GENERALIZED RICCATI EQUATIONS AND STABILIZATION OF STOCHASTIC SYSTEMS

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Abstract: This paper is concerned with mean-square stabilization of stochastic linear systems by linear state feedback. We derive a generalized Riccati-type matrix equation to characterize stabilizability and develop a method to solve this equation. Numerical examples are given to illustrate our results.

Keywords: stochastic systems, mean-square stabilization; Newton's method

1. INTRODUCTION

We address the problem of stabilizing stochastic linear systems with state- and input-dependent noise. While in the deterministic case one can reduce linear systems to normal forms, from which their stabilizability properties can be read, to our knowledge there are no general stabilizabilty criteria available in the stochastic case. On the other hand the problem of *optimal* stabilization of stochastic systems (in the LQ-sense see e.g. (Wonham, 1967), (Haussmann, 1971), or in the H^∞-sense see e.g. (Hinrichsen and Pritchard, 1998), (Ugrinovskii, 1998)) has been studied over more then thirty years; but in addressing the optimal stabilization problem one usually assumes a non-optimal stabilization to be already given.
Our aim is to derive an algorithm that can be applied without any a priori knowledge on the stabilizability of the system to compute a stabilizing feedback-gain matrix if the system is stabilizable. To this end we consider the Riccati equation of the stochastic LQ-problem. Thus, actually, we are concerned with an optimal stabilization problem, too. To solve the Riccati equation we present a nonlocal convergence result for Newton's method in a partially ordered space. The idea is, that after an appropriate rational transformation, we can solve the Riccati equation by Newton's method starting at a scalar multiple of the identity matrix. We illustrate our results by some numerical examples from the literature. In particular we give an example, that our method can also be applied to solve Riccati-type matrix equations occuring in the suboptimal H^∞-type stabilization problem.

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2. STOCHASTIC CONTROL SYSTEMS

Regard the linear Itô differential equation

$$dx(t) = Ax(t)dt + Bu(t)dt + \sum_{i=1}^{N} A_0^i x(t)dw_i(t) + \sum_{i=1}^{N} B_0^i(t)u(t)dw_i(t) \qquad (1)$$

where $(A,\, B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$ and

$$(A_0^i, B_0^i) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}, \quad i = 1,\dots,N.$$

The $(w_i(t))_{t\in\mathbb{R}_+}$ are uncorrelated normed zero mean real Wiener processes on a probability space $(\Omega, \mathcal{F}, \mu)$ with respect to an increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$.
Let $L^2_w(\mathbb{R}_+, \mathbb{K}^m)$ denote the corresponding space of non-anticipating stochastic processes u with

$$\|u(\cdot)\|^2_{L^2_w} = \mathcal{E}\left(\int_0^\infty \|u(t)\|^2 dt\right) < \infty,$$

where $\mathcal{E}$ denotes expectation.
The process u is regarded as the control input of the given system.
It is known from Itô-theory, that for all $(x_0, u) \in \mathbb{K}^n \times L^2_w(\mathbb{R}_+, \mathbb{K}^m)$ there exists a unique solution $x(\cdot, x_0, u)$ of (1) with initial value $x(0, x_0, u) = x_0$.

Remark 1. By our assumption the covariance matrix Q of the N-dimensional Wiener process $[w_1(t), \ldots, w_N(t)]^T$ is the identity. This is not a restriction, for if $Q \neq I$, we can regard uncorrelated and normed linear combinations of the w_i, if we transform the matrices A_0^i and B_0^i as follows:

$$\left[\, \tilde{A}_0^i \;\; \tilde{B}_0^i \,\right]_{i=1}^N = (\sqrt{Q} \otimes I_n) \left[\, A_0^i \;\; B_0^i \,\right]_{i=1}^N ,$$

where $\otimes$ denotes the Kronecker product, and $\left[\, A_0^i \;\; B_0^i \,\right]_{i=1}^N \in \mathbb{K}^{nN \times (n+m)}$.

