3.1 Valid Inequalities and Faces of Polyhedra

Definition 3.1 (Valid Inequality)
Let \( w \in \mathbb{R}^n \) and \( t \in \mathbb{R} \). We say that the inequality \( w^T x \leq t \) is valid for a set \( S \subseteq \mathbb{R}^n \) if
\[
S \subseteq \{ x : w^T x \leq t \}.
\]
Sometimes we write short \((w,t)\) for the inequality \( w^T x \leq t \).

Definition 3.2 (Face)
Let \( P \subseteq \mathbb{R}^n \) be a polyhedron. The set \( F \subseteq P \) is called a face of \( P \), if there is a valid inequality \((w,t)\) for \( P \) such that
\[
F = \{ x \in P : w^T x = t \}.
\]
If \( F \neq \emptyset \) we say that \((w,t)\) supports the face \( F \) and call \( \{ x : w^T x = t \} \) the corresponding supporting hyperplane. If \( F \neq \emptyset \) and \( F \neq P \), then we call \( F \) a nontrivial or proper face.

Observe that any face of \( P(A,b) \) has the form
\[
F = \{ x : Ax \leq b, w^T x \leq t, -w^T x \leq -t \}
\]
which shows that any face of a polyhedron is again a polyhedron.

Example 3.3
We consider the polyhedron \( P \subseteq \mathbb{R}^2 \), which is defined by the inequalities
\[
\begin{align*}
x_1 + x_2 & \leq 2 \quad \text{(3.1a)} \\
x_1 & \leq 1 \quad \text{(3.1b)} \\
x_1, x_2 & \geq 0. \quad \text{(3.1c)}
\end{align*}
\]
We have \( P = P(A,b) \) with
\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.
\]
The line segment $F_1$ from $(0,2)$ to $(1,1)$ is a face of $P$, since $x_1 + x_2 \leq 2$ is a valid inequality and

$$F_1 = P \cap \{x \in \mathbb{R}^2 : x_1 + x_2 = 2\}.$$ 

The singleton $F_2 = \{(1,1)\}$ is another face of $P$, since

$$F_2 = P \cap \{x \in \mathbb{R}^2 : 2x_1 + x_2 = 3\}$$

$$F_2 = P \cap \{x \in \mathbb{R}^2 : 3x_1 + x_2 = 4\}.$$ 

Both inequalities $2x_1 + x_2 \leq 3$ and $3x_1 + x_2 \leq 4$ induce the same face of $P$. In particular, this shows that the same face can be induced by completely different inequalities. The inequalities $x_1 + x_2 \leq 2$, $2x_1 + x_2 \leq 3$ and $3x_1 + x_2 \leq 4$ induce nonempty faces of $P$. They support $F_1/F_2$. In contrast, the valid inequality $x_1 = \frac{5}{2}$ has

$$F_3 = P \cap \{x \in \mathbb{R}^2 : x_1 = \frac{5}{2}\} = \emptyset,$$

and thus $x_1 = \frac{5}{2}$ is not a supporting hyperplane of $P$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure3.1.png}
\caption{Polyhedron for Example 3.3}
\end{figure}

**Remark 3.4**

(i) Any polyhedron $P \subseteq \mathbb{R}^n$ is a face of itself, since $P = P \cap \{x \in \mathbb{R}^n : 0^T x = 0\}$.

(ii) $\emptyset$ is a face of any polyhedron $P \subseteq \mathbb{R}^n$, since $\emptyset = P \cap \{x \in \mathbb{R}^n : 0^T x = 1\}$.

(iii) If $F = P \cap \{x \in \mathbb{R}^n : c^T x = \gamma\}$ is a nontrivial face of $P \subseteq \mathbb{R}^n$, then $c \neq 0$, since otherwise we are either in case (i) or (ii) above.

Let us consider Example 3.3 once more. Face $F_1$ can be obtained by turning inequality (3.1a) into an equality:
3.1 Valid Inequalities and Faces of Polyhedra

\[
F_1 = \left\{ x \in \mathbb{R}^2 : \begin{array}{c}
x_1 + x_2 = 2 \\
x_1 \leq 1 \\
x_1, x_2 \geq 0 \end{array} \right\}
\]

Likewise \( F_2 \) can be obtained by making (3.1a) and (3.1b) equalities

\[
F_2 = \left\{ x \in \mathbb{R}^2 : \begin{array}{c}
x_1 + x_2 = 2 \\
x_1 = 1 \\
x_1, x_2 \geq 0 \end{array} \right\}
\]

Let \( P = \mathbb{P}(A, b) \subseteq \mathbb{R}^n \) be a polyhedron and \( M \) be the index set of the rows of \( A \). For a subset \( I \subseteq M \) we consider the set

\[ fa(I) := \{ x \in P : A_I x = b_I \} \tag{3.2} \]

Since any \( x \in P \) satisfies \( A_I x \leq b_I \), we get by summing up the rows of (3.2) for

\[ c^T := \sum_{i \in I} A_i, \quad \text{and} \quad \gamma := \sum_{i \in I} b_i \]

a valid inequality \( c^T x \leq \gamma \) for \( P \). For all \( x \in P \setminus fa(I) \) there is at least one \( i \in I \), such that \( A_i x < b_i \). Thus \( c^T x < \gamma \) for all \( x \in P \setminus fa(I) \) and

\[ fa(I) = \{ x \in P : c^T x = \gamma \} \]

is a face of \( P \).

**Definition 3.5 (Face induced by index set)**

The set \( fa(I) \) defined in (3.2) is called the face of \( P \) induced by \( I \).

In Example 3.3 we have \( F_1 = fa(\{1\}) \) and \( F_2 = fa(\{1, 2\}) \). The following theorem shows that in fact all faces of a polyhedron can be obtained this way.

**Theorem 3.6** Let \( P = \mathbb{P}(A, b) \subseteq \mathbb{R}^n \) be a nonempty polyhedron and \( M \) be the index set of the rows of \( A \). The set \( F \subseteq \mathbb{R}^n \) with \( F \neq \emptyset \) is a face of \( P \) if and only if \( F = fa(I) = \{ x \in P : A_I x = b_I \} \) for a subset \( I \subseteq M \).

**Proof:** We have already seen that \( fa(I) \) is a face of \( P \) for any \( I \subseteq M \). Assume conversely that \( F = P \cap \{ x \in \mathbb{R}^n : c^T x = t \} \) is a face of \( P \). Then, \( F \) is precisely the set of optimal solutions of the Linear Program

\[ \max \left\{ c^T x : Ax \leq b \right\} \tag{3.3} \]

(here we need the assumption that \( P \neq \emptyset \)). By the Duality Theorem of Linear Programming (Theorem 2.9), the dual Linear Program for (3.3)

\[ \min \left\{ b^T y : A^T y = c, y \geq 0 \right\} \]

also has an optimal solution \( y^* \) which satisfies \( b^T y^* = t \). Let \( I := \{ i : y^*_i > 0 \} \). By the complementary slackness (Theorem 2.10) the optimal solutions of (3.3) are precisely those \( x \in P \) with \( A_i x = b_i \) for \( i \in I \). This gives us \( F = fa(I) \).

This result implies the following consequence:
Corollary 3.7 Every polyhedron has only a finite number of faces.

Proof: There is only a finite number of subsets \( I \subseteq M = \{1, \ldots, m\} \).

We can also look at the binding equations for subsets of polyhedra.

Definition 3.8 (Equality set) Let \( P = P(A, b) \subseteq \mathbb{R}^n \) be a polyhedron. For \( S \subseteq P \) we call
\[
eq(S) := \{i \in M : A_i x = b_i \text{ for all } x \in S\},
\]
the equality set of \( S \).

Clearly, for subsets \( S, S' \) of a polyhedron \( P = P(A, b) \) with \( S \subseteq S' \) we have \( \text{eq}(S) \supseteq \text{eq}(S') \). Thus, if \( S \subseteq P \) is a nonempty subset of \( P \), then any face \( F \) of \( P \) which contains \( S \) must satisfy \( \text{eq}(F) \subseteq \text{eq}(S) \). On the other hand, \( \text{fa}(\text{eq}(S)) \) is a face of \( P \) containing \( S \). Thus, we have the following observation:

Observation 3.9 Let \( P = P(A, b) \subseteq \mathbb{R}^n \) be a polyhedron and \( S \subseteq P \) be a nonempty subset of \( P \). The smallest face of \( P \) which contains \( S \) is \( \text{fa}(\text{eq}(S)) \).

Corollary 3.10 (i) The polyhedron \( P = P(A, b) \) does not have any proper face if and only if \( \text{eq}(P) = M \), that is, if and only if \( P \) is an affine subspace \( P = \{x : Ax = b\} \).

(ii) If \( A\bar{x} < b \), then \( \bar{x} \) is not contained in any proper face of \( P \).

Proof:

(i) Immediately from the characterization of faces in Theorem 3.6.

(ii) If \( A\bar{x} < b \), then \( \text{eq}(\{\bar{x}\}) = \emptyset \) and \( \text{fa}(\emptyset) = P \).

3.2 Inner and Interior Points

Definition 3.11 (Inner point, interior point) The vector \( \bar{x} \in P = P(A, b) \) is called an inner point, if it is not contained in any proper face of \( P \). We call \( \bar{x} \in P \) an interior point, if \( A\bar{x} < b \).

By Corollary 3.10(ii) an interior point \( \bar{x} \) is not contained in any proper face.

Lemma 3.12 Let \( F \) be a face of the polyhedron \( P(A, b) \) and \( \bar{x} \in F \). Then \( \bar{x} \) is an inner point of \( F \) if and only if \( \text{eq}(\{\bar{x}\}) = \text{eq}(F) \).

Proof: Let \( G \) be an inclusionwise smallest face of \( F \) containing \( \bar{x} \). Then, \( \bar{x} \) is an inner point of \( F \) if and only if \( F = G \). By Observation 3.9 we have \( G = \text{fa}(\text{eq}(\{\bar{x}\})) \). And thus, \( \bar{x} \) is an inner point of \( F \) if and only if \( \text{fa}(\text{eq}(\{\bar{x}\})) = F \) as claimed.

Thus, Definition 3.11 can be restated equivalently as: \( \bar{x} \in P = P(A, b) \) is an inner point of \( P \) if \( \text{eq}(\{\bar{x}\}) = \text{eq}(P) \).

Lemma 3.13 Let \( P = P(A, b) \) be a nonempty polyhedron. Then, the set of inner points of \( P \) is nonempty.
3.3 The Fundamental Theorem of Linear Inequalities

**Proof:** Let \( M = \{1, \ldots, m\} \) be the index set of the rows of \( A \), \( I := \text{eq}(P) \) and \( J := M \setminus I \).

If \( J = \emptyset \), that is, if \( I = M \), then by Corollary 3.10(i) the polyhedron \( P \) does not have any proper face and any point in \( P \) is an inner point.

If \( J \neq \emptyset \), then for any \( j \in J \) we can find an \( x^j \in P \) such that \( Ax^j \leq b \) and \( A_j, x^j < b_j \). Since \( P \) is convex, the vector \( y \), defined as

\[
y := \frac{1}{|J|} \sum_{j \in J} x^j
\]

(which is a convex combination of the \( x^j, j \in J \)) is contained in \( P \). Then, \( A_j, y < b_j \) and \( A_1, y = b_1 \). So, eq(\{y\}) = eq(\{P\}) and the claim follows. \( \Box \)

### 3.3 The Fundamental Theorem of Linear Inequalities

**Theorem 3.14 (Fundamental Theorem of Linear Inequalities):** Suppose that \( a_1, \ldots, a_k \in \mathbb{R}^n \) and \( b \in \mathbb{R}^n \) such that \( t := \text{rank}(a_1, \ldots, a_k, b) \). Then exactly one of the following statements holds:

(i) \( b \) is a nonnegative linear combination of linear independent vectors from \( a_1, \ldots, a_k \), i.e., \( b \in \text{cone}(\{a_{i_1}, \ldots, a_{i_t}\}) \) for linear independent vectors \( a_{i_1}, \ldots, a_{i_t} \).

(ii) There is a vector \( c \neq 0 \) such that the hyperplane \( \{x : c^T x = 0\} \) contains \( t - 1 \) linear independent vectors from \( a_1, \ldots, a_k \) and \( c^T a_i \geq 0 \) for \( i = 1, \ldots, k \) and \( c^T b < 0 \).

