Stabilization of linear systems by rotation

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Abstract

We introduce the concept of 'stabilization by rotation' for deterministic linear systems with negative trace. This concept encompasses the well known concept of "vibrational stabilization" introduced by Meerkov in the 1970s and is a deterministic version of 'stabilization by noise' for stochastic systems as introduced by Arnold and coworkers in the 1980s. It is shown that a linear system with negative trace can be stabilized by adding a skew-symmetric matrix, multiplied by a suitable scalar so-called 'gain function' (possibly a constant) which is sufficiently large. To overcome the problem of what is "sufficiently large", we also present a servo mechanism which tunes the gain function by learning from the trajectory until finally the trajectory tends to zero. This approach allows to show that one of Meerkov's assumptions for vibrational stabilization is superfluous. Moreover, while Meerkov as well as Arnold and coworkers assume that a stabilizing periodic function or the noise has sufficiently large frequency and amplitude, we also provide a servo mechanism to determine this function dynamically in a deterministic setup.

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1 Introduction

The problem of stabilization by vibration or by oscillatory inputs goes back to the 1930s if not earlier and is a longstanding problem. See the survey article "Open-loop control using oscillatory inputs" by Baillieul and Lehman [3], where both theoretical results and applications are discussed. In the present paper, we consider a system of the form

$$\dot{x} = A\,x + u \qquad \text{for} \quad A \in \mathbb{R}^{n\times n} \quad \text{with} \quad \operatorname{tr} A < 0\,, \tag{1.1}$$

and study the problem of stabilizing (1.1) by state feedback $u(t) = S(t)x(t)$ where $S(t)$ is constrained to be skew-symmetric. Since the fundamental matrix of the system $\dot{x} = S(t)x$ is confined to the group of orthogonal matrices, which can be rephrased colloquially by saying that the system basically produces rotations of the state space, this concept is called *stabilization by rotation.* It can be regarded as a method of stabilization without introducing dissipation. More precisely, if the norm $x \mapsto V(x) = \|x\|^2$ is regarded as an energy functional, then

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stabilization by rotation does not affect the change of energy along the trajectories of (1.1), i.e. $\frac{\mathrm{d}}{\mathrm{d}t}V(x(t))\big|_{u=S(t)x} = \frac{\mathrm{d}}{\mathrm{d}t}V(x(t))\big|_{u=0}$. The intuition behind this method of stabilization is that $S(t)$ mixes stable and unstable modes which, together with the assumption that $\operatorname{tr} A < 0$, implies that the stable modes dominate as soon as the mixing is strong enough.

The concept of *stabilization by vibration*, which goes back to Meerkov [12] (see also [11]), has some parallels with the present approach. Under an additional observability assumption, Meerkov proves that the system $\dot{x} = \big(A + B(t)\big)x$ can be stabilized by a zero mean periodic function $B(\cdot)$ if, and only if, $\operatorname{tr} A < 0$. Here, the frequency and the amplitude have to be sufficiently large. In general $B(t)$ is not assumed to be skew-symmetric. We introduce the concept of stabilization by vibration in Definition 2.8 and compare it with stabilization by rotation.

The idea of stabilization by rotation has been investigated in the context of *stabilization by noise* for random and for stochastic linear differential equations by Arnold, Crauel and Wihstutz in [1] (see also [2]). They show that the system $\dot{x} = \big(A+S(t)\big)x$ can be stabilized by zero mean random parameter vibrations $S(\cdot)$ if, and only if, $\operatorname{tr} A < 0$. In their approach, $S(\cdot)$ is a stochastic 'noise' process taking values in the space of skew-symmetric matrices. An essential assumption is sufficient intensity and 'richness' of the noise in the sense that enough rotations have to be excited.

Our approach interpolates – in a sense – between these two. We investigate deterministic systems

$$\dot{x} = \big(A + k(t)\Sigma_A\big)x \tag{1.2}$$

with $\operatorname{tr} A < 0$, time-varying $k : \mathbb{R} \to \mathbb{R}$, and $\Sigma_A = -\Sigma_A^T$. First, we show the existence of a skew-symmetric matrix Σ_A, such that $A + k\Sigma_A$ is stable for all constant k with $|k|$ sufficiently large. Then we give sufficient stability criteria for the time-varying system $\dot{x} = \big(A+k(t)\Sigma_A\big)x$ and, moreover, provide a servo mechanism to determine a stabilizing parameter function $k(\cdot)$ tuned by $\|x(\cdot)\|$. Finally, we generalize Meerkov's result by showing that there exist periodic functions $p(\cdot)$ with zero-mean such that $\dot{x} = \big(A + k\,p(t)\Sigma_A\big)x$ is stable for sufficiently large constant k. This shows that the observability assumption in Meerkov's existence result is superfluous. Moreover, we provide a servo mechanism $x(\cdot) \mapsto k(\cdot)$ to determine a stabilizing function k so that the solution of $\dot{x} = \big(A + k(t)\,p(t)\Sigma_A\big)x$ tends to zero for t tending to ∞. In this sense, the concepts of stabilization by random vibrations and by deterministic vibrations are encompassed in the concept of stabilization by rotation.

Contributions related to the present paper are by Morgan and Narendra [13] and Čelikovský [6], who analyze stability of systems $\dot{x} = \big(A + S(t)\big)x$ with a particular skew-symmetric function $S(\cdot)$.