Definition 2. A solution $x(\cdot, x_0, u)$ (with given initial vector x_0 and control law $u \in L^2_w(\mathbb{R}_+, \mathbb{K}^m)$ is said to be *(exponentially mean square) stable* if $x(\cdot, x_0, u) \in L^2_w(\mathbb{R}_+, \mathbb{K}^n)$ for all $x_0 \in \mathbb{K}^n$, or equivalently if there exist κ, $\omega > 0$

$$\forall x_0 \in \mathbb{K}^n,\, t \geq 0 : \; \mathcal{E}\|x(t)\|^2 \leq \kappa e^{-2\omega t}\|x_0\|^2 .$$

The system (1) is said to be *internally (exponentially mean square) stable* if for all $x_0 \in \mathbb{K}^n$ the uncontrolled solution $x(\cdot, x_0, 0)$ is stable.
We call system (1) *(open-loop) stabilizable* if for all $x_0 \in \mathbb{K}^n$ there exists a control $u_{x_0} \in L^2_w(\mathbb{R}_+, \mathbb{K}^m)$ such that $x(\cdot, x_0, u_{x_0})$ is stable.
Finally the system (1) is called *stabilizable by static linear state-feedback* if there exists a matrix $F \in \mathbb{K}^{m \times n}$ such that the closed loop system

$$dx = (A + BF)x dt + \sum_{i=1}^N (A_0^i + B_0^i F) x dw_i$$

is internally stable.

In the theory of *deterministic* linear systems it is well-known, that open-loop stabilizability implies stabilizability by static linear state-feedback and both are equivalent to the existence of LQ-optimal stabilizing state-feedback controls. The latter condition again can be expressed equivalently via the solvability of a certain algebraic Riccati equation.

To state an analogous result for the stochastic case we introduce the Riccati-type rational matrix operator

$$\mathcal{R}(X) = P(X) - S(X)Q(X)^{-1}S(X)^* . \qquad (2)$$

Here P, S and Q are affine linear operators on the real vector space $\mathcal{H}^n \subset \mathbb{K}^{n \times n}$ of $n \times n$ Hermitian matrices (with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) of the form

$$\begin{aligned} P(X) &= A^*X + XA + \sum_{i=1}^N A_0^{i*} X A_0^i + P_0 , \\ S(X) &= XB + \sum_{i=1}^N A_0^{i*} X B_0^i + S_0 , \\ Q(X) &= \sum_{i=1}^N B_0^{i*} X B_0^i + Q_0 . \end{aligned}$$

The matrices A, B, A_0^i and B_0^i are the same as in (1), whereas $P_0 \in \mathcal{H}^n$, $S_0 \in \mathbb{K}^{n \times m}$ and $Q_0 \in \mathcal{H}^m$ are the blocks of a given weight matrix

$$M = \begin{bmatrix} P_0 & S_0 \\ S_0^* & Q_0 \end{bmatrix} > 0 .$$

It is clear that the rational matrix operator $\mathcal{R}$ is well-defined and analytic on the domain

$$\operatorname{dom} \mathcal{R} = \{X \in \mathcal{H}^n \mid \det Q(X) \neq 0\}.$$

Theorem 3. The following are equivalent:

(i) System (1) is open-loop stabilizable.
(ii) System (1) is stabilizable by static linear state-feedback.
(iii) The Riccati-type matrix equation $\mathcal{R}(X) = 0$ has a positive definite solution $X > 0$.

Moreover, if $X > 0$ solves $\mathcal{R}(X) = 0$, then the static linear feedback control $u = Q(X)^{-1}S(X)^*x$ minimizes the cost functional

$$J(x_0, u) = \mathcal{E} \int_0^\infty \begin{bmatrix} x(t; x_0, u) \\ u(t) \end{bmatrix}^* M \begin{bmatrix} x(t; x_0, u) \\ u(t) \end{bmatrix} dt .$$

This result in an infinite-dimensional Hilbert space setting can be found in (Tessitore, 1992). For the convenience of the reader we provide a sketch of the proof in our finite-dimensional case in the Appendix.

In the following we will deal with the matrix equation $\mathcal{R}(X) = 0$, which we call the *stochastic algebraic Riccati equation.* This equation combines features of both the continuous-time and the discrete-time algebraic Riccati equation from deterministic LQ-control and, in fact, contains these as special cases.

