Zero crossings, overshoot and initial undershoot in the step and impulse responses of linear systems

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Abstract—We consider functions in the time and the frequency domain, and show how the occurrence of real zeros in the frequency domain is related to the number of zero crossings, overshoot and initial undershoot in the time domain. This leads to simple proofs of known facts for linear time-invariant systems and their generalizations in various directions. In particular, we consider time-delay systems and differential systems of fractional order.

I. INTRODUCTION

For asymptotically stable linear time invariant single-input single-output systems

\[ \dot{x} = Ax + bu, \quad y = cx \]

with transfer function \( G(s) = c(sI - A)^{-1}b \) the following facts are well-known.

(i) The number of zero crossings of the step and impulse response is bounded from below by the number of positive zeros of the transfer function, see [1].

(ii) If \( G(s) - G(0) \) has a positive zero, then the step response exhibits overshoot, [2], [3].

(iii) If the transfer function has an odd number of positive zeros (counting multiplicities), then the step response exhibits initial undershoot, see [4].

Our goal is to establish these facts in a more abstract setting where we just consider signals in time- and frequency domain. We will then give examples of larger classes of control systems, to which the results can be applied. This leads to a number of generalizations. For instance, in (i) we can consider unstable systems, multiple zeros and transfer functions which are not strictly proper. We also discuss neutral delay equations with non-analytic step and impulse responses, as well as fractional order differential systems.

Thus we add some new facets to the rich literature on the step and impulse response. Our investigations were mainly driven by some questions raised in the survey article [5], which also contains many references relevant to our subject. In particular, there are close relations to [6], [7], where, like in our approach, the analysis is based on the properties of the Laplace transformation, and can be carried over to other classes of systems. Feedback-design methods to avoid overshooting and undershooting of the step response have been considered recently in [8] where also non-strictly proper systems are treated.

The structure of this paper is as follows: Section II is devoted to the basic definitions and auxiliary results. Our main contributions are presented in Section III for general functions and their Laplace transformations. In Section IV, these results are applied to different types of linear systems with and without delay terms.

II. PRELIMINARIES: ZEROS OF EXPONENTIAL POLYNOMIALS

In our main results, we relate properties of a function \( y \) to the number of positive zeros of its Laplace-transform \( Y \). Hence let us first specify the notion of a zero and its multiplicity.

Definition II.1. Let \( K = \mathbb{R} \) or \( K = \mathbb{C} \) and \( g : K \rightarrow K \). A number \( z_0 \in K \) is called a zero of \( g \), if \( g(z_0) = 0 \). If \( g(z_0) = 0 \), then we call \( z_0 \) a zero of multiplicity \( \mu \geq 1 \), if \( \mu \in \mathbb{N} \cup \{\infty\} \) is the largest number, so that \( g \) is at least \( \mu - 1 \) times differentiable in \( z_0 \) with derivatives \( g'(z_0) = \ldots = g^{(\mu-1)}(z_0) = 0 \). We call \( z_0 \in K \) a zero of multiplicity 0, if \( g(z_0) \neq 0 \).

Whenever we will speak of the number of zeros of a function, we will also count the multiplicities, even if we do not state it explicitly. Now, we recall some basic facts concerning exponential polynomials of the form

\[ f(t) = \sum_{j=1}^{\ell} \alpha_j e^{-s_j t}. \]

Here the \( \alpha_j \) are polynomials of given degree \( \deg \alpha_j \in \mathbb{N} \cup \{-1\} = \{-1, 0, 1, 2, \ldots\} \), where for consistency we have to set \( \deg \alpha_j = -1 \) if \( \alpha_j \equiv 0 \). In particular, \( \deg \alpha_j \geq 0 \) implies \( \alpha_j \neq 0 \).

Lemma II.2. [9, Assertions 75 and 76 in Ch. 5] Let \( n_1, \ldots, n_\ell \in \mathbb{N} \) with \( n = n_1 + \cdots + n_\ell \), and distinct real numbers \( s_1, \ldots, s_\ell \) be given.