If all the \( a_i \) and \( b \) are rational, then \( c \) can be chosen to be a rational vector.

**Proof:** First, observe that at most one of the two cases can occur, since otherwise

\[
0 > c^T b = \sum_{i=1}^{k} \lambda_i c^T a_i \geq 0,
\]

which is a contradiction.

We now prove that at least one of the cases occurs. If \( b \) is not in the subspace \( U \) spanned by \( a_1, \ldots, a_k \), then we can write \( b = b_1 + b_2 \) where \( b_1 \in U \) and \( b_2 \neq 0 \) is in the orthogonal complement of \( U \). The vector \( -b_2 \) satisfies \( -b_2^T a_i = 0 \) for \( i = 1, \ldots, k \) and \( -b_2^T b = -\|b_2\|^2 < 0 \). In this case we are done.

Thus for the remainder of the proof we can assume that \( b \) is contained in \( U \) which means that \( t := \text{rank}(a_1, \ldots, a_k, b) = \text{rank}(a_1, \ldots, a_k) \). We can also assume without loss of generality that the vectors \( a_1, \ldots, a_k \) span the whole \( \mathbb{R}^n \), i.e., that \( t = n \). The argument is as follows: if \( t < n \) we consider an isomorphism \( \varphi : U \to \mathbb{R}^t \), which is described by a matrix \( T \in \mathbb{R}^{1 \times n} \). If all the \( a_i \) are rational, then \( T \) is also a rational matrix. If we prove the theorem for the vectors \( \varphi(a_1), \ldots, \varphi(a_k) \) and \( \varphi(b) \), we have one of the following situations:

(i) \( \varphi(b) = \lambda_1 \varphi(a_{i_1}) + \cdots + \lambda_t \varphi(a_{i_t}) = \varphi(\lambda_{i_1} a_{i_1} + \cdots + \lambda_{i_t} a_{i_t}) \) for linear independent vectors \( \varphi(a_{i_t}) \). Then by the fact that \( \varphi \) is an isomorphism, we have \( b = \lambda_{i_1} a_{i_1} + \cdots + \lambda_{i_t} a_{i_t} \) and we are done.

(ii) There is a vector \( c \neq 0 \) such that the hyperplane \( \{x : c^T x = 0\} \) contains \( t - 1 \) linear independent vectors from \( \varphi(a_1), \ldots, \varphi(a_k) \) and \( c^T \varphi(a_i) \geq 0 \) for \( i = 1, \ldots, k \) and \( c^T \varphi(b) < 0 \). Since \( \varphi(x) = Tx \) for any vector in \( U \) we have that \( c^T x = (T^T c)^T x \) and the vector \( T^T c \) is as required.
So, for the rest of the proof we assume that the vectors \( a_1, \ldots, a_k \) span \( \mathbb{R}^n \). We choose linear independent vectors \( a_{i_1}, \ldots, a_{i_\ell} \) and set \( D = \{ a_{i_1}, \ldots, a_{i_\ell} \} \). We then apply the following procedure:

1. Let \( b = \lambda_{i_1} a_{i_1} + \cdots + \lambda_{i_\ell} a_{i_\ell} \).
2. If \( \lambda_{i} \geq 0 \) for all \( i \), then we are in case (i).
3. Otherwise, choose the smallest \( h \) such that \( \lambda_{i_h} < 0 \). We let \( H \) be the subspace spanned by \( D \setminus \{ a_{i_h} \} \), then \( H \) has an orthogonal complement \( H \) dimension 1 and thus there exists a vector \( c \) such that \( H = \{ x : c^T x = 0 \} \). We normalize \( c \) such that \( c^T a_{i_h} = 1 \) and hence
   \[
   \lambda_{i_h} = c^T b < 0. \tag{3.4}
   \]

   (Observe that in case of rational \( a_{i_h} \) the vector \( c \) can be chosen to be rational: Use the Gram-Schmidt orthogonalization applied to \( D \) and \( a_{i_h} \) (where \( a_{i_h} \) is the last vector to be handled). This yields a rational orthonormal basis of \( \mathbb{R}^n \) and \( H \) is the orthogonal complement of the last vector in the orthonormal basis).
4. If \( c^T a_i \geq 0 \) for \( i = 1, \ldots, k \), then we are in case (ii).
5. Otherwise, there is \( s \) such that \( c^T a_s < 0 \). Choose the smallest such \( s \). We set \( D := D \setminus \{ a_{i_1} \cup \{ a_s \} \} \) and continue in the first step. (Observe that the new set \( D \) forms a set of linearly independent vectors, since \( a_s \) can not be a linear combination of the other vectors in \( D \), since \( c^T a_s < 0 \) and \( c^T a_i = 0 \) for the other ones).

The proof is complete, if we can show that the above process must terminate. Suppose that this is not the case. Then, since there are only finitely many choices for the set \( D \), we must have that \( D_p = D_{p + m} \), where \( D_t \) is the set \( D \) at the beginning of iteration \( t \).

Let \( r \) be the highest index, such that \( a_r \) has been removed in one of the iterations \( p, p + 1, \ldots, p + m \) and suppose that this is iteration \( \ell \). Then \( a_r \) must also have been added in another iteration \( q \), where \( \ell \leq q < p + m \). By construction we have:

\[
D_{\ell} \cap \{ a_{r+1}, \ldots, a_m \} = D_q \cap \{ a_{r+1}, \ldots, a_m \}. \tag{3.5}
\]

Let \( D_{\ell} = \{ a_{i_1}, \ldots, a_{i_\ell} \} \), \( b = \lambda_{i_1} a_{i_1} + \cdots + \lambda_{i_{\ell}} a_{i_{\ell}} \), and \( \bar{c} \) be the vector found in iteration \( q \). Then, we have:

- \( \lambda_{i_j} \geq 0 \) for \( i_j < r \), since \( r \) was the smallest index \( h \) in iteration \( \ell \) such that \( \lambda_{i_h} < 0 \).
- \( \bar{c}^T a_{i_j} \geq 0 \) for \( i_j < r \), since \( r \) was the smallest index \( s \) in iteration \( q \) such that \( \bar{c}^T a_i < 0 \).
- This gives us
  \[
  \lambda_{i_j} \bar{c}^T a_{i_j} \geq 0 \quad \text{for } i_j < r. \tag{3.6}
  \]
- \( \lambda_{i_j} < 0 \) and \( \bar{c}^T a_{i_j} < 0 \) for \( i_j = r \), since \( a_r \) was removed in iteration \( \ell \) and added in iteration \( q \):
  \[
  \lambda_{i_j} \bar{c}^T a_{i_j} > 0 \quad \text{for } i_j = r. \tag{3.7}
  \]
- \( \bar{c}^T a_{i_j} = 0 \) for \( i_j > r \), since we have (3.5) and the way we constructed \( \bar{c} \):
  \[
  \lambda_{i_j} \bar{c}^T a_{i_j} = 0 \quad \text{for } i_j > r. \tag{3.8}
  \]

Using (3.6), (3.7) and (3.8) yields:

\[
0 > \bar{c}^T b = \bar{c}^T (\lambda_{i_1} a_{i_1} + \cdots + \lambda_{i_\ell} a_{i_\ell}) = \lambda_{i_1} \bar{c}^T a_{i_1} + \cdots + \lambda_{i_\ell} \bar{c}^T a_{i_\ell} > 0,
\]

which is a contradiction. \( \square \)
3.4 The Decomposition Theorem

Recall that in Section 2.4 we called
\[ \text{cone}(S) := \{ \lambda_1 x_1 + \cdots + \lambda_k x_k : x_i \in S, \lambda_i \geq 0 \text{ for } i = 1, \ldots, k \} \]
the cone generated by the set S. We call a cone C \textit{finitely generated}, if there is a finite set S such that C = cone(S).

\textbf{Theorem 3.15 (Theorem of Weyl-Minkowski-Farkas)} A convex cone C is polyhedral if and only if it is finitely generated.

\textit{If} C \textit{is a rational polyhedral cone, then it is generated by a finite set of integral vectors and each cone generated by rational vectors is a rational polyhedron.}

\textbf{Proof:} Suppose that \( x_1, \ldots, x_k \) are vectors in \( \mathbb{R}^n \). We must prove that cone(\( \{x_1, \ldots, x_k\} \)) is polyhedral. To this end, assume in the first step that \( x_1, \ldots, x_k \) span \( \mathbb{R}^n \).

By the Fundamental Theorem of Linear Inequalities 3.14 we have: \( y \in \text{cone}(\{x_1, \ldots, x_k\}) \) if and only if for all hyperplanes \( \{x : c^T x = 0\} \) containing \( n-1 \) linearly independent vectors from \( x_1, \ldots, x_k \) and such that \( c^T x_i \geq 0 \) for \( i = 1, \ldots, k \) we have \( c^T y \geq 0 \). In other words, \( y \in \text{cone}(\{x_1, \ldots, x_k\}) \) if and only if \( y \) is contained in any halfspace \( \{x : c^T x \geq 0\} \) such that \( \{x : c^T x = 0\} \) contains \( n-1 \) linear vectors from \( x_1, \ldots, x_k \) and such that \( c^T x_i \geq 0 \) for \( i = 1, \ldots, k \). There are only finitely many such halfspaces (since there are only finitely many ways to select the \( n-1 \) linearly independent vectors). So cone(\( \{x_1, \ldots, x_k\} \)) is the solution set of a finite number of inequalities, i.e., a polyhedron. Observe that in the case of rational vectors \( x_i \) each c is rational (see the Fundamental Theorem of Linear Inequalities) and, hence, the cone is a rational polyhedron.

If \( x_1, \ldots, x_k \) do not span \( \mathbb{R}^n \), we follow the above construction by extending each halfspace \( H \) of the subspace \( U \) spanned by \( x_1, \ldots, x_k \) to a halfspace \( H' \) of \( \mathbb{R}^n \) such that \( H' \cap U = H \).

We now prove the other direction. To this end, assume that
\[ C = \{x : Ax \leq 0\} = \{x : A_i x \leq 0, i = 1, \ldots, m\}. \]
We have already shown that each finitely generated cone is polyhedral, so
\[ \text{cone}(\{A_1, \ldots, A_m\}) = \{x : Bx \leq 0\} = \{x : B_j x \leq 0, j = 1, \ldots, p\}. \quad (3.9) \]
(in case of a rational matrix \( A \) by our proof above also all the entries in \( B \) can be chosen to be rational). We are done, if we can show that \( C = \text{cone}(\{B_1, \ldots, B_p\}) \).

By (3.9) we have \( A_i^T B_j < 0 \) for all \( i \) and all \( j \). This gives us \( B_j, \in C \) for all \( j \) and hence cone(\( \{B_1, \ldots, B_p\}\)) \( \subseteq C \).

Assume that there exists some \( y \in C \setminus \text{cone}(\{B_1, \ldots, B_p\}) \). Then, by the Fundamental Theorem of Linear Inequalities, there exists some \( c \neq 0 \) such that \( c^T y < 0 \) and \( c^T B_j \geq 0 \) for \( j = 1, \ldots, m \). By (3.9) we have \( -c \in \text{cone}(\{A_1, \ldots, A_m\}) \), say \( -c^T = \sum_{i=1}^{m} \lambda_i A_i \) for nonnegative scalars \( \lambda_i \).

If \( x \in C \), then we have \( A_i x \leq 0 \) for \( i = 1, \ldots, m \) and hence also
\[ 0 \geq \sum_{i=1}^{m} \lambda_i A_i x = -c^T x. \]
Thus, it follows that \( (-c)^T x \leq 0 \) for all \( x \in C \). But this is a contradiction to the fact that \( y \in C \) and \( -c^T y > 0 \).
If \( C \) is a rational cone, then we have seen above that \( C \) is generated by rational vectors \( B_1, \ldots, B_p \). Since we can multiply the entries in each rational vector by any positive number without changing the generated cone, it follows that the vectors \( B_1, \ldots, B_p \) can in fact be chosen to be integral.

**Theorem 3.16 (Decomposition Theorem for Polyhedra)** A set \( P \subseteq \mathbb{R}^n \) is a polyhedron if and only if there exists a finite set \( X \) of points such that

\[
P = \text{conv}(X) + C
\]

(3.10)

for some polyhedral cone \( C \). If \( P \) is rational, then the points in \( X \) can be chosen to be rational and \( C \) is a rational cone.

