Let $P = \{ x \in \mathbb{R}^n : Ax \leq b \}$ be a rational polyhedron and $P_I = \text{conv}(P \cap \mathbb{Z}^n)$. We have seen in Section 3.9 that $P_I$ is a rational polyhedron and that we can solve the integer program

$$\max \left\{ c^T x : x \in P \cap \mathbb{Z}^n \right\}$$

(9.1)

by solving the Linear Program

$$\max \left\{ c^T x : x \in P_I \right\}.$$  

(9.2)

In this chapter we will be concerned with the question how to find a linear description of $P_I$ (or an adequate superset of $P_I$) which enables us to solve (9.2) and (9.1).

Recall that by Theorem 3.48 in order to describe a polyhedron we need exactly its facets.

### 9.1 Cutting-Plane Proofs

Suppose that we are about to solve an integer program

$$\max \left\{ c^T x : A x \leq b, x \in \mathbb{Z}^n \right\}.$$ 

Let $P = \{ x \in \mathbb{R}^n : Ax \leq b \}$ and $X = P \cap \mathbb{Z}^n$. If we want to establish optimality of a solution (or at least provide an upper bound) this task is equivalent to proving that $c^T x \leq t$ is valid for all points in $X$. Without the integrality constraints we could prove the validity of the inequality by means of a variant of Farkas’ Lemma (cf. Theorem 2.11).

**Lemma 9.1 (Farkas’ Lemma (Variant))** Let $P = \{ x \in \mathbb{R}^n : Ax \leq b \} \neq \emptyset$. The following statements are equivalent:

(i) The inequality $c^T x \leq t$ is valid for $P$.

(iii) There exists $y \geq 0$ such that $A^T y = c$ and $b^T y \leq t$.

**Proof:** By Linear Programming duality, $\max \left\{ c^T x : x \in P \right\} \leq t$ if and only if $\min \left\{ b^T y : A^T y = c, y \geq 0 \right\} \leq t$. So, $c^T x \leq t$ is valid for $P \neq \emptyset$ if and only if there exists $y \geq 0$ such that $A^T y = c$ and $b^T y \leq t$. \qed
As a consequence of the above lemma, if an inequality \( c^T x \leq t \) is valid for \( P = \{ x : Ax \leq b \} \), then we can derive the validity of the inequality. Namely, we can find \( y \geq 0 \) such that for \( c = A^T y \) and \( t' = y^T b \) the inequality \( c^T x \leq t' \) is valid for \( P \) and \( t' \leq t \). This clearly implies that \( c^T x \leq t \) is valid.

How can we prove validity in the presence of integrality constraints? Let us start with an example. Consider the following linear system

\[
\begin{align*}
2x_1 + 3x_2 &\leq 27 \quad (9.3a) \\
2x_1 - 2x_2 &\leq 7 \quad (9.3b) \\
-6x_1 - 2x_2 &\leq -9 \quad (9.3c) \\
-2x_1 - 6x_2 &\leq -11 \quad (9.3d) \\
-6x_1 + 8x_2 &\leq 21 \quad (9.3e)
\end{align*}
\]

Figure 9.1 shows the polytope \( P \) defined by the inequalities in (9.3) together with the convex hull of the points from \( X = P \cap \mathbb{Z}^2 \).

As can be seen from Figure 9.1, the inequality \( x_2 \leq 5 \) is valid for \( X = P \cap \mathbb{Z}^2 \). However, we cannot use Farkas’ Lemma to prove this fact from the linear system (9.3), since the point \((9/2, 6) \in P\) has second coordinate 6.

Suppose we multiply the last inequality (9.3e) of the system (9.3) by 1/2. This gives us the valid inequality

\[
-3x_1 + 4x_2 \leq 21/2. \quad (9.4)
\]

For any integral vector \((x_1, x_2) \in X\) the left hand side of (9.4) will be integral, so we can round down the right hand side of (9.4) to obtain the valid inequality (for \( X \)):

\[
-3x_1 + 4x_2 \leq 10. \quad (9.5)
\]

We now multiply the first inequality (9.3a) by 3 and (9.5) by 2 and add those inequalities. This gives us a new valid inequality:

\[
17x_2 \leq 101.
\]
Dividing this inequality by 17 and rounding down the resulting right hand side gives us the valid inequality $x_2 \leq 5$.

The procedure used in the example above is called a cutting plane proof. Suppose that our system $Ax \leq b$ is formed by the inequalities

$$a_i^T x \leq b_i, \quad i = 1, \ldots, m$$

(9.6)

and let $P = \{ x : Ax \leq b \}$. Let $y \in \mathbb{R}^m_+$ and set

$$c := (A^T y) = \sum_{i=1}^{m} y_i a_i$$

$$t := b^T y = \sum_{i=1}^{m} y_i b_i.$$ 

As we have already seen, every point in $P$ satisfies $c^T x \leq t$. But we can say more. If $c$ is integral, then for every integral vector in $P$ the quantity $c^T x$ is integral, so it satisfies the stronger inequality

$$c^T x \leq \lfloor t \rfloor.$$ 

The inequality (9.7) is called a Gomory-Chvátal cutting plane. The term “cutting plane” stems from the fact that (9.7) cuts off part of the polyhedron $P$ but not any of the integral vectors in $P$.

**Definition 9.2 (Cutting-Plane Proof)**

Let $Ax \leq b$ be a system of linear inequalities and $c^T x \leq t$ be an inequality. A sequence of linear inequalities

$$c_1^T x \leq t_1, c_2^T x \leq t_2, \ldots, c_k^T x \leq t_k$$

is called a cutting-plane proof of $c^T x \leq t$ (from $Ax \leq b$), if each of the vectors $c_1, \ldots, c_k$ is integral, $c_k = c$, $t_k = t$, and if for each $i = 1, \ldots, k$ the following statement holds: $c_i^T x \leq t'_i$ is a nonnegative linear combination of the inequalities $Ax \leq b$, $c_1^T x \leq t_1, \ldots, c_i^T x \leq t_{i-1}$ for some $t'_i$ with $\lfloor t'_i \rfloor \leq t_i$.

Clearly, if $c^T x \leq t$ has a cutting-plane proof from $Ax \leq b$, then $c^T x \leq t$ is valid for each integral solution of $Ax \leq b$. Moreover, a cutting plane proof is a clean way to show that the inequality $c^T x \leq t$ is valid for all integral vectors in a polyhedron.

**Example 9.3 (Matching Polytope)**

The matching polytope $M(G)$ of a graph $G = (V, E)$ is defined as the convex hull of all incidence vectors of matchings in $G$. It is equal to the set of solutions of

$$x(\delta(v)) \leq 1 \quad \text{for all } v \in V$$

$$x \in \mathcal{B}^E$$

Alternatively, if we let $P$ denote the polytope obtained by replacing $x \in \mathcal{B}^E$ by $0 \leq x$, then $M(G) = P_1$.

Let $T \subseteq V$ be a set of nodes of odd cardinality. As the edges of a matching do not share an endpoint, the number of edges of a matching having both endpoints in $T$ is at most $\frac{|T|-1}{2}$. Thus,

$$x(\gamma(T)) \leq \frac{|T|-1}{2}$$

(9.8)

is a valid inequality for $M(G) = P_1$. Here, $\gamma(T)$ denotes the set of edges which have both endpoints in $T$. We now give a cutting-plane proof of (9.8).
For \( v \in T \) take the inequality \( x(\delta(v)) \leq 1 \) with weight 1/2 and sum up the resulting \(|T|\) inequalities. This gives:
\[
x(\gamma(T)) + \frac{1}{2} x(\delta(T)) \leq \frac{|T|}{2}.
\]
(9.9)

For each \( e \in \delta(T) \) we take the inequality \(-x_e \leq 0\) with weight 1/2 and add it to (9.9). This gives us:
\[
x(\gamma(T)) \leq \frac{|T|}{2}.
\]
(9.10)

Rounding down the right hand side of (9.10) yields the desired result (9.8).

In the sequel we are going to show that cutting-plane proofs are always possible, provided \( P \) is a polytope.

**Theorem 9.4** Let \( P = \{ x : Ax \leq b \} \) be a rational polytope and let \( c^T x \leq t \) be a valid inequality for \( X = P \cap \mathbb{Z}^n \), where \( c \) is integral. Then, there exists a cutting-plane proof of \( c^T x \leq t' \) from \( Ax \leq b \) for some \( t' \leq t \).

We will prove Theorem 9.4 by means of a special case (Theorem 9.6). We need another useful equivalent form of Farkas’ Lemma:

**Theorem 9.5 (Farkas’ Lemma for inequalities)** The system \( Ax \leq b \) has a solution \( x \) if and only if there is no vector \( y \geq 0 \) such that \( y^T A = 0 \) and \( y^T b < 0 \).

**Proof:** See standard textbooks about Linear Programming, e.g. [?].

From this variant of Farkas’ Lemma we see that \( P = \{ x : Ax \leq b \} \) is empty if and only if we can derive a contradiction \( 0^T x \leq -1 \) from the system \( Ax \leq b \) by means of taking a nonnegative linear combination of the inequalities. The following theorem gives the analogous statement for integral systems:

**Theorem 9.6** Let \( P = \{ x : Ax \leq b \} \) be a rational polytope and \( X = P \cap \mathbb{Z}^n \) be empty: \( X = \emptyset \). Then there exists a cutting-plane proof of \( 0^T x \leq -1 \) from \( Ax \leq b \).

Before we embark upon the proofs (with the help of a technical lemma) let us derive another look at Gomory-Chvátal cutting-planes. By Farkas’ Lemma, we can derive any valid inequality \( c^T x \leq t \) (or a stronger version) for a polytope \( P = \{ x : Ax \leq b \} \) by using a nonnegative linear combination of the inequalities. In view of this fact, we can define Gomory-Chvátal cutting-planes also directly in terms of the polyhedron \( P \): we just take a valid inequality \( c^T x \leq t \) for \( P \) with \( c \) integral which induces a nonempty face and round down \( t \) to obtain the cutting plane \( c^T x \leq \lfloor t \rfloor \).

The proof of Theorems 9.4 and 9.6 is via induction on the dimension of the polytope. The following lemma allows us to translate a cutting-plane proof on a face \( F \) to a proof on the entire polytope \( P \).