A first analysis of the stochastic algebraic Riccati equation seems to have been undertaken in (Wonham, 1968) whose results were refined later e.g. in (Haussmann, 1971). Basically Wonham proposed a non-local Newton iteration to solve the equation $\mathcal{R}(X) = 0$ under the assumption that the system (1) is stabilizable and a stabilizing (in the sense of Def. 10 below) initial matrix X_0 is given. The same method for the deterministic Riccati equation was proposed in (Kleinman, 1968) and developed further e.g. in (Coppel, 1974).

But unlike in the deterministic case there are no simple criteria available, whether a given *stochastic* system is stabilizable at all; and if the system is stabilizable it is also not clear, how to find a stabilizing matrix and thus an initial matrix for the iteration.
To circumvent these difficulties, we present a general framework for the non-local Newton iteration to be applicable. Then we show that after a certain transformation the equation $\mathcal{R}(X) = 0$ can be solved by this method without any previous knowledge on the stabilizability of (1).

3. NEWTON'S METHOD IN A PARTIALLY ORDERED VECTOR SPACE

In (Damm and Hinrichsen, 1999) we have shown that the non-local Newton iteration for the different types of Riccati equations relies only on a few properties of a certain class of concave operators in a partially ordered vector space; therefore this method can be applied to more general matrix equations.
We briefly summarize the main facts and definitions; for convenience we restrict ourselves to the partially ordered space of Hermitian matrices.

Let $\mathcal{H}^n \subset \mathbb{K}^{n\times n}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) as above denote the real space of real or complex $n \times n$ Hermitian matrices. By $\mathcal{H}^n_+ = \{X \in \mathcal{H}^n \mid X \geq 0\}$ we denote the closed convex cone of nonnegative definite matrices and by $\operatorname{int}\left(\mathcal{H}^n_+\right)$ its interior, i.e. the open cone of positive definite matrices. The cone $\mathcal{H}^n_+$ induces a partial ordering on $\mathcal{H}^n$: we write $X \geq Y$, if $X - Y \in \mathcal{H}^n_+$.

3.1 *Resolvent positive operators*

An important tool in our approach is the theory of positive operators in ordered vector spaces based on the work (Krein and Rutman, 1950).

Definition 4. A linear operator $\mathcal{T} : \mathcal{H}^n \to \mathcal{H}^m$ is called *positive* ($\mathcal{T} \geq 0$) if it maps $\mathcal{H}^n_+$ to $\mathcal{H}^m_+$. A linear operator $\mathcal{T} : \mathcal{H}^n \to \mathcal{H}^n$ is called *inverse positive* if it is invertible and $\mathcal{T}^{-1}$ is positive; it is called *resolvent positive*, if for all sufficiently large $\alpha > 0$ the resolvent operator $(\alpha I - \mathcal{T})^{-1}$ is positive.
By $\sigma(\mathcal{T})$ we denote the spectrum of $\mathcal{T}$, and by $\rho(\mathcal{T}) = \max\{|\lambda|; \lambda \in \sigma(\mathcal{T})\}$ the spectral radius.

Example 5.
(i) Let $A_0 \in \mathbb{K}^{n\times n}$, then the operator $\Pi : \mathcal{H}^n \to \mathcal{H}^n$ defined by $\Pi(X) = A_0^* X A_0$ is positive. If A_0 is nonsingular, then it is also inverse positive.
(ii) All positive operators Π are also resolvent positive, since for $\alpha > \rho(\Pi)$ the resolvent $(\alpha I - \Pi)^{-1} = \sum_{k=0}^{\infty} \alpha^{-(k+1)} \Pi^k$ is positive.
(iii) Given $A \in \mathbb{K}^{n\times n}$, the associated *Lyapunov operator* $\mathcal{L}_A : \mathcal{H}^n \to \mathcal{H}^n$, $\mathcal{L}_A(X) = A^* X + XA$, is resolvent positive but, in general, not positive.

Theorem 6. (Schneider, 1965) Let $\mathcal{L} : \mathcal{H}^n \to \mathcal{H}^n$ be resolvent positive and $\Pi : \mathcal{H}^n \to \mathcal{H}^n$ be positive. Then the following are equivalent:

(i) $\mathcal{L} + \Pi$ is stable, i.e. $\sigma(\mathcal{L} + \Pi) \subset \mathbb{C}_-$.
(ii) $-(\mathcal{L} + \Pi)$ is inverse positive.
(iii) $\sigma(\mathcal{L}) \subset \mathbb{C}_-$ and $\rho\left(\mathcal{L}^{-1}\Pi\right) < 1$.
(iv) $\exists X > 0 :\quad (\mathcal{L} + \Pi)(X) < 0$.

