lambda U = \mathbb{S}$ , and *separated* if for some compact neighborhood  $Q$  of  $e$  we have  $(a_i, s_i, t_i)Q \cap (a_j, s_j, t_j)Q = \emptyset, i \neq j$ , and *relatively separated* if  $X$  is a finite union of separated sets.

**Lemma 3.3.** *Let  $U$  be a neighborhood of the identity in  $\mathbb{S}$ , and let  $\alpha > 1$  and  $\beta, \gamma > 0$  be defined such that*

$$[\alpha^{\frac{1}{n}-1}, \alpha^{\frac{1}{n}}) \times [-\frac{\beta}{2}, \frac{\beta}{2})^{n-1} \times [-\frac{\gamma}{2}, \frac{\gamma}{2})^n \subseteq U. \quad (20)$$

Then the sequence

$$\{(\epsilon \alpha^j, \beta \alpha^{j(1-\frac{1}{n})} k, S_{\beta \alpha^{j(1-\frac{1}{n})} k} A_{\alpha^j} \gamma m) : j \in \mathbb{Z}, k \in \mathbb{Z}^{n-1}, m \in \mathbb{Z}^n, \epsilon \in \{-1, 1\}\} \quad (21)$$

is  $U$ -dense and relatively separated.

*Proof.* Set

$$U_0 := [\alpha^{\frac{1}{n}-1}, \alpha^{\frac{1}{n}}) \times [-\frac{\beta}{2}, \frac{\beta}{2})^{n-1} \times [-\frac{\gamma}{2}, \frac{\gamma}{2})^n.$$

It is sufficient to prove that the sequence (21) is  $U_0$ -dense.

For this, fix any  $(x, y, z) \in \mathbb{S}$ . In the following we assume that  $x \in \mathbb{R}^+$  in which case we have to set  $\epsilon = 1$ . If  $x < 0$ , the same arguments apply while choosing  $\epsilon = -1$ . We have that

$$(\alpha^j, \beta \alpha^{j(1-\frac{1}{n})} k, S_{\beta \alpha^{j(1-\frac{1}{n})} k} A_{\alpha^j} \gamma m) \circ U_0 = \{(\alpha^j u, \alpha^{j(1-\frac{1}{n})} (\beta k + v), S_{\beta \alpha^{j(1-\frac{1}{n})} k} A_{\alpha^j} (\gamma m + w)) : (u, v, w) \in U_0\}.$$

Then  $[\log_\alpha x + (\frac{1}{n} - 1), \log_\alpha x + \frac{1}{n})$  contains a unique integer  $j$ , and there exists a unique  $u \in [\alpha^{\frac{1}{n}-1}, \alpha^{\frac{1}{n}})$  such that  $\log_\alpha x = \log_\alpha u + j$ . Further, there exist unique  $k \in \mathbb{Z}^{n-1}$  and  $v \in [-\frac{\beta}{2}, \frac{\beta}{2})^{n-1}$  so that  $\beta k + v = \alpha^{j(\frac{1}{n}-1)} y$ . Finally, we have that

$$S_{\beta \alpha^{j(1-\frac{1}{n})} k} A_{\alpha^j} (\gamma m + w) = \begin{pmatrix} \alpha^j (\gamma m_1 + w_1) + \beta \langle k, \gamma \tilde{m} + \tilde{w} \rangle \\ \alpha^{\frac{j}{n}} (\gamma \tilde{m} + \tilde{w}) \end{pmatrix}.$$

There exist unique  $\tilde{m} \in \mathbb{Z}^{n-1}$  and  $\tilde{w} \in [-\frac{\gamma}{2}, \frac{\gamma}{2})^{n-1}$  such that  $\gamma \tilde{m} + \tilde{w} = \alpha^{-\frac{j}{n}} \tilde{z}$  and unique  $m_1 \in \mathbb{Z}$  and  $w_1 \in [-\frac{\gamma}{2}, \frac{\gamma}{2})$  such that  $\gamma m_1 + w_1 = \alpha^{-j} z_1 - \beta \alpha^{-\frac{j}{n}} \langle k, \tilde{z} \rangle$ . We have shown that there exist unique  $(u, v, w) \in U_0$  and  $j \in \mathbb{Z}, k \in \mathbb{Z}^{n-1}, m \in \mathbb{Z}^n$  such that

$$(x, y, z) = (\alpha^j u, \alpha^{j(1-\frac{1}{n})} (\beta k + v), S_{\beta \alpha^{j(1-\frac{1}{n})} k} A_{\alpha^j} (\gamma m + w)).$$

Finally, the uniqueness of the decomposition proves immediately that the chosen sequence is relatively separated.  $\square$

Next we define the  $U$ -oscillation as

$$\text{osc}_U(a, s, t) := \sup_{u \in U} |\mathcal{SH}_\psi(\psi)(u \circ (a, s, t)) - \mathcal{SH}_\psi(\psi)(a, s, t)|. \quad (22)$$

Then, the following decomposition theorem, which was proved in a general setting in [10, 11, 12, 13, 15], says that discretizing the representation by means of an  $U$ -dense set produces an atomic decomposition for  $\mathcal{SC}_{p,w}$ .

**Theorem 3.4.** *Assume that the irreducible, unitary representation  $\pi$  is  $w$ -integrable and let an appropriately normalized  $\psi \in L_2(\mathbb{R}^n)$  which fulfills*

$$M\langle\psi, \pi(a, s, t)\rangle := \sup_{u \in (a, s, t)U} |\langle\psi, \pi(u)\psi\rangle| \in L_{1,w}(\mathbb{S}) \quad (23)$$

be given. Choose a neighborhood  $U$  of  $e$  so small that

$$\|\text{osc}_U\|_{L_{1,w}(\mathbb{S})} < 1. \quad (24)$$

Then for any  $U$ -dense and relatively separated set  $X = ((a, s, t)_\lambda)_{\lambda \in \Lambda}$  the space  $\mathcal{SC}_{p,w}$  has the following atomic decomposition: If  $f \in \mathcal{SC}_{p,w}$ , then

$$f = \sum_{\lambda \in \Lambda} c_\lambda(f) \pi((a, s, t)_\lambda) \psi \quad (25)$$

where the sequence of coefficients depends linearly on  $f$  and satisfies

$$\|(c_\lambda(f))_{\lambda \in \Lambda}\|_{\ell_{p,w}} \leq C \|f\|_{\mathcal{SC}_{p,w}} \quad (26)$$

with a constant  $C$  depending only on  $\psi$  and with  $\ell_{p,w}$  being defined by

$$\ell_{p,w} := \{c = (c_\lambda)_{\lambda \in \Lambda} : \|c\|_{\ell_{p,w}} := \|cw\|_{\ell_p} < \infty\},$$

where  $w = (w((a, s, t)_\lambda))_{\lambda \in \Lambda}$ . Conversely, if  $(c_\lambda(f))_{\lambda \in \Lambda} \in \ell_{p,w}$ , then  $f = \sum_{\lambda \in \Lambda} c_\lambda \pi((a, s, t)_\lambda) \psi$  is in  $\mathcal{SC}_{p,w}$  and

$$\|f\|_{\mathcal{SC}_{p,w}} \leq C' \|(c_\lambda(f))_{\lambda \in \Lambda}\|_{\ell_{p,w}}. \quad (27)$$