(a) For \( j = 1, \ldots, \ell \) consider polynomials \( \alpha_j(t) \) with degrees \( \deg \alpha_j = n_j - 1 \). Then either \( n = 0 \), i.e. all \( \alpha_j \) vanish, or \( f \) defined by (1) has at most \( n - 1 \) real zeros.

(b) For \( n \) arbitrarily given data points \( (t_1, f_1) \in \mathbb{R}^2 \) satisfying \( t_1 < t_2 < \ldots < t_n \), there exist unique polynomials \( \alpha_j, j = 1, \ldots, \ell \) with \( \deg \alpha_j \leq n_j - 1 \) so that \( f \) defined by (1) interpolates \( f(t_j) = f_j \).

Actually, the properties (a) and (b) are equivalent and express the fact that the terms \( \alpha_j(t) e^{-s_j t} \) form a Chebyshev system, see [10]. For convenience, we repeat the simple proof.

Proof: (a) For \( \ell = 1 \) the assertion just concerns polynomials and is well-known (e.g. [11]). So, as an induction hypothesis, assume that it holds for \( \ell - 1 \) terms. Consider

\[ e^{s_j t} f(t) = \alpha_j(t) + \sum_{j=1}^{\ell-1} \alpha_j(t) e^{(s_j-s_{j+1}) t}, \]

which has the same zeros as \( f \). Since \( \frac{d^n}{dt^n} \alpha_j(t) = 0 \), the \( n_j \)-th derivative of (2) is

\[ \frac{d^n}{dt^n} \left( e^{s_j t} f(t) \right) = \sum_{j=1}^{\ell-1} \alpha_j(t) e^{(s_j-s_{j+1}) t}, \]

with polynomials \( \alpha_j \) of degrees \( n_j - 1, j = 1, \ldots, \ell - 1 \). By the induction hypothesis, (3) has at most \( n_1 + n_2 + \cdots + n_{\ell-1} - 1 = n - n_\ell - 1 \) zeros, and by Rolle’s theorem \( f \) has at most \( n - 1 \) zeros.

(b) We consider the \( n \)-dimensional subspace

\[ V = \text{span} \{ e^{-s_1 t}, \ldots, e^{-s_{\ell-1} t}, e^{-s_\ell t} \} \]

of the real vector space of continuous functions from \( \mathbb{R} \) to \( \mathbb{R} \) and the mapping \( \Phi : V \rightarrow \mathbb{R}^n \) defined by \( \Phi(f) = [f(t_1), \ldots, f(t_n)]^T \).

From (a) we know that \( \Phi(f) = 0 \) implies \( f = 0 \). Hence, as a linear mapping between \( n \)-dimensional spaces, \( \Phi \) is nonsingular (in particular surjective) which proves (b).

III. MAIN RESULTS

Throughout this section we make the following assumption, which will be strengthened successively.

Assumption III.1. Consider a piecewise continuous non-constant function \( y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \), where at each point \( t \geq 0 \) is at least left or
right continuous. Fix some $t_0 \geq 0$ with $y(t_0) \neq 0$. Let there exist an $s_0 \in \mathbb{R}$ so that the Laplace transform

$$Y(s) = \mathcal{L}y(s) = \int_0^\infty y(t)e^{-st} \, dt$$

is defined for all $s \in \mathbb{C}_{> s_0} = \{ s \in \mathbb{C} \mid \Re s > s_0 \}$.

Note that $Y$ is analytic on $\mathbb{C}_{> s_0}$ (see [12, Thm. 3.1]) and not identically equal to zero. Hence, its zeros are of finite multiplicity and isolated, i.e. they don’t have any accumulation points (cf. [13, Thm. 3.2.8]). Note further, that the function $y$ may have step discontinuities but no single exception points. In particular, $y(t_0) \neq 0$ implies the existence of $a,b \in \mathbb{R}_0$ with $a < b$ and $a \leq t_0 < b$, so that $y(t) \neq 0$ for all $t$ with $a < t < b$.

