HIGHER ORDER MIXED-MOMENT APPROXIMATIONS FOR THE FOKKER–PLANCK EQUATION IN ONE SPACE DIMENSION

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Abstract. We study mixed-moment models (full zeroth moment, half higher moments) for a Fokker–Planck equation in one space dimension. Mixed-moment minimum-entropy models are known to overcome the zero net-flux problem of full-moment minimum-entropy \( M_n \) models. A realizability theory for these mixed moments of arbitrary order is derived, as well as a new closure, which we refer to as Kershaw closure. They provide nonnegative distribution functions combined with an analytical closure. Numerical tests are performed with standard first-order finite volume schemes and compared with a finite difference Fokker–Planck scheme.

Key words. partial differential equations, Fokker–Planck equations, methods of moments, minimum entropy, partial moments, hyperbolic partial differential equation

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1. Introduction. In recent years many approaches have been proposed for the solution of time-dependent kinetic transport equations—for example in electron radiation therapy or radiative heat transfer problems. Since transport equations are very high-dimensional, one tries to approximate them, for example, by so-called moment models. Typical representatives are polynomial \( P_n \) methods [26, 15, 6] and their simplifications, the \( \text{SP}_n \) [23] methods. These models are computationally inexpensive since they form an analytically closed system of hyperbolic differential equations. However, they suffer from severe drawbacks: The \( P_n \) methods are generated by closing the balance equations with a distribution function which is a polynomial in the angular variable. This implies that this distribution function might have negative values resulting in nonphysical values like a negative density. Additionally, in many cases a very high number of moments is needed for a reasonable approximation of the transport solution. This is particularly true in beam cases, where the exact transport solution forms a Dirac delta.

The entropy minimization \( M_n \) models [35, 12, 5, 37, 1] are expected to overcome this problem since the moment equations are always closed with a positive distribution. In many situations these models perform very well, but they can produce unphysical steady-state shocks due to a zero net-flux problem. It has been shown by Hauck that these shocks exist for every odd order [25]. Perturbation theory has been applied to the \( M_1 \) closure which, in some cases, removes the drawback of shocks [20]. To improve this situation, half- or partial-moment models were introduced in [13, 19]. These models work especially well in one space dimension, because they capture the...
potential discontinuity of the probability density in the angular variable which in one
dimension is well located. If, however, a Fokker–Planck operator is used instead of the
standard integral scattering operator, so that scattering is extremely forward-peaked
[38], these half-moment approximations do not work for reasons we will explain below.

To improve this situation a new model with mixed moments was proposed in [21]
which is able to avoid this problem. Instead of choosing full or half moments, a mixed-
moment approximation is used. Contrary to a typical half-moment approximation,
the lowest order moment (density) is kept as a full moment while all higher moments
are half moments. This ensures the continuity of the underlying distribution function
while maintaining the flexibility of the half-moment approach.

The fact that this modification of the moment model gives reasonable results
encourages us to study the general case for an arbitrary number of moments. A
problem that was left open in the previous work [21] was the question of realizability
of these mixed moments. We will define realizability precisely below.

The first part of this paper deals with the investigation of necessary and sufficient
realizability conditions for the mixed-moment problem of arbitrary order. We note
that, in contrast to the full-moment problem, the mixed-moment problem has not
been discussed in the literature before now.

In the second part of the paper, higher-order mixed-moment closures of the bal-
ance equations are derived, investigated, and compared to each other, continuing the
work done in [21], where mixed moments of order one were introduced. These clos-
ures, which we call Kershaw closures because they are inspired by ideas outlined in
[27], give an explicit, analytical closure for the moment system and are therefore very
efficient for higher orders since no nonlinear system has to be solved. The paper is
organized as follows.

In section 2, we give an overview of the basic Fokker–Planck equation and the
different moment models which can be derived from it. In particular, half-moment
approximations and mixed half-full moment approximations are discussed. Section
4 contains the necessary and sufficient conditions for realizability of mixed moments
of arbitrary order. Several higher order mixed-moment closures of the balance equa-
tions including polynomial, minimum-entropy, and Kershaw closures are developed in
section 5. The resulting equations are numerically compared with each other and the
Fokker–Planck equation in section 6. We conclude with a discussion of the results
and present open problems for future work in section 7.

2. Moment approximations and realizability. We consider the one-
dimensional Fokker–Planck equation for $t > 0$, $x \in \mathbb{R}$, $\mu \in [-1, 1]$,

$$
\partial_t \psi(t, x, \mu) + \mu \partial_x \psi(t, x, \mu) + \sigma_a(t, x) \psi(t, x, \mu) = \frac{T(t, x)}{2} \Delta_\mu \psi(t, x, \mu) + Q(t, x, \mu),
$$

(2.1)

where $\sigma_a$ is the absorption coefficient, $T$ is the so-called transport coefficient, $Q$ is a
source, and $\Delta_\mu$ is given by the one-dimensional projection of the Laplace–Beltrami
operator on the unit sphere

$$
\Delta_\mu \psi = \frac{\partial}{\partial \mu} \left( (1 - \mu^2) \frac{\partial \psi}{\partial \mu} \right).
$$

(2.2)

Equation (2.1) can be derived from the standard linear transport equation in the
limit of forward-peaked anisotropic scattering [38]. The transport coefficient is the
difference of the zeroth and the first moment of the scattering kernel.
We define the moments of the probability density $\psi$ as

$$
\psi^{(j)} = \int_{-1}^{1} \mu^j d\mu, \quad j \geq 0.
$$

We multiply (2.1) by 1 and $\mu$, respectively, integrate over $[-1, 1]$, and obtain

$$
\partial_t \psi^{(0)} + \partial_x \psi^{(1)} + \sigma_a \psi^{(0)} = Q^{(0)},
$$

(2.3)

$$
\partial_t \psi^{(1)} + \partial_x \psi^{(2)} + \sigma_a \psi^{(1)} = -\frac{T}{2} \psi^{(1)} + Q^{(1)}.
$$

(2.4)

This is a system of moment equations, which could in principle be continued. If, however, we want to truncate this system at a finite order, we see that the system is underdetermined (two equations for the three unknowns $\psi^{(0)}$, $\psi^{(1)}$, $\psi^{(2)}$). Thus we need to devise an additional relation which is called moment closure. Typically, this is done by assuming a specific form for the density, known as an ansatz, which depends on the known moments. For instance, we could assume that $\psi$ is a linear function in $\mu$. It matches the known moments $\psi^{(0)}$ and $\psi^{(1)}$ if

$$
\psi_{P_1}(\mu) = \frac{1}{2} \psi^{(0)} + \frac{3}{2} \psi^{(1)} \mu.
$$

(2.5)

This distribution can be integrated to obtain $\psi^{(2)}$ in terms of $\psi^{(0)}$ and $\psi^{(1)}$.

When dealing with closures, it is always crucial to ask which moments can be realized by a physical (that is, nonnegative) distribution. In the case of two moments, we have as a necessary condition that

$$
\psi^{(0)} = \int_{-1}^{1} \psi d\mu \geq 0 \quad \text{and} \quad |\psi^{(1)}| = \left| \int_{-1}^{1} \mu \psi d\mu \right| \leq \psi^{(0)}.
$$

(2.6)

Here these conditions are also sufficient; that is, for each pair of moments $(\psi^{(0)}, \psi^{(1)})$ that satisfies (2.6) one can find a nonnegative density that reproduces these moments. This is the realizability problem mentioned in the introduction. (Note that realizability of course does not guarantee the positivity of any representing distribution: the linear closure (2.5) can become negative even when $\psi^{(0)}$ and $\psi^{(1)}$ satisfy (2.6).)

This paragraph follows the definitions and results from [9]. For more details on the original truncated Hausdorff moment problem, see, e.g., [10, 8, 28].

**Definition 2.1.** Given a vector of real numbers $(\psi^{(0)}, \psi^{(1)}, \ldots, \psi^{(n)})$ and an interval $[a, b] \subset \mathbb{R}$, the truncated Hausdorff moment problem entails finding a positive Borel measure $\lambda$ on $\mathbb{R}$ such that

$$
\int \mu^j d\lambda(\mu) = \psi^{(j)}, \quad 0 \leq j \leq n,
$$

(2.7)

and $\text{supp} \lambda \subseteq [a, b]$.

We also define the realizability domain $\mathcal{R}_n$ as the set of vectors $(\psi^{(0)}, \psi^{(1)}, \ldots, \psi^{(n)})$ in $\mathbb{R}^{n+1}$ such that the truncated moment problem (2.7) has a solution for $[a, b] = [-1, 1]$.

**Remark 2.2.** From now on we assume that $\lambda$ has a distribution so that the truncated moment problem can be translated as follows: Find a nonnegative distribution $\psi(\mu) \geq 0$ such that

$$
\int_{a}^{b} \mu^j \psi(\mu) \, d\mu = \psi^{(j)}, \quad 0 \leq j \leq n.
$$
Such a density $\psi$ is said to be a representing distribution of the moment vector $(\psi(0), \psi(1), \ldots, \psi(n))$.

We allow (at least formally) the case when $\psi$ includes a finite linear combination of Dirac $\delta$-functions. Distributions $\psi$ which consist only of a linear combination of Dirac $\delta$-functions are called atomic.

**Definition 2.3.** In the following, the inequality $A \geq 0$ indicates that a matrix $A$ is positive semidefinite. For Hermitian matrices $A, B \in \mathbb{R}^{n \times n}$ we define the partial ordering “$\geq$” by $A \geq B$ if $A - B \geq 0$, that is, iff $A - B$ is positive semidefinite.