Given such an atomic decomposition, the problem arises under which conditions a function  $f$  is completely determined by its moments  $\langle f, \pi((a, s, t)_\lambda) \psi \rangle$  and how  $f$  can be reconstructed from these moments. This is answered by the following theorem which establishes the existence of Banach frames.

**Theorem 3.5.** *Impose the same assumptions as in Theorem 3.4. Choose a neighborhood  $U$  of  $e$  such that*

$$\|\text{osc}_U\|_{L_{1,w}(\mathbb{S})} < 1 / \|\mathcal{SH}_\psi(\psi)\|_{L_{1,w}(\mathbb{S})}. \quad (28)$$

Then, for every  $U$ -dense and relatively separated family  $X = ((a, s, t)_\lambda)_{\lambda \in \Lambda}$  in  $G$  the set  $\{\pi((a, s, t)_\lambda) \psi : \lambda \in \Lambda\}$  is a Banach frame for  $\mathcal{SH}_{p,w}$ . This means that

- i)  $f \in \mathcal{SC}_{p,w}$  if and only if  $(\langle f, \pi((a, s, t)_\lambda) \psi \rangle_{\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}})_{\lambda \in \Lambda} \in \ell_{p,w}$ ;
- ii) there exist two constants  $0 < D \leq D' < \infty$  such that

$$D \|f\|_{\mathcal{SC}_{p,w}} \leq \|(\langle f, \pi((a, s, t)_\lambda) \psi \rangle_{\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}})_{\lambda \in \Lambda}\|_{\ell_{p,w}} \leq D' \|f\|_{\mathcal{SC}_{p,w}}; \quad (29)$$

- iii) there exists a bounded, linear reconstruction operator  $\mathcal{S}$  from  $\ell_{p,w}$  to  $\mathcal{SC}_{p,w}$  such that

$$\mathcal{S} \left( (\langle f, \psi((a, s, t)_\lambda) \rangle_{\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}})_{\lambda \in \Lambda} \right) = f.$$

It remains to check how the conditions (23), (24) and (28) can be ensured. To this end, we need the following lemma which was proved in a general setting in [15].

**Lemma 3.6.** *Let  $\mathcal{SH}_\psi(\psi) \in L_{1,w}(\mathbb{S})$  and  $\text{osc}_U \in L_{1,w}(\mathbb{S})$  for one compact neighborhood  $U$  of  $e$ . Then we have that  $\psi$  fulfills (23). If, in addition,  $\mathcal{SH}_\psi(\psi)$  is continuous, then*

$$\lim_{U \rightarrow \{e\}} \|\text{osc}_U\|_{L_{1,w}(\mathbb{S})} = 0. \quad (30)$$

To apply the whole machinery of Theorems 3.4 and 3.5 to our shearlet group setting it remains to prove that  $\|\text{osc}_U\|_{L_{1,w}(\mathbb{S})}$  becomes arbitrarily small for a sufficiently small neighborhood  $U$  of  $e$ .

**Theorem 3.7.** *Let  $\psi$  be a function contained in the Schwartz space  $\mathcal{S}$  with  $\text{supp } \hat{\psi} \subseteq ([-a_1, -a_0] \cup [a_0, a_1]) \times Q_b$ . Then, for every  $\varepsilon > 0$ , there exists a sufficiently small neighborhood  $U$  of  $e$  so that*

$$\|\text{osc}_U\|_{L_{1,w}(\mathbb{S})} \leq \varepsilon. \quad (31)$$

*Proof.* By Theorem 3.2 we have that  $\mathcal{SH}_\psi(\psi) \in L_{1,w}(\mathbb{S})$ . Moreover, it is easy to check that  $\mathcal{SH}_\psi(\psi)$  is continuous on  $\mathbb{S}$ . Thus, by Lemma 3.6, it remains to show that  $\text{osc}_U \in L_{1,w}(\mathbb{S})$  for some compact neighborhood of  $e$ . By definition of  $\text{osc}_U$  and Parseval's identity we have that

$$\begin{aligned} \text{osc}_U(a, s, t) &= \sup_{(\alpha, \beta, \gamma) \in U} \left| \langle \hat{\psi}, \hat{\psi}_{a,s,t} \rangle - \langle \hat{\psi}, \hat{\psi}_{(\alpha, \beta, \gamma)(a,s,t)} \rangle \right| \\ &= \sup_{(\alpha, \beta, \gamma) \in U} \left| |a|^{1-\frac{1}{2n}} \mathcal{F} \left( \hat{\psi}(A_a S_s^T \cdot) \tilde{\psi} \right) (t) - |a\alpha|^{1-\frac{1}{2n}} \mathcal{F} \left( \hat{\psi}(A_{a\alpha} S_{\beta+\alpha^{1-1/n}s}^T \cdot) \tilde{\psi} \right) (\gamma + S_\beta A_\alpha t) \right|, \end{aligned}$$

where we can assume that  $\alpha > 0$ . By Lemma 3.1, we see that for  $(\alpha, \beta)$  in a sufficiently small neighborhood of  $(1, 0_{n-1})$ , the function  $\hat{\psi}(A_{a\alpha} S_{\beta+\alpha^{1-1/n}s}^T \cdot) \tilde{\psi}$  becomes zero except for values  $a$  contained in two finite intervals away from zero and values  $s$  in a finite interval. Thus, it remains to show that  $\int_{\mathbb{R}^n} \text{osc}_U(a, s, t) w(a, s) dt \leq C(a, s)$  with a finite constant  $C(a, s)$ . The main idea is to split the integral into three parts

$$\int_{\mathbb{R}^n} \text{osc}_U(a, s, t) w(a, s) dt = |a|^{1-\frac{1}{2n}} (I_1 + I_2 + I_3),$$