A. Zero Crossings

Loosely speaking, we define a zero crossing (compare [5]) as a place, where a function changes its sign. This can be a single point or a whole interval, if the given function is constantly equal to zero in the interior of this interval.

**Definition III.2.** Let $y: \mathbb{R}_0 \to \mathbb{R}$ be as in Assumption III.1. For $t_1, t_2, \in \mathbb{R}_0$ with $0 < t_1 < t_2$ consider the interval or the point $T = [t_1, t_2]$ and assume that $y(t) = 0$ for all $t$ with $t_1 < t < t_2$. We say that $y$ has a zero crossing in $T$, if there exists an $\epsilon > 0$ so that $y(t - \delta)y(t + \delta) < 0$ for all $\delta$ with $0 < \delta < \epsilon$.

**Theorem III.3.** Let $y$ and $Y$ be given as in Assumption III.1. If $Y$ possesses isolated real zeros $s_1, s_2, \ldots, s_n \in \mathbb{R}_0$ of multiplicities $n_1, \ldots, n_n$, respectively, with $n = n_1 + \cdots + n_n$ and $y$ possesses $n$ disjoint zero crossings $T_1, T_2, \ldots, T_m$, then $m \geq n$.

**Proof:** By the differentiation rule (e.g. [12, Thm. 2.7]) for all $k = 0, \ldots, n_j - 1$ we have

$$0 = \frac{d^k}{ds^k} Y(s) \bigg|_{s = s_j} = \mathcal{L}(t^k y(t)) (s_j) = \int_0^\infty y(t)t^k e^{-st} \, dt ,$$

which, by linear combination, implies

$$\int_0^\infty y(t)\alpha_j(t)e^{-st} \, dt = 0 \quad (4)$$

for all polynomials $\alpha_j$ with $\deg \alpha_j < n_j$. Assume now that $m < n$ and let $t_j \in T_j$ for $j = 1, \ldots, m$. By Lemma II.2 (b) we can choose polynomials $\alpha_j$, $j = 1, \ldots, \ell$ with $\deg \alpha_j = m_j - 1$ where $m_j \leq n_j$ and $\sum_{j=1}^\ell m_j = m + 1 \leq n$ so that

$$f(t) = \sum_{j=1}^\ell \alpha_j(t)e^{-\gamma_j t}$$

satisfies $f(t_k) = 0$ for $k = 1, \ldots, m$ and $f(t_0) = y(t_0) \neq 0$. By Lemma II.2 (a) the function $f(t)$ has exactly $m$ zeros, counting multiplicities. Since $t_k \neq t_0$ for $k \neq i$, this means $f(t) \neq 0$ for all $t \not\in \{ t_1, \ldots, t_m \}$ and $f'(t_0) \neq 0$ for $k = 1, \ldots, m$. (Note that $f'(t_k) = 0$ would increase the multiplicity of the zero $t_k$.)

Thus $y(t)f(t) \geq 0$ for all $t \geq 0$ and $y(t_0)f(t_0) > 0$. Together with (4) and the piecewise continuity of $t \mapsto y(t)f(t)$ we obtain

$$0 < \int_0^\infty y(t)f(t) \, dt = \int_0^\infty y(t) \sum_{j=1}^\ell \alpha_j(t)e^{-s_j t} \, dt = 0$$

which is a contradiction. Hence $m \geq n$.

B. Overshoot

Now we have to strengthen our assumption the first time.

**Assumption III.4.** Let $y$ and $Y$ be given as in Assumption III.1 with the following properties:

(a) The limit $y_\infty = \lim_{t \to \infty} y(t)$ exists and $y(0) \neq y_\infty$.