**Lemma 9.7** Let \( P = \{ x : Ax \leq b \} \) be a rational polytope and \( F \) be a face of \( P \). If \( c^T x \leq \lfloor t \rfloor \) is a Gomory-Chvátal cutting-plane for \( F \), then there exists a Gomory-Chvátal cutting-plane \( c^T x \leq \lfloor t \rfloor \) for \( P \) such that
\[
F \cap \left\{ x : c^T x \leq \lfloor t \rfloor \right\} = F \cap \left\{ x : c^T x \leq \lfloor t \rfloor \right\}.
\]
(9.11)
9.1 Cutting-Plane Proofs

Proof: By Theorem\ref{thm:gomory-chvatal}, we can write \( P = \{ x : A'x \leq b', A''x = b'' \} \) and \( F = \{ x : A'x \leq b', A''x = b'' \} \), where \( A' \) and \( A'' \) are integral. Let \( t^* = \max \{ c^T x : x \in F \} \). Since \( c^T x \leq t \) is valid for \( F \) we must have \( t \geq t^* \). So, the following system does not have a solution:

\[
\begin{align*}
A'x & \leq b' \\
A''x & \leq b'' \\
-A''x & \leq -b'' \\
c^T x & > t
\end{align*}
\]

By the Farkas’ Lemma there exist vectors \( y' \geq 0 \), \( y'' \) such that

\[
\begin{align*}
(y')^T A' + (y'')^T A'' = c^T \\
(y')^T b' + (y'')^T b'' = t
\end{align*}
\]

This looks like a Gomory-Chvátal cutting-plane \( c^T x \leq \lfloor t^* \rfloor \) for \( P \) with the exception that \( y'' \) is not necessarily nonnegative. However, the vector \( y'' - \lfloor y'' \rfloor \) is nonnegative and it turns out that replacing \( y'' \) by this vector will work. Let

\[
\begin{align*}
\tilde{c}^T := (y')^T A' + (y'' - \lfloor y'' \rfloor)^T A'' = c^T - (\lfloor y'' \rfloor)^T A'' \\
\tilde{t} := (y')^T b' + (y'' - \lfloor y'' \rfloor)^T b'' = t - (\lfloor y'' \rfloor)^T b''.
\end{align*}
\]

Observe that \( \tilde{c} \) is integral, since \( c \) is integral, \( A'' \) is integral and \( \lfloor y'' \rfloor \) is integral. The inequality \( \tilde{c}^T x \leq \tilde{t} \) is a valid inequality for \( P \), since we have taken a nonnegative linear combination of the constraints. Now, we have

\[
\lfloor t \rfloor = \lfloor y'' \rfloor^T b' + (\lfloor y'' \rfloor)^T b'' = \lfloor \tilde{t} \rfloor + (\lfloor y'' \rfloor)^T b'', \quad (9.12)
\]

where the last equality follows from the fact that \( \lfloor y'' \rfloor \) and \( b'' \) are integral. This gives us:

\[
\begin{align*}
F \cap \{ x : \tilde{c}^T x \leq \lfloor \tilde{t} \rfloor \} \\
= & F \cap \{ x : c^T x \leq \lfloor t \rfloor, A''x = b'' \} \\
= & F \cap \{ x : c^T x \leq \lfloor t \rfloor, (\lfloor y'' \rfloor)^T A''x = (\lfloor y'' \rfloor)^T b'' \} \\
= & F \cap \{ x : c^T x \leq \lfloor t \rfloor plus (\lfloor y'' \rfloor)^T b'', (\lfloor y'' \rfloor)^T A''x = (\lfloor y'' \rfloor)^T b'' \} \\
= & F \cap \{ x : c^T x \leq \lfloor t \rfloor \}.
\end{align*}
\]

This completes the proof. \( \square \)

Proof of Theorem\ref{thm:gomory-chvatal}. We use induction on the dimension of \( P \). If \( \dim(P) = 0 \), then the claim obviously holds. So, let us assume that \( \dim(P) \geq 1 \) and that the claim holds for all polytopes of smaller dimension.

Let \( c^T x \leq \delta \) with \( c \) integral be an inequality which induces a proper face of \( P \). Then, by Farkas’ Lemma, we can derive the inequality \( c^T x \leq \delta \) from \( Ax \leq b \) and \( c^T x \leq \lfloor \delta \rfloor \) is a Gomory-Chvátal cutting-plane for \( P \). Let

\[
P' := \{ x \in P : c^T x \leq \lfloor \delta \rfloor \}
\]

be the polytope obtained from \( P \) by applying the Gomory-Chvátal cut \( c^T x \leq \lfloor \delta \rfloor \).

Case 1: \( P = \emptyset \): By Farkas’ Lemma we can derive the inequality \( 0^T x \leq -1 \) from the inequality system \( Ax \leq b \), \( c^T x \leq \lfloor \delta \rfloor \) which defines \( P \). Since \( c^T x \leq \lfloor \delta \rfloor \) was a Gomory-Chvátal cutting-plane
for \( P \) (and thus was derived itself from \( Ax \leq b \)) this means, we can derive the contradiction from \( Ax \leq b \).

**Case 2:** \( \bar{P} \neq \emptyset \):

Define the face \( F \) of \( \bar{P} \) by

\[
F := \left\{ x \in \bar{P} : c^T x = [\delta] \right\} = \left\{ x \in P : c^T x = [\delta] \right\}.
\]

If \( \delta \) is integral, then \( F \) is a proper face of \( P \), so \( \dim(F) < \dim(P) \) in this case. If \( \delta \) is not integral, then \( P \) contains points which do not satisfy \( c^T x = [\delta] \) and so also in this case we have \( \dim(F) < \dim(P) \).

By the induction hypothesis, there is a cutting-plane proof of \( 0^T x \leq -1 \) for \( F \), that is, from the system \( Ax \leq b, c^T x = [\delta] \). By Lemma 9.7 there is a cutting-plane proof from \( Ax \leq b, c^T x \leq [\delta] \) for an inequality \( w^T x \leq d \) such that

\[
\emptyset = F \cap \left\{ x : 0^T x \leq -1 \right\} = F \cap \left\{ x : w^T x \leq [d] \right\}.
\]

We have

\[
\emptyset = F \cap \left\{ x : w^T x \leq [d] \right\} = \bar{P} \cap \left\{ x : c^T x = [\delta], w^T x \leq [d] \right\}.
\]  \hspace{1cm} (9.13)

Let us restate our result so far: We have shown that there is a cutting plane proof from \( Ax \leq b, c^T x \leq [\delta] \) for an inequality \( w^T x \leq d \) which satisfies (9.13).

Thus, the following linear system does not have a solution:

\[
\begin{align*}
Ax & \leq b \\
c^T x & \leq [\delta] \\
-c^T x & \leq -[\delta] \\
w^T x & \leq [d].
\end{align*}
\]  \hspace{1cm} (9.14a)

By Farkas’ Lemma for inequalities there exist \( y, \lambda_1, \lambda_2, \mu \geq 0 \) such that

\[
\begin{align*}
y^T A + \lambda_1 c^T - \lambda_2 c^T + \mu w^T &= 0 \quad \text{ (9.15a)} \\
y^T b + \lambda_1 [\delta] - \lambda_2 [\delta] + \mu [d] &< 0. \quad \text{ (9.15b)}
\end{align*}
\]

If \( \lambda_2 = 0 \), then (9.15) means that already the system obtained from (9.14) by dropping \( c^T x \geq [\delta] \) does not have a solution, that is

\[
\emptyset = \left\{ x : Ax \leq b, c^T x \leq [\delta], w^T x \leq [d] \right\}.
\]

So, by Farkas’ Lemma we can derive \( 0^T x \leq -1 \) from this system which consists completely of Gomory-Chvátal cutting-planes for \( P \).

So, it suffices to handle the case that \( \lambda_2 > 0 \). In this case, we can divide both lines in (9.15) by \( \lambda_2 \) and get that there exist \( y', \lambda' \geq 0, \lambda' \geq 0 \) and \( \mu' \geq 0 \) such that

\[
\begin{align*}
y'^T A + (\lambda') c^T + (\mu') w^T &= c^T \quad \text{ (9.16a)} \\
y'^T b + (\lambda') [\delta] + (\mu') [d] &= \theta < [\delta]. \quad \text{ (9.16b)}
\end{align*}
\]

Now, (9.16) states that we can derive an inequality \( c^T x \leq \theta \) from \( Ax \leq b, c^T x \leq [\delta], w^T x \leq [d] \) with \( \theta < [\delta] \). Since all the inequalities in the system were Gomory-Chvátal cutting-planes this implies that

\[
c^T x \leq [\delta] - \tau \quad \text{ for some } \tau \in \mathbb{Z}, \tau \geq 1
\]  \hspace{1cm} (9.17)
is a Gomory-Chvátal cutting-plane for $P$.
Since $P$ is bounded, the value $z = \min \{ c^T x : x \in P \}$ is finite. If we continue as above, starting with $\bar{P} = \{ x \in P : c^T x \leq |\delta| - \tau \}$, at some point we will obtain a cutting-plane proof of some $c^T x \leq t$ where $t < z$ so that $P \cap \{ x : c^T x \leq t \} = \emptyset$. Then, by Farkas’ Lemma we will be able to derive $0^T x \leq -1$ from $Ax \leq b$, $c^T x \leq t$.

Proof of Theorem 9.4 Case 1: $P \cap \mathbb{Z}^n = \emptyset$

By Theorem 9.6 there is a cutting-plane proof of $0^T x \leq -1$ from $Ax \leq b$. Since $P$ is bounded, $\ell := \max \{ c^T x : x \in P \}$ is finite. By Farkas’ Lemma, we can derive $c^T x \leq \ell$ and thus we have the Gomory-Chvátal cutting plane $c^T x \leq |\ell|$. Adding an appropriate multiple of $0^T x \leq -1$ to $c^T x \leq |\ell|$ gives an inequality $c^T x \leq t'$ for some $t' \leq \ell$ which yields the required cutting-plane proof.

Case 2: $P \cap \mathbb{Z}^n \neq \emptyset$

Again, let $\ell := \max \{ c^T x : x \in P \}$ which is finite, and define $\bar{P} := \{ x \in P : c^T x \leq |\ell| \}$, that is, $\bar{P}$ is the polytope obtained by applying the Gomory-Chvátal cutting-plane $c^T x \leq |\ell|$ to $P$.

If $|\ell| \leq t$ we already have a cutting-plane proof of an inequality with the desired properties. So, assume that $|\ell| > t$. Consider the face

$$F = \{ x \in \bar{P} : c^T x = |\ell| \}$$

of $\bar{P}$. Since $c^T x \leq t$ is valid for all integral points in $P$ and by assumption $t < |\ell|$, the face $F$ cannot contain any integral point. By Theorem 9.6 there is a cutting-plane proof of $0^T x \leq -1$ from $Ax \leq b$, $c^T x \leq |\ell|$. We now use Lemma 9.7 as in the proof of Theorem 9.6. The lemma shows that there exists a cutting-plane proof of some inequality $w^T x \leq |d|$ from $Ax \leq b$, $c^T x \leq |\ell|$ such that $\bar{P} \cap \{ x : c^T x = |\ell|, w^T x \leq |d| \} = \emptyset$.