**Proof:** Suppose that \( P = P(A, b) = \{ x : Ax \leq b \} \) is a polyhedron. We consider the polyhedral cone

\[
C' := \left\{ \left( \begin{array}{c} x \\ 1 \end{array} \right) \in \mathbb{R}^{n+1} : Ax - \lambda b \leq 0, \lambda \geq 0 \right\}
\]

(3.11)

By Theorem 3.15 this cone \( C' \) is generated by finitely many vectors \( (\frac{x_1}{\lambda_1}), \ldots, (\frac{x_k}{\lambda_k}) \). Observe that since \( \lambda \geq 0 \) for all \( (\frac{x_i}{\lambda_i}) \in C' \) we have \( \lambda_i \geq 0 \) for \( i = 1, \ldots, k \). We set

\[
Q := \text{conv}\{[x_1 : \lambda_1 > 0]\}
\]

\[
C := \text{cone}\{[x_1 : \lambda_1 = 0]\}.
\]

We then have:

\[
x \in P \iff Ax \leq b
\]

\[
\iff \left( \begin{array}{c} x \\ 1 \end{array} \right) \in C'
\]

\[
\iff \left( \begin{array}{c} x \\ 1 \end{array} \right) \in \text{cone}\left\{ \left( \begin{array}{c} x_1 \\ \lambda_1 \end{array} \right), \ldots, \left( \begin{array}{c} x_k \\ \lambda_k \end{array} \right) \right\}
\]

\[
\iff x \in Q + C.
\]

Assume now conversely that \( P = \text{conv}(X) + C \) for a finite set \( X = \{ x_1, \ldots, x_k \} \) and a polyhedral cone \( C \). By Theorem 3.15, the cone \( C \) is finitely generated, i.e., \( C = \text{cone}\{[y_1, \ldots, y_p]\} \). We have

\[
x \in P \iff x = \sum_{i=1}^{k} \lambda_i x_i + \sum_{j=1}^{p} \mu_j y_j \text{ for some } \lambda_i \geq 0, \mu_j \geq 0 \text{ with } \sum_{i=1}^{k} \lambda_i = 1
\]

\[
\iff \left( \begin{array}{c} x \\ 1 \end{array} \right) \in \text{cone}\left\{ \left( \begin{array}{c} x_1 \\ 1 \end{array} \right), \ldots, \left( \begin{array}{c} x_k \\ 1 \end{array} \right), \left( \begin{array}{c} y_1 \\ 0 \end{array} \right), \ldots, \left( \begin{array}{c} y_p \\ 0 \end{array} \right) \right\}.
\]

By Theorem 3.15 the last cone in the above calculation is polyhedral, i.e., there is a matrix \((A, b)\) such that

\[
\text{cone}\left\{ \left( \begin{array}{c} x_1 \\ 1 \end{array} \right), \ldots, \left( \begin{array}{c} x_k \\ 1 \end{array} \right), \left( \begin{array}{c} y_1 \\ 0 \end{array} \right), \ldots, \left( \begin{array}{c} y_p \\ 0 \end{array} \right) \right\} = \left\{ \left( \begin{array}{c} x \\ \lambda \end{array} \right) : [A \ b] \left( \begin{array}{c} x \\ \lambda \end{array} \right) \leq 0 \right\}
\]

\[
= \left\{ \left( \begin{array}{c} x \\ \lambda \end{array} \right) : Ax + \lambda b \leq 0 \right\}.
\]

Thus

\[
x \in P \iff \left( \begin{array}{c} x \\ 1 \end{array} \right) \in \text{cone}\left\{ \left( \begin{array}{c} x_1 \\ 1 \end{array} \right), \ldots, \left( \begin{array}{c} x_k \\ 1 \end{array} \right), \left( \begin{array}{c} y_1 \\ 0 \end{array} \right), \ldots, \left( \begin{array}{c} y_p \\ 0 \end{array} \right) \right\}
\]

\[
\iff \{x : Ax \leq b\}.
\]

Thus, \( P \) is a polyhedron.

\[\square\]
Corollary 3.17 (Characterization of polytopes) A set $P \subseteq \mathbb{R}^n$ is a polytope if and only if it is the convex hull of finitely many points.

Proof: Let $P$ be a polytope. By the Decomposition Theorem we have $P = \text{conv}(X) + C$ for a finite set $X$ and a polyhedral cone $C$. Since $P$ is bounded we must have $C = \{0\}$ and $P = \text{conv}(X)$.

Conversely, if $P = \text{conv}(X)$, then $P$ is a polyhedron (again by the Decomposition Theorem) and also bounded, thus a polytope. \hfill \Box

Example 3.18 A stable set (or independent set) in an undirected graph $G = (V,E)$ is a subset $S \subseteq V$ of the vertices such that none of the vertices in $S$ are joined by an edge. We can formulate the problem of finding a stable set of maximum cardinality as an IP:

$$\max \sum_{v \in V} x_v \quad (3.12a)$$

$$x_u + x_v \leq 1 \quad \text{for all edges } (u,v) \in E \quad (3.12b)$$

$$x_v \geq 0 \quad \text{for all vertices } v \in V \quad (3.12c)$$

$$x_v \leq 1 \quad \text{for all vertices } v \in V \quad (3.12d)$$

$$x_v \in \mathbb{Z} \quad \text{for all vertices } v \in V \quad (3.12e)$$

As an application of Corollary 3.17 we consider the so-called stable-set polytope $\text{STAB}(G)$, which is defined as the convex hull of the incidence vectors of stable sets in an undirected graph $G$.

$$\text{STAB}(G) = \text{conv}( \left\{ x \in \mathbb{B}^V : x \text{ is an incidence vector of a stable set in } G \right\} ). \quad (3.13)$$

By Corollary 3.17, $\text{STAB}(G)$ is a polytope and we can solve (3.12) by Linear Programming over a polytope $\{x : Ax \leq b\}$. Of course, the issue is how to find $A$ and $b$! \hfill \Box

### 3.5 Dimension

Intuitively the notion of dimension seems clear by considering the degrees of freedom we have in moving within a given polyhedron (cf. Figure 3.2).

Definition 3.19 (Affine Combination, affine independence, affine hull)

An affine combination of the vectors $v^1, \ldots, v^k \in \mathbb{R}^n$ is a linear combination $x = \sum_{i=1}^{k} \lambda_i v^i$ such that $\sum_{i=1}^{k} \lambda_i = 1$.

Given a set $X \subseteq \mathbb{R}^n$, the affine hull of $X$, denoted by $\text{aff}(X)$ is defined to be the set of all affine combinations of vectors from $X$, that is

$$\text{aff}(X) := \{ x = \sum_{i=1}^{k} \lambda_i v_i : \sum_{i=1}^{k} \lambda_i = 1 \text{ and } v_1, \ldots, v_k \in X \}$$

The vectors $v^1, \ldots, v^k \in \mathbb{R}^n$ are called affinely independent, if $\sum_{i=1}^{k} \lambda_i v_i = 0$ and $\sum_{i=1}^{k} \lambda_i = 0$ implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$.

Lemma 3.20 The following statements are equivalent

(i) The vectors $v^1, \ldots, v^k \in \mathbb{R}^n$ are affinely independent.
(ii) The vectors $v^2 - v^1, \ldots, v^k - v^1 \in \mathbb{R}^n$ are linearly independent.

(iii) The vectors $(v^1_1), \ldots, (v^k_1) \in \mathbb{R}^{n+1}$ are linearly independent.

Proof:

(i)⇔(ii) If $\sum_{i=2}^{k} \lambda_i (v^i - v^1) = 0$ and we set $\lambda_1 := -\sum_{i=2}^{k} \lambda_i$, this gives us $\sum_{i=1}^{k} \lambda_i v^i = 0$ and $\sum_{i=1}^{k} \lambda_i = 0$. Thus, from the affine independence it follows that $\lambda_1 = \cdots = \lambda_k = 0$.

Assume conversely that $v^2 - v^1, \ldots, v^k - v^1$ are linearly independent and $\sum_{i=1}^{k} \lambda_i v^i = 0$ with $\sum_{i=1}^{k} \lambda_i = 0$. Then $\lambda_1 = -\sum_{i=2}^{k} \lambda_i$ which gives $\sum_{i=1}^{k} \lambda_i (v^i - v^1) = 0$. The linear independence of $v^2 - v^1, \ldots, v^k - v^1$ implies $\lambda_2 = \cdots = \lambda_k = 0$ which in turn also gives $\lambda_1 = 0$.

(ii)⇔(iii) This follows immediately from

$$\sum_{i=1}^{k} \lambda_i (v^i_1) = 0 \iff \left\{ \begin{array}{c} \sum_{i=1}^{k} \lambda_i v^i = 0 \\ \sum_{i=1}^{k} \lambda_i = 0 \end{array} \right\}$$

Definition 3.21 (Dimension of a polyhedron, full-dimensional polyhedron)

The dimension $\dim P$ of a polyhedron $P \subseteq \mathbb{R}^n$ is one less than the maximum number of affinely independent vectors in $P$. We set $\dim \emptyset = -1$. If $\dim P = n$, then we call $P$ full-dimensional.

Example 3.22

Consider the polyhedron $P \subseteq \mathbb{R}^2$ defined by the following inequalities (see Figure 3.3):

\begin{align*}
x &\leq 2 \quad (3.14a) \\
x + y &\leq 4 \quad (3.14b) \\
x + 2y &\leq 10 \quad (3.14c) \\
x + 2y &\leq 6 \quad (3.14d) \\
x + y &\geq 2 \quad (3.14e) \\
x, y &\geq 0 \quad (3.14f)
\end{align*}
Figure 3.3: A fulldimensional polyhedron in $\mathbb{R}^2$.

The polyhedron $P$ is full dimensional, since $(2,0)$, $(1,1)$ and $(2,2)$ are three affinely independent vectors.

Example 3.23
Consider the stable set polytope from Example 3.18. Let $P$ be the polytope determined by the inequalities in (3.12). We claim that $P$ is full dimensional. To see this, consider the $n$ unit vectors $e_i = (0, \ldots, 1, 0, \ldots, 0)^T$, $i = 1, \ldots, n$ and $e_0 := (0, \ldots, 0)^T$. Then $e_0, e_1, \ldots, e_n$ are affinely independent and thus $\dim P = n$.

Theorem 3.24 (Dimension Theorem) Let $F \neq \emptyset$ be a face of the polyhedron $P(A, b) \subseteq \mathbb{R}^n$. Then we have

$$\dim F = n - \text{rank} A_{eq(F)}.$$

Proof: By Linear Algebra we know that

$$\dim \mathbb{R}^n = n = \text{rank} A_{eq(F)} + \dim \ker A_{eq(F)}.$$

Thus, the theorem follows, if we can show that $\dim \ker (A_{eq(F)}) = \dim F$. We abbreviate $l := \text{eq}(F)$ and set $r := \dim \ker A_{L_1}$, $s := \dim F$.

$r \geq s$:
Select $s + 1$ affinely independent vectors $x^0, x^1, \ldots, x^s \in F$. Then, by Lemma 3.20 $x^1 - x^0, \ldots, x^s - x^0$ are linearly independent vectors and $A_{L_1}(x^j - x^0) = b_1 - b_1 = 0$ for $j = 1, \ldots, s$. Thus, the dimension of $\ker A_{L_1}$ is at least $s$.

$s \geq r$:
Since we have assumed that $F \neq \emptyset$, we have $s = \dim F \geq 0$. Thus, in the sequel we can assume that $r \geq 0$ since otherwise there is nothing left to prove.

By Lemma 3.13 there exists an inner point $\bar{x}$ of $F$ which by Lemma 3.12 satisfies $\text{eq}([\bar{x}]) = \text{eq}(F) = l$. Thus, for $J := M \setminus I$ we have

$$A_{L_1} \bar{x} = b_1 \quad \text{and} \quad A_{J_1} \bar{x} < b_1.$$

Let $\{x^1, \ldots, x^r\}$ be a basis of $\ker A_{L_1}$. Then, since $A_{J_1} \bar{x} < b_1$ we can find $\varepsilon > 0$ such that $A_{J_1} (\bar{x} + \varepsilon x^k) < b_1$ and $A_{L_1} (\bar{x} + \varepsilon x^k) = b_1$ for $k = 1, \ldots, r$. Thus, $\bar{x} + \varepsilon x^k \in F$ for $k = 1, \ldots, r$. 