(b) $y$ is differentiable on $\mathbb{R}_0$ and the derivative is piecewise continuous.

Together with $Y(s)$ we consider the function $G(s) = sY(s)$.

Note that (a) implies $s_0 = 0$ and by the terminal value theorem (e.g. [12, Thm. 2.36]), we have $y_\infty = \lim_{s \to 0} G(s) =: G(0)$.

**Definition III.5.** We say that $y$ exhibits overshoot, if for some $t > 0$ the signs of $y_\infty - y(0)$ and $y_\infty - y(t)$ are different, that is $(y_\infty - y(0))(y_\infty - y(t)) < 0$. To quantify the overshooting behaviour further, we say that $y$ has $n$ crossings of the limit value if $y_\infty - y(t)$ has $n$ zero crossings on $\mathbb{R}_0 > 0$ (see Figure 3).

As a consequence of Theorem III.3 we have the following sufficient criteria.

**Theorem III.6.** Let $y$ and $Y$ be given as in Assumption III.4.

(i) If $G(s) - G(0)$ has $n$ zeros in $\mathbb{R}_0 \setminus \{ 0 \}$, then $y$ has at least $n$ crossings of the limit value.

(ii) Assume that there exists an $s_0 \leq 0$ so that the Laplace transform $\mathcal{L}(y - y_\infty)(s)$ is defined for all $s \in \mathbb{C}_{> s_0}$. If $G'(0) = G''(0) = \ldots = G^{(n)}(0) = 0$ with $n_0 \geq 0$ and $G(s) - G(0)$ has $n_1$ zeros in $\mathbb{R}_{> s_0} \setminus \{ 0 \}$ then $y$ has at least $n = n_0 + n_1$ crossings of the limit value.

**Proof:** (i) We have $G(s) - G(0) = s\mathcal{L}(y - y_\infty)(s)$.

(ii) Let $F(s) = \frac{G(s) - G(0)}{s} = \mathcal{L}(y - y_\infty)(s)$ for $s \in \mathbb{C}_{> s_0}$. Then $F^{(k-1)}(0) = \frac{1}{k!} G^{(k)}(0)$ for $k = 1, \ldots, n_0$. Thus $\mathcal{L}(y - y_\infty)(s)$ has $n$ zeros in $\mathbb{R}_{> s_0}$.

By Theorem III.3, in both cases $y - y_\infty$ has $n$ zero crossings.

C. Initial undershoot

We strengthen our assumptions again.

**Assumption III.7.** Let $y$ and $Y$ be given as in Assumption III.4 and let

$$t_\tau = \inf \{ t > 0 \mid y(t) \neq y(0) \} .$$

Furthermore for a given number $\mu \in \mathbb{N}$, $\mu \geq 1$ assume the following:

(c) $y$ is at least $\mu + 1$ times differentiable on $\mathbb{R}_0$, and $y^{(\mu+1)}$ is piecewise continuous,

(d) $y^{(\mu)}(t_\tau) = 0$ for $k = 1, 2, \ldots, \mu - 1$ and $y^{(\mu)}(t_\tau) \neq 0$.

The following definition is a variation of the one given in [4].

**Definition III.8.** We say that $y$ exhibits initial undershoot, if $\sgn y^{(\mu)}(t_\tau) \neq \sgn(y_\infty - y(0))$.

Note that $y^{(\mu)}$ is the derivative of lowest-order, which is non-zero in $t_\tau$. Graphically speaking, $t_\tau$ denotes the first moment in time at which $y$ starts to move away from $y(0)$, and initial undershoot means that this movement takes place in the opposite direction of the limit value of $y$.

**Theorem III.9.** Let $y$ and $Y$ be given as in Assumption III.7. The function $y$ has initial undershoot if and only if $G(s) - y(0)$ has an odd number of positive zeros (counting multiplicities).