The main result is the following theorem.

**Theorem 2.4.** Define the Hankel matrices

$$A(k) := \left(\psi(i+j)\right)_{i,j=0}^k, \quad B(k) := \left(\psi(i+j+1)\right)_{i,j=0}^k, \quad C(k) := \left(\psi(i+j)\right)_{i,j=1}^k.$$

Then the truncated Hausdorff moment problem has a solution iff

- for $n = 2k + 1$, $bA(k) \geq B(k) \geq aA(k)$;
- for $n = 2k$, $A(k) \geq 0$ and
  
  $$(a + b)B(k - 1) \geq abA(k - 1) + C(k).$$

**Proof.** See [9] for a detailed proof of this theorem. □

The following remark will be important later.

**Remark 2.5.** In [9] the authors showed that, starting from the previous theorem, there exists a minimal atomic representing distribution $\psi$ (in the sense that it contains the fewest possible number of Dirac $\delta$ functions while still representing the moments) and that one can directly find this distribution with the help of its generating function. This is the consequence of a recursiveness property of the Hankel matrices $A(k)$ and $B(k)$.

Let $\Phi = (\phi_0, \ldots, \phi_{r-1}) := A(r - 1)^{-1}v(r, r - 1)$, where $v(i, j) = \left(\psi(i+j)\right)_{j=0}^i$ and $r$ is the smallest integer such that $A(r - 1)$ is nonsingular;¹ the generating function is defined by

$$g_\phi(\mu) = \mu^r - \sum_{i=0}^{r-1} \phi_i \mu^i.$$

The roots of this polynomial give the atoms $\mu_i$ of the distribution $\psi$, whereas the densities are calculated afterward from the Vandermonde system

$$\rho_0 \mu_0^i + \cdots + \rho_{r-1} \mu_{r-1}^i = v(i, 0), \quad i = 0, \ldots, r - 1.$$

Furthermore, when the moment vector $(\psi(0), \psi(1), \ldots, \psi(n))$ is on the boundary of the realizability domain (which is equivalent to $r < k + 1$, where $k$ is as defined in Theorem 2.4), the minimal atomic representing measure is the unique representing measure.

Note that finding the atoms can be done in a robust way. Although the condition of the Vandermonde system is very bad (namely, exponential in $r$ [22]), efficient and robust algorithms for the solution of this linear system exist [4]. In our paper we will only provide results where the solution of this system can be obtained analytically using symbolic calculations.

¹ This integer $r$ is the Hankel rank defined in [9].
3. Moment closures. Several moment approximations have been extensively studied. Here we briefly recall some of them and also add a new closure strategy in section 3.3.

We first remark that all closures provided here have benefits and drawbacks. The polynomial closure $P_n$ (section 3.1) as well as minimum-entropy closures (for full, half, and mixed moments, sections 3.2, 3.5, and 3.6, respectively) are strictly hyperbolic. This is not the case for the quadrature method of moments (QMOM), which we describe in detail in section 3.4. For arbitrary-order Kershaw closures (section 3.3) it is not known if the system is strictly hyperbolic. Higher order minimum-entropy models provide a good approximation of the underlying kinetic problem but suffer from high numerical costs for the inversion of the moment problem. On the other hand, they can be stated and implemented and can preserve realizability without the knowledge of the realizability conditions. If one knows the realizability conditions, the numerical costs can be avoided using corresponding Kershaw closures which are based on analytical closures of the higher moments without the need of numerical inversion. They are therefore comparable in speed with $P_n$ models while having the same realizability domain as the true solution. However, this is (especially in more than one spatial dimension) not yet always possible.

3.1. $P_n$ equations. The $P_n$ equations can be most easily understood as a Galerkin semidiscretization in the angle $\mu$. We expand the distribution function in terms of a truncated series expansion,

$$\psi_{P_n}(t, x, \mu) = \sum_{l=0}^{N} \psi^{(l)}(t, x) \frac{2l + 1}{2} P_l(\mu),$$

where $P_l$ are the Legendre polynomials. The expansion coefficients are the moments with respect to the Legendre polynomials:

$$\psi^{(l)}(t, x) = \int_{-1}^{1} \psi(t, x, \mu) P_l(\mu) d\mu.$$

Note that we use the index $l$ to distinguish these moments from the moments taken with respect to the monomials. The Galerkin projection is done by testing the Fokker–Planck equation with $P_l$, integrating with respect to $\mu$ over $[-1, 1]$, and using the recursion relation of the Legendre polynomials to obtain the $P_n$ equations

$$\partial_t \psi^{(l)} + \partial_x \left( \frac{l + 1}{2l + 1} \psi^{(l+1)} + \frac{l}{2l + 1} \psi^{(l-1)} \right) + \sigma_a \psi^{(l)} = -\frac{T}{2} l (l + 1) \psi^{(l)} + Q^{(l)}$$

for $l = 0, \ldots, N$. Recall that $\psi^{(N+1)} = 0$. We have also used the fact that the Legendre polynomials are eigenfunctions of the Laplace–Beltrami operator.

3.2. Minimum-entropy closure. The approximations based on the expansion of the distribution function into a polynomial suffer from several drawbacks [12]. As mentioned above, the distribution function can become negative, and thus the moments computed from this distribution can become unphysical. One way to overcome this problem is to use an entropy minimization principle to obtain the constitutive equation to close the moment equations. The first step to derive the minimum entropy ($M_1$) model [12, 2, 5] is identical to the $P_n$ method. We test with the monomials and
\[ \frac{\partial_t \psi^{(0)}}{\partial x} + \partial_x \psi^{(1)} + \sigma_a \psi^{(0)} = Q^{(0)}, \tag{3.4} \]
\[ \frac{\partial_t \psi^{(1)}}{\partial x} + \partial_x \psi^{(2)} + \sigma_a \psi^{(1)} = -T \psi^{(1)} + Q^{(1)}. \tag{3.5} \]

Now, instead of using the Galerkin ansatz, we determine a distribution function \( \psi_M \) that minimizes a functional, the entropy \( H \), under the constraint that it reproduces the lower order moments,

\[ \int_{-1}^{1} \psi_M d\mu = \psi^{(0)} \quad \text{and} \quad \int_{-1}^{1} \mu \psi_M d\mu = \psi^{(1)}. \tag{3.6} \]

If the entropy is

\[ H(\psi) = \int_{-1}^{1} \psi \log \psi - \psi d\mu, \]

then the minimizer can formally be written as [35]

\[ \psi_M(\mu) = e^{\alpha + \beta \mu}. \tag{3.7} \]

This is a Maxwell–Boltzmann-type distribution, and \( \alpha, \beta \) are Lagrange multipliers enforcing the constraints.

It is not possible to express the highest moment \( \psi^{(2)} \) explicitly in terms of \( \psi^{(0)} \) and \( \psi^{(1)} \), but we can write the flux function as

\[ \psi^{(2)} = \chi \left( \frac{\psi^{(1)}}{\psi^{(0)}} \right) \psi^{(0)}. \tag{3.8} \]

The so-called Eddington factor \( \chi \) can be computed numerically; see [35, 5].

### 3.3. Kershaw closure.

We now want to develop another closure strategy which we call Kershaw closure. The key idea of Kershaw in [27] is to derive a closure which preserves the realizability conditions. First we recall that on the boundary of the realizability domain the distribution function must be atomic, that is, a linear combination of Dirac delta functions. Since the realizable set \( R_n \) is a convex cone, one can linearly interpolate between the boundary distributions to find a solution which realizes all moments. The resulting distribution is therefore exact on the boundary of the realizable set. We additionally want it to be exact in the equilibrium limit, where \( \psi \) is independent of the angular variable.

We use the \( n = 2 \) case to introduce this method. Here the realizability conditions are

\[ \psi^{(0)} \geq 0, \quad -\psi^{(0)} \leq \psi^{(1)} \leq \psi^{(0)}, \quad (\psi^{(1)})^2 \leq \psi^{(0)} \psi^{(2)} \leq \psi^{(0)}. \]

If we define the normalized moments \( \phi^{(j)} := \frac{\psi^{(j)}}{\psi^{(0)}} \), the above conditions can be rewritten as

\[ -1 \leq \phi^{(1)} \leq 1, \quad (\phi^{(1)})^2 \leq \phi^{(2)} \leq 1. \]

Thus the relative moment \( \phi^{(2)} \) is bounded from above and below by two functions depending on \( \phi^{(1)} \). The distribution on the lower second order realizability boundary \( (\phi^{(2)} = (\phi^{(1)})^2) \) is given by

\[ \psi_{\text{low}} = \psi^{(0)} \delta \left( \phi^{(1)} - \mu \right), \]
while the upper second order realizability boundary distribution \((\phi^{(2)} = 1)\) is given by

\[
\psi_{\text{up}} = \psi^{(0)} \left(\frac{1 - \phi^{(1)}}{2} \delta(1 + \mu) + \frac{1 + \phi^{(1)}}{2} \delta(1 - \mu)\right).
\]

By linearity of the problem, every convex combination \(\psi_{K_1} = \alpha \psi_{\text{low}} + (1 - \alpha) \psi_{\text{up}}, \alpha \in [0,1]\), will reproduce all first moments and satisfies \(\psi_{K_1} \geq 0\). We choose \(\alpha\) so that our closure is exact for moments of the constant distribution \(\psi_{\text{const}} \equiv \psi^{(0)}/2\). Calculating moments up to order 2 of \(\psi_{\text{const}}\) gives normalized moments \((\phi^{(1)}, \phi^{(2)}) = (0, \frac{1}{3})\). Plugging this into the equations above, we end up with

\[
\phi^{(2)} = \alpha (\phi^{(1)})^2 + (1 - \alpha) \frac{\phi^{(1)}=0}{(1 - \alpha) \cdot 1} = \frac{1}{3}.
\]

This implies \(\alpha = \frac{2}{3}\). Altogether we obtain

\[
\phi^{(2)} = \frac{1}{3} (2(\phi^{(1)})^2 + 1).
\]

This relation is demonstrated in Figure 1. The value of \(\phi^{(2)}\) on the upper boundary (dashed black line), \(\phi^{(2)}_{\text{up}} = 1\), is convexly combined with its value on the lower boundary (blue solid line), \(\phi^{(2)}_{\text{low}} = (\phi^{(1)})^2\), to obtain the realizable convex combination (red dashed-dotted line) which interpolates the equilibrium point \((0, \frac{1}{3})\) (red point).