where

$$\begin{aligned} I_1 &:= \int_{\mathbb{R}^n} \sup_{(\alpha, \beta, \gamma) \in U} \left| |1 - \alpha^{1-\frac{1}{2n}}| \mathcal{F} \left( \hat{\psi}(A_a S_s^T \cdot) \tilde{\psi} \right) (t) w(a, s) dt \right. \\ I_2 &:= \int_{\mathbb{R}^n} \sup_{(\alpha, \beta, \gamma) \in U} \left| \alpha^{1-\frac{1}{2n}} \left| \mathcal{F} \left( \hat{\psi}(A_a S_s^T \cdot) \tilde{\psi} \right) (t) - \mathcal{F} \left( \hat{\psi}(A_a S_s^T \cdot) \tilde{\psi} \right) (\gamma + S_\beta A_\alpha t) \right| w(a, s) dt \right. \\ I_3 &:= \int_{\mathbb{R}^n} \sup_{(\alpha, \beta, \gamma) \in U} \left| \alpha^{1-\frac{1}{2n}} \left| \mathcal{F} \left( \hat{\psi}(A_a S_s^T \cdot) \tilde{\psi} \right) (\gamma + S_\beta A_\alpha t) \right. \right. \\ &\quad \left. \left. - \mathcal{F} \left( \hat{\psi}(A_{a\alpha} S_{\beta+\alpha^{1-1/n}s}^T \cdot) \tilde{\psi} \right) (\gamma + S_\beta A_\alpha t) \right| w(a, s) dt. \right. \end{aligned}$$

The integrals  $I_1$  and  $I_3$  can be exactly handled as in the proof of [6, Theorem 3.7]. The same holds true for  $I_2$  except that we have to replace  $t_2 \in \mathbb{R}$  in the proof [6, Theorem 3.7] by  $\tilde{t} \in \mathbb{R}^{n-1}$ , where  $t = (t_1, \tilde{t}^T)^T$ . The corresponding estimates are detailed in [8].  $\square$

#### 4. ANALYSIS OF SINGULARITIES

In this section, we deal with the decay of the shearlet transform at hyperplane singularities in  $\mathbb{R}^n$  and at special simplex singularities in  $\mathbb{R}^3$ . For the behaviour of the shearlet transform at singularities in  $\mathbb{R}^2$  we refer to [20, 23].

**4.1. Hyperplane Singularities.** We consider  $(n - m)$ -dimensional hyperplanes in  $\mathbb{R}^n$ ,  $m = 1, \dots, n - 1$  through the origin given by

$$\underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}}_{x_A} + P \underbrace{\begin{pmatrix} x_{m+1} \\ \vdots \\ x_n \end{pmatrix}}_{x_E} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad P := \begin{pmatrix} p_1^T \\ \vdots \\ p_m^T \end{pmatrix} \in \mathbb{R}^{m, n-m}. \quad (32)$$

Note that this setting excludes some special hyperplanes, e.g., for  $n = 3$  and  $m = 1$  planes containing the  $x_1$ -axis and for  $n = 3$  and  $m = 2$  lines contained within the  $x_1x_2$ -plane. To detect such hyperplane singularities one has to perform a simple variable exchange in the shearlet setting or to define „shearlets on the cone” similar to [20].

Let  $\delta$  denote the Delta distribution. Then we obtain for

$$\nu_m := \delta(x_A + Px_E)$$

that

$$\begin{aligned} \hat{\nu}_m(\omega) &= \int_{\mathbb{R}^n} \delta(x_A + Px_E) e^{-2\pi i(\langle x_A, \omega_A \rangle + \langle x_E, \omega_E \rangle)} dx \\ &= \int_{\mathbb{R}^{n-m}} e^{-2\pi i(-\langle Px_E, \omega_A \rangle + \langle x_E, \omega_E \rangle)} dx_E \\ &= \delta(\omega_E - P^T \omega_A). \end{aligned} \quad (33)$$

The following theorem describes the decay of the shearlet transform at hyperplane singularities. We use the notation  $\mathcal{SH}_\psi f(a, s, t) \sim |a|^r$  as  $a \rightarrow 0$ , if there exist constants  $0 < c \leq C < \infty$  such that

$$c|a|^r \leq \mathcal{SH}_\psi f(a, s, t) \leq C|a|^r \text{ as } a \rightarrow 0.$$

**Theorem 4.1.** *Let  $\psi \in L_2(\mathbb{R}^n)$  be a shearlet satisfying  $\hat{\psi} \in C^\infty(\mathbb{R}^n)$ . Assume further that  $\hat{\psi}(\omega) = \hat{\psi}_1(\omega_1)\hat{\psi}_2(\tilde{\omega}/\omega_1)$ , where  $\text{supp } \hat{\psi}_1 \in [-a_1, -a_0] \cup [a_0, a_1]$  for some  $a_1 > a_0 \geq \alpha > 0$ ,  $\int_{\mathbb{R}} \hat{\psi}_1(\xi)|\xi|^{m-1} d\xi \neq 0$ ,  $m = 1, \dots, n-1$  and  $\text{supp } \hat{\psi}_2 \in Q_b$ . If*

$$(s_m, \dots, s_{n-1}) = (-1, s_1, \dots, s_{m-1})P \quad \text{and} \quad (t_1, \dots, t_m) = -(t_{m+1}, \dots, t_n)P^T, \quad (34)$$

then

$$\mathcal{SH}_\psi \nu_m(a, s, t) \sim |a|^{\frac{1-2m}{2n}} \quad \text{as } a \rightarrow 0. \quad (35)$$

Otherwise, the shearlet transform  $\mathcal{SH}_\psi \nu_m$  decays rapidly as  $a \rightarrow 0$ .

The condition (34) requires that the the shearlet is aligned with the hyperplane (32) and that  $t$  lies within the hyperplane.

The condition on  $\hat{\psi}_1$  and  $\hat{\psi}_2$  can be relaxed toward a rapid decay of the functions.