**Proof:** Let $\tilde{y}(t) = y(t - t_\tau) - y(t_\tau) = y(t + t_\tau) - y(0)$ and (see e.g. [12, Thm. 1.31])

$$\tilde{Y}(s) = \mathcal{L}(\tilde{y})(s) = e^{t_\tau s} \mathcal{L}(y)(s) - e^{t_\tau s} G(s) = y(0) .$$
Obviously $\tilde{Y}$ and $G(s)-y(0)$ have the same positive zeros. Moreover, $\tilde{y}(0) = 0$ and $\tilde{y}^{(k)}(0) = y^{(k)}(t_r)$ for all $k = 1, \ldots, \mu$. Since $y_\infty - y(0) = \tilde{Y}(0)$, the function $y$ has initial undershoot, if and only if
\[
\text{sgn } y^{(\mu)}(0) = -\text{sgn } \tilde{Y}(0) .
\] (5)

By the differentiation theorem and the initial value theorem [12, Thm. 2.34] we get
\[
\tilde{y}^{(\mu)}(0) = \lim_{r \to \infty} L\tilde{y}^{(\mu)}(r) = \lim_{r \to \infty} r^\mu \tilde{Y}(r) .
\]

In particular, since $\tilde{y}^{(\mu)}(0) \neq 0$, there exists $r_0 \geq 0$, so that $\tilde{Y}(r)$ has constant sign for all $r \geq r_0$. Thus the number $m$ of positive zeros (counting multiplicities) of $\tilde{Y}$ is finite. As every zero of odd multiplicity leads to a sign change, we have $\text{sgn } \tilde{y}^{(\mu)}(0) = (-1)^m \text{sgn } \tilde{Y}(0)$. Hence (5) holds if and only if $m$ is odd.

IV. APPLICATIONS: STEP AND IMPULSE RESPONSE OF LINEAR SYSTEMS

Our results can easily be applied to linear single-input single-output control systems. We will consider different classes of state-space systems (to be specified below) with a proper transfer function $G$ which is analytic on some right half plane $\mathbb{C}_{> s_0}$, where $s_0 \in \mathbb{R}$. In the time-domain, the step response $y_{\text{step}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is the output obtained for zero initial conditions and the step input $u_{\text{step}} \equiv 1$ for $t \geq 0$. The impulse response $y_{\text{impulse}} : \mathbb{R}_{> 0} \to \mathbb{R}$ is the output corresponding to zero initial data and Dirac-input $u_{\text{impulse}} = \delta$. The Laplace-transform of the step response is $G(s)$ and the Laplace-transform of the impulse response is $G(s)$. We will discuss the impulse response only for systems without direct feedthrough terms, i.e. for strictly proper transfer functions, so that $y_{\text{impulse}}$ does not contain impulses.

A. Finite-dimensional linear time-invariant systems

Let us first consider a linear time-invariant systems of the form
\[
\dot{x} = Ax + bu , \quad y = cx + du
\] (6)

with $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$, $c \in \mathbb{R}^{1 \times n}$, $d \in \mathbb{R}^{1 \times 1}$, zero initial data $x(0) = 0$, and transfer function
\[
G(s) = c(sI - A)^{-1}b + d .
\]

The following theorems are immediate consequences of our results in Section III.

Theorem IV.1. (i) Assume that $G$ has $n$ zeros in $\mathbb{R}_{> s_0}$. Assume further that $n_k$ of these zeros are positive, and 0 is a zero of multiplicity $n_0 \geq 0$. If $n_0 < 2$, then $y_{\text{step}}$ has at least $n_0 + 1$ zero crossings. If $n_0 \geq 2$, then $y_{\text{step}}$ has at least $n_0 - 1$ zero crossings. Let now the stability assumption $s_0 < 0$ hold.

(ii) Assume that $G(s) - G(0)$ has $n_+ = n_0$ positive zeros, and $G'(0) = G''(0) = \cdots = G^{(n_0)}(0) = 0$ with $n_0 \geq 0$. Then $y_{\text{step}}$ has at least $n_0 - n_+ + n_0$ crossings of the limit value.