Remark 7. If $\mathcal{L} = \mathcal{L}_A(X) = A^*X + XA$ and $\Pi(X) = \Pi_{A_0}(X) = \sum_{i=1}^{N} A_0^{i\,*} X A_0^i$ with matrices A and A_0^i as in (1) then any of the conditions in the theorem is equivalent to the internal stability of system (1), (Khasminskij, 1980).

3.2 *Concavity, stabilizability, and Newton's method*

Let $\mathcal{G}$ be a nonlinear Fréchet-differentiable operator from some open domain $\operatorname{dom}\mathcal{G} \subset \mathcal{H}^n$ to $\mathcal{H}^n$. Let further $\operatorname{dom}_+\mathcal{G}$ be some nonempty open convex subset of $\operatorname{dom}\mathcal{G}$. By $\mathcal{G}'_X(H)$ we denote the derivative of $\mathcal{G}$ at X in direction H.

Definition 8. The operator $\mathcal{G}$ is said to be $\operatorname{dom}_+\mathcal{G}$-*concave* on $\operatorname{dom}\mathcal{G}$ if for all $Y \in \operatorname{dom}\mathcal{G}$ and $Z \in \operatorname{dom}_+\mathcal{G}$

$$\mathcal{G}(Y) - \mathcal{G}(Z) + \mathcal{G}'_Y(Z - Y) \geq 0.$$

In geometric terms $\mathcal{G}$ is $\operatorname{dom}_+\mathcal{G}$-concave on $\operatorname{dom}\mathcal{G}$, if the graph of $\mathcal{G}$ over $\operatorname{dom}_+\mathcal{G}$ lies below all tangents to the graph of $\mathcal{G}$ at arbitrary points in $\operatorname{dom}\mathcal{G}$. Thus $\operatorname{dom}_+\mathcal{G}$-concavity on $\operatorname{dom}\mathcal{G}$ implies concavity on $\operatorname{dom}_+\mathcal{G}$.

Example 9. Regard the Riccati operator $\mathcal{R}$ from (2) and let $\operatorname{dom}_+\mathcal{R} = \{X \in \mathcal{H}^n \mid Q(X) > 0\} \subset \operatorname{dom}\mathcal{R}$. Then the operator $\mathcal{R}$ is $\operatorname{dom}_+\mathcal{R}$-concave on $\operatorname{dom}\mathcal{R}$, (Damm and Hinrichsen, 1999).

Definition 10. The operator $\mathcal{G}$ is said to be *stabilizable* if there exists a matrix $X \in \operatorname{dom}\mathcal{G}$, such that $\sigma(\mathcal{G}'_X) \subset \mathbb{C}_-$. The matrix X is then called *stabilizing* (for $\mathcal{G}$).

Example 11. The Riccati operator $\mathcal{R}$ from (2) is stabilizable if and only if the underlying system (1) is stabilizable. A stabilizing matrix X yields the stabilizing feedback-gain matrix $F = Q(X)^{-1}S(X)^*$, (Damm and Hinrichsen, 2000).

Theorem 12. Let $\mathcal{G}$ and the sets $\operatorname{dom}\mathcal{G}$ and $\operatorname{dom}_+\mathcal{G}$ be given as above and assume that the following conditions hold:

(a) The set $\operatorname{dom}_+ \mathcal{G}$ is *saturated above*, i.e. $\operatorname{dom}_+ \mathcal{G} = \operatorname{dom}_+ \mathcal{G} + \mathcal{H}_+^n$.
(b) The operator $\mathcal{G}$ is $\operatorname{dom}_+ \mathcal{G}$-concave on $\operatorname{dom} \mathcal{G}$.
(c) For all $X \in \operatorname{dom} \mathcal{G}$ the derivative $\mathcal{G}'_X$ is resolvent positive.