*Proof.* An application of Plancherel’s theorem for tempered distribution together with (33) and (8) yields

$$\begin{aligned} \mathcal{SH}_\psi \nu_m(a, s, t) &:= \langle \nu_m, \psi_{a,s,t} \rangle \\ &= \langle \hat{\nu}_m, \hat{\psi}_{a,s,t} \rangle \\ &= \int_{\mathbb{R}^n} \delta(\omega_E - P^T \omega_A) |a|^{1-\frac{1}{2n}} e^{2\pi i \langle t, \omega \rangle} \bar{\psi} \left( a\omega_1, \text{sgn}(a)|a|^{\frac{1}{n}}(\omega_1 s + \tilde{\omega}) \right) d\omega \\ &= |a|^{1-\frac{1}{2n}} \int_{\mathbb{R}^m} e^{2\pi i \langle t_A + Pt_E, \omega_A \rangle} \bar{\psi} \left( a\omega_1, \text{sgn}(a)|a|^{\frac{1}{n}}(\omega_1 s + \left( \frac{\tilde{\omega}_A}{P^T \omega_A} \right)) \right) d\omega_A \end{aligned}$$

with  $\tilde{\omega}_A = (\omega_2, \dots, \omega_m)^T$ . By definition of  $\hat{\psi}$  this can be rewritten as

$$\mathcal{SH}_\psi \nu_m(a, s, t) = |a|^{1-\frac{1}{2n}} \int_{\mathbb{R}^m} e^{2\pi i \langle t_A + Pt_E, \omega_A \rangle} \bar{\psi}_1(a\omega_1) \bar{\psi}_2 \left( |a|^{\frac{1}{n}-1} \left( s + \frac{1}{\omega_1} \left( \frac{\tilde{\omega}_A}{P^T \omega_A} \right) \right) \right) d\omega_A.$$

Substituting  $\tilde{\xi}_A = (\xi_2, \dots, \xi_m)^\top := \tilde{\omega}_A/\omega_1$ , i.e.,  $d\tilde{\omega}_A = |\omega_1|^{m-1} d\tilde{\xi}_A$ , we get

$$\begin{aligned} \mathcal{SH}_\psi \nu_m(a, s, t) &= |a|^{1-\frac{1}{2n}} \int_{\mathbb{R}} \int_{\mathbb{R}^{m-1}} e^{2\pi i \omega_1 \langle t_A + P t_E, (1, \tilde{\xi}_A^\top)^\top \rangle} \tilde{\psi}_1(a\omega_1) |\omega_1|^{m-1} \\ &\quad \times \tilde{\psi}_2 \left( |a|^{\frac{1}{n}-1} \left( s + \begin{pmatrix} \tilde{\xi}_A \\ P^\top (1, \tilde{\xi}_A^\top)^\top \end{pmatrix} \right) \right) d\tilde{\xi}_A d\omega_1 \end{aligned}$$

and further by substituting  $\xi_1 := a\omega_1$

$$\begin{aligned} \mathcal{SH}_\psi \nu_m(a, s, t) &= |a|^{1-m-\frac{1}{2n}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}} e^{2\pi i \frac{\xi_1}{a} \langle t_A + P t_E, (1, \tilde{\xi}_A^\top)^\top \rangle} |\xi_1|^{m-1} \tilde{\psi}_1(\xi_1) d\xi_1 \\ &\quad \times \tilde{\psi}_2 \left( |a|^{\frac{1}{n}-1} \left( s + \begin{pmatrix} \tilde{\xi}_A \\ P^\top (1, \tilde{\xi}_A^\top)^\top \end{pmatrix} \right) \right) d\tilde{\xi}_A. \end{aligned}$$

Finally, by substituting  $\tilde{\omega}_A := |a|^{\frac{1}{n}-1} (\tilde{\xi}_A + s_a)$ , where  $s_a := (s_1, \dots, s_{m-1})^\top$  and  $s_e := (s_m, \dots, s_{n-1})^\top$ , we obtain

$$\begin{aligned} \mathcal{SH}_\psi \nu_m(a, s, t) &= |a|^{\frac{1-2m}{2n}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}} e^{2\pi i \frac{\xi_1}{a} \langle t_A + P t_E, (1, |a|^{1-1/n} \tilde{\omega}_A^\top - s_a^\top) \rangle} |\xi_1|^{m-1} \tilde{\psi}_1(\xi_1) d\xi_1 \\ &\quad \times \tilde{\psi}_2 \left( |a|^{\frac{1}{n}-1} \left( s_e - P^\top \begin{pmatrix} -1 \\ s_a \end{pmatrix} \right) + P^\top \begin{pmatrix} 0 \\ \tilde{\omega}_A \end{pmatrix} \right) d\tilde{\omega}_A. \end{aligned}$$

If the vector

$$s_e - P^\top \begin{pmatrix} -1 \\ s_a \end{pmatrix} \neq 0_{n-m} \quad (36)$$

then at least one component of its product with  $|a|^{1/n-1}$  becomes arbitrary large as  $a \rightarrow 0$ . On the other hand, by the support property of  $\tilde{\psi}_2$ , we conclude that  $\tilde{\psi}_2(\tilde{\omega}_A, \cdot)$  becomes zero if we  $\tilde{\omega}_A$  is not in  $Q_{(b_1, \dots, b_{m-1})} \subset \mathbb{R}^{m-1}$ . But for all  $\tilde{\omega}_A \in Q_{(b_1, \dots, b_{m-1})}$  at least one component of

$$|a|^{\frac{1}{n}-1} \left( s_e - P^\top \begin{pmatrix} 1 \\ s_a \end{pmatrix} \right) + P^\top \begin{pmatrix} 0 \\ \tilde{\omega}_A \end{pmatrix}$$

is not within the support of  $\hat{\psi}_2$  for  $a$  sufficiently small so that  $\hat{\psi}_2$  becomes zero again. Assume now that we have equality in (36). Then

$$\begin{aligned}
\mathcal{SH}_\psi \nu_m(a, s, t) &= |a|^{\frac{1-2m}{2n}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}} e^{2\pi i \frac{\xi_1}{a} \langle t_A + Pt_E, (1, |a|^{1-1/n} \tilde{\omega}_A^T - s_a^T) \rangle} |\xi_1|^{m-1} \tilde{\psi}_1(\xi_1) d\xi_1 \\
&\quad \times \tilde{\psi}_2 \left( P^T \begin{pmatrix} \tilde{\omega}_A \\ 0 \\ \tilde{\omega}_A \end{pmatrix} \right) d\tilde{\omega}_A \\
&= C |a|^{\frac{1-2m}{2n}} \int_{\mathbb{R}^{m-1}} \tilde{\psi}_1^{(m-1)} \left( \langle t_A + Pt_E, (1, |a|^{1-1/n} \tilde{\omega}_A^T - s_a^T) \rangle / a \right) \\
&\quad \times \tilde{\psi}_2 \left( P^T \begin{pmatrix} \tilde{\omega}_A \\ 0 \\ \tilde{\omega}_A \end{pmatrix} \right) d\tilde{\omega}_A \\
&= C |a|^{\frac{1-2m}{2n}} \int_{\mathbb{R}^{m-1}} \tilde{\psi}_1^{(m-1)} \left( \langle t_A + Pt_E, \left( \tilde{\omega}_A^T - |a|^{1/n-1} s_a \right) \rangle |a|^{-1/n} \right) \\
&\quad \times \tilde{\psi}_2 \left( P^T \begin{pmatrix} \tilde{\omega}_A \\ 0 \\ \tilde{\omega}_A \end{pmatrix} \right) d\tilde{\omega}_A,
\end{aligned}$$