Baxendale and Hennig [5] consider linear stochastic differential equations in $\mathbb{R}^2$ of the form $dx = \left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + u\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)x + \sigma\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x \circ dW_t$ with a control u bounded by K, and show that for K big enough the system can be stabilized in an almost sure as well as in an L^p sense.

Another approach goes back to Kao and Wihstutz [9, 10], who investigate stabilization of linear systems in companion form by noise. Roughly speaking, they consider the scalar equation $y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = 0$ with real coefficients a_j, which are perturbed by mean zero noise. A simplified formulation of their result is the existence of a stationary and ergodic $\mathbb{R}^n$-valued stochastic process $(\xi) = (\xi_k)_{1\le k\le n}$ such that the differential equation with coefficients $(a_k+\xi_k(t/\varepsilon))_{1\le l\le n}$ is stable for ε sufficiently small if, and only if, $a_{n-1} < 0$ (note that a_{n-1} is the trace of the associated companion form matrix). This result is not so closely related to the present approach since a system in companion form does not allow for stabilization by rotation due to the fact that one cannot add a skew-symmetric matrix without destroying the structure of the system.

Stability of stochastic and random linear systems is characterized by Lyapunov exponents. For a survey on asymptotic methods for Lyapunov exponents see Wihstutz [15], in particular with respect to the fact that the impact of noise can result in stabilization as well as in destabilization.

In order to describe the present approach in some more detail, we stress that the only knowledge of the nominal system $\dot{x} = Ax$ needed in order to construct the stabilization by rotation device are the eigenvectors of the symmetric part of A. This information yields a skew-symmetric matrix Σ_A which, when multiplied by a sufficiently large real valued function k (possibly a constant), yields stabilization of (1.2).

The skew-symmetric matrix

$$\Sigma_n \;:=\; \begin{bmatrix} 0 & & -1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix} = (\sigma_{ij})_{1\le i,j\le n} \in \mathbb{R}^{n\times n}, \quad \sigma_{ij} = \begin{cases} 0, & i=j \\ -1, & i<j \\ 1, & i>j \end{cases}, \tag{1.3}$$

plays a central rôle in this approach. If, for $A \in \mathbb{R}^{n\times n}$, the eigenvectors of $A + A^T$ are collected in a matrix U, then U is orthogonal,

$$D = U^T(A + A^T)U \qquad \text{is diagonal,} \tag{1.4}$$

and throughout the paper we write

$$\Sigma_A = U\Sigma_n U^T \quad \text{ and } \quad A_k = A + k\,\Sigma_A \;\text{ for } k \in \mathbb{R}\,. \tag{1.5}$$

Note that Σ_A is not uniquely defined by A, since the columns of U may be reordered; however, for a given matrix A, we will assume Σ_A to be fixed.

The paper is organized as follows. Section 2 states the main results on stability properties of (1.2); the proofs are relegated to Section 6, where we make use of some technical results derived in Sections 3 and 4. In Section 5, dynamic stabilization is illustrated by a numerical example.

We close the introduction with some remarks on notation.

$\operatorname{tr} A$	trace of a square matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $\operatorname{tr} A := \sum_{i=1}^n a_{ii}$
$\sigma(A)$	spectrum of $A \in \mathbb{R}^{n\times n}$
A^T	transpose of $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $A^T := (a_{ji})$
$\lambda_{\max}(A)$	$:= \max \sigma(A + A^T)$ for $A \in \mathbb{R}^{n\times n}$
$\sigma_{\min}(P), \sigma_{\max}(P)$	smallest, largest singular value of $P \in \mathbb{R}^{n\times n}$, respectively
$\kappa_2(P)$	$:= \sigma_{\max}(P)/\sigma_{\min}(P)$, condition number of non-singular $P \in \mathbb{R}^{n\times n}$
x^T, x^*	transpose of $x \in \mathbb{C}^n$, complex conjugate of $x \in \mathbb{C}^n$, respectively

$\|x\|$	$:= \sqrt{x^* x}$ for $x \in \mathbb{C}^n$
$\|A\|$	$:= \max\{\|Ax\| : \|x\| \le 1\}$ for $A \in \mathbb{C}^{n\times n}$
$\mathbb{N}_j$	$:= \{j, j+1, \ldots\}$, for $j \in \{0, 1, 2, \ldots\}$
$\mathbb{N}$	$:= \mathbb{N}_1$
$\mathrm{so}(n, \mathbb{R})$	skew symmetric matrices in $\mathbb{R}^{n\times n}$, characterized by $\Sigma^T = -\Sigma$ for $\Sigma \in \mathrm{so}(n, \mathbb{R})$
$\mathbb{C}_-$	$:= \{s \in \mathbb{C} : \mathrm{Re}\, s < 0\}$
$f(k) = \mathcal{O}(k^p)$	for $p > 0$, the function $f : (0, \infty) \to [0, \infty)$ satisfies that $k \mapsto f(k)/k^p$ is bounded if $k \to 0$, or $k \to \infty$, respectively.

We will often make use of the following technical constants

$$M = M_A := 1 + \frac{2n}{-\mathrm{tr}\, A}\|A\| \qquad \text{and} \qquad \gamma = \gamma_A := \frac{2M-1}{2M+1}$$

for $A \in \mathbb{R}^{n\times n}$ with $\mathrm{tr}\, A \neq 0$, skipping dependence on A notationally in case no confusion can occur.