![Fig. 1. Construction of the K$_1$ closure using a convex combination of the upper (dashed black line) and lower (blue solid line) boundary second moments.](image)

We call this model the Kershaw K$_1$ closure. Note that using this approach it is not possible to provide point-values of the distribution. It is, however, possible to derive this class of models without the explicit representation of the distributions on the upper and lower boundaries [36].

### 3.4. Quadrature method of moments

Similar ideas are also known in hydrodynamics and other fields as the quadrature method of moments (QMOM) [34,
These ideas have been recently applied to radiative transfer [43]. The general idea is to use an \( N \)-atomic discrete measure

\[
\psi_{Q MOM}^N = \sum_{i=1}^{N} \alpha_i \delta(\mu - \mu_i),
\]

where \( N \) is a fixed number that may be independent of the order \( n \). Typically one uses \( n = 2N - 1 \) to obtain \( 2N \) equations and \( 2N \) unknowns (\( N \) weights and abscissas).

Plugging \( \psi_{Q MOM}^N \) into the moment problem (2.7) yields a nonlinear system which can be solved using the so-called Wheeler-algorithm [34, 43], which diagonalizes a tridiagonal matrix to find the weights and abscissas. This results in a robust and efficient algorithm for the inversion of the moment problem. Additionally a major advantage of the QMOM approach is the fact that it can correctly reproduce moments which lie on the realizability boundary [43] since there the distribution is (uniquely) atomic [27].

Although the original \( N \)-node QMOM, tracking \( 2N \) moments, is very efficient and has a lot of advantages, it also has some drawbacks [43]: The reconstructed higher moments will always lie on the (higher order) realizability boundary which immediately implies that it is not possible to correctly reproduce the equilibrium distribution. However, numerical experiments show [43] that the equilibrium limit can be captured with EQMOM (extended QMOM). Another result of \( \psi_{Q MOM}^N \) being on the \( n + 1 \)th order realizability boundary is that the eigenvalues of the Jacobian of the flux are not unique, giving only weak hyperbolicity of the system of partial differential equations. Minimum entropy models, on the other hand, exhibit strict hyperbolicity everywhere in the interior of the realizability domain. It is also not possible to obtain pointwise values of the distribution function itself.

### 3.5. Half-moment approximation.

A model which has been successfully applied to radiative transfer in one dimension, removing some drawbacks of the minimum-entropy model, is the half-moment approximation [39, 42]. A typical disadvantage of the minimum-entropy solution can be seen in the numerical section—for example, in Figure 4. The idea of half-moment models is to average not over all directions but over certain subsets—for example, the sets of particles moving left and those moving right. In one dimension, this means to integrate over \([-1, 0]\) and \([0, 1]\), respectively. We denote the half moments by

\[
\psi_+^{(j)}(t, x) = \int_0^1 \mu^j \psi(t, x, \mu) d\mu \quad \text{and} \quad \psi_-^{(j)}(t, x) = \int_{-1}^0 \mu^j \psi(t, x, \mu) d\mu.
\]

Applying this approach to the Fokker–Planck equation, we obtain

\[
\partial_t \psi_+^{(0)} + \partial_x \psi_+^{(1)} + \sigma_a \psi_+^{(0)} = \frac{T}{2} \int_0^1 \partial_\mu (1 - \mu^2) \partial_\mu \psi d\mu + Q_+^{(0)}.
\]

If we use integration by parts, the integral on the right-hand side becomes

\[
\int_0^1 \partial_\mu (1 - \mu^2) \partial_\mu \psi d\mu = -\partial_\mu \psi(0^+).
\]

Similarly,

\[
\int_{-1}^0 \partial_\mu (1 - \mu^2) \partial_\mu \psi d\mu = \partial_\mu \psi(0^-).
\]
We note that, in contrast to the spherical harmonics approach, on the right-hand side a microscopic term, namely, a value of the distribution itself instead of its moments, appears. A naive approach would be to use entropy minimization on each half space separately. Thus we would use the discontinuous ansatz

\[
\psi_{HM1}(\mu) = \begin{cases} 
  e^{\alpha - \beta - \mu} & \text{for } \mu \in [-1, 0], \\
  e^{\alpha + \beta + \mu} & \text{for } \mu \in [0, 1] 
\end{cases}
\]  

(3.13)

to close both the flux and the right-hand side. This, however, would violate the important conservation property

\[
\int_{-1}^{1} \partial_\mu (1 - \mu^2) \partial_\mu \psi d\mu = 0
\]

of the Fokker–Planck equation, resulting in a wrong approximation of the original equation [21]. One would therefore have to model the boundary terms differently. This is similar to the problems one encounters when the discontinuous Galerkin method is applied to diffusion equations; see, e.g., [3, 7]. Similar interface terms appear that have to be approximated carefully. The deeper theoretical reason for these problems is that the domain of definition of the Laplace–Beltrami operator defines a continuous symmetric bilinear form

\[
a(\lambda, \psi) := -\int_{-1}^{1} (1 - \mu^2) \partial_\mu \lambda(\mu) \partial_\mu \psi d\mu
\]
on the space \(H^1([-1, 1], \mathbb{R})\) [41]. The method of moments is a Galerkin method, which should use subspaces of the domain of definition of the symmetric bilinear form as ansatz spaces. In one dimension this excludes discontinuous functions. Instead of modeling the interface terms, we move to a conforming discretization by demanding continuity of the ansatz.

3.6. A mixed-moment method. To impose continuity, we test (2.1) with 1 and integrate over \([-1, 1]\), and then with \(\mu\) and integrate over \([-1, 0]\) and \([0, 1]\) separately. One obtains balance equations of the form

\[
\begin{align*}
\partial_t \psi^{(0)} + \partial_x \psi_+^{(1)} + \partial_x \psi_-^{(1)} + \sigma_a \psi^{(0)} &= Q^{(0)}, \\
\partial_x \psi_+^{(2)} + \partial_x \psi_-^{(2)} + \sigma_a \psi^{(1)} &= \frac{T}{2} (\psi^{(0)} - 2\psi_+^{(1)}) + Q^{+^{(1)}}, \\
\partial_t \psi_-^{(1)} + \partial_x \psi_+^{(2)} + \sigma_a \psi^{(1)} &= \frac{T}{2} (-\psi^{(0)} - 2\psi_-^{(1)}) + Q_-^{(1)}.
\end{align*}
\]  

(3.14, 3.15, 3.16)

We close this system with an underlying distribution that is continuous, because it is in the domain of definition of the Laplace–Beltrami operator, and will thus also preserve the conservation property of the Fokker–Planck equation. This happens automatically if we use the minimum-entropy principle constrained by the zeroth full moment and the two half moments of first order. In this case, the mixed minimum-entropy (MME) ansatz is [21]

\[
\psi_{MME} = \begin{cases} 
  e^{\alpha_+ + \beta_+ \mu} & \mu \in [0, 1], \\
  e^{\alpha_- + \beta_- \mu} & \mu \in [-1, 0]. 
\end{cases}
\]  

(3.17)
A complete hierarchy of MME methods of arbitrary order (see section 5.2) follows, and so we call this model MM\textsubscript{1}. Here, one could also consider a linear closure, called the mixed-moment MP\textsubscript{n} closure, but this has the drawback that it allows for negative energies and does not adapt to the correct speed of propagation, just as with the full-moment P\textsubscript{n} model.

Again, the system cannot be closed analytically, but the second moments can be written in the form

$$\psi^{(2)}_{\pm} = \chi \pm \left( \frac{\psi^{(1)}_{-}}{\psi^{(0)}_{+}}, \frac{\psi^{(1)}_{+}}{\psi^{(0)}_{-}} \right) \psi^{(0)},$$

where the Eddington factors $\chi_{\pm}$ have to be determined numerically as for the full-moment ansatz.

We note that in [21] the mixed-moment MP\textsubscript{n} or minimum-entropy closures have only been discussed for the lowest order case.

In the following we will study mixed-moment problems of arbitrary order.

4. Realizability of mixed-moment problems. In this section we state necessary and sufficient conditions for the mixed moments to be consistent with a positive distribution function.