where  $\tilde{\psi}_1$  has the Fourier transform  $\hat{\psi}_1(\xi_1) := \tilde{\psi}_1(\xi_1)$  for  $\xi_1 \geq 0$  and  $\hat{\psi}_1(\xi_1) := -\tilde{\psi}_1(\xi_1)$  for  $\xi_1 < 0$ . Since by our assumptions the support of  $\hat{\psi}_1$  is bounded away from the origin, we see that  $\tilde{\psi}_1$  is again in  $C^\infty(\mathbb{R})$ . If  $t_A + Pt_E \neq 0_m$ , then, since  $\tilde{\psi}_1 \in C^\infty$  the function  $\tilde{\psi}_1^{(m-1)}$  decays rapidly as  $a \rightarrow 0$  for all  $\tilde{\omega}_A$  in the bounded domain, where  $\hat{\psi}_2$  doesn't become zero. Consequently, the value of the shearlet transform decays rapidly. If  $t_A + Pt_E = 0_m$  and  $\tilde{\psi}_1^{(m-1)}(0) \neq 0$ , then

$$\mathcal{SH}_\psi \nu_m(a, s, t) = C |a|^{\frac{1-2m}{2n}} \tilde{\psi}_1^{(m-1)}(0) \int_{\mathbb{R}^{m-1}} \tilde{\psi}_2 \left( P^T \begin{pmatrix} \tilde{\omega}_A \\ 0 \\ \tilde{\omega}_A \end{pmatrix} \right) d\tilde{\omega}_A \sim |a|^{\frac{1-2m}{2n}}.$$

This finishes the proof.  $\square$

**Remark 4.2.** Other choices of the dilation matrix are possible, e.g.,

$$A_a := \begin{pmatrix} a & 0_{n-1}^T \\ 0_{n-1} & \operatorname{sgn}(a) \sqrt{|a|} I_{n-1} \end{pmatrix}.$$

Then we have to replace (35) by  $|a|^{\frac{n-2m-1}{4}}$  which increases for  $n < 2m + 1$  as  $a \rightarrow 0$ . Therefore, we prefer our choice.

**4.2. Tetrahedron Singularities.** In the following, we deal with the cone  $\mathcal{C}$  in the first octant of  $\mathbb{R}^3$  given by

$$\mathcal{C} := \{x = Ct : t \geq 0 \text{ componentwise}\}, \tag{37}$$

where

$$C := (p \ q \ r) = \begin{pmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{pmatrix}, \quad p_j, q_j, r_j > 0, \quad j = 1, 2$$

and the vectors  $p, q, r$  are linearly independent. The vector

$$n_{pq} := \left( 1, \frac{p_2 - q_2}{p_1 q_2 - p_2 q_1}, \frac{q_1 - p_1}{p_1 q_2 - p_2 q_1} \right)^T = (1, \tilde{n}_{pq}^T)^T$$

is a multiple of the normal vector of the plane spanned by  $p$  and  $q$ . Similarly, we use the notation  $n_{pr}, n_{qr}$  for the corresponding vectors perpendicular to the  $pr$ -plane and  $qr$ -plane. Let  $\chi_{\mathcal{C}}$  denote the characteristic function of the cone  $\mathcal{C}$ . Since the Fourier transform of the Heavyside function  $H$  is

$$\hat{H}(\omega) = \frac{1}{2\pi i} \text{pv} \left( \frac{1}{\omega} \right) + \sqrt{\frac{\pi}{2}} \delta(\omega),$$

see [14, p. 340], we obtain that

$$\begin{aligned} \hat{\chi}_{\mathcal{C}}(\omega) &= \int_{\mathcal{C}} e^{-2\pi i \langle x, \omega \rangle} dx = |\det C| \int_{\mathbb{R}_+^3} e^{-2\pi i \langle t, C^T \omega \rangle} dt \\ &= c_1 \left( \frac{1}{p^T \omega} \frac{1}{q^T \omega} \frac{1}{r^T \omega} \right) + c_2 \left( \frac{1}{p^T \omega} \frac{1}{q^T \omega} \delta(r^T \omega) + \frac{1}{p^T \omega} \frac{1}{r^T \omega} \delta(q^T \omega) + \frac{1}{q^T \omega} \frac{1}{r^T \omega} \delta(p^T \omega) \right) \\ &\quad + c_3 \left( \frac{1}{p^T \omega} \delta(q^T \omega) \delta(r^T \omega) + \frac{1}{q^T \omega} \delta(p^T \omega) \delta(r^T \omega) + \frac{1}{r^T \omega} \delta(p^T \omega) \delta(q^T \omega) \right) \\ &\quad + c_4 (\delta(p^T \omega) \delta(q^T \omega) \delta(r^T \omega)) \end{aligned} \quad (38)$$

with nonzero constants  $c_j$ ,  $j = 1, 2, 3, 4$ . We have omitted the pv to simplify the notation. This can be used to prove the following theorem.

**Theorem 4.3.** *Let  $\psi \in L_2(\mathbb{R}^3)$  be a shearlet satisfying  $\hat{\psi} \in C^\infty(\mathbb{R}^3)$ . Assume further that  $\hat{\psi}(\omega) = \hat{\psi}_1(\omega_1) \hat{\psi}_2(\tilde{\omega}/\omega_1)$ , where  $\text{supp } \hat{\psi}_1 \in [-a_1, -a_0] \cup [a_0, a_1]$  for some  $a_1 > a_0 \geq \alpha > 0$ ,  $\int_{\mathbb{R}} \hat{\psi}_1(\xi)/\xi d\xi \neq 0$  and  $\hat{\psi}_2$  is a non-negative function with  $\text{supp } \hat{\psi}_2 \in Q_b$ ,  $\hat{\psi}_2(0) \neq 0$ . Let  $a > 0$ . If*

$$s = -\tilde{n}_{pq}, n_{pq}^T t = 0 \quad \text{or} \quad s = -\tilde{n}_{pr}, n_{pr}^T t = 0 \quad \text{or} \quad s = -\tilde{n}_{qr}, n_{qr}^T t = 0$$