2 Stabilization by rotation

The following result is fundamental for the present approach. It exhibits the central rôle played by the skew symmetric matrix Σ_n in stabilizing a matrix A with negative trace.

Theorem 2.1 *For any $A \in \mathbb{R}^{n\times n}$ with $\mathrm{tr}\, A < 0$ there exists $k^* \ge 0$ such that, for all $k \in \mathbb{R}$ with $|k| \ge k^*$, the zero solution of*

$$\dot{x} = (A + k\Sigma_A)\, x \tag{2.1}$$

is exponentially stable, i.e. $\sigma(A_k) \subset \mathbb{C}_-$.
Moreover, the transient bound $T(k) := \max_{t\ge 0} \|e^{tA_k}\|$ satisfies $\lim_{|k|\to\infty} T(k) = 1$.

The proof of Theorem 2.1 (which is given in Section 6) makes use of a careful inspection of the eigenvalues of Σ_n in conjunction with perturbation results on matrices, which is presented in Section 3.

It may be worthwhile to point out that Theorem 2.1 does not yield existence of k^* such that the time-varying system

$$\dot{x} = \big(A + k(t)\Sigma_A\big)x \tag{2.2}$$

is asymptotically stable for every $t \mapsto k(t)$ with $k(t) \ge k^*$, and not even that (2.2) is asymptotically stable for every k with $\lim_{t\to\infty} k(t) = \infty$. This is shown in the following example.

Example 2.2 *For the system* (2.2) *with specific entries*

$$A = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{with} \quad \Sigma_A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{2.3}$$

the following hold (compare Fig. 1).

(i) *For any* $k^* \in \mathbb{R}$*, there exist* $\ell, m > k^*$ *and* $h, \tilde{h} > 0$ *such that the periodic gain function*

$$\begin{array}{rcl} k : [0,\infty) & \to & \{\ell, m\} \\ t & \mapsto & k(t) = \left\{ \begin{array}{ll} m, & t \in [j(h+\tilde{h}), j(h+\tilde{h}) + \tilde{h}) \\ \ell, & t \in [j(h+\tilde{h}) + \tilde{h}, (j+1)(h+\tilde{h})) \end{array} \right. \end{array} \tag{2.4}$$

for $j = 0, 1, 2, \ldots$ *applied to* (2.2) *yields, for initial condition* $x(0) = x^0 = (1,1)^T$ *and some* $\mu > 1$*, a solution* x *with*

$$\forall\, j \in \mathbb{N} : \ \|x(j(h+\tilde{h}))\| = \mu^j \, \|x^0\|.$$

(ii) *There exists a piecewise constant gain function* $k : [0,\infty) \to [0,\infty)$ *with* $\lim_{t\to\infty} k(t) = \infty$ *such that* (2.2) *has an initial condition* $x(0) = x^0 \in \mathbb{R}^2$ *for which the corresponding solution is unbounded.*

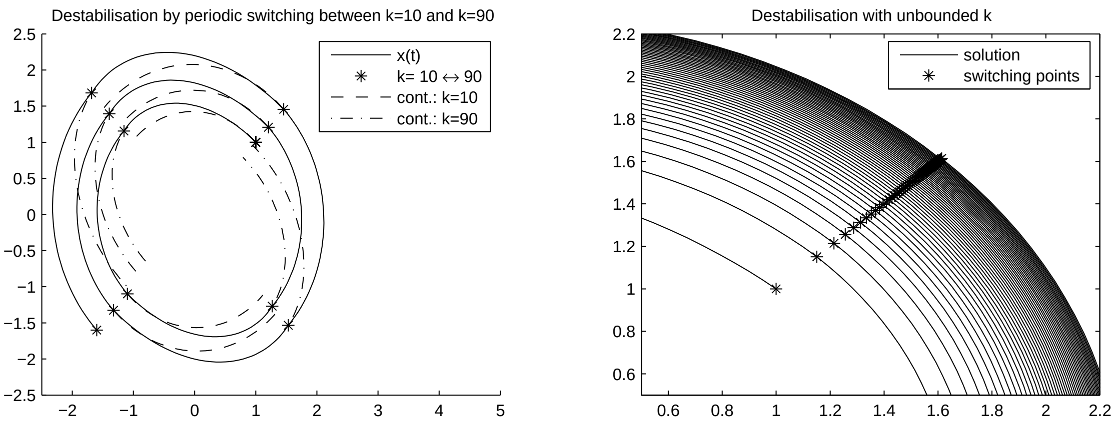

Fig. 1: Destabilisation by switching the gain parameter k. The left figure corresponds to Example 2.2 (i): the points marked by an asterisk denote the switches between $k = 10$ and $k = 90$. The dashed and dash-dotted curves extrapolate the (asymptotically stable) solutions starting in the switching points, if switching is not applied further, i.e. k is kept constant. The right figure illustrates (ii), where $k(t)$ tends to ∞ as $t \to \infty$. All curves are traversed counter-clockwise, starting in $[1,1]^T$. Both solutions are plotted over the same time interval $[0, 0.9]$.

A proof of the assertions (i) and (ii) is given in Section 6. Note that the gain in (2.4) is stabilizing if k is replaced by $\hat{k}$ for sufficiently large $\hat{k}$; this is an immediate consequence of Theorem 2.9.