(iii) $y_{\text{step}}$ has initial undershoot if and only if $G(s) - y_{\text{step}}(0)$ has an odd number of positive zeros.

Proof: (i) We have $\frac{1}{L}G(s) = L(y_{\text{step}}(s))$. If $G$ has a zero of order $n_0 \geq 2$ in 0 then $L(y_{\text{step}}(s))$ is defined on $\mathbb{C}_{> s_0}$ and has a zero of order $n_0 - 1$ in 0. All other zeros of $G$ and $L(y_{\text{step}}(s))$ coincide. If $n_0 < 2$, then $L(y_{\text{step}}(s))$ is only defined on $\mathbb{C}_{> s_0}$ and has $n_0$ positive zeros. Thus the assertion follows from Theorem III.3. Note that the limit $y_{\text{step}}(\infty) = G(0)$ exists, if $s_0 < 0$. Hence, assertions (ii) and (iii) follow from Theorems III.6 and III.9.

Theorem IV.2. Let $G$ be strictly proper and assume that $G$ has $n$ zeros in $\mathbb{R}_{> s_0}$. Then $y_{\text{impulse}}$ has at least $n$ zero crossings.

Proof: Since $L(y_{\text{impulse}}) = G$ this follows from Theorem III.3. ■

Remark IV.3. Assertion (i) of Theorem IV.1 was stated in [5] as an open conjecture. A proof for asymptotically stable systems without explicit consideration of multiple zeros or a direct feedthrough term had already been given in [1]. Since $y_{\text{step}}$ is analytic, all zero crossings can be considered as points.

B. Linear time-delay systems

Now consider the linear single-input single-output time-delay system with zero initial data

\[
\sum_{i=0}^{M} E_i \dot{x}(t - \tau_i) = \sum_{i=0}^{M} A_i x(t - \tau_i) + \sum_{i=0}^{M} b_i u(t - \tau_i) ,
\]

\[
y(t) = \sum_{i=0}^{M} c_i x(t - \tau_i) + \sum_{i=0}^{M} d_i u(t - \tau_i) ,
\]

\[
x(\theta) = 0 \quad \text{for } \theta \in [-\tau_0, 0] .
\] (7)

where $E_i, A_i \in \mathbb{R}^{n \times n}$, $E_0 = I$, $b_i \in \mathbb{R}^{n \times 1}$, $c_i \in \mathbb{R}^{1 \times n}$, $d_i \in \mathbb{R}$ for $i = 0, \ldots, M$. Without loss of generality let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_M$. For the theoretical background see e.g. [14], [15].

Systems with delays in the derivatives, i.e. $E_i \neq 0$ for at least one $i > 0$, are known as neutral time-delay systems. In the case that $E_i = 0$ for all $i = 1, \ldots, M$, the system is called a retarded time-delay system.

The fundamental solution of system (7) is the unique matrix function $K : \mathbb{R}_{> 0} \to \mathbb{R}^{n \times n}$, which solves the matrix delay differential equation

\[
\sum_{i=0}^{M} E_i \dot{K}(t - \tau_i) = \sum_{i=0}^{M} A_i K(t - \tau_i) .
\]

\[
K(0) = I , \quad K(\theta) = 0 \quad \text{for } \theta \in [-\tau_0, 0] .
\] (8)

If there exist numbers $\gamma, \alpha > 0$, so that $\|K(t)\| \leq \gamma e^{-\alpha t}$ for all $t \geq 0$, then system (7) is called exponentially stable.

The fundamental solution of (8) is continuous on $\mathbb{R}_{> 0}$. In case of a neutral time-delay system the derivative $\dot{K}(t)$ of the fundamental solution can have a countable number of finite jump discontinuities. In contrast, the derivative of the solution of a retarded time-delay system can only have a finite number of finite jump discontinuities. This is due to the smoothening property of the solution of a retarded time-delay system, for details see [14].