then

$$\mathcal{SH}_\psi \chi_{\mathcal{C}}(a, s, t) \sim a^{5/6}.$$

If

$$\begin{aligned} 1 - p_1 s_1 - p_2 s_2 = 0, s &\neq \frac{q_2 - p_2}{p_1 q_2 - p_2 q_1}, s \neq \frac{r_2 - p_2}{p_1 r_2 - p_2 r_1} \quad \text{and} \quad t = cp \quad \text{or} \\ 1 - q_1 s_1 - q_2 s_2 = 0, s &\neq \frac{p_2 - q_2}{q_1 p_2 - q_2 p_1}, s \neq \frac{r_2 - q_2}{q_1 r_2 - q_2 r_1} \quad \text{and} \quad t = cq \quad \text{or} \\ 1 - r_1 s_1 - r_2 s_2 = 0, s &\neq \frac{q_2 - r_2}{r_1 q_2 - r_2 q_1}, s \neq \frac{p_2 - r_2}{r_1 p_2 - r_2 p_1} \quad \text{and} \quad t = cr \end{aligned}$$

then

$$\mathcal{SH}_\psi \chi_{\mathcal{C}}(a, s, t) \sim a^{3/2}.$$

If

$$1 - p_1 s_1 - p_2 s_2 \neq 0, 1 - q_1 s_1 - q_2 s_2 \neq 0, 1 - r_1 s_1 - r_2 s_2 \neq 0 \quad \text{and} \quad t = (0, 0, 0)^T,$$

then

$$\mathcal{SH}_\psi \chi_{\mathcal{C}}(a, s, t) \sim a^{13/9}.$$



If

$$\begin{aligned} 1 - p_1 s_1 - p_2 s_2 = 0, 1 - q_1 s_1 - q_2 s_2 \neq 0, 1 - r_1 s_1 - r_2 s_2 \neq 0 & \quad \text{or} \\ 1 - q_1 s_1 - q_2 s_2 = 0, 1 - p_1 s_1 - p_2 s_2 \neq 0, 1 - r_1 s_1 - r_2 s_2 \neq 0 & \quad \text{or} \\ 1 - r_1 s_1 - r_2 s_2 = 0, 1 - p_1 s_1 - p_2 s_2 \neq 0, 1 - q_1 s_1 - q_2 s_2 \neq 0 & \\ \text{and } t_1 - t_2 s_1 - t_3 s_2 = 0 & \end{aligned}$$

then

$$\mathcal{SH}_{\psi\chi\mathcal{C}}(a, s, t) \sim a^{3/2}.$$

Otherwise, the shearlet transform  $\mathcal{SH}_{\psi\chi\mathcal{C}}(a, s, t)$  decays rapidly as  $a \rightarrow 0$ .

*Proof.* To determine the decay of  $\mathcal{SH}_{\psi\chi\mathcal{C}}(a, s, t) = \langle \hat{\chi}_{\mathcal{C}}, \hat{\psi}_{a,s,t} \rangle$  as  $a \rightarrow 0$ , we consider the four parts of (38) separately.

1. Since  $p, q, r$  are linearly independent, we have by the support of  $\hat{\psi}$  that

$$\langle \delta(p^T \cdot) \delta(q^T \cdot) \delta(r^T \cdot), \hat{\psi}_{a,s,t} \rangle = \hat{\psi}_{a,s,t}(0) = 0.$$

2. Next we obtain

$$\begin{aligned} \langle \delta(p^T \cdot) \delta(q^T \cdot) \frac{1}{r^T \cdot}, \hat{\psi}_{a,s,t} \rangle &= a^{5/6} \frac{1}{r^T n_{pq}} \int_{\mathbb{R}} e^{2\pi i \omega_1 \langle t, n_{pq} \rangle} \frac{\tilde{\psi}_1(a\omega_1)}{\omega_1} \tilde{\psi}_2 \left( a^{-2/3} (s + \tilde{n}_{pq}) \right) d\omega_1 \\ &\sim a^{5/6} \tilde{\psi}_2 \left( a^{-2/3} (s + \tilde{n}_{pq}) \right) \int_{\mathbb{R}} e^{2\pi i \xi_1 \langle t, n_{pq} \rangle / a} \frac{\tilde{\psi}_1(\xi_1)}{\xi_1} d\xi_1. \end{aligned} \quad (39)$$

If  $s \neq -\tilde{n}_{pq}$ , then (39) becomes zero for sufficiently small  $a$  since  $\tilde{\psi}_2$  is compactly supported. If  $s = -\tilde{n}_{pq}$ , then

$$\langle \delta(p^T \cdot) \delta(q^T \cdot) \frac{1}{r^T \cdot}, \hat{\psi}_{a,s,t} \rangle \sim a^{5/6} \phi_1(\langle t, n_{pq} \rangle / a),$$

where  $\phi_1$  defined by  $\hat{\phi}_1(\xi) := \tilde{\psi}_1(\xi) / \xi \in \mathcal{S}$  is rapidly decaying. Thus, the above expression decays rapidly as  $a \rightarrow 0$  except for  $n_{pq}^T t = 0$ , i.e.,  $t$  is in the  $pq$ -plane, where the decay is  $a^{5/6}$ .

3. For  $I_3 := \langle \delta(p^T \cdot) \frac{1}{q^T \cdot} \frac{1}{r^T \cdot}, \hat{\psi}_{a,s,t} \rangle$  we get with  $\omega_3 = -(\omega_1 + p_1 \omega_2) / p_2$  that

$$I_3 = a^{5/6} \int_{\mathbb{R}^2} e^{2\pi i \langle t, \omega \rangle} \tilde{\psi}_1(a\omega_1) \tilde{\psi}_2 \left( a^{-2/3} \left( s + \frac{1}{\omega_1} \begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix} \right) \right) \frac{1}{q^T \omega} \frac{1}{r^T \omega} d\omega_1 d\omega_2.$$