Instead of the piecewise constant functions k in Example 2.2 one might construct smooth functions which destabilize the system as well; non-smoothness is not the important feature. Instead, important for a stabilizing effect is the interplay of the following properties of k: (i) monotonicity, (ii) bounded variation, and (iii) high gain. To be more precise, we show that $u = k(t)\Sigma_A x$ stabilizes the system if (i) $k(\cdot)$ is non-decreasing and sufficiently high (Theorem 2.3), (ii) $k(\cdot)$ has a bounded variation property and is unbounded (Theorem 2.4), (iii) $k(t) = kp(t)$ with bounded piecewise monotone p and sufficiently high k (Theorem 2.9). The proofs of these results, given in Section 6, rely heavily on estimates for the solutions of the parameterized Lyapunov equation (3.7).

Theorem 2.3 *For any* $A \in \mathbb{R}^{n\times n}$ *with* $\operatorname{tr} A < 0$ *there exists* $k^* = k^*(A) > 0$ *such that, for every measurable and non-decreasing* $k : [0,\infty) \to [k^*,\infty)$*, the zero solution of* (2.2) *is asymptotically stable.*

Note that the scalar k^* in Theorem 2.3 depends on A, which may be considered as a drawback. However, this problem can be resolved by determining $k(\cdot)$ by a servo mechanism. Loosely speaking, $k(\cdot)$ is tuned adaptively such that $k(t)$ increases as long as $\|x(t)\|$ is "too large", and settles to a finite limit as soon as it is stabilizing.

As a prerequisite we need a variation of Theorem 2.3, where the monotonicity assumption is replaced by a boundedness condition on the derivative of k. In this case, even exponential decay of $\|x\|$ is obtained.

Theorem 2.4 *Let* $A \in \mathbb{R}^{n\times n}$ *with* $\operatorname{tr} A < 0$ *and* $k : [0,\infty) \to \mathbb{R}$ *with* $k(t) \to \infty$ *as* $t \to \infty$*. Assume that* k *is essentially Lipschitz continuous for* $t \to \infty$*, i.e.*

$$\limsup_{t\to\infty} \left(\sup_{h>0} \frac{|k(t+h) - k(t)|}{h} \right) \quad < \quad \infty \,. \tag{2.5}$$

Then there exist $\lambda > 0$ *and, for every* $t_0 \geq 0$*, a number* $M(t_0) > 0$*, such that the solution of the initial value problem* (2.2) *with* $x(0) = x^0$ *satisfies*

$$\forall\, t \geq t_0 \;:\; \|x(t)\| \;\leq\; M(t_0)\, e^{-\lambda\,(t-t_0)}\, \|x^0\| \,.$$

Application of Theorem 2.3 or 2.4 requires either knowledge of a sufficiently large k^* or $k(t) \to \infty$, respectively. The following theorem provides a servo mechanism which finds a bounded stabilizing high-gain parameter function $k(\cdot)$ to ensure stability.

Theorem 2.5 *Suppose that* $A \in \mathbb{R}^{n\times n}$ *has* $\operatorname{tr} A < 0$ *and let* $r \in (0,\infty]$, $p \geq 1$*. Then the gain adaptation*

$$\dot{k} = \min\{r, \|x(t)\|^p\}, \qquad k(0) = k^0, \tag{2.6}$$

in conjunction with

$$\dot{x} = \big(A + k(t)\Sigma_A\big)x, \qquad x(0) = x^0, \tag{2.7}$$

defines, for any $x^0 \in \mathbb{R}^n$, $k^0 > 0$*, an initial value problem which has a unique solution* (x,k) *on the whole of* $[0,\infty)$*, and this solution satisfies*

(i) $\lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R}$,

(ii) $\lim_{t\to\infty} x(t) = 0$.

Remark 2.6 (i) If $r = \infty$, then (2.6) reduces to $\dot{k} = \|x(t)\|^p$. It may, however, be advantageous to choose $r > 0$ small in order to avoid an overshoot of the gain value. The gain adaptation $\dot{k}(t) = \|x(t)\|^2$ is ubiquitous in the area of adaptive high-gain stabilization of input-output systems, see for example the seminal work by Morse [14] and Willems and Byrnes [16]. The gain adaptation (2.6) with $0 < r < \infty$ is due to Ilchmann and Ryan [8].

(ii) Note that Theorem 2.5 does not say that the system $\dot{x} = \big(A + k(t)\,\Sigma_A\big)x$ becomes asymptotically stable, nor is the so called "limit system" $\dot{x} = \big(A + k_\infty\,\Sigma_A\big)x$ necessarily stable. The dynamic gain adaptation (2.6) ensures only that the specific trajectory (x,k) converges: $\lim_{t\to\infty} x(t;x^0,k^0) = 0$ and $\lim_{t\to\infty} k(t;x^0,k^0) = k_\infty \in \mathbb{R}$. We conjecture that for any nonzero initial value x^0 and k^0 arbitrary the limit system is asymptotically stable (the initial value $x^0 = 0$ gives $k(\cdot) \equiv k(0)$, so the assertion does not hold for $x^0 = 0$).

The dynamic stabilization provided by Theorem 2.5 is robust with respect to arbitrary bounded skew-symmetric perturbations of A.