Let
\[
v(t) := \sum_{i=0}^{M} b_i u(t - \tau_i) \quad \text{for } t \geq 0 \quad \text{and } v(t) = 0 \quad \text{for } t < 0 .
\]

The general solution of the inhomogeneous differential equation in (7) for $t \geq 0$ can be written as
\[
x(t) = (K * v)(t) := \int_{-\infty}^{t} K(t - s)v(s) ds ,
\] (9)

where $*$ denotes the convolution operation.

If $v(t)$ is continuous except for finite jump discontinuities, then $x(t)$ is continuous on $\mathbb{R}_{\geq 0}$, see [14]. In particular, the solution $x_{\text{step}}$ to the step input $u_{\text{step}}$ is continuous for $t \geq 0$. The solution $x_{\text{impulse}}$ to the Dirac input $u_{\text{impulse}}$ is only piecewise continuous, since it takes the form
\[
x_{\text{impulse}}(t) = (K * v)(t) := \sum_{i=0}^{M} K(t - \tau_i)b_i .
\]
The frequency domain description of system (7) is obtained by Laplace transformation. This leads to the transfer function

\[ G(s) = \sum_{i=0}^{M} d_i e^{-s\tau_i} + \left( \sum_{i=0}^{M} c_i e^{-s\tau_i} \right) \left( \sum_{i=0}^{M} s E_i e^{-s\tau_i} - \sum_{i=0}^{M} A_i e^{-s\tau_i} \right)^{-1} \left( \sum_{i=0}^{M} b_i e^{-s\tau_i} \right). \]  

We call \( G \) strictly proper, if \( d_i = 0 \) for all \( i \).

**Theorem IV.4.** Theorems IV.1 and IV.2 carry over to this new setting.

*Proof:* Since \( y_{\text{step}} \) and \( y_{\text{impulse}} \) are piecewise continuous on \( \mathbb{R}_{\geq 0} \), the assertions (i) and (ii) are immediately clear. To prove (iii) we use the same procedure as in the proof of Theorem III.9. But since \( y^{(\mu+1)} \) is not necessarily piecewise continuous, we can not use the initial value theorem [12, Thm. 2.34]. To prove that

\[ y^{(\mu)}(0) = \lim_{s \to \infty} s^{\mu} G(s) \]

we use the fact that for \( t \in [0, \tau] \) the step response \( y(t) \) of system (7) coincides with the step response \( \hat{y}(t) \) of the system

\[ \dot{x}(t) = A_0 x(t) + b_0 u(t), \quad \hat{y}(t) = c_0 \hat{x}(t) + d_0 u(t). \]  

Now, we can apply the initial value theorem [12, Thm. 2.34] to \( \hat{G} = \mathcal{L} \hat{y} \), to get

\[ \hat{y}^{(\mu)}(0) = \lim_{s \to \infty} s^{\mu} \hat{G}(s). \]  

By (10) and the corresponding form of the transfer function \( \hat{G}(s) \) of (11), we get

\[ \lim_{s \to \infty} s^{\mu} \hat{G}(s) = \lim_{s \to \infty} s^{\mu} \hat{G}(s). \]

Here we have exploited that for \( s \to \infty \) in (10) all terms \( e^{s\tau_i} \) with \( i > 0 \) tend to zero. From (12) and (13) we now have

\[ y^{(\mu)}(0) = \lim_{t \to 0} y^{(\mu)}(t) = \lim_{t \to 0} \hat{y}^{(\mu)}(t) = \lim_{s \to \infty} s^{\mu} \hat{G}(s) \]

All other arguments are the same as in the proof of Theorem III.9.

**Remark IV.5.** In our approach, the step from systems without delays to systems with delays is immediate. In contrast, the techniques used e.g. in [1] at least would need some nontrivial modifications to cover this case, since they rely on differentiability of the impulse response. Note also that a zero crossing does not have to be a single point anymore and we do not require our system to be exponentially stable or the transfer function to be strictly proper.