-----

The vectors \( \varepsilon x^1, \ldots, \varepsilon x^r \) are linearly independent and, by Lemma 3.20, \( \varepsilon x^1 + \varepsilon x^r + \varepsilon x^r \) form a set of \( r + 1 \) affinely independent vectors in \( F \) which implies \( \dim F \geq r \).

\[ \square \]

**Corollary 3.25** Let \( P = P(A, b) \subseteq \mathbb{R}^n \) be a nonempty polyhedron. Then:

(i) \( \dim P = n - \text{rank} A_{eq(P)} \).

(ii) \( P \) is full dimensional if \( \text{eq}(P) = \emptyset \).

(iii) \( P \) is full dimensional if \( P \) contains an interior point.

(iv) If \( F \) is a proper face of \( P \), then \( \dim F \leq \dim P - 1 \).

\[ \square \]

**Proof:**

(i) Use Theorem 3.24 with \( F = P \).

(ii) Immediate from (i).

(iii) If \( P \) has an interior point, then \( \text{eq}(P) = \emptyset \).

(iv) Let \( I := \text{eq}(P) \) and \( j \in \text{eq}(F) \setminus I \) and \( J := \text{eq}(P) \cup \{j\} \). We show that \( A_{I,} \) is linearly independent of the rows in \( A_{L,} \). This shows that \( \text{rank} A_{eq(F),} \geq \text{rank} A_{J,} > \text{rank} A_{I,} \) and by the Dimension Theorem we have \( \dim F \leq \dim P - 1 \).

Assume that \( A_{I,} \sum \lambda_i A_{I,} x = \sum \lambda_i b_i \).

Take \( x \in F \) arbitrary, then

\[ b_j = A_{I,} x = \sum \lambda_i A_{I,} x = \sum \lambda_i b_i. \]

Since \( j \not\in \text{eq}(P) \), there is \( x \in P \) such that \( A_{I,} x < b_j \). But by the above we have

\[ b_j > A_{I,} x = \sum \lambda_i A_{I,} x = \sum \lambda_i b_i = b_j, \]

which is a contradiction.

\[ \square \]

We derive another important consequence of the Dimension Theorem about the facial structure of polyhedra:

**Theorem 3.26 (Hoffman and Kruskal)** Let \( P = P(A, b) \subseteq \mathbb{R}^n \) be a polyhedron. Then a nonempty set \( F \subseteq P \) is an inclusionwise minimal face of \( P \) if and only if \( F = \{x : A_{I,} x = b_{I}, A_{J,} x \leq b_{J}\} \) for some index set \( I \subseteq M \) and \( \text{rank} A_{I,} = \text{rank} A \).

**Proof:** “⇒”: Let \( F \) be a minimal nonempty face of \( P \). Then, by Theorem 3.6 and Observation 3.9 we have \( F = \text{fa}(I) \), where \( I = \text{eq}(F) \). Thus, for \( J := M \setminus I \) we have

\[ F = \{x : A_{I,} x = b_{I}, A_{J,} x \leq b_{J}\}. \]

We claim that \( F = G \), where

\[ G = \{x : A_{I,} x = b_{I}\}. \]
By (3.15) we have $F \subseteq G$. Suppose that there exists $y \in G \setminus F$. Then, there exists $j \in J$

$$A_1, y = b_1, A_j, y > b_j.$$  \hspace{1cm} (3.17)

Let $\bar{x}$ be any inner point of $F$ which exists by Lemma 3.13. We consider for $\tau \in \mathbb{R}$ the point

$$z(\tau) = \bar{x} + \tau(y - \bar{x}) = (1 - \tau)\bar{x} + \tau y.$$  

Observe that $A_1, z(\tau) = (1 - \tau)A_1, \bar{x} + \tau A_1, y = (1 - \tau)b_1 + \tau b_1 = b_1$, since $\bar{x} \in F$ and $y$ satisfies (3.17). Moreover, $A_j, z(0) = A_j, \bar{x} < b_j$, since $J \subseteq M \setminus I$.

Since $A_j, y > b_j$ we can find $\tau \in \mathbb{R}$ and $j_0 \in J$ such that $A_{j_0}, z(\tau) = b_{j_0}$ and $A_1, z(\tau) \leq b_1$. Then, $\tau \neq 0$ and

$$F' := \{x \in P : A_1, x = b_1, A_{j_0}, x = b_{j_0}\}$$

is a face which is properly contained in $F$ (note that $\bar{x} \in F \setminus F'$). This contradicts the choice of $F$ as inclusionwise minimal. Hence, we have that $F$ can be represented as (3.16).

It remains to prove that $\operatorname{rank} A_1, = \operatorname{rank} A$. If $\operatorname{rank} A_1, < \operatorname{rank} A$, then there exists an index $j \in J = M \setminus I$, such that $A_{j,}$ is not a linear combination of the rows in $A_1,$. Then, we can find a vector $w \neq 0$ such that $A_1, w = 0$ and $A_{j,} w > 0$. For $\theta > 0$ appropriately chosen the vector $y := \bar{x} + \theta w$ satisfies (3.17) and as above we can construct a proper face $F'$ of $F$ contradicting the minimality of $F$.

"$\Leftarrow$": If $F = \{x : A_1, x = b_1\}$, then $F$ is an affine subspace and Corollary 3.10(i) shows that $F$ does not have any proper face. By assumption $F \subseteq P$ and thus $F = \{x : A_1, x = b_1, A_{j_0}, x = b_{j_0}\}$ is a minimal face of $P$.

**Corollary 3.27** All minimal nonempty faces of a polyhedron $P = P(A, b)$ have the same dimension, namely $n - \operatorname{rank} A$.  

**Corollary 3.28** Let $P = P(A, b) \subseteq \mathbb{R}^n$ be a nonempty polyhedron and $\operatorname{rank}(A) = n - k$. Then $P$ has a face of dimension $k$ and does not have a proper face of lower dimension.

**Proof:** Let $F$ be any nonempty face of $P$. Then, $\operatorname{rank} A_{\text{eq}(F),} = \operatorname{rank} A = n - k$ and thus by the Dimension Theorem (Theorem 3.24) it follows that $\dim(F) \geq n - (n - k) = k$. Thus, any nonempty face of $P$ has dimension at least $k$.

On the other hand, by Corollary 3.27 any inclusionwise minimal nonempty face of $P$ has dimension $n - \operatorname{rank} A = n - (n - k) = k$. Thus, $P$ has in fact faces of dimension $k$.

There will be certain types of faces which are of particular interest:

- extreme points (vertices),
- facets.

We will also be interested in extreme rays, which are directions in which we can "move towards infinity" inside a polyhedron.

In the next section we discuss extreme points and their meaning for optimization. Section 3.7 deals with facets and their importance in describing polyhedra by means of inequalities.
3.6 Extreme Points

Definition 3.29 (Extreme point, pointed polyhedron)
The point \( x \in P = P(A, b) \) is called an extreme point of \( P \), if \( x = \lambda x + (1 - \lambda)y \) for some \( x, y \in P \) and \( 0 < \lambda < 1 \) implies that \( x = y = x \).

A polyhedron \( P = P(A, b) \) is pointed, if it has at least one extreme point.

Example 3.30
Consider the polyhedron from Example 3.22. The point \((2, 2)\) is an extreme point of the polyhedron.

Theorem 3.31 (Characterization of extreme points) Let \( P = P(A, b) \subseteq \mathbb{R}^n \) be a polyhedron and \( x \in P \). Then, the following statements are equivalent:

(i) \( \{ x \} \) is a zero-dimensional face of \( P \).

(ii) There exists a vector \( c \in \mathbb{R}^n \) such that \( x \) is the unique optimal solution of the Linear Program \( \max \{ c^T x : x \in P \} \).

(iii) \( x \) is an extreme point of \( P \).

(iv) \( \text{rank} A_{\text{eq}}(x) \). = \( n \).

Proof: \( \text{"(i)\Rightarrow(ii)"} \): Since \( \{ x \} \) is a face of \( P \), there exists a valid inequality \( w^T x \leq t \) such that \( \{ x \} = \{ x \in P : w^T x = t \} \). Thus, \( x \) is the unique optimum of the Linear Program with objective \( c := w \).

(\( \text{"(ii)\Rightarrow(iii)"} \)): Let \( \tilde{x} \) be the unique optimum solution of \( \max \{ c^T x : x \in P \} \). If \( \tilde{x} = \lambda x + (1 - \lambda)y \) for some \( x, y \in P, (x, y) \neq (\bar{x}, \bar{x}) \) and \( 0 < \lambda < 1 \), then we have
\[
\bar{c}^T \tilde{x} = \lambda c^T x + (1 - \lambda)c^T y \leq \lambda c^T \bar{x} + (1 - \lambda)c^T \bar{x} = c^T \bar{x}.
\]
Thus, we can conclude that \( c^T \bar{x} = c^T x = c^T y \) which contradicts the uniqueness of \( \bar{x} \) as optimal solution.

(\( \text{"(iii)\Rightarrow(iv)"} \)): Let \( I := \text{eq}(\{ x \}) \). If \( \text{rank} A_{I, \bar{x}} < n \), there exists \( y \in \mathbb{R}^n \setminus \{0\} \) such that \( A_{I, \bar{x}} y = 0 \). Then, for sufficiently small \( \epsilon > 0 \) we have \( x := \bar{x} + \epsilon y \in P \) and \( y := \bar{x} - \epsilon y \in P \) (since \( A_{I, \bar{x}} \bar{x} < b_I \) for all \( j \notin I \)). But then, \( \bar{x} = \frac{1}{\epsilon} x + \frac{1}{\epsilon} y \) which contradicts the assumption that \( \bar{x} \) is an extreme point.

(\( \text{"(iv)\Rightarrow(i)"} \)): Let \( I := \text{eq}(\{ x \}) \). By (iv), the system \( A_{I, \bar{x}} \bar{x} = b_I \) has a unique solution which must be \( \bar{x} \) (since \( A_{I, \bar{x}} = b_I \) by construction of I). Hence
\[
\{ x \} = \{ x : A_{I, x} = b_I \} = \{ x \in P : A_{I, x} = b_I \}
\]
and by Theorem 3.6 \( \{ x \} \) is a zero-dimensional face of \( P \).

The result of the previous theorem has interesting consequences for optimization. Consider the Linear Program
\[
\max \{ c^T x : x \in P \}, \tag{3.18}
\]
where \( P \) is a pointed polyhedron (that is, it has extreme points). Since by Corollary 3.27 all minimal proper faces of \( P \) have the same dimension, it follows that the minimal proper faces of \( P \) are of the form \( \{ x \} \), where \( x \) is an extreme point of \( P \). Suppose that \( P \neq \emptyset \) and \( c^T x \) is bounded on \( P \). We know that there exists an optimal solution \( x^* \in P \). The set of optimal solutions of (3.18) is a face
\[
F = \{ x \in P : c^T x = c^T x^* \}
\]
which contains a minimal nonempty face \( F' \subseteq F \). Thus, we have the following corollary:
Corollary 3.32 If the polyhedron $P$ is pointed and the Linear Program (3.18) has optimal solutions, it has an optimal solution which is also an extreme point of $P$.

Another important consequence of the characterization of extreme points in Theorem 3.31 is the following:

Corollary 3.33 Every polyhedron has only a finite number of extreme points.

Proof: By the preceding theorem, every extreme point is a face. By Corollary 3.7, there is only a finite number of faces.

Let us now return to the Linear Program (3.18) which we assume to have an optimal solution. We also assume that $P$ is pointed, so that the assumptions of Corollary 3.32 are satisfied. By the Theorem of Hoffman and Kruskal (Theorem 3.26) every extreme point $\bar{x}$ of $P$ is the solution of a subsystem

\[ A_1 \bar{x} = b_1, \text{ where } \text{rank} A_1 = n. \]

Thus, we could obtain an optimal solution of (3.18) by “brute force”, if we simply consider all subsets $I \subseteq M$ with $|I| = n$, test if $\text{rank} A_I = n$ (this can be done by Gaussian elimination) and solve $A_I \bar{x} = b_I$. We then choose the best of the feasible solutions obtained this way. This gives us a finite algorithm for (3.18). Of course, the Simplex Method provides a more sophisticated way of solving (3.18).

Let us now derive conditions which ensure that a given polyhedron is pointed.