By using the same arguments as in the proof of Theorem 9.6 it follows that there is a cutting-plane proof of an inequality $c^T x \leq |\ell| - \tau$ for some $\tau \in \mathbb{Z}$, $\tau \geq 1$ from $Ax \leq b$. Continuing this way, we finally get an inequality $c^T x \leq t'$ with $t' \leq \ell$.

9.2 A Geometric Approach to Cutting Planes: The Chvátal Rank

Let $P = \{ x : Ax \leq b \}$ be a rational polyhedron and $P_1 := \text{conv}(P \cap \mathbb{Z}^n)$. Suppose we want to find a linear description of $P_1$. One approach is to add valid inequalities step by step, obtaining tighter and tighter approximations of $P_1$.

We have already seen that, if $c^T x \leq \delta$ is a valid inequality for $P$ with $c$ integral, then $c^T x \leq |\delta|$ is valid for $P_1$. If $c^T x = \delta$ was a supporting hyperplane of $P$, that is, $P \cap \{ x : c^T x = \delta \}$ is a proper face of $P$, then $c^T x \leq |\delta|$ is a Gomory-Chvátal cutting-plane. Otherwise, the inequality $c^T x \leq \delta$ is dominated by that of a supporting hyperplane. Anyway, we have

$$P_1 \subseteq \left\{ x \in \mathbb{R}^n : c^T x \leq |\delta| \right\}$$

for any valid inequality $c^T x \leq \delta$ for $P$ where $c$ is integral. This suggests to take the intersection of all sets of the form (9.18) as an approximation to $P$.

**Definition 9.8** Let $P$ be a rational polyhedron. Then, $P'$ is defined as

$$P' := \bigcap_{c \text{ is integral and } c^T x \leq \delta \text{ is valid for } P} \left\{ x \in \mathbb{R}^n : c^T x \leq |\delta| \right\}.$$
Observe that (9.19) is the same as taking the intersection over all Gomory-Chvátal cutting-planes for $P$. It is not a priori clear that $P'$ is a polyhedron, since there is an infinite number of cuts.

**Theorem 9.9** Let $P$ be a rational polyhedron. Then $P'$ as defined in (9.19) is also a rational polyhedron.

**Proof:** If $P = \emptyset$ the claim is trivial. So let $P \neq \emptyset$. By Theorem 4.27 there is a TDI-system $Ax \leq b$ with integral $A$ such that $P = \{x : Ax \leq b\}$. We claim that

$$P' = \{x \in \mathbb{R}^n : Ax \leq \lfloor b \rfloor\}.$$  

(9.20)

From this the claim follows, since the set on the right hand side of (9.20) is a rational polyhedron ($A$ and $\lfloor b \rfloor$ are integral, and there are only finitely many constraints).

Since every row $a_i^T x \leq b_i$ of $Ax \leq b$ is a valid inequality for $P$ it follows that $P' \subseteq \{x \in \mathbb{R}^n : Ax \leq \lfloor b \rfloor\}$. So, it suffices to show that the set on the right hand side of (9.20) is contained in $P'$.

Let $c^T x = \delta$ be a supporting hyperplane of $P$ with $c$ integral, $P \subseteq \{x : c^T x \leq \delta\}$. By Linear Programming duality we have

$$\delta = \max \left\{c^T x : x \in P\right\} = \min \left\{b^T y : A^T y = c, y \geq 0\right\}.$$  

(9.21)

Since the system $Ax \leq b$ is TDI and $c$ is integral, the minimization problem in (9.21) has an optimal solution $y^*$ which is integral.

Let $x \in \{x : Ax \leq \lfloor b \rfloor\}$.

$$c^T x = (A^T y^*)^T x \quad \text{(since $y^*$ is feasible for the problem in (9.21))}$$

$$= (y^*)^T (Ax) \leq (y^*)^T \lfloor b \rfloor \quad \text{(since $Ax \leq \lfloor b \rfloor$ and $y^* \geq 0$)}$$

$$= [(y^*)^T \lfloor b \rfloor] \quad \text{(since $y^*$ and $\lfloor b \rfloor$ are integral)}$$

$$\leq [(y^*)^T b] \quad \text{(since $\lfloor b \rfloor \leq b$ and $y^* \geq 0$)}$$

$$= \lfloor \delta \rfloor \quad \text{(by the optimality of $y^*$ for (9.21))}.$$

Thus, we have

$$\{x : Ax \leq \lfloor b \rfloor\} \subseteq \left\{x : c^T x \leq \lfloor \delta \rfloor\right\}.$$

Since $c^T x = \delta$ was an arbitrary supporting hyperplane, we get that

$$\{x : Ax \leq \lfloor b \rfloor\} \subseteq \bigcap_{c, \delta} \left\{x : c^T x \leq \lfloor \delta \rfloor\right\} = P'$$

as required. \qed

We have obtained $P'$ from $P$ by taking all Gomory-Chvátal cuts for $P$ as a first wave. Given that $P'$ is a rational polyhedron, we can take as a second wave all Gomory-Chvátal cuts for $P'$. Continuing this procedure gives us better and better approximations of $P_1$. We let

$$P^{(0)} := P$$

$$P^{(i)} := \left(P^{(i-1)}\right)' \quad \text{for } i \geq 1.$$
This gives us a sequence of polyhedra

\[ P = P^{(0)} \supset P^{(1)} \supset P^{(2)} \supset \cdots \supset P_1, \]

which are generated by the waves of cuts.

We know that \( P_1 \) is a rational polyhedron (given that \( P \) is one) and by Theorem 9.4 every valid inequality for \( P_1 \) will be generated by the waves of Gomory-Chvátal cuts. Thus, we can restate the result Theorem 9.4 in terms of the polyhedra \( P^{(i)} \) as follows:

**Theorem 9.10** Let \( P \) be a rational polytope. Then we have \( P^{(k)} = P_1 \) for some \( k \in \mathbb{N} \).

**Definition 9.11 (Chvátal rank)**

Let \( P \) be a rational polytope. The Chvátal rank of \( P \) is defined to be the smallest integer \( k \) such that \( P^{(k)} = P_1 \).

### 9.3 Cutting-Plane Algorithms

Cutting-plane proofs are usually employed to prove the validity of some classes of inequalities. These valid inequalities can then be used in a cutting-plane algorithm.

Suppose that we want to solve the integer program

\[ z^* = \max \left\{ c^T x : x \in P \cap \mathbb{Z}^n \right\} \]

and that we know a family \( F \) of valid inequalities for \( P_1 = \text{conv}(P \cap \mathbb{Z}^n) \). Usually, \( F \) will not contain a complete description of \( P_1 \) since either such a description is not known or we do not know how to separate over \( F \) efficiently. The general idea of a cutting-plane algorithm is as follows:

- We find an optimal solution \( x^* \) for the Linear Program \( \max \left\{ c^T x : x \in P \right\} \). This can be done by any Linear Programming algorithm (possibly a solver that is available only as a black-box).
- If \( x^* \) is integral, we already have an optimal solution to the IP and we can terminate.
- Otherwise, we search our family (or families) of valid inequalities for inequalities which are violated by \( x^* \), that is, \( w^T x^* > d \) where \( w^T x \leq d \) is valid for \( P_1 \).
- We add the inequalities found to our LP-relaxation and resolve to find a new optimal solution \( x^{**} \) of the improved formulation. This procedure is continued.
- If we are fortunate (or if \( F \) contains a complete description of \( P_1 \)), we terminate with an optimal integral solution. We say “fortunate”, since if \( F \) is not a complete description of \( P_1 \), this depends on the objective function and our family \( F \).
- If we are not so lucky, we still have gained something. Namely, we have found a new formulation for our initial problem which is better than the original one (since we have cut off some non-integral points). The formulation obtained upon termination gives an upper bound \( \bar{z} \) for the optimal objective function value \( z^* \) which is no worse than the initial one (and usually is much better). We can now use \( \bar{z} \) in a branch and bound algorithm.
Algorithm 9.1 Generic cutting-plane algorithm

\textbf{GENERIC-CUTTING-PLANE}

\textbf{Input:} An integer program $\max \{ c^T x : x \in P, x \in \mathbb{Z}^n \}$; a family $\mathcal{F}$ of valid inequalities for $P_1 = \text{conv}(P \cap \mathbb{Z}^n)$

1. repeat
2. Solve the Linear Program $\max \{ c^T x : x \in P \}$. Let $x^*$ be an optimal solution.
3. if $x^*$ is integral then
   4. An optimal solution to the integer program has been found. stop.
   5. else
5. Solve the separation problem for $\mathcal{F}$, that is, try to find an inequality $w^T x \leq d$ in $\mathcal{F}$ such that $w^T x^* > d$.
6. if such an inequality $w^T x \leq d$ cutting off $x^*$ was found then
7. Add the inequality to the system, that is, set $P := P \cap \{ x : w^T x \leq d \}$.
8. else
9. We do not have an optimal solution yet. However, we have a better formulation for the original problem. stop.
10. end if
11. end if
12. until forever

Algorithm 9.1 gives a generic cutting-plane algorithm along the lines of the above discussion. The technique of using improved upper bounds from a cutting-plane algorithm in a branch and bound system is usually referred to as branch-and-cut (cf. the comments about preprocessing in Section 8.4.2).

It is clear that the efficiency of a cutting-plane algorithm depends on the availability of constraints that give good upper bounds. In view of Theorem 5.48 the only inequalities (or cuts) we need are those that induce facets. Thus, one is usually interested in finding (by means of mathematical methods) as many facet-inducing inequalities as possible.

9.4 Gomory’s Cutting-Plane Algorithm

In this section we assume that the integer program which we want to solve is given in the following form:

$$\begin{align*}
\text{max } c^T x & \quad (9.22a) \\
A x &= b \quad (9.22b) \\
x &\geq 0 \quad (9.22c) \\
x &\in \mathbb{Z}^n \quad (9.22d)
\end{align*}$$

where $A$ is an integral $m \times n$-matrix with $\text{rank}(A) = m$ and $b$ is an integral vector in $\mathbb{Z}^m$. As usual we let $P = \{ x \in \mathbb{R}^n : A x = b, x \geq 0 \}$ and $P_I = \text{conv}(P \cap \mathbb{Z}^n)$.