Assume further that $\mathcal{G}$ is stabilizable and let X_0 be stabilizing.
If the inequality $\mathcal{G}(X) \geq 0$ has a solution $\hat{X}$ in $\operatorname{dom}_+ \mathcal{G}$, then the iteration scheme

$$X_{k+1} = X_k - (\mathcal{G}'_{X_k})^{-1}(\mathcal{G}(X_k)) \qquad (3)$$

defines a sequence (X_k) in $\operatorname{dom} \mathcal{G}$ with the following properties:

(i) $\forall k = 1, 2, \ldots : \; X_k \geq X_{k+1} \geq \hat{X}$ and thus $X_k \in \operatorname{dom}_+ \mathcal{G}$. Moreover $\sigma(\mathcal{G}'_{X_k}) \subset \mathbb{C}_-$.
(ii) (X_k) converges to a limit matrix $X_\infty \in \operatorname{dom}_+ \mathcal{G}$ that satisfies $\mathcal{G}(X_\infty) = 0$ and is the largest solution of $\mathcal{G}(X) \geq 0$.

If the inequality $\mathcal{G}(X) \geq 0$ is not solvable in $\operatorname{dom}_+ \mathcal{G}$, then either (i) fails, i.e. for some iterate X_k we have $X_k \notin \operatorname{dom}_+ \mathcal{G}$ or $\sigma(\mathcal{G}'_{X_k}) \not\subset \mathbb{C}_-$; or (ii) fails i.e. the X_k diverge to ∞ or the limit matrix X_∞ is a boundary point of $\operatorname{dom}_+ \mathcal{G}$.

It can be shown, that the operator $\mathcal{R}$ satisfies the assumptions (a), (b), and (c); but if we want to apply the theorem in order to solve the equation $\mathcal{R}(X) = 0$, we need to find a stabilizing matrix X_0 which might be hard.

3.3 *An algorithmic test for stabilizability*

Instead of the operator $\mathcal{R}$ we consider on
$\operatorname{dom} \mathcal{G} = \{Y \in \mathcal{H}^n \mid \det Y \neq 0, Y^{-1} \in \operatorname{dom} \mathcal{R}\}$
the transformed operator

$$\mathcal{G} : \operatorname{dom} \mathcal{G} \to \mathcal{H}^n, \quad \mathcal{G}(Y) = -Y\mathcal{R}(Y^{-1})Y.$$

If we set $\operatorname{dom}_+ \mathcal{G} = \operatorname{int} \mathcal{H}_+^n$, then it can be shown that the triple $(\mathcal{G}, \operatorname{dom} \mathcal{G}, \operatorname{dom}_+ \mathcal{G})$ satisfies the conditions (a), (b), and (c) of Thm. 12. Since our proof (in particular of (c)) involves lengthy technical computations, the details shall be omitted here.
Moreover a direct calculation (Damm and Hinrichsen, 2000) shows that

$$\lim_{\alpha \to +\infty} \frac{1}{\alpha} \, \mathcal{G}'_{\alpha I} = -\mathcal{L}_{P_0 - S_0 Q_0^{-1} S_0^*} .$$

Since $M > 0$ it follows that also $P_0 - S_0 Q_0^{-1} S_0^* > 0$, whence $-\mathcal{L}_{P_0 - S_0 Q_0^{-1} S_0^*}$ is stable. Thus for large $\alpha > 0$ the matrix $Y_0 = \alpha I$ stabilizes $\mathcal{G}$, i.e. $\sigma(\mathcal{G}'_{\alpha I}) \subset \mathbb{C}_-$ and can serve as a starting point for the Newton iteration.

To stabilize system (1) one can thus proceed as follows:

- Choose a matrix $M > 0$ (e.g. $M = I$) and regard the corresponding operator $\mathcal{G}$.
- Find $\alpha > 0$ large enough, s.t. $\sigma(\mathcal{G}'_{\alpha I}) \subset \mathbb{C}_-$.
- Start the standard Newton-iteration (3) for $\mathcal{G}$ with initial matrix $Y_0 = \alpha I$.
- Watch in each step, whether the iterate Y_k is stabilizing for $\mathcal{G}$. If one Y_k is not stabilizing, then (1) is not stabilizable.
- If the Y_k converge to some $Y_\infty \in \operatorname{int} \mathcal{H}_+^n$ then (1) is stabilizable and $F = Q(Y_\infty^{-1}) S(Y_\infty^{-1})^*$ is a stabilizing feedback-gain matrix.
  If otherwise the Y_k leave $\operatorname{int} \mathcal{H}_+^n$ or converge to some $Y_\infty \in \partial \mathcal{H}_+^n$ then (1) is not stabilizable.