Corollary 3.34 A nonempty polyhedron $P = P(A, b) \subseteq \mathbb{R}^n$ is pointed if and only if $\text{rank} A = n$.

Proof: By Corollary 3.27 the minimal nonempty faces of $P$ are of dimension 0 if and only if $\text{rank} A = n$. If $\text{rank} A < n$, then we can find $y \in \mathbb{R}^n$ with $y \neq 0$ such that $Ay = 0$. But then $x + \theta y \in P$ for all $\theta \in \mathbb{R}$ which contradicts the assumption that $P$ is bounded.

Corollary 3.35 Any nonempty polytope is pointed.

Proof: Let $P = P(A, b)$ be a polytope and $\bar{x} \in P$ arbitrary. By Corollary 3.34 it suffices to show that $\text{rank} A = n$. If $\text{rank} A < n$, then we can find $y \in \mathbb{R}^n$ with $y \neq 0$ such that $Ay = 0$. But then $x + \theta y \in P$ for all $\theta \in \mathbb{R}$ which contradicts the assumption that $P$ is bounded.

Corollary 3.36 Any nonempty polyhedron $P \subseteq \mathbb{R}^n_+$ is pointed.

Proof: If $P = P(A, b) \subseteq \mathbb{R}^n_+$, we can write $P$ alternatively as

\[ P = \left\{ x : \begin{pmatrix} A \\ -1 \end{pmatrix} x \leq \begin{pmatrix} b \\ 0 \end{pmatrix} \right\} = P(\bar{A}, \bar{b}). \]

Since $\text{rank} \bar{A} = n$, we see again that the minimal faces of $P$ are extreme points.

On the other hand, Theorem 3.31(ii) is a formal statement of the intuition that by optimizing with the help of a suitable vector over a polyhedron we can “single out” every extreme point. We now derive a stronger result for rational polyhedra:
**Theorem 3.37** Let $P = P(A, b)$ be a rational polyhedron and let $\bar{x} \in P$ be an extreme point of $P$. There exists an integral vector $c \in \mathbb{Z}^n$ such that $\bar{x}$ is the unique optimal solution of \( \max \{ c^T x : x \in P \} \).

**Proof:** Let $I := \text{eq}(\{\bar{x}\})$ and $M := \{1, \ldots, m\}$ be the index set of the rows of $A$. Consider the vector $\bar{c} = \sum_{i \in M} A_i$. Since all the $A_i$ are rational, we can find a $\theta > 0$ such that $c := \theta \bar{c} \in \mathbb{Z}^n$ is integral. Since $\text{fa}(I) = \{\bar{x}\}$ (cf. Observation 3.9), for every $x \in P$ with $x \neq \bar{x}$ there is at least one $i \in I$ such that $A_i x < b_i$. Thus, for all $x \in P \setminus \{\bar{x}\}$ we have $c^T x = \theta \sum_{i \in M} A_i^T x < \theta \sum_{i \in M} b_i = \theta c^T \bar{x}$.

This proves the claim. \(\square\)

Consider the polyhedron $P^m(A, b) := \{x : Ax = b, x \geq 0\}$, where $A$ is an $m \times n$ matrix. A **basis** of $A$ is an index set $B \subseteq \{1, \ldots, n\}$ with $|B| = m$ such that the square matrix $A_B$ formed by the columns from $B$ is nonsingular. The **basic solution** corresponding to $B$ is the vector $(x_B, x_N)$ with $x_B = A_B^{-1} b$, $x_N = 0$. The basic solution is called **feasible**, if it is contained in $P^m(A, b)$.

The following theorem is a well-known result from Linear Programming:

**Theorem 3.38** Let $P = P^m(A, b) = \{x : Ax = b, x \geq 0\}$ and $\bar{x} \in P$, where $A$ is an $m \times n$ matrix of rank $m$. Then, $\bar{x}$ is an extreme point of $P$ if and only if $\bar{x}$ is a basic feasible solution for some basis $B$.

**Proof:** Suppose that $\bar{x}$ is a basic solution for $B$ and $\bar{x} = \lambda x + (1 - \lambda)y$ for some $x, y \in P$. It follows that $x_N = y_N = 0$. Thus $x_B = y_B = A_B^{-1} b = \bar{x}_B$. Thus, $\bar{x}$ is an extreme point of $P$.

Assume now conversely that $\bar{x}$ is an extreme point of $P$. Let $B := \{i : \bar{x}_i > 0\}$. We claim that the matrix $A_B$ consists of linearly independent columns. Indeed, if $A_B y_B = 0$ for some $y_B \neq 0$, then for some $\varepsilon > 0$ we have $x_B \pm \varepsilon y_B \geq 0$. Hence, if we set $N := \{1, \ldots, m\} \setminus B$ and $y = (y_B, y_N)$ we have $\bar{x} \pm \varepsilon y \in P$ and hence we can write $x$ as a convex combination $x = \frac{1}{2} (\bar{x} + \varepsilon y) + \frac{1}{2} (\bar{x} - \varepsilon y)$ contradicting the fact that $\bar{x}$ is an extreme point.

Since $A_B$ has linearly independent columns, it follows that $|B| \leq m$. Since rank $A = m$ we can augment $B$ to a basis $B'$. Then, $\bar{x}$ is the basic solution for $B'$.

We close this section by deriving structural results for polytopes. We need one auxiliary result:

**Lemma 3.39** Let $X \subseteq \mathbb{R}^n$ be a finite set and $v \in \mathbb{R}^n \setminus \text{conv}(X)$. There exists an inequality that separates $v$ from $\text{conv}(X)$, that is, there exist $w \in \mathbb{R}^n$ and $t \in \mathbb{R}$ such that $w^T x \leq t$ for all $x \in \text{conv}(X)$ and $w^T v > t$.

**Proof:** Let $X = \{x_1, \ldots, x_k\}$. Since $v \notin \text{conv}(X)$, the system

\[
\sum_{i=1}^{k} \lambda_k x_k = v
\]

\[
\sum_{i=1}^{k} \lambda_k = 1
\]

\[
\lambda_i \geq 0 \quad \text{for } i = 1, \ldots, k
\]
does not have a solution. By Farkas’ Lemma (Theorem 2.11), there exists a vector \( \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^{n+1} \) such that
\[
\begin{align*}
y^T x_i + z & \leq 0 \quad \text{for } i = 1, \ldots, k \\
y^T v + z & > 0.
\end{align*}
\]

If we choose \( w := -y \) and \( t := -z \) we have \( w^T x_i \leq t \) for \( i = 1, \ldots, k \).

If \( x = \sum_{i=1}^k \lambda_i x_i \) is a convex combination of the \( x_i \), then as in Section 2.2 we have:
\[
w^T x = \sum_{i=1}^k \lambda_i w^T x_i \leq \max\{w^T x_i : i = 1, \ldots, k\} \leq t.
\]
Thus, \( w^T x \leq t \) for all \( x \in \text{conv}(X) \).

**Theorem 3.40** A polytope is equal to the convex hull of its extreme points.

**Proof:** The claim is trivial, if the polytope is empty. Thus, let \( P = P(A, b) \) be a nonempty polytope. Let \( X = \{x_1, \ldots, x_k\} \) be the extreme points of \( P \) (which exist by Corollary 3.35 and whose number is finite by Corollary 3.33). Since \( P \) is convex and \( x_1, \ldots, x_k \in P \), we have \( \text{conv}(X) \subseteq P \). We must show that \( \text{conv}(X) \supseteq P \). Assume that there exists \( v \in P \setminus \text{conv}(X) \). Then, by Lemma (3.39) we can find an inequality \( w^T x \leq t \) such that \( w^T x \leq t \) for all \( x \in \text{conv}(X) \) but \( w^T v > t \). Since \( P \) is bounded and nonempty, the Linear Program \( \max \{ w^T x : x \in P \} \) has a finite solution value \( t^* \in \mathbb{R} \). Since \( v \in P \) we have \( t^* > t \). Thus, none of the extreme points of \( P \) is an optimal solution, which is impossible by Corollary 3.32.

**Corollary 3.41** If \( X \subseteq \mathbb{R}^n \) is a finite nonempty set of rational vectors, then \( \text{conv}(X) \) is a rational polytope.

**Proof:** Immediately from Theorem 3.40 and Corollary 3.17

The results of Corollary 3.17 and Corollary 3.41 are the two major driving forces behind polyhedral combinatorics. Let \( X \subseteq \mathbb{R}^n \) be a nonempty finite set, for instance, let \( X \) be the set of incidence vectors of stable sets of a given graph \( G \) as in the above example. Then, by the preceeding theorem we can represent \( \text{conv}(X) \) as a pointed polytope:
\[
\text{conv}(X) = P = P(A, b) = \{x : Ax \leq b\}.
\]
Since \( P \) is bounded and nonempty, for any given \( c \in \mathbb{R}^n \) the Linear Program
\[
\max \{ c^T x : x \in P \} = \max \{ c^T x : x \in \text{conv}(X) \} \quad (3.19)
\]
has a finite value which by Observation 2.2 coincides with \( \max \{ c^T x : x \in X \} \). By Corollary 3.32 an optimal solution of (3.19) will always be obtained at an extreme point of \( P \), which must be a point in \( X \) itself. So, if we solve the Linear Program (3.19) we can also solve the problem of maximizing \( c^T x \) over the discrete set \( X \).

### 3.7 Facets

In the preceeding section we proved that for a finite set \( X \subseteq \mathbb{R}^n \) its convex hull \( \text{conv}(X) \) is always a polytope and thus has a representation
\[
\text{conv}(X) = P(A, b) = \{x : Ax \leq b\}.
\]
This motivates the questions which inequalities are actually needed in order to describe a polytope, or more general, to describe a polyhedron.
Definition 3.42 (Facet)  
A nontrivial face $F$ of the polyhedron $P = P(A, b)$ is called a facet of $P$, if $F$ is not strictly contained in any proper face of $P$.

Example 3.43  
Consider again the polyhedron from Example 3.22. The inequality $x \leq 3$ is valid for $P$. Of course, also all inequalities from (3.14) are also valid. Moreover, the inequality $x + 2y \leq 6$ defines a facet, since (3.3) and (2.2) are affinely independent. On the other hand, the inequality $x + y \leq 4$ defines a face that consists only of the point $[2,2]$.

Theorem 3.44 (Characterization of facets) Let $P = P(A, b) \subseteq \mathbb{R}^n$ be a polyhedron and $F$ be a face of $P$. Then, the following statements are equivalent:

(i) $F$ is a facet of $P$.
(ii) $\text{rank } A_{\text{eq}(F)} = \text{rank } A_{\text{eq}(P)} + 1$
(iii) $\dim F = \dim P - 1$.

Proof: The equivalence of (ii) and (iii) is an immediate consequence of the Dimension Theorem (Theorem 3.24).

“(i)$\Rightarrow$(iii)”: Suppose that $F$ is a facet but $k = \dim F < \dim P - 1$. By the equivalence of (ii) and (iii) we have $\text{rank } A_{\text{eq}(I)} > \text{rank } A_{\text{eq}(P)} + 1$, where $I = \text{eq}(F)$. Chose $i \in I$ such that for $J := I \setminus \{i\}$ we have $\text{rank } A_{\text{eq}(J)} = \text{rank } A_{\text{eq}(I)} - 1$. Then $\text{fa}(J)$ is a face which contains $F$ and which has dimension $k + 1 \leq \dim P - 1$. So $\text{fa}(J)$ is a proper face of $P$ containing $F$ which contradicts the maximality of $F$.

“(iii)$\Rightarrow$(i)”: Suppose that $G$ is any proper face of $P$ which strictly contains $F$. Then $F$ is a proper face of $G$ and by Corollary 3.25(iv) applied to $F$ and $P' = G$ we get $\dim F \leq \dim G - 1$ which together with $\dim F = \dim P - 1$ gives $\dim G = \dim P$. But then, again by Corollary 3.25(iv), $G$ cannot be a proper face of $P$.

Example 3.45  
As an application of Theorem 3.44 we consider again the stable-set polytope, which we have seen to be full-dimensional in Example 3.18.

For any $v \in V$, the inequality $x_v \geq 0$ defines a facet of $\text{STAB}(G)$, since the $n - 1$ unit vectors with ones at places other than position $v$ and the zero vector form a set of $n$ affinely independent vectors from $\text{STAB}(G)$ which all satisfy the inequality as equality.