Definition 4.1. Given a vector of real numbers $(\psi^{(0)}_{+}, \psi^{(1)}_{+}, \ldots, \psi^{(n)}_{+}, \psi^{(1)}_{-}, \ldots, \psi^{(n)}_{-})$, the truncated Hausdorff mixed-moment problem on $[-1, 1]$ entails finding a nonnegative distribution function $\psi$ such that

\begin{align}
(4.1a) \quad & \int_{-1}^{1} \psi(\mu) \, d\mu = \psi^{(0)}_{+}, \\
(4.1b) \quad & \int_{0}^{1} \mu^{j} \psi(\mu) \, d\mu = \psi^{(j)}_{+}, \quad 0 \leq j \leq n, \\
(4.1c) \quad & \int_{-1}^{0} \mu^{j} \psi(\mu) \, d\mu = \psi^{(j)}_{-}, \quad 0 \leq j \leq n.
\end{align}

Furthermore, we denote the realizable domain of vectors for which a solution to this problem exists by $\text{MR}_{n} \subset \mathbb{R}^{2n+1}$.

To adapt Theorem 2.4 to our mixed-moment problem, we use a basic fact and an elementary lemma.

Fact 1. The mixed-moment data $\gamma$ is realizable iff there exist $\psi^{(0)}_{\pm}$ and $\psi^{(0)}_{\mp}$ such that $\psi^{(0)} = \psi^{(0)}_{+} + \psi^{(0)}_{-}$, and the moments $(\psi^{(0)}_{+}, \psi^{(1)}_{+}, \ldots, \psi^{(n)}_{+})$ and $(\psi^{(0)}_{-}, \psi^{(1)}_{-}, \ldots, \psi^{(n)}_{-})$ are realizable under the Hausdorff conditions for $[0, 1]$ and $[-1, 0]$, respectively.

The lemma we use appears in a slightly different form as Lemma 2.3 in [9]. We can use it to show that, according to the realizability conditions in Theorem 2.4, the moments of order $1, \ldots, n$ for each half interval $[0, 1]$ and $[-1, 0]$ define lower bounds for the quantities $\psi^{(0)}_{+}$ and $\psi^{(0)}_{-}$, respectively.\footnote{If we consider the more general truncated mixed-moment Hausdorff problem over two adjacent intervals $[\mu_{-}, \mu_{0}]$ and $[\mu_{0}, \mu_{+}]$ (instead of specifically $[-1, 0]$ and $[0, 1]$), depending on the signs of $\mu_{-}$, $\mu_{0}$, and $\mu_{+}$, the conditions of Lemma 4.2 can also lead to upper bounds on $\psi^{(0)}_{\mp}$. This does not occur when $\mu_{0} = 0$, as in our case.} Below we use $R(M)$ to indicate the linear space spanned by the columns of the matrix $M$.

Lemma 4.2. Let $A \in \mathbb{R}^{(k+1) \times (k+1)}$ be symmetric, $C \in \mathbb{R}^{k \times k}$ be symmetric,
\(b \in \mathbb{R}^k\), and \(a \in \mathbb{R}\) so that
\[
A = \begin{pmatrix}
a & b^T \\
b & C
\end{pmatrix}.
\]

(i) If \(A \geq 0\), then \(C \geq 0\), \(b \in R(C)\), and \(a \geq w^T C w\), where \(b = C w\).

(ii) If \(C \geq 0\) and \(b \in R(C)\), then \(A \geq 0\) iff \(a \geq w^T C w\), where \(b = C w\).

Remark 4.3. Since \(C\) is invertible on its range, a more explicit formula for the bound on \(a\) in Lemma 4.2 can be written using the pseudoinverse \(C^\dagger\) of \(C\). That is, for all \(w\) such that \(b = Cw\), we have \(w^T C w = b^T C^\dagger b\). Thus the bound on \(a\) is well defined even when \(C\) is singular.

Thus the mixed-moment data \((\psi(0), \psi(1), \ldots, \psi(n), \psi(1), \ldots, \psi(n))\) is realizable only if \(\psi(0)\) is large enough that it can be split up into \(\psi(0)^+\) and \(\psi(0)^-\) which each satisfy the lower bounds imposed by \((\psi(1), \ldots, \psi(n))\) and \((\psi(1), \ldots, \psi(n))\), respectively, through Lemma 4.2 and the appropriate Hausdorff conditions. This gives the following result.

**Theorem 4.4.** Let \(B_{\pm}(k)\) and \(C_{\pm}(k)\) be defined as in Theorem 2.4 but with \(\psi(i)\) replaced by \(\psi_{\pm}^i\) (respectively) for \(i \geq 1\). We also let \(D(k)_{\pm} := (\psi_{\pm}^{(i+j)+1})_{i,j=1}^k\) and define the mixed-moment matrices \(A_{\pm}(k)\) using both the full moment \(\psi(0)\) and the partial moments by
\[
\left( A_{\pm}(k) \right)_{ij} = \begin{cases} 
\psi(i) & \text{if } i = j = 0, \\
\psi(0) & \text{otherwise,} \\
\psi(0) & i, j \in \{0, 1, \ldots, k\}.
\end{cases}
\]

Then the mixed-moment data \(\gamma\) is realizable iff

(i) when \(N = 2k\) is even,
\[
\begin{align*}
& (4.2a) & \pm B_{\pm}(k-1) - C_{\pm}(k) \geq 0, \\
& (4.2b) & A_{\pm}(k) \geq 0, \\
& (4.2c) & \psi(0) \geq b_+^T C_{\pm}(k)^\dagger b_+ + b_-^T C_{\pm}(k)^\dagger b_-,
\end{align*}
\]
where \(b_{\pm} := (\psi_{\pm}^{(1)}, \ldots, \psi_{\pm}^{(k)})^T\) are simply the first columns of \(A_{\pm}(k)\), but omitting the top element, respectively, and \(C_{\pm}(k)^\dagger\) represent the pseudoinverses of \(C_{\pm}(k)\), respectively;

(ii) when \(N = 2k + 1\) is odd,
\[
\begin{align*}
& (4.3a) & \pm B_{\pm}(k) \geq 0, \\
& (4.3b) & A_{\pm}(k) \mp B_{\pm}(k) \geq 0, \\
& & \psi_{\pm}(0) \geq \psi_{\pm}^{(1)} + b_+^T (C_{\pm}(k) - D_{\pm}(k))^\dagger b_+ \\
& & - \psi_{\pm}^{(1)} + b_-^T (C_{\pm}(k) - D_{\pm}(k))^\dagger b_-,
\end{align*}
\]
where here \(b_{\pm} := (\psi_{\pm}^{(1)} - \psi_{\pm}^{(2)}, \ldots, \psi_{\pm}^{(k)} - \psi_{\pm}^{(k+1)})^T\) are simply the first columns of \(A_{\pm}(k) \mp B_{\pm}(k)\), but omitting the top element, respectively, and \((C_{\pm}(k) + D_{\pm}(k))^\dagger\) represent the pseudoinverses of \(C_{\pm}(k) + D_{\pm}(k)\), respectively.

**Proof.** We first prove the necessity and sufficiency of the following conditions, which follow more directly from Lemma 4.2:
(i) when $N = 2k$ is even,
\begin{align}
(4.4a) & \quad \pm B_\pm (k - 1) - C_\pm (k) \geq 0, \\
(4.4b) & \quad C_\pm (k) \geq 0, \\
(4.4c) & \quad b_\pm \in R (C_\pm (k)), \\
(4.4d) & \quad \psi^{(0)} \geq b_+^T C_+ (k) b_+ + b_-^T C_- (k) b_-;
\end{align}

(ii) when $N = 2k + 1$ is odd,
\begin{align}
(4.5a) & \quad \pm B_\pm (k) \geq 0, \\
(4.5b) & \quad C_\pm (k) \mp D_\pm (k) \geq 0, \\
(4.5c) & \quad b_\pm \in R (C_\pm (k) \mp D_\pm (k)), \\
& \quad \psi^{(0)} \geq \psi_+^{(1)} + b_+^T (C_+ (k) - D_+ (k)) b_+ \\
& \quad \quad - \psi_-^{(1)} + b_-^T (C_- (k) + D_- (k)) b_-.
\end{align}

We first quickly note that both sets of conditions have the following form: they first include the Hausdorff conditions which do not involve zeroth-order moments, followed by the conditions of Lemma 4.2, from which the final condition for each side is summed to give a total lower bound on $\psi^{(0)}$. We prove just the even case, as the proof in the odd case is analogous.

For necessity of (4.4), assume that we have a density $\psi$ which represents $\gamma$. Then let $\psi_+^{(0)} := \int_0^1 \psi (\mu) d\mu$ and $\psi_-^{(0)} := \int_0^1 \psi (\mu) d\mu$, so that clearly $\psi^{(0)} = \psi_+^{(0)} + \psi_-^{(0)}$. Then (4.4a) follows immediately from (2.10). Conditions (4.4b)–(4.4d) follow from (2.9) and Lemma 4.2, where $a = \psi_\pm^{(0)}$, and (4.4d) is obtained simply by summing the final conditions Lemma 4.2 for each half interval.

For sufficiency (4.4), we choose $\psi_+^{(0)} = b_+^T C_+ (k - 1) b_+$ and $\psi_-^{(0)} = \psi^{(0)} - b_+^T C_+ (k - 1) b_+ \geq b_-^T C_- (k) b_-$. With the values and conditions (4.4b)–(4.4d), we can use Lemma 4.2 to construct positive-definite matrices $A_\pm (k)$. These matrices together with condition (4.4a) are the Hausdorff conditions. Therefore, we have densities defined on both $[0,1]$ and $[-1,0]$ which we can concatenate to represent $\gamma$.