Any integer program with rational data can be brought into this form by elementary transformations (see textbooks about Linear Programming [3, 4, 5] where those methods are used for Linear Programs): if $x_j$ is not sign restricted, we replace $x_j$ by two new variables $x_j = x_j^+ - x_j^-$ where $x_j^+, x_j^- \geq 0$. Any inequality $a_j^T x \leq b_1$ can be transformed into an equality by introducing a slack variable $s \geq 0$ which yields $a_j^T x + s = b_1$. Since $A$ and $b$ are integral, the new slack variable will also be an integral variable.
Suppose that we solve the LP-relaxation
\[
\begin{align*}
\max & \quad c^T x \\
Ax & = b \quad (9.23a) \\
x & \geq 0 \quad (9.23b)
\end{align*}
\]
of (9.22) by means of the Simplex method.\(^1\)

Recall that a basis for (9.23) is an index set \(B \subseteq \{1, \ldots, n\}\) with \(|B| = m\) such that the corresponding submatrix \(A_B\) of \(A\) is nonsingular. The basis is termed feasible if \(x_B := A_B^{-1} b \geq 0\). Clearly, in this case \((x_B, x_N)\) with \(x_N := 0\) is a feasible solution of (9.23). It is a well-known fact that (9.23) has an optimal solution if and only if there is an optimal basic solution [?, ?, ?].

Suppose that we are given an optimal basis \(B\) and a corresponding optimal basic solution \(x^*\) for (9.23). As in Section 8.4.1 we can parametrize \(x^*\) by the nonbasic variables:
\[
\begin{align*}
x_B^* &= A_B^{-1} b - A_B^{-1} A_N x_N^* =: \tilde{b} - \tilde{A}_N x_N^* \\
x_N^* &= 0.
\end{align*}
\]
This gives the equivalent statement of the problem (9.23) in the basis representation:
\[
\begin{align*}
\max & \quad c^T x_B^* - c^T_N x_N \\
x_B^* + \tilde{A}_N x_N &= \tilde{b} \\
x & \geq 0
\end{align*}
\]
\[
\tag{9.26}
\]
If \(x^*\) is integral, then \(x^*\) is an optimal solution for our integer program (9.22). Otherwise, there is a basic variable \(x_i^*\) which has a fractional value, that is, \(x_i^* = \bar{b}_i \notin \mathbb{Z}\). We will now use the equation in (9.24) which defines \(x_i^*\) to derive a valid inequality for \(P_1\). We will then show that the inequality derived is in fact a Gomory-Chvátal cutting-plane.

Let \(\bar{A} = (\bar{a}_{1k})\). Any feasible solution \(x\) of the integer program (9.22) satisfies (9.24). So, we have
\[
\begin{align*}
x_i &= \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \in \mathbb{Z} \quad (9.27) \\
&\quad -[\bar{b}_i] \quad \in \mathbb{Z} \\
&\quad \sum_{j \in N} |\bar{a}_{ij}| x_j \in \mathbb{Z}. \quad (9.28)
\end{align*}
\]
Adding (9.27), (9.28) and (9.29) results in:
\[
\begin{align*}
(\bar{b}_i - [\bar{b}_i]) - \sum_{j \in N} (\bar{a}_{ij} - [\bar{a}_{ij}]) x_j \in \mathbb{Z}.
\end{align*}
\]
\[
\tag{9.30}
\]
Since \(0 < (\bar{b}_i - [\bar{b}_i]) < 1\) and \(\sum_{j \in N} (\bar{a}_{ij} - [\bar{a}_{ij}]) x_j \geq 0\), the value on the left hand side of (9.30) can only be integral, if it is nonpositive, that is, we must have
\[
(\bar{b}_i - [\bar{b}_i]) - \sum_{j \in N} (\bar{a}_{ij} - [\bar{a}_{ij}]) x_j \leq 0
\]
for every \(x \in P_1\). Thus, the following inequality is valid for \(P_1\):
\[
\sum_{j \in N} (\bar{a}_{ij} - [\bar{a}_{ij}]) x_j \geq (\bar{b}_i - [\bar{b}_i]).
\]
\[
\tag{9.31}
\]
\(^1\)Basically, one could also use an interior point method. The key point is that in the sequel we need an optimal basis.
Moreover, the inequality (9.31) is violated by the current basic solution \( x^* \), since \( x^*_N = 0 \) (which means that the left hand side of (9.31) is zero) and \( x^*_i = \bar{b}_i \not\in \mathbb{Z} \), so that \( \bar{b}_i - \lfloor \bar{b}_i \rfloor = (x^*_i - \lfloor x^*_i \rfloor) > 0 \).

As promised, we are now going to show that (9.31) is in fact a Gomory-Chvátal cutting-plane. By (9.24) the inequality
\[
 x_i + \sum_{j \in N} \bar{a}_{ij} x_j \leq \bar{b}_i 
\] (9.32)
is valid for \( P \). Since \( P \subseteq \mathbb{R}^n_+ \), we have \( \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \sum_{j \in N} \bar{a}_{ij} x_j \) and the inequality
\[
 x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \bar{b}_i 
\] (9.33)
must also be valid for \( P \). In fact, since the basic solution \( x^* \) for the basis \( B \) satisfies (9.32) and (9.33) with equality, the inequalities (9.32) and (9.33) both induce supporting hyperplanes. Observe that all coefficients in (9.33) are integral. Thus,
\[
 x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor, 
\] (9.34)
is a Gomory-Chvátal cutting-plane. We can now use (9.24) to rewrite (9.34), that is, to eliminate \( x_i \) (this corresponds to taking a nonnegative linear combination of (9.34) and the appropriate inequality stemming from the equality (9.24)). This yields (9.31), so (9.31) is (a scalar multiple of) a Gomory-Chvátal cutting-plane. It is important to notice that the difference between the left-hand side and the right-hand side of the Gomory-Chvátal cutting-plane (9.34), hence also of (9.31) is integral, when \( x \) is integral. Thus, if (9.31) is rewritten using a slack variable \( s \geq 0 \) as
\[
 \sum_{j \in N} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j - s = (\bar{b}_i - \lfloor \bar{b}_i \rfloor), 
\] (9.35)
then this slack variable \( s \) will also be a nonnegative integer variable. Gomory’s cutting plane algorithm is summarized in Algorithm 9.2.

**Example 9.12**

We consider the following integer program:

\[
\begin{align*}
\max \quad & 4x_1 - x_2 \\
7x_1 - 2x_2 & \leq 14 \\
x_2 & \leq 3 \\
2x_1 - 2x_2 & \leq 3 \\
x_1, x_2 & \geq 0 \\
x_1, x_2 & \in \mathbb{Z}
\end{align*}
\]

The feasible points and the polyhedron \( P \) described by the inequalities above are depicted in Figure 9.2(a).

Adding slack variables gives the following LP-problem:

\[
\begin{align*}
\max \quad & 4x_1 - x_2 \\
7x_1 - 2x_2 + x_3 & = 14 \\
x_2 + x_4 & = 3 \\
2x_1 - 2x_2 + x_5 & = 3 \\
x_1, x_2, x_3, x_4, x_5 & \geq 0 \\
x_1, x_2, x_3, x_4, x_5 & \in \mathbb{Z}
\end{align*}
\]
Algorithm 9.2 Gomory’s cutting-plane algorithm

GOMORY-CUTTING-PLANE
Input: An integer program max \( \{ c^T x : Ax = b, x \geq 0, x \in \mathbb{Z}^n \} \)

1. repeat
2. Solve the current LP-relaxation max \( \{ c^T x : Ax = b, x \geq 0 \} \). Let \( x^* \) be an optimal basic solution.
3. if \( x^* \) is integral then
4. An optimal solution to the integer program has been found. stop.
5. else
6. Choose one of the basis integer variables which is fractional in the optimal LP-solution, say \( x_i = \bar{b}_i \). This variable is parametrized as follows:

\[
x_i = \bar{b}_i - \sum_{j \in \mathcal{N}} \bar{a}_{ij} x_j
\]

7. Generate the Gomory-Chvátal cut

\[
\sum_{j \in \mathcal{N}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor)x_j \geq (\bar{b}_i - \lfloor \bar{b}_i \rfloor)
\]

and add it to the LP-formulation by means of a new nonnegative integer slack variable \( s \):

\[
\sum_{j \in \mathcal{N}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor)x_j - s = (\bar{b}_i - \lfloor \bar{b}_i \rfloor)
\]

\[
s \geq 0
\]

\[
s \in \mathbb{Z}
\]

8. end if
9. until forever
(a) Inequalities leading to the polytope shown with thick lines together with the convex hull of the integral points.

(b) The first cut produced by Gomory’s algorithm is $x_1 \leq 2$ (shown in red).

(c) The next cut produced is $x_1 - x_2 \leq 1$ (shown in red). After this cut the algorithm terminates with the optimum solution $(2, 1)$.

Figure 9.2: Example problem for Gomory’s algorithm. Thick solid lines indicate the polytope described by the inequalities, the pink shaded region is the convex hull of the integral points (red) which are feasible.
An optimal basis for the corresponding LP-relaxation is $B = \{1, 2, 5\}$ with

$$A_B = \begin{pmatrix} 7 & -2 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \quad x_B^* = \begin{pmatrix} \frac{20}{7} \\ 3 \\ 0 \\ 0 \\ \frac{23}{7} \end{pmatrix}$$

The optimal basis representation (9.26) is given by:

$$\text{max } \frac{59}{7} - \frac{1}{7}x_3 - \frac{1}{7}x_4 = \frac{20}{7} \quad (9.36a)$$

$$x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 = \frac{20}{7} \quad (9.36b)$$

$$x_1, x_2, x_3, x_4, s \geq 0 \quad (9.36c)$$

The variable $x_1^*$ is fractional, $x_1^* = \frac{20}{7}$. From the first line of (9.36) we have:

$$x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 = \frac{20}{7}.$$ 

The Gomory-Chvátal cut (9.31) generated for $x_1$ is

$$\left(\frac{1}{7} - \left(\frac{1}{7}\right)\right)x_3 + \left(\frac{2}{7} - \left[\frac{2}{7}\right]\right)x_4 \geq \left(\frac{20}{7} - \left[\frac{20}{7}\right]\right)$$

$$\iff \frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}$$

Thus, the following new constraint will be added to the LP-formulation:

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 - s = \frac{6}{7},$$

where $s \geq 0$, $s \in \mathbb{Z}$ is a new integer nonnegative slack variable. Before we continue, let us look at the inequality in terms of the original variables, that is, without the slack variables. We have $x_3 = 14 - 7x_1 + 2x_2$ and $x_4 = 3 - x_2$. Substituting we get the cutting-plane

$$\frac{1}{7}(14 - 7x_1 + 2x_2) + \frac{2}{7}(3 - x_2) \geq \frac{6}{7}$$

$$\iff x_1 \leq 2.$$ 

The cutting-plane $x_1 \leq 2$ is shown in Figure 9.2 (b).