As a consequence of the previous theorem we show that for any facet of a polyhedron $P = P(A, b)$ there is at least one inequality in $Ax \leq b$ inducing the facet:

Corollary 3.46 Let $P = P(A, b) \subseteq \mathbb{R}^n$ be a polyhedron and $F$ be a facet of $P$. Then, there exists an $j \in M \setminus \text{eq}(P)$ such that

$$F = \{x \in P : A_j x = b_j\}.$$  (3.20)

Proof: Let $I = \text{eq}(P)$. Choose any $j \in \text{eq}(F) \setminus I$ and set $J := I \cup \{j\}$. Still $J \subseteq \text{eq}(F)$, since $I \subseteq \text{eq}(F)$ and $j \in \text{eq}(F)$. Thus, $F \subseteq \text{fa}(J) \subseteq P$ (we have $\text{fa}(J) \neq P$ since any inner point $\bar{x}$ of $P$ has $A_j \bar{x} < b_j$ since $j \in \text{eq}(F) \setminus I$ and by the maximality of $F$ we have $F = \text{fa}(J)$. So,

$$F = \text{fa}(J) = \{x \in P : A_j x = b_j\} = \{x \in P : A_1 x = b_1, A_j x = b_j\} = \{x \in P : A_j x = b_j\},$$

where the last equality follows from $I = \text{eq}(P)$.

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The above corollary shows that, if for a polyhedron \( P \) we know \( A \) and \( b \) such that \( P = P(A, b) \), then all facets of \( P \) are of the form (3.20).

**Definition 3.47 (Redundant constraint, irredundant system)**

Let \( P = P(A, b) \) be a polyhedron and \( I = \text{eq}(P) \). The constraint \( A_I x \leq b_I \) is called redundant with respect to \( Ax \leq b \), if \( P(A, b) = P(A_{M \setminus I}, b_{M \setminus I}) \), that is, if we can remove the inequality without changing the solution set.

We call \( Ax \leq b \) irredundant or minimal, if it does not contain a redundant constraint.

Observe that removing a redundant constraint may make other redundant constraints irredundant.

The following theorem shows that in order to describe a polyhedron we need an inequality for each of its facets and that, conversely, a list of all facet defining inequalities suffices.

**Theorem 3.48 (Facets are necessary and sufficient to describe a polyhedron)** Let \( P = P(A, b) \) be a polyhedron with equality set \( I = \text{eq}(P) \) and \( J := M \setminus I \). Suppose that no inequality in \( A_I x \leq b_I \) is redundant. Then, there is a one-to-one correspondence between the facets of \( P \) and the inequalities in \( A_I x \leq b_I \):

For each row \( A_I, \) of \( A \), the inequality \( A_I x \leq b_I \) defines a distinct facet of \( P \). Conversely, for each facet \( F \) of \( P \) there exists exactly one inequality in \( A_I x \leq b_I \) which induces \( F \).

**Proof:** Let \( F \) be a facet of \( P \). Then, by Corollary 3.46 there exists \( j \in J \) such that

\[
F = \{ x \in P : A_J x = b_J \}. \tag{3.21}
\]

Thus, each facet is represented by an inequality in \( A_I x \leq b_I \).

Moreover, if \( F_1 \) and \( F_2 \) are facets induced by rows \( j_1 \in J \) and \( j_2 \in J \) with \( j_1 \neq j_2 \), then we must have \( F_1 \neq F_2 \), since \( \text{eq}(F_I) = \text{eq}(P) \cup \{ j_i \} \) for \( i = 1, 2 \) by Corollary 3.46. Thus, each facet is induced by exactly one row of \( A_I \).

Conversely, consider any inequality \( A_I x \leq b_I \) where \( j \in J \). We must show that the face \( F \) given in (3.21) is a facet. Clearly, \( F \neq P \), since \( j \in \text{eq}(F) \setminus \text{eq}(P) \). So \( \dim F \leq \dim P - 1 \). We are done, if we can show that \( \text{eq}(F) = \text{eq}(P) \cup \{ j \} \), since then \( \text{rank } A_{\text{eq}(F)} \leq \text{rank } A_{\text{eq}(P)} + 1 \) which gives \( \dim F \geq \dim P - 1 \) and Theorem 3.44 proves that \( F \) is a facet.

Take any inner point \( \bar{x} \) of \( P \). This point satisfies

\[
A_I \bar{x} = b_I \text{ and } A_{J'} \bar{x} < b_I.
\]

Let \( J' := J \setminus \{ j \} \). Since \( A_I, x \leq b_I \) is not redundant in \( Ax \leq b \), there exists \( y \) such that

\[
A_I y = b_I, A_{J'} y \leq b_{J'}, \text{ and } A_{J'} y > b_{J'}.
\]

Consider \( z = \lambda y + (1 - \lambda) \bar{x} \). Then for an appropriate choice of \( \lambda \in (0, 1) \) we have

\[
A_I z = b_I, A_{J'} z < b_{J'}, \text{ and } A_{J'} z = b_{J'}.
\]

Thus, \( z \in F \) and \( \text{eq}(F) = \text{eq}(P) \cup \{ j \} \) as required. \( \square \)

**Corollary 3.49** Each face of a polyhedron \( P \), except for \( P \) itself, is the intersection of facets of \( P \).

**Proof:** Let \( K := \text{eq}(P) \). By Theorem 3.6, for each face \( F \), there is an \( I \subseteq M \) such that

\[
F = \{ x \in P : A_I x = b_I \} = \{ x \in P : A_{I \setminus K} x = b_{I \setminus K} \} = \bigcap_{j \in I \setminus K} \{ x \in P : A_j x = b_j \},
\]

where by Theorem 3.48 each of the sets in the intersection above defines a facet. \( \square \)
Corollary 3.50 Any defining system for a polyhedron must contain a distinct facet-inducing inequality for each of its facets.

Lemma 3.51 Let \( P = P(A, b) \) with \( I = \text{eq}(P) \) and let \( F = \{ x \in P : w^T x = t \} \) be a proper face of \( P \). Then, the following statements are equivalent:

(i) \( F \) is a facet of \( P \).

(ii) If \( c^T x = \gamma \) for all \( x \in F \), then \( c^T \) is a linear combination of \( w^T \) and the rows in \( A_{I,J} \).

**Proof:** “(i)⇒(ii)”: We can write \( F = \{ x \in P : w^T x = t, c^T x = \gamma \} \), so we have for \( J := \text{eq}(F) \) by Theorem 3.44 and the Dimension Theorem

\[
\dim P = n - \text{rank} A_{I,J} = 1 + \dim F \leq n - \text{rank} \begin{pmatrix} A_{I,J} & w^T \\ c^T \end{pmatrix} + 1.
\]

Thus

\[
\text{rank} \begin{pmatrix} A_{I,J} \\ w^T \\ c^T \end{pmatrix} \leq \text{rank} A_{I,J} + 1.
\]

Since \( F \) is a proper face, we have \( \text{rank} A_{I,J} = \text{rank} A_{I,J} + 1 \) which means that \( \text{rank} \begin{pmatrix} A_{I,J} \\ w^T \\ c^T \end{pmatrix} = \text{rank} \begin{pmatrix} A_{I,J} \\ w^T \\ c^T \end{pmatrix} \). So, \( c \) is a linear combination of \( w^T \) and the vectors in \( A_{I,J} \).

“(ii)⇒(i)”: Let \( J = \text{eq}(F) \). By assumption, \( \text{rank} A_{I,J} = \text{rank} \begin{pmatrix} A_{I,J} \\ w^T \\ c^T \end{pmatrix} = \text{rank} A_{I,J} + 1 \). So \( \dim F = \dim P - 1 \) by the Dimension Theorem and by Theorem 3.44 we get that \( F \) is a facet.

Suppose that the polyhedron \( P = P(A, b) \) is of full dimension. Then, for \( I := \text{eq}(P) \) we have \( \text{rank} A_{I,J} = 0 \) and we obtain the following corollary:

Corollary 3.52 Let \( P = P(A, b) \) be full-dimensional let \( F = \{ x \in P : w^T x = t \} \) be a proper face of \( P \). Then, the following statements are equivalent:

(i) \( F \) is a facet of \( P \).

(ii) If \( c^T x = \gamma \) for all \( x \in F \), then \( c^T x = \gamma \) is a scalar multiple of \( \binom{w}{1} \).

**Proof:** The fact that (ii) implies (i) is trivial. Conversely, if \( F \) is a facet, then by Lemma 3.51 above, \( c^T \) is a “linear combination” of \( w^T \), that is, \( c = \lambda w \) is a scalar multiple of \( w \). Now, since for all \( x \in F \) we have \( w^T x = t \) and \( \gamma = c^T x = \lambda w^T x = \lambda t \), the claim follows.

Example 3.53
We consider the 0/1-Knapsack polytope

\[
P_{\text{KNAPSACK}} := P_{\text{KNAPSACK}}(N, a, b) := \text{conv} \left\{ x \in \mathbb{B}^N : \sum_{j \in N} a_j x_j \leq b \right\}
\]

which we have seen a couple of times in these lecture notes (for instance in Example 1.3). Here, the \( a_i \) are nonnegative coefficients and \( b \geq 0 \). We use the general index set \( N \) instead
3.7 Facets

of \( N = \{1, \ldots, n\} \) to emphasize that a knapsack constraint \( \sum_{j \in N} a_j x_j \leq b \) might occur as a constraint in a larger integer program and might not involve all variables.

In all what follows, we assume that \( a_j \leq b \) for \( j \in N \) since \( a_j > b \) implies that \( x_j = 0 \) for all \( x \in P_{\text{KnapSack}}(N, a, b) \). Under this assumption \( P_{\text{KnapSack}}(N, a, b) \) is full-dimensional, since \( \chi^S \) and \( \chi^{[i]} \) \((i \in N)\) form a set of \( n+1 \) affinely independent vectors in \( P_{\text{KnapSack}}(N, a, b) \).

Each inequality \( x_j \geq 0 \) for \( j \in N \) is valid for \( P_{\text{KnapSack}}(N, a, b) \). Moreover, each of these nonnegativity constraints defines a facet of \( P_{\text{KnapSack}}(N, a, b) \), since \( \chi^S \) and \( \chi^{[i]} \) \((i \in N \setminus \{j\})\) form a set of \( n \) affinely independent vectors that satisfy the inequality at equality. In the sequel we will search for more facets of \( P_{\text{KnapSack}}(N, a, b) \) and, less ambitious, for more valid inequalities.

A set \( C \subseteq N \) is called a cover, if
\[
\sum_{j \in C} a_j > b.
\]

The cover is called a minimal cover, if \( C \setminus \{j\} \) is not a cover for all \( j \in C \).

Each cover \( C \) gives us a valid inequality \( \sum_{j \in C} x_j \leq |C| - 1 \) for \( P_{\text{KnapSack}} \) (if you do not see this immediately, the proof will be given below). It turns out that this inequality is quite strong, provided that the cover is minimal.

By Observation 2.2 it suffices to show that the inequality is valid for the knapsack set
\[
X := \left\{ x \in B^N : \sum_{j \in N} a_j x_j \leq b \right\}.
\]

Suppose that \( x \in X \) does not satisfy the cover inequality. We have that \( x = \chi^S \) is the incidence vector of a set \( S \subseteq N \). By assumption we have
\[
|C| - 1 < \sum_{j \in C} x_j = \sum_{j \in C \cap S} x_j = |C \cap S|.
\]

So \( |C \cap S| = |C| \) and consequently \( C \subseteq S \). Thus,
\[
\sum_{j \in N} a_j x_j \geq \sum_{j \in C} a_j x_j = \sum_{j \in C \cap S} a_j + \sum_{j \in C \setminus S} a_j > b,
\]

which contradicts the fact that \( x \in X \).

We claim that the cover inequality defines a facet of \( P_{\text{KnapSack}}(C, a, b) \), if the cover is minimal. Suppose that there is a facet-defining inequality \( c^T x \leq \delta \) such that
\[
F := \left\{ x \in P_{\text{KnapSack}}(C, a, b) : \sum_{j \in C} x_j = |C| - 1 \right\}
\]
\[
\subseteq F_c := \left\{ x \in P_{\text{KnapSack}}(C, a, b) : c^T x = \delta \right\}.
\]

We will show that \( c^T x \leq \delta \) is a nonnegative scalar multiple of the cover inequality.