Finally, it is not hard to see that, again via Lemma 4.2, (4.4b)–(4.4d) are equivalent to (4.2b)–(4.2c).

**Example 4.5.** We examine the coupling conditions (4.2c) and (4.3c) for $n = 2$ and $n = 3$ more explicitly.

When $n = 2$, we have $k = 1$, and we have $C_\pm (1) = \psi_\pm^{(2)}$, whose pseudoinverses are 0 when $\psi_\pm^{(2)} = 0$. Therefore,
\[
b_\pm^T C_\pm (1) b_+ = \begin{cases} (\psi_\pm^{(1)})^2 & \text{if } \psi_\pm^{(2)} \neq 0, \\ 0 & \text{otherwise}. \end{cases}
\]

The range conditions (4.4c) are only nontrivial in the singular cases, which for $n = 2$ are when $\psi_\pm^{(2)} = 0$ or $\psi_-^{(2)} = 0$. In these cases the range conditions require $\psi_\pm^{(1)} = 0$, respectively (which are consistent with the fact that here we must have supp $\psi \cap [0,1] \subseteq \{0\}$ or supp $\psi \cap [-1,0] \subseteq \{0\}$, respectively, for any representing $\psi$). But in the nonsingular case, condition (4.2c) reads
\[
\psi^{(0)} \geq \frac{(\psi_+^{(1)})^2}{\psi_+^{(2)}} + \frac{(\psi_-^{(1)})^2}{\psi_-^{(2)}}.
\]
to see, using arguments similar to those for full moments \[30\], that MME models and (strictly inside the realizability domain). Although we do not prove it here, it is easy to check for each model separately. The distribution function to be continuous in \( n \) is a polynomial in every half space in the angular variable \( \gamma \).

The generating functions \( \psi \) are the roots of the generating function with \( \psi(1) = 0 \). As for full moments (see Remark 2.5) atoms \( \mu_{i\pm} \) for the distribution function are the roots of the generating function with \( \gamma_i = (\pm 1)^i \phi_{1\pm} \), respectively. The densities \( \rho_{i\pm} \) can be calculated afterward from the corresponding Vandermonde system. The generating functions \( g_i(k) = \mu^k - (\sum_{i=1}^{k-1} \phi_i \mu^i) \) for \( n \leq 6 \) and \( k = \lceil \frac{n}{2} \rceil \) are given by \( \phi \) in Table 1.

5. Mixed-moment closures. In this section we derive the mixed-moment MP\(_n\) closure for arbitrary order as well as the minimum-entropy closure. A special discrete closure which we call the Kershaw closure is given up to second order.

Remark 5.1. We want to emphasize that all models derived here are hyperbolic (strictly inside the realizability domain). Although we do not prove it here, it is easy to see, using arguments similar to those for full moments \[30\], that MME models and mixed MP\(_n\) models fulfill this property. Additionally all eigenvalues are bounded in absolute value by one.

For (mixed-moment) Kershaw closures we know of no general proof for hyperbolicity, but it is easy to check for each model separately.

5.1. MP\(_n\). The mixed MP\(_n\) closure consists of a continuous basis function which is a polynomial in every half space in the angular variable \( \mu \). We additionally demand the distribution function to be continuous in \( \mu = 0 \). This results in the following ansatz:

\[
\psi_{\text{MP}_n} = \begin{cases} 
\alpha + \sum_{i=1}^{n} \beta^{(i)}_+ \mu^i & \text{for } \mu \in [0, 1], \\
\alpha + \sum_{i=1}^{n} \beta^{(i)}_- \mu^i & \text{for } \mu \in [-1, 0].
\end{cases}
\]
We use monomials here because spherical harmonics (that is, Legendre polynomials) lose their orthogonality on the half spaces and are therefore not superior to the usual monomial basis.

5.2. Mixed-minimum entropy. The general MM\(_n\) is obtained by the same entropy ansatz as for the original MME/MM\(_1\). The entropy minimizer is given by

\[
\psi_{\text{MM}_n} = \begin{cases} 
\exp \left( \alpha + \sum_{i=1}^{n} \beta_+^{(i)} \mu^i \right) & \text{for } \mu \in [0, 1], \\
\exp \left( \alpha + \sum_{i=1}^{n} \beta_-^{(i)} \mu^i \right) & \text{for } \mu \in [-1,0].
\end{cases}
\]

As proposed in [21], a tabulation is used for \(n = 1\) to avoid a nonlinear solution technique such as Newton’s method. However, this seems not appropriate for larger \(n\) since the tableau has dimension \(2^n\). Here a robust algorithm has to be applied, especially at the boundary of the realizability domain; see, e.g., [1] for more details on this topic for the original minimum-entropy models. Additionally, this task is even more challenging for \(n > 2\) because in this case the integrals cannot be solved analytically. Since this is not the main topic of this paper, we will only give numerical examples for \(n = 1\).

5.3. Kershaw closures. To avoid the nonlinear inversion process of the minimum-entropy approach, we want to construct a closure which can be computed analytically. The key idea for the mixed moments is identical to the idea for the full moments explained in section 3.3. Although arbitrary high orders are in principle available due to Theorem 4.4, we only present Kershaw closures up to order 2. As in section 3.3, convex combinations of “upper” and “lower” higher order realizability distributions \(\psi_{\text{up}}, \psi_{\text{low}}\) are needed. However, finding suitable candidates which are symmetric in terms of the two half spaces which reproduce moments on the coupling conditions (4.2c) and (4.3c) is a nontrivial task and may be possible only due to intensive symbolic calculations.

5.3.1. MK\(_1\). We want to construct a distribution function \(\psi_{\text{MK}_1}\), which on one hand is realizable and on the other hand interpolates the equilibrium point

\[
\begin{pmatrix} \phi_+^{(1)} \phi_-^{(1)} \phi_+^{(2)} \phi_-^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \ -1/4 \ 1/6 \ 1/6 \end{pmatrix}.
\]

One idea would be to do a convex combination between the isotropic state \(\psi_{\text{const}}\) and the free streaming limit calculated in Theorem 4.4:

\[
\psi_+^{(2)} = \alpha \int_0^1 \mu^2 \psi_{\text{const}} \, d\mu + (1 - \alpha) \int_0^1 \mu^2 \psi_{\text{up}} \, d\mu
\]

with

\[
\psi_{\text{up}} = \psi^{(0)} \begin{pmatrix} \phi_+^{(1)} \delta (1-\mu) - \phi_-^{(1)} \delta (1+\mu) + (1-\phi_+^{(1)} + \phi_-^{(1)}) \delta (\mu) \end{pmatrix}
\]

and \(\alpha(\phi_+^{(1)}, \phi_-^{(1)})\) such that \(\alpha(1/4, -1/4) = 1\). This works well for full moments (see, e.g., [36]) but fails for mixed moments. In the isotropic limit \(\psi \to \psi_{\text{const}}\) there is no unique description of this state in terms of available moments; that is, the problem is underdetermined. Choosing, e.g., \(\psi_{\text{const}} = \frac{1}{2} \psi^{(0)} = 2 \psi_+^{(1)} = -2 \psi_-^{(1)}\) gives different
closures in (5.1). Correspondingly, different choices of \( \psi_{\text{const}} \) result in systems with different eigenvalues for the Jacobian of the flux function. Choosing naively \( \alpha = 1 \) and \( \psi^{(2)} = \psi^{(0)}_\pm \) in the first example and \( \psi^{(2)} = \pm 2 \psi^{(1)}_\pm \) in the second example gives system matrices

\[
M_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 & 1 \\ \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

which have the following set of eigenvalues: \( \lambda_1 = (-\frac{2}{3}, 0, \frac{2}{3}) \) and \( \lambda_2 = (-\sqrt{\frac{1}{3}}, 0, \sqrt{\frac{1}{3}}) \).

To overcome this problem we use again, as in section 3.3 for the K model, a convex combination of the upper and lower realizability boundaries for \( n = 2 \):

\[
\psi_{\text{low}} = \psi^{(0)} \left( \frac{\phi^{(1)}_+}{\phi^{(1)}_+ - \phi^{(1)}_-} \delta \left( \mu - \left( \phi^{(1)}_+ - \phi^{(1)}_- \right) \right) - \frac{\phi^{(1)}_-}{\phi^{(1)}_+ - \phi^{(1)}_-} \delta \left( \mu + \left( \phi^{(1)}_+ - \phi^{(1)}_- \right) \right) \right).
\]

A convex combination of those implies second moments

\[
\phi^{(2)}_\pm = \pm \alpha \phi^{(1)}_\pm \pm (1 - \alpha) \phi^{(1)}_\pm \left( \phi^{(1)}_+ - \phi^{(1)}_- \right).
\]

Since the equilibrium point should be interpolated, we can conclude \( \alpha = \frac{1}{3} \). However, every nonconstant solution \( 0 \leq \alpha (\phi^{(1)}_+, \phi^{(1)}_-) \leq 1 \) with \( \alpha (\frac{1}{3}, -\frac{1}{3}) = \frac{1}{3} \) would be appropriate since at the lower order boundaries the two distributions coincide.

In Figure 2 these lower (2a) and upper (2b) boundary representations of \( \phi^{(2)}_+ \) are shown, as well as the normalized positive second moment \( \phi^{(2)}_+ \) of the MK model which is then compared with the one calculated for the MME solution. Note that for \( \phi^{(2)}_+ \) the result is just mirrored along the bisecting line of this triangle.