Substituting first  $\xi_2 := a^{-2/3} (s_1 + \omega_2 / \omega_1)$  and then  $\xi_1 := a\omega_1$  this becomes

$$\begin{aligned} I_3 &= a^{3/2} \int_{\mathbb{R}^2} e^{2\pi i \xi_1 \left( t_1 - \frac{t_3}{p_2} - s_1 \left( t_2 - \frac{p_1 t_3}{p_2} \right) \right) / a} e^{2\pi i \xi_1 \xi_2 \left( t_2 - \frac{p_1 t_3}{p_2} \right) / a^{1/3}} \frac{\tilde{\psi}_1(\xi_1)}{\xi_1} \\ &\quad \times \tilde{\psi}_2 \left( a^{-2/3} \left( -\frac{1}{p_2} + \frac{p_1}{p_2} s_1 + s_2 \right) - \frac{p_1}{p_2} \xi_2 \right) \frac{1}{g_{pq}(\xi_2)} \frac{1}{g_{pr}(\xi_2)} d\xi_1 d\xi_2 \end{aligned}$$

where  $g_{pq}(\xi_2) := 1 - \frac{q_2}{p_2} - s_1 \left( q_1 - \frac{p_1 q_2}{p_2} \right) + a^{2/3} \xi_2 \left( q_1 - \frac{p_1 q_2}{p_2} \right)$ . If  $1 - p_1 s_1 - p_2 s_2 \neq 0$ , then  $\tilde{\psi}_2 \left( \left( \xi_2, a^{-2/3} \left( -\frac{1}{p_2} + \frac{p_1}{p_2} s_1 + s_2 \right) - \frac{p_1}{p_2} \xi_2 \right)^T \right)$  becomes zero for sufficiently small  $a$  by the support property of  $\tilde{\psi}_2$ .

Let  $1 - p_1 s_1 - p_2 s_2 = 0$ .

3.1. If  $1 - \frac{q_2}{p_2} - s_1 \left( q_1 - \frac{p_1 q_2}{p_2} \right) \neq 0$ , i.e.,  $s_1 \neq -\frac{p_2 - q_2}{p_1 q_2 - p_2 q_1}$  and  $1 - \frac{r_2}{p_2} - s_1 \left( r_1 - \frac{p_1 r_2}{p_2} \right) \neq 0$ , i.e.,

$s_1 \neq -\frac{p_2-r_2}{p_1r_2-p_2r_1}$ , then the function  $\phi_2$  defined by  $\hat{\phi}_2 := \frac{\tilde{\psi}_2(\xi_2(1, -\frac{p_1}{p_2})^\top)}{g_{pq}(\xi_2)g_{pr}(\xi_2)} \in \mathcal{S}$  is rapidly decaying and we obtain

$$I_3 = a^{3/2} \int_{\mathbb{R}^1} e^{2\pi i \xi_1 (t_1 - \frac{t_3}{p_2} - s_1(t_2 - \frac{p_1 t_3}{p_2})) / a} \frac{\tilde{\psi}_1(\xi_1)}{\xi_1} \phi_2 \left( \frac{\xi_1(t_2 p_2 - p_1 t_3)}{p_2 a^{1/3}} \right) d\xi_1$$

If  $t_2 p_2 - p_1 t_3 \neq 0$ , then

$$\phi_2 \left( \frac{\xi_1(t_2 p_2 - p_1 t_3)}{p_2 a^{1/3}} \right) \leq C \frac{a^{2r/3}}{a^{2r/3} + \|\xi_1(t_2 - p_1 t_3 / p_2)\|^{2r}} \quad \forall r \in \mathbb{N}$$

and since  $\tilde{\psi}_1(\xi_1) = 0$  for  $\xi_1 \in [-a_0, a_0]$ , we see that  $I_3$  is rapidly decaying as  $a \rightarrow 0$ . Note that  $\phi_2(0) \neq 0$  for sufficiently small  $a$  since  $\hat{\psi}_2$  is non-negative. If  $t_2 p_2 - p_1 t_3 = 0$ , then

$$I_3 \sim a^{3/2} \phi_1 \left( \frac{t_1 - \frac{t_3}{p_2}}{a} \right)$$

which decays rapidly as  $a \rightarrow 0$  except for  $t_1 p_2 = t_3$ . Now  $t_2 p_2 - p_1 t_3 = 0$  and  $t_1 p_2 = t_3$  imply that  $t = cp$ ,  $c \in \mathbb{R}$ . In this case we have that  $I_3 \sim a^{3/2}$ .

3.2. If  $s_1 = -\frac{p_2-q_2}{p_1 q_2 - p_2 q_1}$  and consequently  $s = -\tilde{n}_{pq}$ , then

$$\begin{aligned} I_3 &\sim a^{5/6} \int_{\mathbb{R}^2} e^{2\pi i \xi_1 (t_1 - \frac{t_3}{p_2} - s_1(t_2 - \frac{p_1 t_3}{p_2})) / a} e^{2\pi i \xi_1 \xi_2 (t_2 - \frac{p_1 t_3}{p_2}) / a^{1/3}} \frac{\tilde{\psi}_1(\xi_1)}{\xi_1} \frac{\tilde{\psi}_2(\xi_2(1, -p_1/p_2)^\top)}{g_{pr}(\xi_2)} \frac{1}{\xi_2} d\xi_2 d\xi_1 \\ &\sim a^{5/6} \int_{\mathbb{R}} e^{2\pi i \xi_1 (t_1 - \frac{t_3}{p_2} - s_1(t_2 - \frac{p_1 t_3}{p_2})) / a} \frac{\tilde{\psi}_1(\xi_1)}{\xi_1} (\phi_2 * \text{sgn}) \left( \frac{\xi_1(p_2 t_2 - p_1 t_3)}{p_2 a^{1/3}} \right) d\xi_1 \end{aligned}$$

where  $\hat{\phi}_2(\xi_2) := \frac{\tilde{\psi}_2(\xi_2(1, -p_1/p_2)^\top)}{g_{pr}(\xi_2)} \in \mathcal{S}$ . Since  $\phi_2 * \text{sgn}$  is bounded we get

$$I_3 \sim a^{5/6} \phi_1 \left( \frac{t_1 - \frac{t_3}{p_2} - s_1(t_2 - \frac{p_1 t_3}{p_2})}{a} \right),$$

where  $\hat{\phi}_1(\xi_1) := \frac{\tilde{\psi}_1(\xi_1)}{\xi_1} \in \mathcal{S}$ . The last expression decays rapidly as  $a \rightarrow 0$  except for  $t_1 - \frac{t_3}{p_2} - s_1(t_2 - \frac{p_1 t_3}{p_2}) = 0$ , where  $I_3 \sim a^{5/6}$ . Together with the conditions on  $s$  the latter is the case if  $n_{pq}^\top t = 0$ .