Reoptimization of the new LP leads to the following optimal basis representation:

$$\text{max } \frac{15}{2} - \frac{1}{5}x_5 - 3s = 2$$

$$x_1 \quad \quad +s = \frac{1}{2}$$

$$x_2 \quad -\frac{1}{2}x_5 = \frac{1}{2}$$

$$x_3 \quad -x_5 = 1$$

$$x_4 \quad +\frac{1}{2}x_5 = \frac{5}{2}$$

$$x_1, x_2, x_3, x_4, x_5, s \geq 0$$

$$x_1, x_2, x_3, x_4, x_5, s \in \mathbb{Z}$$

The optimal solution $x^* = (\frac{2}{7}, \frac{1}{7}, 1, \frac{5}{7}, 0)$ is still fractional. We choose basic variable $x_2$ which is fractional to generate a new cut. We have

$$x_2 = \frac{1}{2}x_5 + s = \frac{1}{2}.$$
and so the new cut is

\[
\left( -\frac{1}{2} - \left\lfloor -\frac{1}{2} \right\rfloor \right) x_5 \geq \left( \frac{1}{2} - \left\lfloor -\frac{1}{2} \right\rfloor \right)
\]

\[\iff \frac{1}{2} x_5 \geq \frac{1}{2} \]

(observe that \( \left( -\frac{1}{2} - \left\lfloor -\frac{1}{2} \right\rfloor \right) = \frac{1}{2} \), since \( \left\lfloor -\frac{1}{2} \right\rfloor = -1 \)). We introduce a new slack variable \( t \geq 0,\ t \in \mathbb{Z} \) and add the following constraint:

\[
\frac{1}{2} x_5 - t = \frac{1}{2}.
\]

Again, we can translate the new cut \( \frac{1}{2} x_5 \geq \frac{1}{2} \) in terms of the original variables. It amounts to

\[
\frac{1}{2} (2x_1 - 2x_2) \geq \frac{1}{2}
\]

\[\iff x_1 - x_2 \leq 1.\]

The new cutting-plane \( x_1 - x_2 \leq 1 \) is shown in Figure 9.2(c).

After reoptimization we obtain the following situation:

\[
\begin{align*}
\text{max} & \quad 7 -3s - t \\
x_1 & \quad +s \quad = \quad 2 \\
x_2 & \quad +s \quad -t \quad = \quad 1 \\
x_3 & \quad -5s \quad -2t \quad = \quad 2 \\
x_4 & \quad +6s \quad +t \quad = \quad 2 \\
x_5 & \quad -t \quad = \quad 1 \\
\end{align*}
\]

\[
x_1, x_2, x_3, x_4, x_5, s, t \geq 0 \\
x_1, x_2, x_3, x_4, x_5, s, t \in \mathbb{Z}
\]

The optimal basic solution is integral, thus it is also an optimal solution for the original integer program: \((x_1,x_2) = (2,1)\) constitutes an optimal solution of our original integer program.

\[
\]

One can show that Gomory’s cutting-plane algorithm always terminates after a finite number of steps, provided the cuts are chosen appropriately. The proof of the following theorem is beyond the scope of these lecture notes. We refer the reader to [?, ?].

**Theorem 9.13** Suppose that Gomory’s algorithm is implemented in the following way:

(i) We use the lexicographic Simplex algorithm for solving the LPs.

(ii) We always derive the Gomory-Chvátal cut from the first Simplex row in which the basic variable is fractional.

Then, Gomory’s cutting-plane algorithm terminates after a finite number of steps with an optimal integer solution. \(\square\)

### 9.5 Mixed Integer Cuts

In this section we consider the situation of mixed integer programs

\[
\text{(MIP)} \quad \text{max} \quad c^T x \\
Ax = b \quad \text{(9.37a)}
\]

\[
x \geq 0 \quad \text{(9.37b)}
\]

\[
x \in \mathbb{Z}^p \times \mathbb{R}^{n-p}. \quad \text{(9.37c)}
\]
In this case, the approach taken so far does not work: In a nutshell, the basis of the Gomory-Chvátal cuts was the fact that, if \( X = \{ y \in \mathbb{Z}^n : y \leq b \} \), then \( y \leq \lfloor b \rfloor \) is valid for \( X \). More precisely, we saw that all Gomory-Chvátal cuts for \( P_1 = P \cap \mathbb{Z}^n \) are of the form \( c^T x \leq \lfloor \delta \rfloor \) where \( c^T x \leq \delta \) is a supporting hyperplane of \( P \) with integral \( c \). If \( x \) is not required to be integral, we may not round down the right hand side of \( c^T x \leq \delta \) to obtain a valid inequality for \( P_1 \).

The approach taken in Gomory’s cutting-plane algorithm from Section 9.4 does not work either, since for instance

\[
\frac{1}{3} + \frac{1}{3} x_1 - 2 x_2 \in \mathbb{Z}
\]

with \( x_1 \in \mathbb{Z}_+ \) and \( x_2 \in \mathbb{R}_+ \) has a larger solution set than

\[
\frac{1}{3} + \frac{1}{3} x_1 \in \mathbb{Z}.
\]

Thus, we can not derive the validity of (9.31) (since we can not assume that the coefficients of the fractional variables are nonnegative) which forms the basis of Gomory’s algorithm.

The key to obtaining cuts for mixed integer programs is the following disjunctive argument:

**Lemma 9.14** Let \( P_1 \) and \( P_2 \) be polyhedra in \( \mathbb{R}^n \) and \( (a^{(i)})^T x \leq \alpha_i \) be valid for \( P_i \), \( i = 1, 2 \). Then, for any vector \( c \in \mathbb{R}^n \) satisfying \( c \leq \min(a^{(1)}, a^{(2)}) \) componentwise and \( \delta \geq \max(\alpha_1, \alpha_2) \) the inequality

\[
c^T x \leq \delta
\]

is valid for \( X = P_1 \cup P_2 \) and \( \text{conv}(X) \).

**Proof:** Let \( x \in X \), then \( x \in P_1 \) or \( x \in P_2 \). If \( x \in P_1 \), then

\[
c^T x = \sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} a^{(1)}_j x_j \leq \alpha_i \leq \delta,
\]

where the first inequality follows from \( c \leq a^{(1)} \) and \( x \geq 0 \).

Let us go back to the situation in Gomory’s algorithm. We solve the LP-relaxation

\[
\text{(LP)} \quad \max \quad c^T x \quad \text{subject to} \quad Ax = b, \quad x \geq 0 \tag{9.38}\]

of (9.37) and obtain an optimal basic solution \( x^* \). As in Section 9.4 we parametrize the solutions of (9.38) by means of the nonbasic variables:

\[
x^*_b = A_B^{-1} b - \bar{A}_N x^*_N =: \bar{b} - \bar{A}_N x^*_N.
\]

Let \( x_1 \) be an integer variable. Then, any feasible solution to the MIP (9.37) satisfies:

\[
\bar{b}_1 - \sum_{j \in N} \bar{a}_{1j} x_j \in \mathbb{Z}, \tag{9.39}
\]

since the quantity on the left hand side of (9.39) is the value of variable \( x_1 \). Let \( N^+ := \{ j \in N : \bar{a}_{1j} \geq 0 \} \), \( N^- := N \setminus N^+ \). Also, for a shorter notation we set \( f_0 := (\bar{b}_1 - [\bar{b}_1]) \in [0, 1) \). Equation (9.39) is equivalent to

\[
\sum_{j \in N} \bar{a}_{1j} x_j = f_0 + k \quad \text{for some } k \in \mathbb{Z}. \tag{9.40}
\]

If, \( k \geq 0 \), then the quantity on the left hand side of (9.40) is at least \( f_0 \), if \( k \leq -1 \), then it is at most \( f_0 - 1 \). Accordingly, we distinguish between two cases:
Case 1: $\sum_{j \in N} a_{ij} x_j \geq f_0 \geq 0$: In this case, we get from $\sum_{j \in N^+} a_{ij} x_j \geq \sum_{j \in N} a_{ij} x_j$ the inequality:
\[ \sum_{j \in N^+} a_{ij} x_j \geq f_0. \] (9.41)

Case 2: $\sum_{j \in N} a_{ij} x_j \leq f_0 - 1 < 0$: Then, $\sum_{j \in N^+} a_{ij} x_j \leq \sum_{j \in N} a_{ij} x_j \leq f_0 - 1$ which is equivalent to
\[ - \frac{f_0}{1 - f_0} \sum_{j \in N^-} a_{ij} x_j \geq f_0. \] (9.42)

We split $P_1$ into two parts, $P_1 = P_1 \cup P_2$, where
\[ P_1 := P_1 \cap \left\{ x : \sum_{j \in N} a_{ij} x_j \geq 0 \right\} \]
\[ P_2 := P_1 \cap \left\{ x : \sum_{j \in N} a_{ij} x_j < 0 \right\}. \]

Inequality (9.41) is valid for $P_1$, while inequality (9.42) is valid for $P_2$. We apply Lemma 9.14 to get an inequality $\pi^T x \geq \pi_0$. If $j \in N^+$, the coefficient for $x_j$ is $\pi_j := \max \left\{ a_{ij}, 0 \right\} = a_{ij}$.
If $j \in N^-$, the coefficient for $x_j$ is $\pi_j := \max \left\{ 0, -\frac{f_0}{1 - f_0} a_{ij} \right\} = -\frac{f_0}{1 - f_0} a_{ij}$. Thus, we obtain the following inequality which is valid for $P_1$:
\[ \sum_{j \in N^+} a_{ij} x_j - \frac{f_0}{1 - f_0} \sum_{j \in N^-} a_{ij} x_j \geq f_0. \] (9.43)

We can strengthen (9.43) by the following technique: The derivation of (9.43) remains valid even if we add integer multiples of integer variables to the left hand side of (9.40); if $\pi$ is an integral vector, then
\[ \sum_{j \in N^+} a_{ij} x_j \geq f_0 + k \quad \text{for some } k \in \mathbb{Z}. \] (9.40)

Thus, we can achieve that every integer variable is in one of the two sets $M^+ := \{ j : a_{ij} + \pi_j \geq 0 \}$ or $M^- := \{ j : a_{ij} + \pi_j < 0 \}$. If $j \in M^+$, then the coefficient $\pi_j$ of $x_j$ in the new version $\pi^T x \geq \pi_0$ of (9.43) is $a_{ij} + \pi_j$, so the best we can achieve is $\pi_j := f_j := (a_{ij} - \lfloor a_{ij} \rfloor)$. If $j \in M^-$, then the coefficient $\pi_j$ is $-\frac{f_0}{1 - f_0} (a_{ij} + \pi_j)$ and the smallest value we can achieve is $-\frac{f_0}{1 - f_0} (f_j - 1) = \frac{f_0(1-f_j)}{1-f_0}$. In summary, the smallest coefficient we can achieve for an integer variable is
\[ \min \left( f_j, \frac{f_0(1-f_j)}{1-f_0} \right). \] (9.44)

The minimum in (9.44) is $f_j$ if and only if $f_j \leq f_0$. This leads to Gomory’s mixed integer cut:
\[ \sum_{j : f_j \leq f_0} f_j x_j + \sum_{j : f_j > f_0} \frac{f_0(1-f_j)}{1-f_0} x_j, \] (9.45)
\[ + \sum_{j \in N^+ \atop x_j \text{ no integer variable}} a_{ij} x_j - \frac{f_0}{1-f_0} \sum_{j \in N^- \atop x_j \text{ no integer variable}} a_{ij} x_j \geq f_0. \]

Similar to Theorem 9.13 it can be shown that an appropriate choice of cutting-planes leads to an algorithm which solves the MIP (9.37) in a finite number of steps.
9.6 Structured Inequalities

In the previous sections of this chapter we have derived valid inequalities for general integer and mixed-integer programs. Sometimes focussing on a single constraint (or a small subset of the constraints) can reveal that a particular problem has a useful “local structure”. In this section we will explore such local structures in order to derive strong inequalities.