For \( i \in C \) consider the set \( C_i := C \setminus \{i\} \). Since, \( C \) is minimal, \( C_i \) is not a cover. Consequently, each of the \( |C| \) incidence vectors \( \chi^{C_i} \in B^C \) is contained in \( F \), so \( \chi^{C_i} \in F_c \) for \( i \in C \). Thus, for \( i \neq j \) we have
\[
0 = c^T \chi^{C_i} - c^T \chi^{C_j} = c^T (\chi^{C_i} - \chi^{C_j}) = c_j - c_i.
\]

Hence we have \( c_i = \gamma \) for \( i \in C \) and \( c^T x \leq \delta \) is of the form
\[
\gamma \sum_{j \in C} x_j \leq \delta.
\]
Fix $i \in C$. Then, by $F \subseteq F_c$ we have

$$c^T x^{C_i} = \gamma \sum_{j \in C_i} x_j = \delta$$

and we get

$$\sum_{j \in C_i} x_j = |C| - 1,$$

so $\delta = \gamma(|C| - 1)$ and $c^T x \leq \delta$ must be a nonnegative scalar multiple of the cover inequality.

\[\]

**Corollary 3.54** A full-dimensional polyhedron has a unique (up to positive scalar multiples) irredundant defining system.

**Proof:** Let $Ax \leq b$ be an irredundant defining system. Since $P$ is full-dimensional, we have $A_{eq}(P) = \emptyset$. By Theorem 3.48 there is a one-to-one correspondence between the inequalities in $Ax \leq b$ and the facets of $P$. By Corollary 3.52 two valid inequalities for $P$ which induce the same facet are scalar multiples of each other. $\Box$

### 3.8 The Characteristic Cone and Extreme Rays

In the previous section we have learned that each polyhedron can be represented by its facets. In this section we learn another representation of a polyhedron which is via its extreme points and extreme rays.

**Definition 3.55 (Characteristic cone, (extreme) ray)**

Let $P$ be a polyhedron. Then, its **characteristic cone** or **recession cone** $\text{char. cone}(P)$ is defined to be:

$$\text{char. cone}(P) := \{ r : x + r \in P \text{ for all } x \in P \}.$$ 

We call any $r \in \text{char. cone}(P)$ with $r \neq 0$ a ray of $P$. A ray $r$ of $P$ is called an **extreme ray** if there do not exist rays $r^1, r^2$ of $P$, $r^1 \neq \lambda r^2$ for any $\lambda \in \mathbb{R}_+$ such that $r = \lambda r^1 + (1-\lambda)r^2$ for some $\lambda \in [0, 1]$.

In other words, $\text{char. cone}(P)$ is the set of all directions $y$ in which we can go from all $x \in P$ without leaving $P$. This justifies the name “ray” for all vectors in $\text{char. cone}(P)$. Since $P \neq 0$ implies that $0 \in \text{char. cone}(P)$ and $P = \emptyset$ implies $\text{char. cone}(P) = \mathbb{R}^n$, we have that $\text{char. cone}(P) \neq \emptyset$.

**Lemma 3.56** Let $P = P(A, b)$ be a nonempty polyhedron. Then

$$\text{char. cone}(P) = \{ x : Ax \leq 0 \}.$$

**Proof:** If $Ay \leq 0$, then for all $x \in P$ we have $A(x + y) = Ax + Ay \leq Ax \leq b$, so $x + y \in P$. Thus $y \in \text{char. cone}(P)$.

Conversely, if $y \in \text{char. cone}(P)$ we have $A_i y \leq 0$ if there exists an $x \in P$ such that $A_i x = b_i$. Let $J := \{ j : A_j x < b_j \text{ for all } x \in P \}$ be the set of all other indices. We are done, if we can show that $A_j y \leq 0$.

If $A_j y > 0$ for some $j \in J$, take an inner point $x \in P$ and consider $z = x + \lambda y$ for $\lambda \geq 0$. Then, by choosing $\lambda > 0$ appropriately, we can find $j' \in J$ such that $A_{j'} z = b_{j'}$. $A_{j'} z \leq b_{j'}$ and $A_{j'} z \leq b_{j'}$ which contradicts the fact that $j' \in J$. $\Box$
The characteristic cone of a polyhedron \( P(A, b) \) is itself a polyhedron \( f(P) = P(A, 0) \), albeit a very special one. For instance, \( \text{char.cone}(P) \) has at most one extreme point, namely the vector 0. To see this, assume that \( r \neq 0 \) is an extreme point of \( \text{char.cone}(P) \). Then, \( A r \leq 0 \) and from \( r \neq 0 \) we have we have \( r \neq \frac{1}{2} r \in \text{char.cone}(P) \) and \( r \neq \frac{1}{2} r \in \text{char.cone}(P) \). But then \( r = \frac{1}{2} (\frac{1}{2} r) + \frac{1}{2} (\frac{1}{2} r) \) is a convex combination of two distinct points in \( \text{char.cone}(P) \) contradicting the assumption that \( r \) is an extreme point of \( \text{char.cone}(P) \).

Together with Corollary 3.34 we have:

**Observation 3.57** Let \( P = P(A, b) \neq \emptyset \). Then, \( 0 \in \text{char.cone}(P) \) and \( 0 \) is the only potential extreme point of \( \text{char.cone}(P) \). The zero vector is an extreme point of \( \text{char.cone}(P) \) if and only if \( \text{rank } A = n \).

**Lemma 3.58 (Properties of the characteristic cone)** Let \( P = P(A, b) \) be a nonempty polyhedron and \( \text{char.cone}(P) = \{x : Ax \leq 0\} \) be its characteristic cone. Then, the following statements hold:

(i) \( y \in \text{char.cone}(P) \) if and only if there exists some \( x \in P \) such that \( x + \lambda y \in P \) for all \( \lambda \geq 0 \);

(ii) \( P + \text{char.cone}(P) = P \);

(iii) \( P \) is bounded if and only if \( \text{char.cone}(P) = \{0\} \);

(iv) If \( P = Q + C \) where \( Q \) is a polytope and \( C \) is a polyhedral cone, then \( C = \text{char.cone}(P) \).

**Proof:**

(i) Let \( y \in \text{char.cone}(P) \). Then \(Ay \leq 0\), so for for any \( \lambda \geq 0 \) we have \(A(\lambda y) = \lambda Ay \leq 0\) and, hence \( \lambda y \in \text{char.cone}(P) \). Assume conversely that \( x' \in P \) and \( y \) such that \( x' + \lambda y \in P \) for all \( \lambda \geq 0 \). We show that \( Ay \leq 0 \) which implies that \( y \in \text{char.cone}(P) \). We have \( A(x' + \lambda y) \leq b \) for all \( \lambda \geq 0 \) if and only if \( Ax' + \lambda Ay \leq b \) for all \( \lambda \geq 0 \). Assume that there is an index \( i \) such that \( (Ay)_i > 0 \). Then (note that \( (Ax')_i \) is constant):

\[
(Ax')_i + \lambda (Ay)_i \to \infty \quad \text{for } \lambda \to \infty
\]

which contradicts the fact that \( (Ax')_i + \lambda (Ay)_i \leq b_i \).

(ii) Let \( x \in P \), then \( x = x + 0 \in P + \text{char.cone}(P) \), so \( P \subseteq P + \text{char.cone}(P) \). On the other hand for \( x + y \in P + \text{char.cone}(P) \) with \( x \in P \) and \( y \in \text{char.cone}(P) \) we have \( x + y \in P \) by definition of the characteristic cone.

(iii) Let \( P \) be bounded. Then there is an \( M \) such that \( \|x\| < M \) for all \( x \in P \). Assume that there exists \( y \in \text{char.cone}(P) \) with \( y \neq 0 \). Then \( \|x + \lambda y\| \to \infty \) for \( \lambda \to \infty \) contradicting the boundedness of \( P \).

Now assume conversely that \( \text{char.cone}(P) = \{0\} \) and assume that \( P \) is unbounded. Due to the Decomposition Theorem (Theorem 3.16) we can write \( P = Q + C' \) for a polytope \( Q' \) and a polyhedral cone \( C' \). If \( P \) is unbounded, we can follow, that there is a \( c \in C' \) with \( c \neq 0 \) (otherwise \( P = Q' \) and the polytope \( Q' \) is clearly bounded). For all \( \lambda \geq 0 \) also \( \lambda c \) is in the cone \( C' \) and for all \( x \in Q' \), also \( x = x + 0 \in P \) since \( 0 \in C' \). We get:

\[
x + \lambda c \in P \quad \text{for all } \lambda \geq 0
\]

By part (i) we have \( 0 \neq c \in \text{char.cone}(P) \) which is a contradiction.
Thus, the solution set of \( A x = 0 \) and \( y \in \text{char. cone}(P) \). We show that \( y \in C \). Let \( x \in P \). Since \( y \in \text{char. cone}(P) \), we have \( x + ny \in P = Q + C \) for all \( n \in \mathbb{N} \), say \( x + ny = q_n + c_n \) for some \( q_n \in Q \) and \( c_n \in C \). We have
\[
B y = \frac{1}{n} B q_n + \frac{1}{n} B c_n - \frac{1}{n} B x \leq \frac{1}{n} B q_n - \frac{1}{n} B x.
\]

Let \( \varepsilon > 0 \). Choose \( n \) such that: \( \| \frac{1}{n} B q_n - \frac{1}{n} B x \| < \varepsilon \). Then \( B y \leq \varepsilon \). Since \( \varepsilon > 0 \) was arbitrary, we get \( B y \leq 0 \) and, thus, \( y \in C \).

We have to show that \( y \in \text{char. cone}(P) \), i.e., \( p + y \in P \) for all \( p \in P \). We have
\[
p + y = x + c + y = x + (c + y) \in Q + C = P
\]
with \( x \in Q \) and \( c \in C \) and \( c + q \in C \) (since \( B(c + q) = Bc + Bq \leq 0 \) since \( c \) and \( q \) in \( C \))

\[\Box\]

Recall the Decomposition Theorem (Theorem 3.16): Any polyhedron \( P \) can be written as \( P = Q + C \), where \( Q \) is a polytope and \( C \) is a polyhedral cone. By Lemma 3.58 (iv) we see that inevitably we must have \( C = \text{char. cone}(P) \):

**Observation 3.59** Any polyhedron \( P \) can be written as \( P = Q + \text{char. cone}(C) \), where \( Q \) is a polytope.

**Theorem 3.60 (Characterization of extreme rays)** Let \( P = P(A, b) \subseteq \mathbb{R}^n \) be a nonempty polyhedron. Then, the following statements are equivalent:

(i) \( r \) is an extreme ray of \( P \).

(ii) \( \{0r : r \in \mathbb{R}_+\} \) is a one-dimensional face of \( \text{char. cone}(P) = \{x : Ax \leq 0\} \).

(iii) \( r \in \text{char. cone}(P) \setminus \{0\} \) and for \( I := \{i : A_i, r = 0\} \) we have \( \text{rank } A_I = n - 1 \).

**Proof:** Let \( I := \{i : A_i, r = 0\} \).

"(i)\Rightarrow(ii)" : Let \( F \) be the smallest face of \( \text{char. cone}(P) \) containing the set \( \{0r : r \in \mathbb{R}_+\} \). By Observation 3.9 we have
\[
F = \{x \in \text{char. cone}(P) : A_I, x = 0_t\}
\]
and \( \text{eq}(F) = 1 \). If \( \dim F > 1 \), then the Dimension Theorem tells us that \( \text{rank } A_I < n - 1 \).

Thus, the solution set of \( A_I, x = 0_t \) contains a vector \( r^1 \) which is linearly independent from \( r \). For sufficiently small \( \varepsilon > 0 \) we have \( r + \varepsilon r^1 \in \text{char. cone}(P) \), since \( A_I, r = 0_t \), \( A_I, r^1 = 0_t \) and \( A_M \setminus I, r < 0 \). But then \( r = \frac{1}{2}(r + \varepsilon r^1) + \frac{1}{2}(r - \varepsilon r^1) \) contradicting the fact that \( r \) is an extreme ray.

So \( \dim F = 1 \). Since \( r \neq 0 \), the unbounded set \( \{0r : r \in \mathbb{R}_+\} \) which is contained in \( F \) has also dimension 1, hence \( F = \{0r : r \in \mathbb{R}_+\} \) is a one-dimensional face of \( \text{char. cone}(P) \).