Corollary 2.7 *Assume the situation of Theorem 2.5, but instead of* (2.7) *consider*

$$\dot{x} \;=\; \big(A + \Sigma(t) + k\,\Sigma_A\big)x\,, \qquad x(0) = x^0\,,$$

with bounded and measurable $\Sigma : [0,\infty) \to \mathrm{so}(n,\mathbb{R})$*. Then the assertions of Theorem 2.5 remain valid.*

We are now in a position to relate the above approach to the concept of stabilization by vibration as it has been introduced by Meerkov [12].

Definition 2.8 [12, Def. 1] The system $\dot{x} = Ax$ is called *vibrationally stabilizable* if, and only if, there exists a periodic matrix $B(\cdot)$ with zero mean value such that the zero solution of the system $\dot{x} = \big(A + B(t)\big)x$ is asymptotically stable.

Under the assumption that A is 'observable in principle', Meerkov proves that $\dot{x} = Ax$ is vibrationally stabilizable if, and only if, $\mathrm{tr}\,A < 0$. *Observable in principle* means that there exists $c \in \mathbb{R}^{1\times n}$ such that (A,c) is observable; the latter is equivalent to $\mathrm{rk}\,[c^T, A^Tc^T, \ldots, (A^T)^{n-1}c^T] = n$; and this implies that each eigenvalue of A has geometric multiplicity equal to 1.

To see that Meerkov's observability assumption is superfluous, apply $u(t) = B(t)\,x(t)$ with $B(\cdot) = k\,p(\cdot)\,\Sigma_A$ and periodic and piecewise monotone $p : \mathbb{R} \to \mathbb{R}$ to (1.1). (*Piecewise monotone* means that every finite interval can be partitioned into a union of finitely many points and finitely many sub-intervals in such a way that p is monotone on every of the sub-intervals.)

Theorem 2.9 *Let* $A \in \mathbb{R}^{n\times n}$ *with* $\mathrm{tr}\,A < 0$*, and suppose that* $p : \mathbb{R} \to \mathbb{R}$ *is a bounded piecewise monotone periodic function with discrete zeros. Then there exists* $k^* > 0$*, such that for all* k *with* $|k| \geq k^*$ *the system* $\dot{x} = \big(A + kp(t)\Sigma_A\big)x$ *is asymptotically stable.*

For example, the functions $t \mapsto \cos(ct)$ or $t \mapsto \mathrm{sgn}\big(\cos(ct)\big)$, where $c \in \mathbb{R}\setminus\{0\}$, are periodic and piecewise monotone, and they have zero mean. We thus obtain the following corollary.

Corollary 2.10 *The system* $\dot{x} = Ax$ *is vibrationally stabilizable if, and only if,* $\mathrm{tr}\,A < 0$.

Meerkov's method proceeds by choosing $t \mapsto B(t)$ periodic with sufficiently high frequency and sufficiently large amplitude. Theorem 2.9 shows that one may use a periodic function with arbitrary length of the period, one only needs sufficiently large amplitude. The following theorem shows that one does not even have to know how large the amplitude has to be, but choose a dynamic servo mechanism to determine the gain.

Theorem 2.11 *Suppose that* $A \in \mathbb{R}^{n\times n}$ *has* $\mathrm{tr}\,A < 0$*, and let* $r > 0$*,* $p \geq 1$*. Then the gain adaptation*

$$\dot{k} = \min\big\{r, \|x(t)\|^p\big\}, \qquad k(0) = k^0, \tag{2.8}$$

in conjunction with

$$\dot{x} = \big(A + k(t)\,\sin(t)\Sigma_A\big)x, \qquad x(0) = x^0, \tag{2.9}$$

defines, for any $x^0 \in \mathbb{R}^n$*,* $k^0 > 0$*, an initial value problem which has a unique solution* (x,k) *on the whole of* $[0,\infty)$*, and this solution satisfies*

(i) $\lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R}$,

(ii) $\lim_{t\to\infty} x(t) = 0$.

3 Parameterized matrices

In the present section a detailed investigation of the eigenvalues of the skew-symmetric matrix Σ_n, as defined in (1.3), is used in connection with matrix perturbation theory in order to obtain knowledge about the spectrum of $A + k\Sigma_A$ for large $|k|$, where $A \in \mathbb{R}^{n\times n}$, and Σ_A is given by (1.5). We will make essential use of the following well known general result in matrix perturbation theory; see, for example, Hinrichsen and Pritchard [7, Cor. 4.2.3, Prop. 4.2.12, Cor. 4.2.25].

Theorem 3.1 *Let $A, B \in \mathbb{R}^{n\times n}$ and assume that B has distinct eigenvalues $\lambda_1(B), \ldots, \lambda_n(B)$ with corresponding eigenvectors $v_1(B), \ldots, v_n(B)$. Then there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the matrix $\varepsilon A + B$ has n distinct eigenvalues and*

$$\lambda_j(\varepsilon A + B) = \lambda_j(B) + \varepsilon \frac{v_j(B)^* A\, v_j(B)}{v_j(B)^*\, v_j(B)} + \mathcal{O}(\varepsilon^2) \qquad \text{as } \varepsilon \to 0.$$

For appropriate enumeration, the functions

$$\varepsilon \mapsto \lambda_j(\varepsilon A + B) \quad \text{and} \quad \varepsilon \mapsto v_j(\varepsilon A + B) \quad \text{are analytic on } (-\varepsilon_0, \varepsilon_0).$$

In particular, $\lim_{\varepsilon\to 0} v_j(\varepsilon A + B) = v_j(B)$, for $j = 1, \ldots, n$.