9.6.1 Knapsack and Cover Inequalities

We consider the 0/1-Knapsack polytope

\[ P_{\text{KNAPSACK}} := P_{\text{KNAPSACK}}(N, a, b) := \text{conv}\{x \in \mathbb{B}^N : \sum_{j \in N} a_j x_j \leq b \} \]

which we have seen a couple of times in these lecture notes (for instance in Example 1.3). Here, the \( a_i \) are nonnegative coefficients and \( b \geq 0 \). We use the general index set \( N \) instead of \( N = \{1, \ldots, n\} \) to emphasize that a knapsack constraint \( \sum_{j \in N} a_j x_j \leq b \) might occur as a constraint in a larger integer program and might not involve all variables.

In all what follows, we assume that \( a_i \leq b \) for \( j \in N \) since \( a_i > b \) implies that \( x_j = 0 \) for all \( x \in P_{\text{KNAPSACK}}(N, a, b) \). Under this assumption \( P_{\text{KNAPSACK}}(N, a, b) \) is full-dimensional, since \( x^0 \) and \( x^{(j)} \) (\( j \in N \)) form a set of \( n + 1 \) affinely independent vectors in \( P_{\text{KNAPSACK}}(N, a, b) \).

Each inequality \( x_j \geq 0 \) for \( j \in N \) is valid for \( P_{\text{KNAPSACK}}(N, a, b) \). Moreover, each of these nonnegativity constraints defines a facet of \( P_{\text{KNAPSACK}}(N, a, b) \), since \( x^0 \) and \( x^{(i)} \) (\( i \in N \setminus \{j\} \)) form a set of \( n \) affinely independent vectors that satisfy the inequality at equality. In the sequel we will search for more facets of \( P_{\text{KNAPSACK}}(N, a, b) \) and, less ambitious, for more valid inequalities.

**Definition 9.15 (Cover, minimal cover)**

A set \( C \subseteq N \) is called a cover, if

\[ \sum_{j \in C} a_j > b. \]

The cover is called a minimal cover, if \( C \setminus \{j\} \) is not a cover for all \( j \in C \).

Each cover \( C \) gives us a valid inequality \( \sum_{j \in C} x_j \leq |C| - 1 \) for \( P_{\text{KNAPSACK}} \) (if you do not see this immediately, the proof will be given in the following theorem). It turns out that this inequality is quite strong, provided that the cover is minimal.

**Example 9.16**

Consider the knapsack set

\[ X = \{x \in \mathbb{B}^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \} \]

Three covers are \( C_1 = \{1, 2, 6\} \), \( C_2 = \{3, 4, 5, 6\} \) and \( C_3 = \{1, 2, 5, 6\} \) so we have the cover inequalities:

\[
\begin{align*}
  x_1 + x_2 + x_6 &\leq 2 \\
  x_3 + x_4 + x_5 + x_6 &\leq 3 \\
  x_1 + x_2 + x_5 + x_6 &\leq 3
\end{align*}
\]

The cover \( C_3 \) is not minimal, since \( C_1 \subset C_3 \) is also a cover.
Theorem 9.17 Let $C \subseteq N$ be a cover. Then, the cover inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is valid for $P_{\text{KNAPSACK}}(N, a, b)$. Moreover, if $C$ is minimal, then the cover inequality defines a facet of $P_{\text{KNAPSACK}}(C, a, b)$.

Proof: By Observation 2.2 it suffices to show that the inequality is valid for the knapsack set

$$X := \left\{ x \in \mathbb{B}^N : \sum_{j \in N} a_j x_j \leq b \right\}.$$ 

Suppose that $x \in X$ does not satisfy the cover inequality. We have that $x = \chi^S$ is the incidence vector of a set $S \subseteq N$. By assumption we have

$$|C| - 1 < \sum_{j \in C} x_j = \sum_{j \in C \cap S} x_j = |C \cap S|.$$ 

So $|C \cap S| = |C|$ and consequently $C \subseteq S$. Thus,

$$\sum_{j \in N} a_j x_j \geq \sum_{j \in C} a_j x_j = \sum_{j \in C \cap S} a_j = \sum_{j \in C} a_j > b,$$

which contradicts the fact that $x \in X$.

It remains to show that the cover inequality defines a facet of $P_{\text{KNAPSACK}}(C, a, b)$, if the cover is minimal. Suppose that there is a facet-defining inequality $c^T x \leq \delta$ such that

$$\left\{ x \in P_{\text{KNAPSACK}}(C, a, b) : \sum_{j \in C} x_j = |C| - 1 \right\} \subseteq F_c := \left\{ x \in P_{\text{KNAPSACK}}(C, a, b) : c^T x = \delta \right\}. \quad (9.46)$$

We will show that $c^T x \leq \delta$ is a nonnegative scalar multiple of the cover inequality.

For $i \in C$ consider the set $C_i := C \setminus \{i\}$. Since, $C$ is minimal, $C_i$ is not a cover. Consequently, each of the $|C|$ incidence vectors $\chi^{C_i} \in \mathbb{B}^C$ is contained in the set on the left hand side of (9.46), so $\chi^{C_i} \in F_c$ for $i \in C$. Thus, for $i \neq j$ we have

$$0 = c^T \chi^{C_i} - c^T \chi^C = c^T (\chi^{C_i} - \chi^C) = c_i - c_j.$$

Hence we have $c_i = \gamma$ for $i \in C$ and $c^T x \leq \delta$ is of the form

$$\gamma \sum_{j \in C} x_j \leq \delta. \quad (9.47)$$

Fix $i \in C$. Then, by (9.47) we have

$$c^T \chi^{C_i} = \gamma \sum_{j \in C_i} x_j = \delta$$

and by (9.46) we have

$$\sum_{j \in C_i} x_j = |C| - 1,$$

so $\delta = \gamma (|C| - 1)$ and $c^T x \leq \delta$ must be a nonnegative scalar multiple of the cover inequality. \qed
9.6 Structured Inequalities

The proof technique above is a general tool to show that an inequality defines a facet of a full-dimensional polyhedron:

**Observation 9.18 (Proof technique 1 for facets)** Suppose that $P = \text{conv}(X) \subseteq \mathbb{R}^n$ is a full dimensional polyhedron and $\pi^T x \leq \pi_0$ is a valid inequality for $P$. In order to show that $\pi^T x \leq \pi_0$ defines a facet of $P$, it suffices to accomplish the following steps:

(i) Select $t \geq n$ points $x^1, \ldots, x^t \in X$ with $\pi^T x^i = \pi_0$ and suppose that all these points lie on a generic hyperplane $c^T x = \delta$.

(ii) Solve the linear equation system

$$\sum_{j=1}^n c_j x^i_j = \delta \quad \text{for } i = 1, \ldots, t \quad (9.48)$$

in the $n+1$ unknowns $(c, \delta)$.

(iii) If the only solution of $(9.48)$ is $(c, \delta) = \gamma(\pi, \pi_0)$ for some $\gamma \neq 0$, then the inequality $\pi^T x \leq \pi_0$ defines a facet of $P$.

In the proof of Theorem 9.17 we were dealing with a polytope $P_{\text{KNAPSACK}}(C, a, b) \subseteq \mathbb{R}^C$ and we choose $|C|$ points $x^{C_i}$ ($i \in C$) satisfying the cover inequality with equality.

Let us return to the cover inequalities. We have shown that for a minimal cover $C$ the cover inequality $\sum_{i \in C} x_i \leq |C| - 1$ defines a facet of $P_{\text{KNAPSACK}}(C, a, b)$. However, this does not necessarily mean that the inequality is also facet-defining for $P_{\text{KNAPSACK}}(N, a, b)$. Observe that there is a simple way to strengthen the basic cover inequalities:

**Lemma 9.19** Let $C$ be a cover for $X = \left\{ x \in \mathbb{B}^N : \sum_{j \in N} a_j x_j \leq b \right\}$. We define the extended cover $E(C)$ by

$$E(C) := C \cup \left\{ j \in N : a_j \geq a_i \text{ for all } i \in C \right\}.$$

The extended cover inequality

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

is valid for $P_{\text{KNAPSACK}}(N, a, b)$.

**Proof:** Along the same lines as the validity of the cover inequality in Theorem 9.17. \qed

**Example 9.20 (Continued)**

In the knapsack set of Example 9.16 the extended cover inequality for $C = \{3, 4, 5, 6\}$ is

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3.$$

So, the cover inequality $x_3 + x_4 + x_5 + x_6 \leq 3$ is dominated by the extended cover inequality. On the other hand, the extended cover inequality in turn is dominated by the inequality $2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$, so it can not be facet-defining (cf. Theorem 5.48). \qed

We have just seen that even extended cover inequalities might not give us a facet of the knapsack polytope. Nevertheless, under some circumstances they are facet-defining:

**Theorem 9.21** Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $C = \{j_1, \ldots, j_r\}$ with $j_1 < j_2 < \cdots < j_r$ be a minimal cover. Suppose that at least one of the following conditions is satisfied:
(i) \( C = \mathbb{N} \)

(ii) \( E(C) = \mathbb{N} \) and \( (C \setminus \{j_1, j_2\}) \cup \{1\} \) is not a cover.

(iii) \( C = E(C) \) and \( (C \setminus \{j_1\}) \cup \{p\} \) is a cover, where \( p = \min \{j : j \in N \setminus E(C)\} \).

(iv) \( C \subseteq E(C) \subseteq \mathbb{N} \) and \( (C \setminus \{j_1, j_2\}) \cup \{1\} \) is a cover and \( (C \setminus \{j_1\}) \cup \{p\} \) is a cover, where \( p = \min \{j : j \in N \setminus E(C)\} \).