**Example IV.6.** (a) Consider the system

\[ \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t-1) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \]

\[ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \]

with transfer function \( G_1(s) = \frac{s - e^{-s\tau}}{s^2}. \) Since \( G_1 \) has one real positive zero \( s_1 \), both the step response and the impulse response have at least one zero crossing, see Figure 1.

(b) Consider the system

\[ \dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} u(t), \]

\[ y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t). \]

Let us now consider systems with fractional derivatives of the form

\[ \sum_{i=0}^{m} p_i D^\alpha_i y(t) = \sum_{i=0}^{m} q_i D^\beta_i u(t) \]  

where \( \alpha_r > \alpha_{r-1} > \cdots > \alpha_1 \geq 0 \) and \( \beta_m > \beta_{m-1} > \cdots > \beta_1 \geq 0 \) are real numbers. Interpreting this equation in the Riemann-Liouville (or Grünwald-Letnikov) sense and applying the Laplace transform, we obtain the transfer function

\[ G(s) = \frac{Q(s)}{P(s)} = \sum_{i=0}^{m} q_i s^{\beta_i}. \]

Note that \( P \) and \( Q \) are well-defined on the positive real axis and can be extended analytically to \( \mathbb{C} \setminus \{0\} \). We assume that \( p_r \neq 0 \) and \( \alpha_r > \beta_m \) so that \( G(s) \) is strictly proper, and that the system is asymptotically stable, i.e. \( P \) has no zeros in \( \mathbb{C}_{\geq 0} \), see [16], [17].

Both, the corresponding step and impulse response can be represented as a sum of Mittag-Leffler functions and thus is analytic on \( \mathbb{R}_{>0} \) (see [18], [17], [19]).

Hence Theorems IV.1 and IV.2 also hold for system (14). In particular, Theorem IV.1 extends [19, Theorem 1], which states that the step response exhibits overshoot, if \( G'(0) = 0 \).

To illustrate this, we consider the fractional transfer functions

\[ G_3(s) = \frac{s^2 + 1}{s^3 + 3s^2 + 3} \quad \text{and} \quad G_4(s) = \frac{-s^2 + s^2 + 1}{s^3 + 3s^2 + 3}. \]
Since $G_{3/4}$ both are strictly proper, $y_{\text{step}}(0) = 0$ in both cases. Moreover $y_{\text{osc}} = G_{3/4}(0) = \frac{1}{3}$ and $G'_{3/4}(0) = G''_{3/4}(0) = 0$. Obviously $G_{1}$ does not have a positive zero, while $G_{4}$ has exactly one. Consequently, both step responses (computed using [20]) have multiple crossings of the limit value, but only the second exhibits initial undershoot, see Figure 3.

![Step response](image)

Fig. 3. Multiple crossings of the limit value for $G_{3}$ (left) and $G_{4}$ (right); initial undershoot only for $G_{4}$

**Remark IV.7.** Again, the step from integer order systems to fractional order systems was immediate in our approach. But note that the technique used e.g. in [4] would not apply directly, since it is based on the factorization of polynomials.

It is also straightforward now to combine our results and to consider fractional order systems with time delay, but we omit further details.

## V. Conclusion

Our investigations show that many relations between properties of the transfer function and the step and impulse response of a linear time-invariant system can be viewed more generally as relations between a function and its Laplace-transform. This leads to alternative proofs of known results (compared e.g. to [1], [4]), which can easily be transferred to other situations, including unstable systems, multiple zeros, direct feedthrough terms, systems with time delay and fractional systems. While some of these generalizations could also be obtained by adapting previous approaches, the lack of analyticity in the system responses of time-delay systems is not so obvious to overcome. It requires, amongst others, a new definition of zero crossings.

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**References**