Recalling the definitions of Σ_n, Σ_A, and $A_k = A + k\Sigma_A$ in (1.3)–(1.5), we have the following.

Lemma 3.2 *The matrix Σ_n has n distinct eigenvalues $i\omega_j$, where ω_j is given by*

$$\omega_j = \frac{\sin\varphi_j}{\cos\varphi_j - 1}\,, \quad \text{with} \quad \varphi_j = \frac{\pi + 2(j-1)\pi}{n}\,, \tag{3.1}$$

with corresponding normalized eigenvectors

$$v_j(\Sigma_n) \;=\; \frac{1}{\sqrt{n}} \left(1, \frac{i\omega_j + 1}{i\omega_j - 1}, \ldots, \left(\frac{i\omega_j + 1}{i\omega_j - 1}\right)^{n-1}\right)^T; \qquad j = 1, \ldots, n\,. \tag{3.2}$$

Consequently,

(i) $\Sigma_n = V\, i\,\Omega_n\, V^*$ *and* $V^*V = I$

for $V := [v_1(\Sigma_n), \ldots, v_n(\Sigma_n)]$ *and* $\Omega_n = \operatorname{diag}(\omega_1, \ldots, \omega_n)$*;*

(ii) $\operatorname{rk}\ \Sigma_n = \begin{cases} n, & n \text{ even}, \\ n-1, & n \text{ odd}; \end{cases}$

(iii) *all entries of the eigenvectors in* (3.2) *have the same modulus, namely* $1/\sqrt{n}$*;*

(iv) *for any* $A = \operatorname{diag}[a_1, \ldots, a_n] \in \mathbb{R}^{n\times n}$ *and all* $j = 1, \ldots, n$ *one has*

$$v_j(\Sigma_n)^* A\, v_j(\Sigma_n) \;=\; \frac{\operatorname{tr} A}{n}.$$

Proof: Assertion (i) is immediate since Σ_n is skew symmetric.

We proceed in several steps.
Step 1: We show (3.2) and assertion (iii).
Suppose that

$$\Sigma_n v \;=\; i\omega\, v \qquad \text{for some } \omega \in \mathbb{R} \text{ and } v = (\xi_1, \ldots, \xi_n)^T \in \mathbb{C}^n \setminus \{0\}. \tag{3.3}$$

Then, for all $\ell = 1, \ldots, n-1$, we have

$$\begin{aligned} i\omega\, \xi_{\ell+1} - i\omega\, \xi_\ell \;&=\; \big(\Sigma_n\, v\big)_{\ell+1} - \big(\Sigma_n\, v\big)_\ell \\ &=\; \left(\sum_{j=1}^{\ell} \xi_j - \sum_{j=\ell+2}^{n} \xi_j\right) - \left(\sum_{j=1}^{\ell-1} \xi_j - \sum_{j=\ell+1}^{n} \xi_j\right) \;=\; \xi_\ell + \xi_{\ell+1}, \end{aligned}$$

and so

$$\xi_{\ell+1} \;=\; \frac{i\omega + 1}{i\omega - 1}\, \xi_\ell\,. \tag{3.4}$$

Since $\xi_1 = 0$ yields $v = 0$, this proves that every eigenvector of Σ_n is of the form (3.2). Obviously, all entries of $v_j(\Sigma_n)$ have modulus $1/\sqrt{n}$, which proves Assertion (iii). Furthermore, $\|v_j(\Sigma_n)\| = 1$.

Step 2: We show that the eigenvalues are pairwise distinct. If one of the eigenvalues had multiplicity larger than one, then, by (3.4), any two eigenvectors associated with this eigenvalue would be linearly dependent, contradicting the fact that Σ_n, being a skew symmetric matrix and therefore diagonalizable, has n linearly independent eigenvectors.

Step 3: We show that whenever $i\omega$ is an eigenvalue of Σ_n for some $\omega \in \mathbb{R}$, then

$$\left(\frac{i\omega + 1}{i\omega - 1}\right)^n = -1. \tag{3.5}$$

Substituting (3.4) in Assertion (i) yields

$$\big(\Sigma_n\, v(\Sigma_n)\big)_n = i\omega \left(\frac{i\omega + 1}{i\omega - 1}\right)^{n-1} = \sum_{k=1}^{n-1} \left(\frac{i\omega + 1}{i\omega - 1}\right)^{k-1}$$

and thus

$$\frac{1 - \left(\frac{i\omega+1}{i\omega-1}\right)^{n-1}}{1 - \frac{i\omega+1}{i\omega-1}} = i\omega \left(\frac{i\omega + 1}{i\omega - 1}\right)^{n-1},$$

from which (3.5) follows by straightforward calculation.

Step 4: Finally, straightforward calculation shows that, for $\varphi \in \mathbb{R} \setminus 2\mathbb{N}\pi$,

$$e^{i\varphi} = \frac{i\omega + 1}{i\omega - 1} \quad \text{if, and only if,} \quad \omega = \frac{\sin\varphi}{\cos\varphi - 1}\,.$$

Since $e^{i\,n\varphi} = -1$ if, and only if, $\varphi = \frac{\pi + 2\ell\pi}{n}$ for some $\ell \in \{0, \ldots, n-1\}$, (3.1) follows from (3.5), which proves the claim and completes the proof of the lemma. □

Lemma 3.3 *For any $A \in \mathbb{R}^{n\times n}$ with $\operatorname{tr} A < 0$ there exists $k^* \geq 0$ such that*

$$S_\cdot : \{k \in \mathbb{R} : |k| > k^*\} \to \{S \in \mathbb{C}^{n\times n} : \det S \neq 0\}, \quad k \mapsto S_k = [v_1(A_k), \ldots, v_n(A_k)]$$

is analytic, where $v_1(A_k), \ldots, v_n(A_k)$ denote the eigenvectors of A_k in appropriate ordering. Moreover, we have the following.