In extreme cases it is, of course, numerically hard to decide whether $X_\infty \in \partial \mathcal{H}_+^n$ or not; one can use a stopping criterion, and regard the system as not stabilizable if e.g. $\det Y_k < \varepsilon$ for some k.

4. EXAMPLES

We first regard some simple examples that illustrate differences between the stochastic and the deterministic case. Then we apply our algorithm to some problems from the literature.

4.1 *The scalar case*

Consider the scalar stochastic system

$$dx = ax\, dt + bu\, dt + a_0 x\, dw + b_0 u\, dw$$

with real numbers a, b, a_0, and b_0. By Thm. 3 this system is stabilizable if and only if there exists an $f \in \mathbb{R}$ such that the closed-loop system

$$dx = (a + bf)x\, dt + (a_0 + b_0 u)\, dw$$

is stable i.e. $2(a+bf)+(a_0+b_0 f)^2 < 0$ (by Thm. 6). But such an f cannot exist if $b_0 \neq 0$ and the discrimant of the quadratic equation is negative, i.e. $-2ab_0^2 + 2ba_0b_0 + b^2 < 0$. Thus stabilizability in the presence of control dependent noise is not a generic property.

4.2 *A non-stabilizable system*

Now we regard a system with state-dependent noise only:

$$dx = Ax\, dt + Bu\, dt + A_0 x\, dw .$$

One might ask, whether controllability of the pair (A, B) is sufficient for the system to be stabilizable. This is not the case. As an example

we choose $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The pair (A, B) is controllable, but the system is not stabilizable. To see this we regard the operator $\mathcal{L}_{A+BF} + \Pi_{A_0} : \mathcal{H}^2 \to \mathcal{H}^2$ as in Rem. 7, with $F = [f_1, f_2] \in \mathbb{R}^{1\times 2}$.
Since $\mathcal{H}^2 \subset \mathbb{R}^{2\times 2}$ is a three-dimensional space, we can represent this operator by the matrix

$$M_F = \begin{bmatrix} 2 & 2 & 1 \\ f_1 & 3+f_2 & 1 \\ 1 & 2f_1 & 2+2f_2 \end{bmatrix} \in \mathbb{R}^{3\times 3},$$

with respect to the basis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
But $\det(M_F - I) = 2(f_1 - f_2 - 1)^2 \geq 0$, whence for all F the matrix M_F (and thus the operator $\mathcal{L}_{A+BF} + \Pi_{A_0}$) has an eigenvalue λ with $\operatorname{Re}\lambda \geq 1$. Though we can move the eigenvalues of $\mathcal{L}_{A+BF}$ as far as we wish into the left half plane, we cannot stabilize the operator $\mathcal{L}_{A+BF} + \Pi_{A_0}$.
If, however, we decrease the noise intensity and replace A_0 by σA_0, then our test produces a stabilizing feedback for $\sigma^2 < 1/2$. For $\sigma^2 \geq 1/2$ our test suggests, that the system is not stabilizable: The iterates converge to 0.
We also might alter the matrix A to $A - \alpha I$. By the above discussion it is clear, that for $\alpha < 1/2$ the system cannot be stabilizable. For $\alpha \geq 1/2 + 10^{-10}$, however, our algorithm (starting at $Y_0 = I$, if $M = -I$) finds a stabilizing feedback, whereas for $\alpha = 1/2$ the iterates converge to 0.
If e.g. $\alpha = 1/2 + 10^{-6}$, then for $F = [1225.7, 1226.7]$ we have $\max \operatorname{Re}\sigma\left(\mathcal{L}_{A-\alpha I+BF} + \Pi_{A_0}\right) \approx -2\cdot 10^{-6}$.