### 5.3.2. MK

As before, we are looking for two distribution functions which lie on the lower and upper realizability boundaries of order 3, respectively. The corresponding realizability conditions are

\[
\begin{align*}
(5.2a) \quad & \frac{\phi^{(2)}_\pm^2}{\phi^{(3)}_\pm} \leq \phi^{(3)}_\pm \leq \frac{\phi^{(2)}_\pm^2}{1 + \phi^{(1)}_\pm}, \\
(5.2b) \quad & \phi^{(1)}_+ + \frac{\left( \phi^{(1)}_+ - \phi^{(2)}_+ \right)^2}{\phi^{(3)}_+ - \phi^{(3)}_+} - \phi^{(1)}_- + \frac{\left( -\phi^{(1)}_- - \phi^{(2)}_- \right)^2}{\phi^{(3)}_- + \phi^{(3)}_-} \leq 1.
\end{align*}
\]

We observe that

\[
\psi_{\text{low}} = \psi^{(0)} \left( \frac{\phi^{(1)}_+^2}{\phi^{(2)}_+^2} \delta \left( \mu - \frac{\phi^{(2)}_+}{\phi^{(1)}_+} \right) + \frac{\phi^{(1)}_-^2}{\phi^{(2)}_-^2} \delta \left( \mu - \frac{\phi^{(2)}_-}{\phi^{(1)}_-} \right) + \left( 1 - \frac{\phi^{(1)}_+^2}{\phi^{(2)}_+^2} - \frac{\phi^{(1)}_-^2}{\phi^{(2)}_-^2} \right) \delta (\mu) \right)
\]

reproduces

\[
\psi^{(3)}_{\text{low}} = \int \mu^3 \psi d\mu = \psi^{(0)} \frac{\phi^{(1)}_+^2}{\phi^{(2)}_+^2} \left( \frac{\phi^{(2)}_+}{\phi^{(1)}_+} \right)^3 = \psi^{(0)} \frac{\phi^{(2)}_+^2}{\phi^{(1)}_+^2}.
\]
Since the formula for $\psi_{up}$ is very complicated, we only state the generated normalized third-order moments which satisfy the upper boundary/coupling conditions symmetrically in terms of the half intervals:

$$\phi^{(3)}_{+ \ up} = \phi^{(2)}_{+} - \frac{\left( \phi^{(1)}_{+} - \phi^{(2)}_{+} \right)^2}{\phi^{(1)}_{+} - \phi^{(1)}_{-}} - \frac{\phi^{(1)}_{+} \phi^{(2)}_{+} - \phi^{(1)}_{-} \phi^{(2)}_{-}}{\phi^{(2)}_{+} \phi^{(2)}_{-} + \phi^{(1)}_{+} \phi^{(1)}_{-} - \phi^{(2)}_{+} \phi^{(2)}_{-}} + 1,$$

$$\phi^{(3)}_{- \ up} = \phi^{(1)}_{-} - \frac{\left( \phi^{(1)}_{+} + \phi^{(2)}_{+} \right)^2}{\phi^{(1)}_{+} + \phi^{(1)}_{-}} - \frac{\phi^{(1)}_{+} \phi^{(2)}_{+} + \phi^{(1)}_{-} \phi^{(2)}_{-}}{\phi^{(2)}_{+} \phi^{(2)}_{-} + \phi^{(1)}_{+} \phi^{(1)}_{-} + \phi^{(2)}_{+} \phi^{(2)}_{-}} + 1.$$

These moments satisfy the third order coupling condition with equality.
In the equilibrium point we have $\phi_{\pm}^{(3)} = \pm \frac{1}{8}$. Therefore,

$$
\phi_{+}^{(3)} = \alpha \frac{\phi_{+}^{(2)^2}}{\phi_{+}^{(1)}} + (1 - \alpha) \left( \phi_{+}^{(2)} - \frac{\left( \phi_{+}^{(1)} - \phi_{+}^{(2)} \right)^2}{\phi_{+}^{(1)} - \phi_{+}^{(2)}} - 1 \right),
$$

and

$$
\phi_{-}^{(3)} = \alpha \frac{\phi_{-}^{(2)^2}}{\phi_{-}^{(1)}} + (1 - \alpha) \left( \phi_{-}^{(1)} + \frac{\left( \phi_{-}^{(1)} - \phi_{-}^{(2)} \right)^2}{\phi_{-}^{(1)} - \phi_{-}^{(2)}} - 1 \right)
$$

with $\alpha = \frac{1}{2}$.


6.1. Implementation. As reference we use a standard finite-difference approximation of the Fokker–Planck equation (2.1). Moment approximations are calculated using variants of a standard finite volume scheme [40, 29],

$$
u_{i}^{j+1} = \nu_{i}^{j} - \frac{1}{h} \int_{t_{j}}^{t_{j+1}} f \left( u \left( t, x_{i+\frac{1}{2}} \right) \right) dt - \frac{1}{h} \int_{t_{j}}^{t_{j+1}} f \left( u \left( t, x_{i-\frac{1}{2}} \right) \right) dt,
$$

where $\lambda = \frac{\Delta t}{\Delta x}$ and $\nu_{i}^{j}$ denotes the solution at cell center $i$ at time $t_{j}$ of the general hyperbolic system of conservation laws

$$
u_{i} + f \left( u \right)_{x} = 0.
$$

For first order numerical approximation this can be rewritten in conservative form as

$$
u_{i}^{j+1} = \nu_{i}^{j} - \lambda \left[ h(u_{i-1}^{j}, u_{i+1}^{j}) - h(u_{i}^{j}, u_{i}^{j}) \right]
$$

with appropriate numerical fluxes $h(u, v)$. The source terms are approximated consistently using cell-averages of $\sigma_{a}, T,$ and $Q$.

In all examples we use $n_{x} = 1000$ points for the spatial discretization, while additionally $n_{\mu} = 800$ points in the angular variable are used for the Fokker–Planck solution.

6.1.1. Full-moment $P_{n}$. The full-moment spherical harmonics are discretized with a Godunov/Upwind scheme:

$$
h(u, v) = \frac{1}{2} \left( Au + Av - |A| \left( v - u \right) \right),
$$

where $A$ is the Jacobian of the flux function $f$ (which is linear for the $P_{n}$ equations) and $|A| = A^{+} - A^{-}$ with $A = TDT^{-1}$, $D = \text{diag}(\lambda_{1}, \ldots, \lambda_{n})$,

$$
\lambda_{i}^{\pm} = \pm \max(\pm \lambda_{i}, 0), \quad D^{\pm} = \text{diag}(\lambda_{i}^{\pm}, \ldots, \lambda_{n}^{\pm}), \quad A^{\pm} = T D^{\pm} T^{-1}.
$$
6.1.2. Full-moment $K_1/M_1$. The full-moment $K_1/M_1$ model is solved using an HLL solver (see, e.g., [40]):

$$h(u_L, u_R) = \begin{cases} 
F_L & \text{if } 0 \leq S_L, \\
\frac{S_R F_L - S_L F_R + S_L S_R (u_R - u_L)}{S_R - S_L} & \text{if } S_L \leq 0 \leq S_R, \\
F_R & \text{if } 0 \geq S_R
\end{cases}$$

with $F_{L/R} = f(u_{L/R})$ and $S_{L/R}$ the approximate wave speeds at the left and the right cell center, respectively. We choose $S_R = 1 = -S_L$ since the eigenvalues for $M_1$ and $K_1$ are bounded in absolute value by 1 [12].

For the minimum-entropy model we use the numerical approximation from [14] which uses a rational fit for $\phi^{(2)} = \chi (\phi^{(1)})$. For more details see [12, 21, 36].

6.1.3. Mixed-moment methods. All mixed-moment methods are solved using a kinetic scheme; see, e.g., [19]. It performs a trivial upwinding for the “±”-variables, respectively. The only interesting equation is the “full moment” equation for the density. Here the semidiscretized scheme at cell $j$ looks like

$$\partial_t \psi^{(0)}_j + \left( \psi^{(1)}_{j+1} - \psi^{(1)}_{j-1} \right) + \left( \psi^{(1)}_{j+1} - \psi^{(1)}_{j-1} \right) \frac{1}{\Delta x} = S(t, x, \gamma)_j.$$

6.1.4. Time discretization and boundary conditions. The time integration for all schemes is done using The Mathworks’s MATLAB [33] explicit adaptive second order Runge–Kutta integrator ode23. The Fokker–Planck solution for the Source-Beam test case is calculated using the integrator for stiff equations ode23s. Note that both integrators mentioned here are not strongly stability preserving; see, e.g., [24] for details.

Boundary conditions for moment problems are always problematic. We model them using ghost cells at the boundary where we directly prescribe the underlying kinetic distribution. From this we can consistently calculate the moments for all models of arbitrary order.

Under some assumptions the spherical harmonics are equivalent to a special discrete ordinates Fokker–Planck discretization. With this, the spherical harmonics solution obeys the desired positivity for the distribution function. However, necessary for this are Mark boundary conditions [31, 32], which we do not use in our test cases. Therefore, the spherical harmonics solution may oscillate into the negative, as shown below.