(i) $S_k^{-1} A_k S_k \;=\; i\,k\,\Omega_n + \frac{\operatorname{tr} A}{n} I + \operatorname{diag}(\delta_1(k), \ldots, \delta_n(k))$, *where $\delta_j(k) = \mathcal{O}(1/|k|)$ as $|k| \to \infty$ for $j = 1, \ldots, n$.*

(ii) *For U as in (1.4) and V as in Lemma 3.2 (i), we have $S_k = U^T V + \mathcal{O}(1/|k|)$ as $|k| \to \infty$. Consequently, $S_\infty := U^T V$ satisfies*

$$S_\infty^* S_\infty = I \quad \textit{and} \quad S_\infty^* \Sigma_A S_\infty = i\,\Omega_n.$$

(iii) $S_k^* S_k = I + \mathcal{O}(1/|k|)$ *as $|k| \to \infty$.*

Proof: We show assertion (i). By (1.5), Σ_A and Σ_n are similar, and so Σ_A has eigenvalues $i\omega_j$ with corresponding eigenvectors $v_j(\Sigma_A) = U v_j(\Sigma_n)$ as defined in (3.1) and (3.2). Thus, by Theorem 3.1 with $\varepsilon = 1/k$, there exists $k_1 \geq 0$ so that, for all $k \in \mathbb{R}$ with $|k| \geq k_1$, the matrix $A_k = k(\frac{1}{k}A + \Sigma_A)$ has n distinct eigenvalues $\lambda_j(A_k)$ satisfying

$$\begin{aligned}
\lambda_j(A_k) &= k\,i\,\omega_j + v_j(\Sigma_n)^* U^T A\, U v_j(\Sigma_n) + \mathcal{O}(1/|k|) \\
&= k\,i\,\omega_j + \frac{1}{2} v_j(\Sigma_n)^* U^T (A + A^T) U v_j(\Sigma_n) + \mathcal{O}(1/|k|) \\
&= k\,i\,\omega_j + \frac{1}{2} v_j(\Sigma_n)^* D\, v_j(\Sigma_n) + \mathcal{O}(1/|k|) \\
&= k\,i\,\omega_j + \tfrac{\operatorname{tr} A}{n} + \mathcal{O}(1/|k|)\,,
\end{aligned}$$

where D is defined in (1.4), and the last equality follows from Lemma 3.2 (iv). Since $A_k = A + k\Sigma_A$ and $\frac{1}{k}A + \Sigma_A$ have the same eigenvectors – we write $v_j(A_k) = v_j(\frac{1}{k}A + \Sigma_A)$ –, we may, invoking Theorem 3.1, choose $k^* \geq k_1$, such that

$$\tfrac{1}{k} \mapsto v_j(\tfrac{1}{k}A + \Sigma_A) \quad \text{is analytic on } (-1/k^*, 1/k^*), \quad \text{for } j = 1, \ldots, n,$$

and

$$\lim_{|k|\to\infty} v_j(A_k) = \lim_{|k|\to\infty} v_j(\tfrac{1}{k}A + \Sigma_A) = v_j(\Sigma_A) = U^T v_j \quad \text{ for } j = 1, \ldots, n, \tag{3.6}$$

proving assertion (i).

We show assertion (ii). By (3.6) and the definition of $\Sigma_A = U\Sigma_n U^T$ we have, for every j with $1 \leq j \leq n$, $\lim_{|k|\to\infty} v_j(A_k) = v_j(\Sigma_A) = U^T v_j(\Sigma_n)$, and thus $\lim_{|k|\to\infty} S_k = S_\infty$. Since V and U are orthogonal, so is S_∞. The assertion $S_\infty^* \Sigma_A S_\infty = V^T U \Sigma_A U^T V = V^T \Sigma_n V = i\Omega_n$ follows from Lemma 3.2 (i).

We show assertion (iii). Applying Theorem 3.1, for $\varepsilon = 1/k$, to $v_j(A_k) = v_j(\frac{1}{k}A + \Sigma_n)$ yields analyticity of $k \mapsto S_k$ at $k = \pm\infty$, and thus $S_k = S_\infty + \mathcal{O}(1/|k|)$ for $|k| \to \infty$, whence

$$S_k^* S_k = (S_\infty + \mathcal{O}(1/|k|))^* (S_\infty + \mathcal{O}(1/|k|)) = S_\infty^* S_\infty + \mathcal{O}(1/|k|) = I + \mathcal{O}(1/|k|) \quad \text{for } |k| \to \infty.$$