4.3 *A linear oscillator with uncertain parameters*

In (Biswas, 1998) the following model of a harmonic linear oscillator with random damping and restoring forces was analyzed:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1(t) & a_2(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ a_3(t) \end{bmatrix} u .$$

The parameters a_1, a_2 and a_3 are uncertain and modelled as uncorrelated random Gaussian processes with mean values $\bar{a}_1 = -1$, $\bar{a}_2 = 2$, and $\bar{a}_3 = 1$ and covariance σ^2. The model can thus be written in the form (1) with $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A_0^1 = \begin{bmatrix} 0 & 0 \\ \sigma & 0 \end{bmatrix}$, $A_0^2 = \begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix}$, $B_0 = \begin{bmatrix} 0 \\ \sigma \end{bmatrix}$.
Biswas derives a sufficient criterion for stabilizability, which allows him to stabilize the system up to noise intensities $\sigma^2 < 0.2247$.
Our method yields a slight improvement and produces a stabilizing feedback for $\sigma^2 = 0.2265$.

4.4 *An example of satellite dynamics*

Also in (Biswas, 1998) a model for the dynamics of a satellite is regarded. For details we refer to the original paper. The linear model has the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -c(1+\sigma^2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \xi_1(t) & \xi_2(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1+\eta(t) \end{bmatrix} u .$$

Here ξ_1 and ξ_2 are correlated zero-mean Gaussian processes with covariances $\mathcal{E}(\xi_1^2) = \sigma^2$, $\mathcal{E}(\xi_2^2) = (4\sigma^2 + 2\sigma^4)c^2$, and $\mathcal{E}(\xi_1\xi_2) = 2\sigma^2 c$; the parameter c is taken as $c = 0.4266$, and η is a zero-mean Gaussian process with covariance 0.1.
Note that in this example the noise terms are correlated. We can, however, transform the system as indicated in Rem. 1 and apply our results to the arising system with uncorrelated noise terms.

Biswas shows that the system can be stabilized for noise intensities $\sigma^2 < 3.865$, whereas our algorithm produces stabilizing feedback-gain matrices up to the value $\sigma^2 = 5.306$.

4.5 *A matrix equation from robust control*

To demonstrate the applicability of our method in robust stochastic control, we consider a linear model (of a two-mass spring system) from (Ugrinovskii, 1998), where further details can be found:

$$\begin{aligned} dx &= (Ax + B_1 v + B_2 u)dt + A_0 x dw , \\ z &= Cx + Du . \end{aligned}$$

We want to find a feedback-law $u = Fx$, such that the closed-loop system is internally (i.e. for $v \equiv 0$) stable and the effect of the disturbance $v \in L^2(\mathbb{R}_+, \mathbb{R}^\ell)$ on the output $z \in L^2(\mathbb{R}_+, \mathbb{R}^q)$ is small; see also (Hinrichsen and Pritchard, 1998). This problem is a stochastic analog of the H^∞-control problem and leads to the parametrized matrix equation (where the attenuation value γ is an upper bound for the effect of v on z):

$$\begin{aligned} \mathcal{R}_\gamma(X) &= A^*X + XA + A_0^* X A_0 + C^*C \\ &\quad - X\left(B_2(D^*D)^{-1}B_2^* - \gamma^{-2}B_1B_1^*\right)X = 0 , \end{aligned} \tag{4}$$

to be solved in $\operatorname{int}\mathcal{H}_+^n$.
As in Sec. 3.3 we regard the transformed operator

$$\mathcal{G}(Y) = \mathcal{G}_\gamma(Y) = -Y\mathcal{R}_\gamma(Y^{-1})Y$$

on $\operatorname{dom}\mathcal{G} = \{Y \in \mathcal{H}^n \mid \det Y \neq 0\}$, and with $\operatorname{dom}_+\mathcal{G} = \operatorname{int}\mathcal{H}_+^n$ the triple $(\mathcal{G}, \operatorname{dom}\mathcal{G}, \operatorname{dom}_+\mathcal{G})$ satisfies (a), (b), and (c) in Thm. 12.