Then the extended cover inequality defines a facet.

**Proof:** We construct \( n \) affinely independent vectors in \( X \) that satisfy the extended cover inequality at equality. Then, it follows that the proper face induced by the inequality has dimension at least \( n - 1 \), which means that it constitutes a facet.

We use the incidence vectors of the following subsets of \( N \):

(i) the \( |C| \) sets \( C_i := C \setminus \{j_i\} \) for \( j_i \in C \).

(ii) the \( |E(C) \setminus C| \) sets \( C'_k := (C \setminus \{j_1, j_2\}) \cup \{k\} \) for \( k \in E(C) \setminus C \). Observe that \( |C'_k \cap E(C)| = |C| - 1 \) and that \( C'_k \) is not a cover by the assumptions of the theorem.

(iii) the \( |N \setminus E(C)| \) sets \( \tilde{C}_j := C \setminus \{j_1\} \cup \{j\} \) for \( j \in N \setminus E(C) \); again \( |E(C) \cap \tilde{C}_j| = |C| - 1 \) and \( \tilde{C}_j \) is not a cover by the assumptions of the theorem.

It is straightforward to verify that the \( n \) vectors constructed above are in fact affinely independent.

On the way to proving Theorem 9.21 we saw another technique to prove that an inequality is facet defining, which for obvious reasons is called the direct method:

**Observation 9.22 (Proof technique 2 for facets)** In order to show that \( \pi^T x \leq \pi_0 \) defines a facet of \( P \), it suffices to present \( \dim(P) - 1 \) affinely independent vectors in \( P \) that satisfy \( \pi^T x = \pi_0 \) at equality.

Usually we are in the situation that \( P = \text{conv}(X) \) and we will be able to exploit the combinatorial structure of \( X \). Let us diverge for a moment and illustrate this one more time for the matching polytope.

**Example 9.23**

Let \( G = (V, E) \) be an undirected graph and \( M(G) \) be the convex hull of the incidence vectors of all matchings of \( G \). Then, all vectors in \( M(G) \) satisfy the following inequalities:

\[
\begin{align*}
x(\delta(v)) & \leq 1 \quad \text{for all } v \in V \quad (9.49a) \\
x(\gamma(T)) & \leq \frac{|T| - 1}{2} \quad \text{for all } T \subseteq V, |T| \geq 3 \text{ odd} \quad (9.49b) \\
x_e & \geq 0 \quad \text{for all } e \in E \quad (9.49c)
\end{align*}
\]

Inequalities (9.49a) and (9.49b) are obvious and the inequalities (9.49b) have been shown to be valid in Example 9.21.

The polytope \( M(G) \) is clearly full-dimensional. Moreover, each inequality \( x_e \geq 0 \) defines a facet of \( M(G) \). To see this, take the \( |E| \) incidence vectors of the matchings \( \emptyset \) and \( \{e\} \) where \( e \in E \setminus \{e'\} \) which are clearly independent and satisfy the inequality at equality.
9.6 Structured Inequalities

9.6.2 Lifting of Cover Inequalities

We return from our short excursion to matchings to the extended cover inequalities. In Example 9.20 we saw that the extended cover inequality \( x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3 \) is dominated by the inequality \( 2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3 \). How could we possibly derive the latter inequality?

Consider the cover inequality for the cover \( C = \{3, 4, 5, 6\} \)

\[
x_3 + x_4 + x_5 + x_6 \leq 3
\]

which is valid for our knapsack set

\[
X = \{ x \in \mathbb{B}^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \}
\]

from Example 9.16. We may also say that the cover inequality is valid for the set

\[
X' := \{ x \in \mathbb{B}^4 : 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 \}
\]

which is formed by the variables in \( C \). Since the cover is minimal, by Theorem 9.17 the cover inequality defines a facet of \( \text{conv}(X') \), so it is as strong as possible. We would like to transfer the inequality and its strength to the higher dimensional set \( \text{conv}(X) \).

As a first step, let us determine the coefficients \( \beta_1 \) such that the inequality

\[
\beta_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3 \quad (9.50)
\]

is valid for

\[
X'' := \{ x \in \mathbb{B}^5 : 11x_1 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 \}
\]

In a second step, we will choose \( \beta_1 \) as large as possible, making the inequality as strong as possible.

For all \( x \in X'' \) with \( x_1 = 0 \), the inequality (9.50) is valid for all values of \( \beta_1 \). If \( x_1 = 1 \), then (9.50) is valid if and only if

\[
\beta_1 + x_3 + x_4 + x_5 + x_6 \leq 3
\]

is valid for all \( x \in \mathbb{B}^4 \) satisfying

\[
6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 11 = 8.
\]

Thus, (9.50) is valid if and only if

\[
\beta_1 + \max \{ x_3 + x_4 + x_5 + x_6 : x \in \mathbb{B}^4 \text{ and } 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8 \} \leq 3.
\]

This is equivalent to saying that \( \beta_1 \leq 3 - z_1 \), where

\[
z_1 = \max \{ x_3 + x_4 + x_5 + x_6 : x \in \mathbb{B}^4 \text{ and } 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8 \} \quad (9.51)
\]

The problem in (9.51) is itself a KNAPSACK problem. However, the objective function is particularly simple and in our example we can see easily that \( z_1 = 1 \) (we have \( z_1 \geq 1 \) since \( \{1, 0, 0, 0\} \) is feasible for the problem; on the other hand not two items fit into the knapsack of size 8). Thus, (9.50) is valid for all values \( \beta \leq 3 - 1 = 2 \). Setting \( \beta_1 = 2 \) gives the strongest inequality.

The technique that we have seen above is called lifting: we “lift” a lower-dimensional (facet-defining) inequality to a higher-dimensional polyhedron. The fact that this lifting is possible gives another justification for studying “local structures” in integer programs such as knapsack inequalities.
In our example we have lifted the cover inequality one dimension. In order to lift the inequality to the whole polyhedron, we need to solve a more general problem. Namely, we wish to find the best possible values $\beta_j$ for $j \in N \setminus C$ such that the inequality

$$\sum_{j \in N \setminus C} \beta_j x_j + \sum_{j \in C} x_j \leq |C| - 1$$

is valid for $X = \{ x \in B^N : \sum_{j \in N} a_j x_j \leq b \}$. The procedure in Algorithm 9.3 accomplishes this task.

**Algorithm 9.3** Algorithm to lift cover inequalities.

**Input:** The data $N$, $a$, $b$ for a knapsack set $X = \{ x \in B^N : \sum_{j \in N} a_j x_j \leq b \}$, a minimal cover $C$

**Output:** Values $\beta_j$ for $j \in N \setminus C$ such that

$$\sum_{j \in N \setminus C} \beta_j x_j + \sum_{j \in C} x_j \leq |C| - 1$$

is valid for $X$

1. Let $j_1, \ldots, j_r$ be an ordering of $N \setminus C$.
2. for $t = 1, \ldots, r$ do
3. The valid inequality

$$\sum_{i=1}^{t-1} \beta_{j_i} x_{j_i} + \sum_{j \in C} x_j \leq |C| - 1$$

has been obtained so far.
4. To calculate the largest value $\beta_j$ for which

$$\beta_{j_1} x_{j_1} + \sum_{i=1}^{t-1} \beta_{j_i} x_{j_i} + \sum_{j \in C} x_j \leq |C| - 1$$

is valid, solve the following KNAPSACK problem:

$$z_t = \max \sum_{i=1}^{t-1} \beta_{j_i} x_{j_i} + \sum_{j \in C} x_j$$

$$\sum_{i=1}^{t-1} a_{j_i} x_{j_i} + \sum_{j \in C} a_j x_j \leq b - a_{j_t}$$

$x \in B^{|C|+t-1}$

5. Set $\beta_{j_t} := |C| - 1 - z_t$.
6. end for

**Example 9.24 (Continued)**

We return to the knapsack set of Example 9.16 and 9.20. Take the minimal cover $C = \{3,4,5,6\}$

$$x_3 + x_4 + x_5 + x_6 \leq 3$$

and set $j_1 = 1$, $j_2 = 2$ and $j_3 = 7$. We have already calculated the value $\beta_1 = 2$. For $\beta_{j_2} = \beta_2$, the coefficient for $x_2$ in the lifted inequality we need to solve the following
Theorem 9.25

Derive facet-defining inequalities:

We prove that the lifting technique in a more general setting provides us with a tool to derive facet-defining inequalities:

**Theorem 9.25** Suppose that \( X \subseteq \mathbb{B}^n \) and let \( X^\delta = X \cap \{ x \in \mathbb{B}^n : x_1 = \delta \} \) for \( \delta \in \{0, 1\} \).

(i) Suppose that the inequality

\[
\sum_{j=2}^{n} \pi_j x_j \leq \pi_0 \tag{9.52}
\]

is valid for \( X^0 \). If \( X^1 = \emptyset \), then \( x_1 \leq 0 \) is valid for \( X \). If \( X^1 \neq \emptyset \), then the inequality

\[
\beta_1 x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0 \tag{9.53}
\]

is valid for \( X \) if \( \beta_1 \leq \pi_0 - z \), where

\[
z = \max \left\{ \sum_{i=2}^{n} \pi_i x_i : x \in X^1 \right\}.
\]

Moreover, if \( \beta_1 = \pi_0 - z \) and \( 9.52 \) defines a face of dimension \( k \) of \( \text{conv}(X^0) \), then the lifted inequality \( 9.53 \) defines a face of dimension \( k + 1 \) of \( \text{conv}(X) \). In particular, if \( 9.52 \) is facet-defining for \( \text{conv}(X^0) \), then \( 9.53 \) is facet-defining for \( \text{conv}(X) \).

(ii) Suppose that the inequality \( 9.53 \) is valid for \( X^1 \). If \( X^0 = \emptyset \), then \( x_1 \geq 1 \) is valid for \( X \). If \( X^0 \neq \emptyset \), then

\[
\gamma_1 x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0 + \gamma_1 \tag{9.54}
\]

is valid for \( X \) if \( \gamma_1 \geq z' - \pi_0 \), where

\[
z' = \max \left\{ \sum_{i=2}^{n} \pi_i x_i : x \in X^0 \right\}.
\]

Moreover, if \( \gamma_1 = \pi_0 - z' \) and \( 9.52 \) defines a face of dimension \( k \) of \( \text{conv}(X^1) \), then the lifted inequality \( 9.54 \) defines a face of dimension \( k + 1 \) of \( \text{conv}(X) \). In particular, if \( 9.52 \) is facet-defining for \( \text{conv}(X^1) \), then \( 9.54 \) is facet-defining for \( \text{conv}(X) \).

\[\text{\footnotesize 135}\]
Proof: We only prove the first part of the theorem. The second part can be proved along the same lines.