"(ii)\Rightarrow(iii)" : The Dimension Theorem applied to \( \text{char. cone}(P) \) implies that \( \text{rank } A_I = n - 1 \). Since \( \{0r : r \in \mathbb{R}_+\} \) has dimension 1 it follows that \( r \neq 0 \).

"(iii)\Rightarrow(i)" : By Linear Algebra, the solution set of \( A_I, x = 0_t \) is one-dimensional. Since \( A_I, r = 0_t \) and \( r \neq 0 \), for any \( y \) with \( A_I, y = 0_t \) we have \( y = 0r \) for some \( \theta \in \mathbb{R} \).
Suppose that \( r = \lambda r^1 + (1 - \lambda) r^2 \) for some \( r^1, r^2 \in \text{char. cone}(P), \lambda \in (0, 1) \). Then
\[
0_I = A_{I, r} = \lambda A_{I, r^1} + (1 - \lambda) A_{I, r^2} \leq 0_I \leq 0_I
\]
implies that \( A_{I, r^j} = 0 \) for \( j = 1, 2 \). So \( r^j = \theta_j r_j \) for appropriate scalars, and both rays \( r^1 \) and \( r^2 \) must be scalar multiples of each other. \( \square \)

**Corollary 3.61** Every polyhedron \( P \) has only a finite number of extreme rays. Moreover, if \( P \) is pointed, then \( \text{char. cone}(P) \) is finitely generated by the extreme rays of \( P \).

**Proof:** By the preceding theorem, every extreme ray is determined by a subsystem of rank \( n - 1 \), of which there are only finitely many. \( \square \)

Combining this result with Corollary 3.33 we have:

**Corollary 3.62** Every polyhedron has only a finite number of extreme points and extreme rays.

Consider once more the Linear Program
\[
\max \left\{ c^T x : x \in P \right\},
\]
where \( P \) is a pointed polyhedron. We showed in Corollary 3.32, that if the Linear Program (3.18) has optimal solutions, it has an optimal solution which is also an extreme point of \( P \). What happens, if the Linear Program (3.22) is unbounded?

**Theorem 3.63** Let \( P = P(A, b) \) be a pointed nonempty polyhedron.

(i) If the Linear Program (3.22) has optimal solutions, it has an optimal solution which is also an extreme point of \( P \).

(ii) If the Linear Program (3.22) is unbounded, then there exists an extreme ray \( r \) of \( P \) such that \( c^T r > 0 \).

**Proof:** Statement (i) is a restatement of Corollary 3.32. So, we only need to prove (ii). By the Duality Theorem of Linear Programming (Theorem 2.9) the set
\[
\left\{ y : A^T y = c, y \geq 0 \right\}
\]
(which is the feasible set for the dual to (3.22)) must be empty. By Farkas’ Lemma (Theorem 2.11), there exists \( r \) such that \( A^T r \geq 0 \) and \( c^T r < 0 \). Let \( r' := -r \), then \( A^T r' \leq 0 \) and \( c^T r' > 0 \). In particular, \( r' \) is a ray of \( P \).

We now consider the Linear Program
\[
\max \left\{ c^T x : Ax \leq 0, c^T x \leq 1 \right\} = \max \left\{ c^T x : x \in P' \right\}.
\]
Since rank \( A = n \), it follows that \( P' \) is pointed. Moreover, \( P' \neq \emptyset \) since \( r'/c^T r' \in P' \). Finally, the constraint \( c^T x \leq 1 \) ensures that (3.23) is bounded. In fact, the optimum value of (3.23) is 1, since this value is achieved for the vector \( r'/c^T r' \).

By (i) an optimal solution of (3.23) is attained at an extreme point of \( P' \subseteq \text{char. cone}(P) \), say at \( r^* \in \text{char. cone}(P) \). So
\[
c^T r^* = 1.
\]
Let \( I = \{ i : A_i \cdot r^* = 0 \} \). By the Dimension Theorem we have \( \text{rank} \left( \frac{A_I}{c^T} \right) = n \) (since \( r^* \) is a zero-dimensional face of the polyhedron \( P' \) by Theorem 3.31).

If \( \text{rank} A_I = n - 1 \), then by Theorem 3.60 we have that \( r^* \) is an extreme ray of \( P \) as needed.

If \( \text{rank} A_I = n \), then \( r^* \neq 0 \) is a vertex of \( \text{char} \cdot \text{cone}(P) \) by Theorem 3.31 which is a contradiction to Observation 3.57.

Now let \( P = \text{P}(A, b) \) be a rational pointed polyhedron where \( A \) and \( b \) are already chosen to have rational entries. By Theorem 3.31 the extreme points of \( P \) are the 0-dimensional faces. By the Theorem of Hoffmann and Kruskal (Theorem 3.26) each extreme point \( \{ \bar{x} \} \) is the unique solution of a subsystem \( A_I \cdot x = b_I \). Since \( A \) and \( b \) are rational, it follows that each extreme point has also rational entries (Gaussian elimination applied to the linear system \( A_I \cdot x = b_I \) does not leave the rationals). Similarly, by Theorem 3.60 an extreme ray \( r \) is determined by a system \( A_I \cdot r^* = 0 \), where \( \text{rank} A = n - 1 \). So again, \( r \) must be rational. This gives us the following observation:

**Observation 3.64** The extreme points and extreme rays of a rational polyhedron are rational vectors.

### 3.9 Most IPs are Linear Programs

This section is dedicated to establishing the important fact that if \( P \) is a rational polyhedron, then

\[
P_I := \text{conv}(P \cap \mathbb{Z}^n)
\]

is again a rational polyhedron. Clearly, we always have \( P_I \subseteq P \), since \( P \) is convex.

**Lemma 3.65** For any rational polyhedral cone \( C = \{ x : Ax \leq 0 \} \) we have

\[
C_I = C.
\]

**Proof:** By the Theorem of Weyl-Minkowski-Farkas (Theorem 3.15), \( C \) is generated by finitely many integral vectors. \( \square \)

**Theorem 3.66 (Most IPs are LPs)** Let \( P = \text{P}(A, b) \) be a rational polyhedron, then \( P_I = \text{conv}(P \cap \mathbb{Z}^n) \) is a rational polyhedron. If \( P_I \neq \emptyset \), then \( \text{char} \cdot \text{cone}(P) = \text{char} \cdot \text{cone}(P_I) \).

**Proof:** Let \( P \) be a rational polyhedron, then we have \( P = Q + \text{char} \cdot \text{cone}(P) \) (see Observation 3.59). By Theorem 3.16 \( C := \text{char} \cdot \text{cone}(P) \) is a rational cone which by Theorem 3.15 is generated by finitely many integral vectors, say \( \text{char} \cdot \text{cone}(P) = \text{cone}(y_1, \ldots, y_k) \). We let

\[
R := \left\{ \sum_{i=1}^{k} \mu_i y_i : 0 \leq \mu_i \leq 1 \text{ for } i = 1, \ldots, k \right\}.
\]  \hspace{1cm} (3.25)

Then \( R \) is a bounded set, and so is \( Q + R \) (since \( Q \) is bounded, too). Thus, \( (Q + R) \cap \mathbb{Z}^n \) consists of finitely many points and so by Corollary 3.17

\[
(Q + R)_I = \text{conv}((Q + R) \cap \mathbb{Z}^n)
\]

is a polytope. By the Decomposition Theorem 3.16 we done, if we can prove that \( P_I = (Q + R)_I + C \).
3.9 Most IPs are Linear Programs

"P₁ ⊆ (Q + R)₁ + C": Suppose that x ∈ P ∩ Zⁿ, then x = q + c where q ∈ P and c ∈ C.
We have c = ∑ₖ i=1 λᵢ yᵢ for some λᵢ ≥ 0 and, hence,

\[ c = \sum_{i=1}^{k} \lambda_i y_i + \sum_{i=1}^{k} (\lambda_i - \lfloor \lambda_i \rfloor) y_i = c' + r, \text{ where } c' \in C \cap \mathbb{Z}^n, \ r \in R. \]

It follows that x = q + c = (q + r) + c'. Note that q + r = x − c' ∈ Zⁿ, since x ∈ Zⁿ and c' ∈ Zⁿ. Hence x ∈ (Q + R)₁ + C.

We have shown that any integral point in P is contained in (Q + R)₁ + C. Since (Q + R)₁ + C is a polyhedron and, hence, convex, also conv(P ∩ Zⁿ) is contained in (Q + R)₁ + C.

"(Q + R)₁ + C ⊆ P₁": We have

\[ (Q + R)₁ + C \subseteq P₁ + C \]

Lemma 3.65: P₁ + C₁ ⊆ (P + C)₁ = P₁.

This completes the proof.

The result of Theorem 3.66 has far reaching consequences: Any Integer Program

\[ \text{max } c^T x \]
\[ A x \leq b \]
\[ x \in \mathbb{Z}^n \]

with a rational matrix A and a rational vector b can be written as a Linear Program

\[ \text{max } c^T x \]
\[ \tilde{A} x \leq \tilde{b}. \]

Of course, the issue is how to find \( \tilde{A} \) and \( \tilde{b} \)!

It should be noted that the above result can be extended rather easily to mixed integer programs. Moreover, as a byproduct of the proof we obtain the following observation:

Observation 3.67 Let \( P = P(A, b) \) with rational A and b. If \( P₁ \neq \emptyset \), then the extreme rays of P and P₁ coincide.

Theorem 3.68 Let \( P = P(A, b) \) with rational A and b such that \( P₁ \) is nonempty. Let \( c \in \mathbb{R}^n \) be arbitrary. We consider the two optimization problems:

\[ (IP) \quad z^{IP} = \max \left\{ c^T x : x \in P \right\} \]
\[ (LP) \quad z^{LP} = \max \left\{ c^T x : x \in P₁ \right\}. \]

Then, the following statements hold:

(i) The objective value of IP is bounded from above if and only if the objective value of LP is bounded from above.

(ii) If LP has a bounded optimal value, then it has an optimal solution (namely, an extreme point of conv(X)), that is an optimal solution to IP.
(iii) If \( x^\ast \) is an optimal solution to \( IP \), then \( x^\ast \) is also an optimal solution to \( LP \).

**Proof:** Let \( X = P \cap Z^n \). Since \( X \subseteq \text{conv}(X) = P \), it trivially follows that \( z^{LP} \geq z^{IP} \).

(i) If \( z^{IP} = +\infty \) it follows that \( z^{LP} = +\infty \). On the other hand, if \( z^{LP} = +\infty \), there is an integral extreme point \( x^0 \in \text{conv}(X) \) and an integral extreme ray \( r \) of \( \text{conv}(X) \) such that \( c^T x^0 + \mu r \in \text{conv}(X) \) for all \( \mu \geq 0 \) and \( c^T r > 0 \). Thus, \( x^0 + \mu r \in X \) for all \( \mu \in \mathbb{N} \). Thus, we also have \( z^{IP} = +\infty \).

(ii) By Theorem 3.66 we know that \( P = \text{conv}(X) \) is a rational polyhedron. Hence, if \( LP \) has an optimal solution, there exists also an optimal solution which is an extreme point of \( \text{conv}(X) \), say \( x^0 \). But then \( x^0 \in X \) and \( z^{IP} = c^T x^0 = z^{LP} \geq z^{IP} \). Hence, \( x^0 \) is also an optimal solution for \( IP \).

(iii) Since \( x^\ast \in X \subseteq \text{conv}(X) = P \), the point \( x^\ast \) is also feasible for \( LP \). The claim now follows from (ii) and (iii).

\[ \square \]

Theorems 3.66 and 3.68 are particularly interesting in conjunction with the polynomial time equivalence of the separation and optimization (see Theorem 5.24 later on). A general method for showing that an Integer Linear Program \( \max \{ c^T x : x \in X \} \) with \( X = P(A, b) \cap Z^n \) can be solved in polynomial time is as follows:

(i) Find a description of \( P = \text{conv}(X) \), that is, \( \text{conv}(X) = P' = \{ x : A'x \leq b' \} \).

(ii) Give a polynomial time separation algorithm for \( P' \).

(iii) Apply Theorem 5.24.

Although this procedure does usually not yield algorithms that appear to be the most efficient in practice, the equivalence of optimization and separation should be viewed as a guide for searching for more efficient algorithms. In fact, for the vast majority of problems that were first shown to be solvable in polynomial time by the method outlined above, later algorithms were developed that are faster both in theory and practice.