4. Finally, we examine  $I_4 := \langle \frac{1}{p^\top}, \frac{1}{q^\top}, \frac{1}{r^\top}, \hat{\psi}_{a,s,t} \rangle$ . We obtain

$$I_4 = a^{5/6} \int_{\mathbb{R}^3} e^{2\pi i \langle t, \omega \rangle} \tilde{\psi}_1(a\omega_1) \tilde{\psi}_2 \left( a^{-2/3} \left( s + \frac{1}{\omega_1} \begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix} \right) \right) \frac{1}{p^\top \omega} \frac{1}{q^\top \omega} \frac{1}{r^\top \omega} d\omega$$

and further by substituting  $\xi_j := a^{-2/3}(s_{j-1} + \omega_j/\omega_1)$ ,  $j = 2, 3$  and  $\xi_1 := a\omega_1$

$$I_4 = a^{13/6} \int_{\mathbb{R}^3} e^{2\pi i \xi_1 (t_1 + t_2(a^{2/3}\xi_2 - s_1) + t_3(a^{2/3}\xi_3 - s_2)) / a} \frac{\tilde{\psi}_1(\xi_1)}{\xi_1} \frac{\tilde{\psi}_2((\xi_2, \xi_3)^\top)}{g_p(\xi_2, \xi_3) g_q(\xi_2, \xi_3) g_r(\xi_2, \xi_3)} d\xi,$$

where  $g_p(\xi_2, \xi_3) := 1 - p_1 s_1 - p_2 s_2 + a^{2/3}(\xi_2 p_1 + \xi_3 p_2)$ .

4.1. If  $1 - p_1 s_1 - p_2 s_2 \neq 0$ ,  $1 - q_1 s_1 - q_2 s_2 \neq 0$  and  $1 - r_1 s_1 - r_2 s_2 \neq 0$ , then  $\phi_2$  defined by

$\hat{\phi}_2(\xi_2, \xi_3) := \frac{\tilde{\psi}_2((\xi_2, \xi_3)^\top)}{g_p(\xi_2, \xi_3) g_q(\xi_2, \xi_3) g_r(\xi_2, \xi_3)} \in \mathcal{S}$  is rapidly decaying and

$$I_4 = a^{13/6} \int_{\mathbb{R}} e^{2\pi i \xi_1 (t_1 - t_2 s_1 - t_3 s_2) / a} \frac{\tilde{\psi}_1(\xi_1)}{\xi_1} \phi_2(\xi_1(t_2, t_3) / a^{1/3}) d\xi_1,$$

Similarly as before, we see that  $I_4$  decays rapidly as  $a \rightarrow 0$  if  $(t_2, t_3) \neq (0, 0)$ . For  $t_2 = t_3 = 0$  we conclude that  $I_4 \sim a^{13/6} \phi_1((t_1 - t_2 s_1 - t_3 s_2)/a)$ . The right-hand side is rapidly decaying as  $a \rightarrow 0$  except for  $t_1 - t_2 s_1 - t_3 s_2 = 0$ , i.e., for  $t = (0, 0, 0)^T$ , where  $I_4 \sim a^{13/6}$ .

4.2. If  $1 - p_1 s_1 - p_2 s_2 = 0$  and  $1 - q_1 s_1 - q_2 s_2 \neq 0$ ,  $1 - r_1 s_1 - r_2 s_2 \neq 0$ , we obtain with  $\hat{\phi}_2(\xi_2, \xi_3) := \frac{\tilde{\psi}_2((\xi_2, \xi_3)^T)}{g_q(\xi_2, \xi_3) g_r(\xi_2, \xi_3)} \in \mathcal{S}$  that

$$\begin{aligned} I_4 &= a^{3/2} \int_{\mathbb{R}^3} e^{2\pi i \xi_1 (t_1 - t_2 s_1 - t_3 s_2)/a} e^{2\pi i \xi_1 (t_2 \xi_2 + t_3 \xi_3)/a^{1/3}} \frac{\tilde{\psi}_1(\xi_1)}{\xi_1} \hat{\phi}_2(\xi_2, \xi_3) \frac{1}{p_1 \xi_2 + p_2 \xi_3} d\xi \\ &\sim a^{3/2} \int_{\mathbb{R}} e^{2\pi i \xi_1 (t_1 - t_2 s_1 - t_3 s_2)/a} \frac{\tilde{\psi}_1(\xi_1)}{\xi_1} (\phi_2 * h)(\xi_1(t_2, t_3)/a^{1/3}) d\xi_1 \\ &\sim a^{3/2} \phi_1(t_1 - t_2 s_1 - t_3 s_2)/a, \end{aligned}$$

where  $h(u, v) := \operatorname{sgn}(-v/p_2) \delta_u(t_2 - p_1 t_3/p_2)$ . Thus  $I_4$  decays rapidly as  $a \rightarrow 0$  except for  $t_1 - t_2 s_1 - t_3 s_2 = 0$ .

4.3. Let  $1 - p_1 s_1 - p_2 s_2 = 0$  and  $1 - q_1 s_1 - q_2 s_2 = 0$ , i.e.,  $s = -\tilde{n}_{pq}$ . Then we obtain with  $\hat{\phi}_2(\xi_2, \xi_3) := \frac{\tilde{\psi}_2((\xi_2, \xi_3)^T)}{g_r(\xi_2, \xi_3)} \in \mathcal{S}$  that

$$\begin{aligned} I_4 &= a^{5/6} \int_{\mathbb{R}^3} e^{2\pi i \xi_1 (t_1 - t_2 s_1 - t_3 s_2)/a} e^{2\pi i \xi_1 (t_2 \xi_2 + t_3 \xi_3)/a^{1/3}} \frac{\tilde{\psi}_1(\xi_1)}{\xi_1} \hat{\phi}_2(\xi_2, \xi_3) \frac{1}{p_1 \xi_2 + p_2 \xi_3} \frac{1}{q_1 \xi_2 + q_2 \xi_3} d\xi \\ &= a^{5/6} \int_{\mathbb{R}} e^{2\pi i \xi_1 (t_1 - t_2 s_1 - t_3 s_2)/a} \frac{\tilde{\psi}_1(\xi_1)}{\xi_1} (\phi_2 * h)(\xi_1(t_2, t_3)/a^{1/3}) d\xi_1 \\ &\sim a^{5/6} \phi_1(t_1 - t_2 s_1 - t_3 s_2)/a, \end{aligned}$$

where  $h(u, v) := \operatorname{sgn} \frac{p_2 u - p_1 v}{p_1 q_2 - q_1 p_2} \operatorname{sgn} \frac{q_2 u - q_1 v}{p_1 q_2 - q_1 p_2}$ . If  $t_1 - t_2 s_1 - t_3 s_2 = 0$ , i.e.,  $n_{pq}^T t = 0$ , then  $I_4 \sim a^{5/6}$ , otherwise we have a rapid decay as  $a \rightarrow 0$ . This finishes the proof.  $\square$

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