6.1.5. Laplace–Beltrami operator. The Laplace–Beltrami operator is closed consistently using the corresponding distribution functions. The only exceptions are the mixed-moment Kershaw closures. Using integration by parts gives

$$\int_0^1 \mu^m \partial_\mu \left( (1 - \mu^2) \partial_\mu \psi \right) d\mu = \left[ \mu^m \left( (1 - \mu^2) \partial_\mu \psi \right) \right]_0^1 - \int_0^1 m \mu^{m-1} (1 - \mu^2) \partial_\mu \psi d\mu$$

$$= - \left[ m \mu^{m-1} (1 - \mu^2) \psi \right]_0^1 + \int_0^1 m (m - 1) \mu^{m-2} \psi - m (m + 1) \mu^m \psi d\mu$$

$$= m (m - 1) \psi_{+}^{(m-2)} - m (m + 1) \psi_{+}^{(m)}$$
and analogously
\[ \int_{-1}^{0} \mu^m \partial_{\mu} \left( (1 - \mu^2) \partial_{\mu} \psi \right) d\mu = m \left( m - 1 \right) \psi_{m-2}^{(m)} - m \left( m + 1 \right) \psi_{m}^{(m)}. \]

For \( m = 1 \) we additionally get the microscopic term \( \psi(0) \). This is obviously a problem since the Kershaw closure consists of Dirac delta functions. We therefore close this operator using the \( \text{MP}_n \) approximation.

6.1.6. Realizability projection. Sometimes the numerical approximations leave their realizability domain. This is a problem since then some closures cannot be evaluated anymore. The first order schemes used in this work preserve realizability since they are convex combinations of realizable vectors.

Problems still may occur near the realizability boundary. This preserving property obviously holds only for exact arithmetic. If the evaluation of the flux is inexact (e.g., the tabulation for \( \text{MM}_1 \) or simple numerical errors), the scheme may lose this property. Where necessary we apply a suitable projection back into the corresponding realizability domain. This is done by doing a line search along the ray \( \alpha u + (1 - \alpha) u_{eq} \) where \( u \) is the current vector of moments and \( u_{eq} \) is the equilibrium solution, that is, the moments of the constant distribution \( \psi = \alpha \). Then we solve for \( \alpha \) such that the corrected solution lies on the boundary (if evaluation on the boundary is possible, e.g., for Kershaw closures). This somehow corresponds to a limiting procedure in the angular variable, where our local (possibly nonlinear) ansatz is limited toward a constant solution.

6.1.7. Comparison of methods. For every example we present pictures as well as tables for comparison. In the tables, we always calculate for models “\( m_1 \)” and “\( m_2 \)” the difference in the corresponding \( L^p \) sense for the densities \( \psi^{(0)} \):

\[ E_p^{(\psi^{(0)}_{m_1}, \psi^{(0)}_{m_2})} = \int_{0}^{T} \int_{D} \left| \psi^{(0)}_{m_1}(x,t) - \psi^{(0)}_{m_2}(x,t) \right|^p dx \, dt, \]

where \( T \) is the final time of our calculations and \( D \) the spatial domain. Characteristic errors are the \( L^p \) difference at a specific time:

\[ E_p^{(\psi^{(0)}_{m_1}, \psi^{(0)}_{m_2}, t)} = \int_{D} \left| \psi^{(0)}_{m_1}(x,t) - \psi^{(0)}_{m_2}(x,t) \right|^p dx. \]

6.2. Beam in vacuum hitting an absorbing object. In this test we model a beam hitting an object in vacuum. We therefore set for \( x \in [-0.5, 0.5] \)

\[ \sigma_a(x) = \begin{cases} 10 & \text{if } x \in [-0.1, 0.2], \\ 0 & \text{else}, \end{cases} \quad T(x) = Q(x) = 0, \]

and initial and boundary conditions

\[ \psi(x, \mu, 0) = 10^{-4}, \quad x \in (-0.5, 0.5), \]
\[ \psi(-0.5, \mu > 0, t) = \frac{3e^{3\mu+3}}{e^\theta - 1}, \quad \psi(0.5, \mu < 0, t) = 10^{-4}. \]

Note that we do not choose a completely forward-peaked solution (that is, a Dirac delta) because here all minimum entropy and Kershaw models are exact. As
shown in Figure 3, all models have some difficulties reproducing the exact shape of the Fokker–Planck solution for this specific choice of boundary data. One can nicely see the different wave packages arising from this Riemann problem at the left boundary. Fortunately all models recover the correct stationary solution.

Table 2 confirms these observations. $M_1$ and $MM_1$ perform equally well. What is interesting is the large deviation between $MM_1$ and $MK_1$. As expected, $MK_2$ provides better results than $MK_1$. Going to a high number of modes ($P_5$), the error becomes reasonably small.

![Fig. 3. Solutions of the one-beam test case.](image)

**Table 2**

Relative $L^p$ errors, $p \in \{1, 2, \infty\}$, for different models with respect to the Fokker–Planck solution $n_x = 1000$, $n_\mu = 800$. One-beam test case. Characteristic errors evaluated at $t = 2$.

<table>
<thead>
<tr>
<th>Model</th>
<th>$L^1$</th>
<th>$L^2$</th>
<th>$L^\infty$</th>
<th>Char. $L^1$</th>
<th>Char. $L^2$</th>
<th>Char. $L^\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$MM_1$</td>
<td>0.066339</td>
<td>0.033333</td>
<td>0.099496</td>
<td>0.01773</td>
<td>0.022299</td>
<td>0.041594</td>
</tr>
<tr>
<td>$MK_1$</td>
<td>0.16669</td>
<td>0.071425</td>
<td>0.21744</td>
<td>0.049446</td>
<td>0.053638</td>
<td>0.075535</td>
</tr>
<tr>
<td>$MK_2$</td>
<td>0.10197</td>
<td>0.04527</td>
<td>0.19809</td>
<td>0.033389</td>
<td>0.037324</td>
<td>0.052803</td>
</tr>
<tr>
<td>$M_1$</td>
<td>0.07234</td>
<td>0.037757</td>
<td>0.11092</td>
<td>0.020511</td>
<td>0.023531</td>
<td>0.050847</td>
</tr>
<tr>
<td>$P_7$</td>
<td>0.042149</td>
<td>0.022403</td>
<td>0.0851</td>
<td>0.015638</td>
<td>0.020247</td>
<td>0.049295</td>
</tr>
<tr>
<td>$P_{51}$</td>
<td>0.0059108</td>
<td>0.0026413</td>
<td>0.0055583</td>
<td>0.0023004</td>
<td>0.002608</td>
<td>0.0050124</td>
</tr>
</tbody>
</table>
6.3. **Two beams.** This test case models two beams entering into an absorbing medium. Nearly no particles are in the domain initially:

\[ \psi(x, \mu, 0) = 10^{-4}, \quad x \in \left( -\frac{1}{2}, \frac{1}{2} \right), \]

and the equation is supplemented with boundary conditions

\[ \psi\left(-\frac{1}{2}, \mu > 0, t\right) = 100 \cdot \delta(\mu - 1), \quad \psi\left(\frac{1}{2}, \mu < 0, t\right) = 100 \cdot \delta(\mu + 1). \]

Additionally we set the absorption parameter \( \sigma_a = 4 \) and the transport coefficient \( T = 0 \). This is the classical setting where full moment \( M_1 \) (and \( K_1 \) as well) fails due to a zero net-flux during the collision of the two beams. This is shown in Figure 4.

**Table 3**

<table>
<thead>
<tr>
<th>Model</th>
<th>( L^1 )</th>
<th>( L^2 )</th>
<th>( L^\infty )</th>
<th>Char. ( L^1 )</th>
<th>Char. ( L^2 )</th>
<th>Char. ( L^\infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>MM_1</td>
<td>0.0001528</td>
<td>0.0009495</td>
<td>0.005959</td>
<td>0.0038968</td>
<td>0.0042576</td>
<td>0.0055318</td>
</tr>
<tr>
<td>MK_1</td>
<td>0.00010846</td>
<td>0.0025032</td>
<td>0.0025031</td>
<td>0.0025032</td>
<td>0.0025069</td>
<td>0.0025102</td>
</tr>
<tr>
<td>MK_2</td>
<td>0.00010816</td>
<td>0.0025029</td>
<td>0.0044977</td>
<td>0.0024997</td>
<td>0.0025063</td>
<td>0.0027055</td>
</tr>
<tr>
<td>MP_2</td>
<td>0.010856</td>
<td>0.39845</td>
<td>2.0879</td>
<td>0.22026</td>
<td>0.38489</td>
<td>2.087</td>
</tr>
<tr>
<td>MP_3</td>
<td>0.0041039</td>
<td>0.21726</td>
<td>1.5572</td>
<td>0.071395</td>
<td>0.20711</td>
<td>1.5519</td>
</tr>
<tr>
<td>MP_{10}</td>
<td>0.0015805</td>
<td>0.11101</td>
<td>0.9418</td>
<td>0.023389</td>
<td>0.10519</td>
<td>0.93944</td>
</tr>
<tr>
<td>M_1</td>
<td>0.0057931</td>
<td>0.18592</td>
<td>0.25789</td>
<td>0.16074</td>
<td>0.19928</td>
<td>0.21276</td>
</tr>
<tr>
<td>K_1</td>
<td>0.0055333</td>
<td>0.17446</td>
<td>0.25725</td>
<td>0.15275</td>
<td>0.1857</td>
<td>0.21545</td>
</tr>
<tr>
<td>P_5</td>
<td>0.0035746</td>
<td>0.10779</td>
<td>0.29795</td>
<td>0.062191</td>
<td>0.09103</td>
<td>0.23609</td>
</tr>
<tr>
<td>P_{11}</td>
<td>0.0015977</td>
<td>0.056482</td>
<td>0.16725</td>
<td>0.025458</td>
<td>0.047333</td>
<td>0.16291</td>
</tr>
<tr>
<td>P_{21}</td>
<td>0.0006964</td>
<td>0.030168</td>
<td>0.12362</td>
<td>0.010751</td>
<td>0.025912</td>
<td>0.12202</td>
</tr>
</tbody>
</table>

Note that in this case the full-moment \( P_n \) model converges faster toward the Fokker–Planck solution than the mixed \( MP_n \) model. Here we always compare approximately the same number of variables (e.g., \( P_{21} \) with \( MP_{10} \) to avoid instabilities in the \( P_n \) model). MME (MM_1) and mixed Kershaw closures (MK_1 and MK_2) perform well.