To solve (4) we thus need to find an $Y_0 \in \operatorname{dom} \mathcal{G}$, such that the derivative operator

$$\mathcal{G}'_{Y_0} = \mathcal{L}_{A^*+C^*CY_0+A_0^*Y_0^{-1}A_0Y_0} + \Pi_{Y_0^{-1}A_0Y_0}$$

(Π and $\mathcal{L}$ as in Rem. 7) is stable. If (A, C) is observable, it can be shown, that such an Y_0 exists. Obviously Y_0 can be chosen independently of γ. For the data in (Ugrinovskii, 1998):

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5/4 & 5/4 & 0 & 0 \\ 5/4 & -5/4 & 0 & 0 \end{bmatrix}, \; A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/4 & 1/4 & 0 & 0 \\ 1/4 & -1/4 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

we found $Y_0 = 2I - (e_1^T e_4 + e_4^T e_1)$ (with canonical unit vectors e_i) to be stabilizing.

For $\gamma = 2$ our algorithm reproduces (in 10 steps) the solution obtained by Ugrinovskii; by a bisection search we found the optimal attenuation value to be $\gamma_* \approx 1.8293$. For $\gamma = 1.8293$ the solution

$$X = Y_\infty^{-1} \approx \begin{bmatrix} 4.6440 & -6.4837 & -4.7299 & -3.3828 \\ -6.4837 & 19.1359 & 17.4201 & 10.4627 \\ -4.7299 & 17.4201 & 18.9092 & 9.8687 \\ -3.3828 & 10.4627 & 9.8687 & 7.0731 \end{bmatrix} > 0$$

of $\mathcal{R}_\gamma(X) = 0$ is obtained in 13 steps, whereas for $\gamma = 1.8292$ the 11th iterate Y_{11} is not stabilizing (i.e. by Thm. 12 in this case $\mathcal{G}_\gamma(Y) = 0$ is not solvable in $\operatorname{int} \mathcal{H}_+^n$).

5. APPENDIX: PROOF OF THEOREM 3

We only sketch (i)⇒(iii). For $T > 0$ we regard the finite-horizon cost-functional

$$J_T(x_0, u) = \mathcal{E} \int_0^T \left[\cdots\right]^* M \begin{bmatrix} x(t; x_0, u) \\ u(t) \end{bmatrix} dt ,$$

and on $\operatorname{dom}_+ \mathcal{R} \supset \mathcal{H}_+^n$ (see Ex. 9) the differential Riccati-equation $\dot{P} = \mathcal{R}(P)$ with initial condition $P(T) = 0 \in \operatorname{dom}_+ \mathcal{R}$. By time-invariance there exists a number $\Delta > 0$ independent of T, such that the solution $P_T(t) = P(t; T, 0) \in \operatorname{dom}_+ \mathcal{R}$ exists on $]T - \Delta, T]$. For $0 \le t \le T < \Delta$ we can therefore regard a control of the form

$$u(t) = -Q(P_T(t))^{-1} S(P_T(t))^* x(t) + u_1(t) ,$$

where $u_1(t) \in L_w^2([0, T])$ is arbitrary. An application of Itô's formula, like e.g. in (Hinrichsen and Pritchard, 1998), yields for all $T \in [0, \Delta[$

$$J_T(x_0, u) = x_0^* P_T(0) x_0 + \mathcal{E} \int_0^T u_1^*(t) Q(P_T(t)) u_1(t)\, dt .$$

As $Q(P) > 0$ for $P \in \operatorname{dom}_+ \mathcal{R}$ the cost functional is minimized if $u_1 \equiv 0$. Obviously $P_T(0) > 0$, since $J_T(x_0, u) > 0$ for all x_0. Moreover $P_T(t) = P_{T-t}(0)$ is uniformly bounded for $t \in]T - \Delta, T]$, since $\min J_T(x_0, u) \le J(x_0, u_{x_0})$ for all $T > 0$, $x_0 \in \mathbb{K}^n$, and stabilizing $u_{x_0} \in L^2(\mathbb{R}_+, \mathbb{K}^m)$. Therefore the solution P_T can be extended to an open neighbourhood of $T - \Delta$, and it follows that for all $T > 0$ the solution P_T exists in fact on $]-\infty, T]$. For $T' > T > t$ we have $P_{T'}(t) > P_T(t)$, since $J_T(x_0, u)$ increases with T. Therefore $P_T(t)$ converges to some $P_\infty > 0$ as $T \to \infty$, where P_∞ is independent of t. By continuity $X = P_\infty$ solves $\mathcal{R}(X) = 0$ and yields an LQ-optimal feedback control.

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