We first show that the lifted inequality (9.53) is valid for $X$ for all $\beta_1 \leq \pi_0 - z$. We have $X = X^0 \cup X^1$. If $x \in X^0$, then

$$\beta_1 x_1 + \sum_{j=2}^{n} \pi_j x_j = \beta_1 \leq \pi_0 - z.$$

If $x \in X^1$, then

$$\beta_1 x_1 + \sum_{j=2}^{n} \pi_j x_j = \beta_1 + \sum_{j=2}^{n} \pi_j x_j \leq \beta_1 + z \leq (\pi_0 - z) + z = \pi_0,$$

by definition of $z$. Thus, the validity follows.

If (9.52) defines a $k$-dimensional face of $\text{conv}(X^0)$, then there are $k + 1$ affinely independent vectors $x^1, l = 1, \ldots, k + 1$ that satisfy (9.52) at equality. Everyone of those vectors has $x^i_1 = 0$ and so also (9.53) at equality. Choose $x^* \in X^1$ such that $z = \sum_{j=2}^{n} \pi_j x^*_j$. If $\beta_1 = \pi_0 - z$, then $x^*$ satisfies (9.53) also at equality. Moreover, $x^*$ must be affinely independent from all the vectors $x^1$, since the first component of $x^*$ is 1 while all the vectors $x^1$ have first component 0. Thus, we have found $k + 2$ affinely independent vectors satisfying (9.53) at equality and it follows that the face induced has dimension $k + 1$. 

Theorem 9.25 can be used iteratively as in our lifting procedure (Algorithm 9.3): Given $N_1 \subset N = \{1, \ldots, n\}$ and an inequality $\sum_{j \in N_1} \pi_j x_j \leq \pi_0$ which is valid for

$$X \cap \{x \in \mathbb{B}^n : x_j = 0 \text{ for } j \in N \setminus N_1\}$$

we can lift one variable at a time to obtain a valid inequality

$$\sum_{j \in N \setminus N_1} \beta_j x_j + \sum_{j \in N_1} \pi_j x_j \leq \pi_0 \quad (9.55)$$

for $X$. The coefficients $\beta_j$ in (9.55) are independent of the order in which the variables are lifted. The corresponding lifting procedure is a straightforward generalization of our lifting procedure for the cover inequalities. From Theorem 9.25 we obtain the following corollary:

Corollary 9.26 Let $C$ be a minimal cover for $X = \{x \in \mathbb{B}^n : \sum_{j \in N} a_j x_j \leq b\}$. The lifting procedure in Algorithm 9.3 determines a facet-defining inequality for $\text{conv}(X)$.

9.6.3 The Set-Packing Polytope

Integer and mixed integer programs often contain inequalities that have all coefficients from $B = \{0,1\}$. In particular, many applications require logical inequalities of the form

$$\sum_{j \in N} x_j \leq 1$$

(packing constraint: at most one of the $j$s is chosen) or

$$\sum_{j \in N} x_j \geq 1$$

(covering constraint: at least one of the $j$s is picked). This motivates the study of packing, covering problems, cf. Example ??.

Definition 9.27 (Set-Packing Problem and Set Covering Problem)

Let $A \in \mathbb{B}^{m \times n}$ be an $m \times n$-matrix with entries from $B = \{0,1\}$ and $c \in \mathbb{R}^n$. The integer
problems

\[
\begin{align*}
\max \left\{ c^T x : Ax \leq 1, x \in B^n \right\} \\
\max \left\{ c^T x : Ax \geq 1, x \in B^n \right\} \\
\max \left\{ c^T x : Ax = 1, x \in B^n \right\}
\end{align*}
\]

are called the set-packing problem, the set-covering problem and the set-covering partitioning problem, respectively.

In this section we restrict ourselves to the set-packing problem and the set-packing polytope:

\[ P_{\text{PACKING}}(A) := \text{conv}\{x \in B^n : Ax \leq 1\}. \]

For the set-packing problem, there is a nice graph-theoretical interpretation of feasible solutions. Given the matrix \( A \), define an undirected graph \( G(A) \) as follows: the vertices of \( G(A) \) correspond to the columns of \( A \). There is an edge between \( i \) and \( j \) if there is a common nonzero entry in columns \( i \) and \( j \). The graph \( G(A) \) is called the conflict graph or intersection graph.

Obviously, each feasible binary vector for the set-packing problem corresponds to a stable set in \( G(A) \). Conversely, each stable set in \( G(A) \) gives a feasible solution for the set-packing problem. Thus, we have a one-to-one correspondence and it follows that

\[ P_{\text{PACKING}}(A) = \text{conv}\{x \in B^n : x_i + x_j \leq 1 \text{ for all } (i,j) \in G(A)\}. \]

In other words, \( P_{\text{PACKING}}(A) \) is the stable-set polytope \( \text{STAB}(G(A)) \) of \( G(A) \). If \( G \) is a graph, then incidence vectors of the \( n+1 \) sets \( \emptyset \) and \( \{v\} \), where \( v \in V \) are all affinely independent and contained in \( \text{STAB}(G) \) whence \( \text{STAB}(G) \) has full dimension.

We know from Theorem 4.14 that the node-edge incidence matrix of a bipartite graph is totally unimodular (see also Example 4.15). Thus, if \( G(A) \) is bipartite, then by the Theorem of Hoffmann and Kruskal (Corollary 4.12) we have that \( P_{\text{PACKING}}(A) \) is completely described by the linear system:

\[
\begin{align*}
x_i + x_j & \leq 1 \text{ for all } (i,j) \in G(A) \tag{9.56a} \\
x & \geq 0. \tag{9.56b}
\end{align*}
\]

We also know that for a general graph, the system (9.56) does not suffice to describe the convex hull of its stable sets, here \( P_{\text{PACKING}}(A) \). A graph is bipartite if and only if it does not contain an odd cycle (see Lemma 4.6). Odd cycles gives us new valid inequalities:

**Theorem 9.28** Let \( C \) be an odd cycle in \( G \). The odd-cycle inequality

\[
\sum_{i \in C} x_i \leq \frac{|C| - 1}{2}
\]

is valid for \( \text{STAB}(G) \). The above inequality defines a facet of \( \text{STAB}(V(C), E(C)) \) if and only if \( C \) is an odd hole, that is, a cycle without chords.

**Proof:** Any stable set \( x \) can contain at most every second vertex from \( C \), thus \( x(C) \leq (|C| - 1)/2 \) since \( |C| \) is odd. So, the odd-cycle inequality is valid for \( \text{STAB}(G) \).
Suppose the $C$ is an odd hole with $V(C) = \{0, 1, \ldots, k-1\}$, $k \in \mathbb{N}$ odd and let $c^T x \leq \delta$ be a facet-defining inequality with

$$F_C = \left\{ x \in \text{STAB}(V(C), E(C)) : \sum_{i \in C} x_i = \frac{|C|-1}{2} \right\} \subseteq F_c = \left\{ x \in \text{STAB}(V(C), E(C)) : c^T x = \delta \right\}$$

(a) The stable set $S_1 = \{i+2, i+4, \ldots, i-3, i\}$ in the odd cycle $C$.

(b) The stable set $S_2 = \{i+2, i+4, \ldots, i-3, i-1\}$ in the odd cycle $C$.

Figure 9.3: Construction of the stable sets $S_1$ and $S_2$ in the proof of Theorem 9.28. Here, node $i = 0$ is the anchor point of the stable sets. The stable sets are indicated by the black nodes.

Fix $i \in C$ and consider the two stable sets

$$S_1 = \{i+2, i+4, \ldots, i-3, i\}$$

$$S_2 = \{i+2, i+4, \ldots, i-3, i-1\}$$

where all indices are taken modulo $k$ (see Figure 9.3 for an illustration). Then, $\chi^{S_1} \in F_C \subseteq F_c$, so we have

$$0 = c^T \chi^{S_1} - c^T \chi^{S_2} = c^T (\chi^{S_1} - \chi^{S_2}) = c_i - c_{i-1}.$$

Since we can choose $i \in C$ arbitrarily, this implies that $c_i = \gamma$ for all $i \in C$ for some $\gamma \in \mathbb{R}$. As in the proof of Theorem 9.17 we can now conclude that $c^T x \leq \delta$ is a positive scalar multiple of the odd-hole inequality (observe that we used proof technique 1 for facets).

Finally, suppose that $C$ is a cycle with at least one chord. We can find an odd hole $H$ that is contained in $C$ (see Figure 9.4). There are $|C| - |H|$ vertices in $C \setminus H$, and we can find $\frac{|C| - |H|}{2}$ edges $(i_k, j_k) \in C$ where both endpoints are in $C \setminus H$. Consider the following valid inequalities:

$$\sum_{i \in H} x_i \leq \frac{|H|-1}{2}$$

$$x_{i_k} + x_{j_k} \leq 1 \text{ for } k = 1, \ldots, \frac{|C| - |H|}{2}.$$

Summing up those inequalities yields $\sum_{i \in C} x_i \leq \frac{|C|-1}{2}$, which is the odd-cycle inequality for $C$. Hence, $C$ is redundant and can not induce a facet of $\text{STAB}(V(C), E(C))$. This completes the proof.

The final class of inequalities we consider here are the so-called clique-inequalities:
Figure 9.4: If $C$ is an odd cycle with chords, then there is an odd hole $H$ contained in $C$ (indicated by the thick lines).

**Theorem 9.29** Let $Q$ be a clique in $G$. The **clique inequality**

$$\sum_{i \in Q} x_i \leq 1$$

is valid for $\text{STAB}(G)$. The above inequality defines a facet of $\text{STAB}(G)$ if and only if $Q$ is a maximal clique, that is, a clique which is maximal with respect to inclusion.

**Proof:** The validity of the inequality is immediate. Assume that $Q$ is maximal. We find $n$ affinely independent vectors that satisfy $x(Q) = 1$. For $v \in Q$, we take the incidence vector of $[v]$. For $u \notin Q$, we choose a node $v \in Q$ which is not adjacent to $u$. Such a node exists, since $Q$ is maximal. We add the incidence vector of $[u,v]$ to our set. In total we have $n$ vectors which satisfy $x(Q) \leq 1$ with equality. They are clearly affinely independent.

Assume conversely that $Q$ is not maximal. So, there is a clique $Q' \supset Q$, $Q' \neq Q$. The clique inequality $x(Q') \leq 1$ dominates $x(Q) \leq 1$, so $x(Q) \leq 1$ is not necessary in the description of $\text{STAB}(G)$ and $x(Q) \leq 1$ can not define a facet. \qed