As shown in Figure 4 the MM_1 solution is indistinguishable from the Fokker–Planck solution, while M_1 and P_5 behave differently. The \( L^1 \) errors in Table 3 show that (up to numerical deviations of order \( 0.1h = \frac{1}{10n_x} \)) MM_1, MK_1, and MK_2 give the same results as Fokker–Planck. This is to be expected since the Fokker–Planck solution is a linear combination of two Dirac deltas in \( \mu \) which can be arbitrarily closely prescribed by the MM_1 model and exactly prescribed by MK_1 and MK_2.

6.4. **Rectangular initial condition (IC).** In this test case we start with an isotropic distribution where nearly all mass is concentrated in the middle of the domain \( X = [0, 7] \):

\[ \psi(x, \mu, 0) = \begin{cases} 10 & \text{if } x \in [3, 4], \\ 10^{-4} & \text{else}. \end{cases} \]

At the boundary we have

\[ \psi(0, \mu > 0, t) = 10^{-4}, \quad \psi(7, \mu < 0, t) = 10^{-4}. \]
Fig. 4. Solutions of the two-beam test case. $MM_1$ is approximately identical to Fokker–Planck.
Fig. 5. Solutions of the rectangular IC test case.
Table 4
Relative $L^p$ errors, $p \in \{1, 2, \infty\}$, for different models with respect to the Fokker–Planck solution with $n_x = 1000$, $n_\mu = 800$. Rectangular IC test case. Characteristic errors evaluated at $t = 1$.

<table>
<thead>
<tr>
<th>Model</th>
<th>$L^1$</th>
<th>$L^2$</th>
<th>$L^\infty$</th>
<th>Char. $L^1$</th>
<th>Char. $L^2$</th>
<th>Char. $L^\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MM1</td>
<td>0.019549</td>
<td>0.23133</td>
<td>0.16337</td>
<td>0.16749</td>
<td>0.17692</td>
<td>0.2757</td>
</tr>
<tr>
<td>MK1</td>
<td>0.022142</td>
<td>0.28221</td>
<td>0.3131</td>
<td>0.2479</td>
<td>0.27771</td>
<td>0.62157</td>
</tr>
<tr>
<td>MK2</td>
<td>0.016665</td>
<td>0.25969</td>
<td>0.22311</td>
<td>0.096056</td>
<td>0.11219</td>
<td>0.2806</td>
</tr>
<tr>
<td>MP5</td>
<td>0.0027983</td>
<td>0.038741</td>
<td>0.025006</td>
<td>0.012051</td>
<td>0.01356</td>
<td>0.021774</td>
</tr>
<tr>
<td>MP10</td>
<td>0.00017611</td>
<td>0.0028254</td>
<td>0.0024361</td>
<td>0.0019917</td>
<td>0.0021764</td>
<td>0.0036513</td>
</tr>
<tr>
<td>M1</td>
<td>0.03298</td>
<td>0.39211</td>
<td>0.22989</td>
<td>0.14476</td>
<td>0.14726</td>
<td>0.17734</td>
</tr>
<tr>
<td>P11</td>
<td>0.0037782</td>
<td>0.051546</td>
<td>0.0377</td>
<td>0.022936</td>
<td>0.029121</td>
<td>0.055617</td>
</tr>
<tr>
<td>P21</td>
<td>0.00020889</td>
<td>0.0047516</td>
<td>0.0090014</td>
<td>0.0040468</td>
<td>0.0058232</td>
<td>0.016255</td>
</tr>
</tbody>
</table>

Fig. 6. Cut of the solutions of the rectangular IC test case evaluated at the characteristic time $t = 1$.

We use a slightly scattering material without absorption; therefore, $\sigma_a = 0$ and $T = 10^{-2}$.

Here, the mixed polynomial $MP_n$ models perform better than the full spherical harmonics $P_n$ models. $MM_1$ performs slightly better than $MK_1$ but worse than $MK_2$. This can be expected since more waves for the solution are available. By the same argument the $M_1$ model performs worse than the other nonlinear models. As one can see, in the space-time plots in Figure 5 the nonlinear closures are exact at the beginning for a short period of time. This is true until the Fokker–Planck distribution
is no longer isotropic. In Figure 6 the different wave packages of the models can be seen. Convergence errors for this test case can be found in Table 4.

6.5. Source-Beam. This test has been used in [20]. However, we do not use the smoothed version but the discontinuous one with $X = [0, 3]$, 

$$
\begin{align*}
\sigma_a(x) &= \begin{cases} 
1 & \text{if } x \leq 2, \\
0 & \text{else,}
\end{cases} \\
T(x) &= \begin{cases} 
0 & \text{if } x \leq 1, \\
2 & \text{if } 1 < x \leq 2, \\
10 & \text{else,}
\end{cases} \\
Q(x) &= \begin{cases} 
1 & \text{if } 1 \leq x \leq 1.5, \\
0 & \text{else},
\end{cases}
\end{align*}
$$

and initial and boundary conditions

$$
\begin{align*}
\psi(x, \mu, 0) &= 10^{-4}, \\
\psi(0, \mu > 0, t) &= \delta(\mu - 1), \\
\psi(3, \mu < 0, t) &= 10^{-4}.
\end{align*}
$$

As shown in Figure 7, the full-moment models are not able to reproduce the Fokker–Planck solution, not even in steady state ($t = 4$). The MK$_2$ model provides reasonably better results than MM$_1$ and MK$_1$. This behavior is also visible in the convergence errors shown in Table 5.

<table>
<thead>
<tr>
<th>Table 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative $L^p$ errors, $p \in {1, 2, \infty}$, for different models with respect to the Fokker–Planck solution $n_x = 1000$, $n_\mu = 800$. Source-Beam test case. Characteristic errors evaluated at $t = 2$.</td>
</tr>
<tr>
<td>Model</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>MM$_1$</td>
</tr>
<tr>
<td>MK$_1$</td>
</tr>
<tr>
<td>MK$_2$</td>
</tr>
<tr>
<td>MP$_{10}$</td>
</tr>
<tr>
<td>M$_1$</td>
</tr>
<tr>
<td>P$_{21}$</td>
</tr>
</tbody>
</table>

7. Conclusions and outlook. We have established a theory for a mixed-moment realizability approach in one space dimension. The resulting minimum-entropy models as well as the corresponding Kershaw closures perform well in the numerical tests that we performed. The zero net-flux problem of full-moment minimum-entropy and Kershaw closures can be overcome. Additionally a consistent approximative model for the Fokker–Planck equation with Laplace–Beltrami operator has been derived.

Using the techniques provided in this paper an approximation using arbitrary numbers of moments can be derived.

Compared with the corresponding minimum-entropy model (which provides the same advantages), the MK$_n$ models can be evaluated much more efficiently since the closure can be evaluated analytically without the need of many nonlinear solves. Thus it can compete in speed with the corresponding polynomial MP$_n$ model.

We want to emphasize that it is in principle possible to use the QMOM approach not only for full moments but also for other partial or mixed moments. This can be done by simply plugging $\psi_{QMOM}$ into the corresponding moment problem, ensuring that the atoms are within the right support of the measure (e.g., for half moments in $[-1, 0]$ and $[0, 1]$, respectively). The influence of the moment problem on the atoms
Until now there exists no general realizability theory for the moment problems in three dimensions. Explicit necessary and sufficient conditions have only been provided for up to $n = 2$ [27].

A lot of work has been done to derive similar partial-moment models in three dimensions. One general approach is the minimum-entropy quarter-moment approximation [18] which performs well for the transport equation. However, it has the same problems as the half-moment approximation in one dimension—namely, that the Fokker–Planck operator must be closed consistently. The typical strategy which leads to decoupling provides unphysical results and does not reproduce the correct solution. Again a formulation using a continuous distribution function is expected to provide reasonably good results in many situations. However, realizability theory

and densities in the QMOM algorithm is to the best of the authors’ knowledge not investigated yet and may be subject to future research.

Fig. 7. Solutions of the Source-Beam test case.
for these mixed moments with $n > 2$ still has to be developed. We assume that the procedure works as in this paper, but without a general theory for partial moments it seems hard to formulate the correct conditions.

MME moments of arbitrary order suffer from the problem of numerical inversion of the closure, similar to full-moment $M_n$ [25]. Therefore, mixed Kershaw closures (which can be closed analytically) are of general interest because they can be evaluated with the same effort as spherical harmonics while guaranteeing the realizability (and especially the positivity) of the solution.

REFERENCES