This completes the proof of the lemma. □

If $A \in \mathbb{R}^{n\times n}$ with $\operatorname{tr} A < 0$, then Lemma 3.3 (i) ensures the existence of some $k^* \geq 0$ so that $\sigma(A_k) \subset \mathbb{C}_-$ holds for all $k \in \mathbb{R}$ with $|k| \geq k^*$. Therefore (see, for example, Hinrichsen and Pritchard [7, Cor. 3.3.46])

$$P_k := \int_0^\infty \mathrm{e}^{A_k^T s}\mathrm{e}^{A_k s}\,\mathrm{d}s \tag{3.7}$$

is the unique positive definite solution of

$$A_k^T P_k + P_k A_k = -I\,. \tag{3.8}$$

Lemma 3.4 *For any $A \in \mathbb{R}^{n\times n}$ with $\operatorname{tr} A < 0$, the matrix P_k as defined in (3.7) satisfies*

$$P_k = \frac{n}{-2\operatorname{tr} A} I + \mathcal{O}(1/|k|) \qquad \textit{for } k \textit{ with } |k| \to \infty. \tag{3.9}$$

Proof: Let $k^* \geq 0$ be given as in Lemma 3.3 and set, for $k \in \mathbb{R}$ with $|k| \geq k^*$,

$$D_k := S_k^{-1}\, A_k\, S_k, \qquad E_k := S_k^*\, S_k - I\,.$$

Then

$$\begin{aligned} P_k + \frac{n}{2\mathrm{tr}\, A} I &= \int_0^\infty S^{-1*}{}_k \mathrm{e}^{D_k^* s}\, S_k^*\, S_k\, \mathrm{e}^{D_k s}\, S_k^{-1}\,\mathrm{d}s - \int_0^\infty \mathrm{e}^{\frac{2\mathrm{tr}\, A}{n}\, Is}\,\mathrm{d}s \\ &= S_k{}^{-1*} \int_0^\infty \left(\mathrm{e}^{D_k^* s}\, E_k\, \mathrm{e}^{D_k s} - E_k \mathrm{e}^{\frac{2\mathrm{tr}\, A}{n} s}\right) \mathrm{d}s\; S_k^{-1} \\ &\quad + \; S^{-1*}{}_k \int_0^\infty \left(\mathrm{e}^{D_k^* s}\, \mathrm{e}^{D_k s} - \mathrm{e}^{\frac{2\mathrm{tr}\, A}{n}\, s} I\right)\, \mathrm{d}s\; S_k^{-1}\,. \end{aligned} \tag{3.10}$$

By Lemma 3.3 (iii), it remains to show that the integrals in (3.10) are of order $1/|k|$ as $|k| \to \infty$. Invoking Lemma 3.3 (i) yields, for all $s \geq 0$,

$$\mathrm{e}^{D_k s} = \mathrm{e}^{s\operatorname{tr} A/n}\mathrm{diag}\left(\mathrm{e}^{(ik\omega_1 + \delta_1(k))s}, \ldots, \mathrm{e}^{(ik\omega_n + \delta_n(k))s}\right)$$

with

$$\delta_j(k) = \mathcal{O}(1/|k|) \quad \text{as } |k| \to \infty \;\; \text{for } j = 1, \ldots, n.$$

Thus,

$$\exists\, k_1 \geq 0\; \forall\, s \geq 0\; \forall\, k \in \mathbb{R} \text{ with } |k| \geq k_1 \;:\; \|\mathrm{e}^{D_k s}\| = \mathrm{e}^{(\operatorname{tr} A/n + \mathcal{O}(1/|k|))s} \leq \mathrm{e}^{\frac{\operatorname{tr} A}{2n} s}\,.$$

Hence, by Lemma 3.3 (iii),

$$\begin{aligned} \left\| \int_0^\infty \left(\mathrm{e}^{D_k^* s}\, E_k\, \mathrm{e}^{D_k s} - E_k \mathrm{e}^{\frac{2\mathrm{tr}\, A}{n} s}\right) \mathrm{d}s \right\| &\leq \int_0^\infty \left(\|\mathrm{e}^{D_k s}\|^2 + \mathrm{e}^{\frac{\mathrm{tr}\, A}{n} s}\right) \mathrm{d}s\, \|E_k\| \\ &\leq 2\int_0^\infty \mathrm{e}^{\frac{\mathrm{tr}\, A}{n} s}\,\mathrm{d}s\, \|E_k\| \\ &= \frac{2n}{-\mathrm{tr}\, A}\|E_k\| \\ &= \mathcal{O}(1/|k|)\,, \end{aligned}$$

where the latter equality follows from Lemma 3.3 (iii). Furthermore, there exists $k_2 > k_1$ so that, for all $k \in \mathbb{R}$ with $|k| \geq k_2$ and for all $j = 1, \ldots, n$, we have $\frac{\operatorname{tr} A}{n} + \delta_j(k) < 0$, and therefore

$$\begin{aligned}\int_0^\infty \mathrm{e}^{D_k^* s}\mathrm{e}^{D_k s}\,\mathrm{d}s &= \int_0^\infty \operatorname{diag}\left(\mathrm{e}^{2\left(\frac{\operatorname{tr} A}{n} + \operatorname{Re}\delta_1(k)\right)s}, \ldots, \mathrm{e}^{2\left(\frac{\operatorname{tr} A}{n} + \operatorname{Re}\delta_n(k)\right)s}\right)\mathrm{d}s \\ &= \left(2\frac{\operatorname{tr} A}{n} I + \operatorname{Re}\operatorname{diag}\left(\delta_1(k), \ldots, \delta_n(k)\right)\right)^{-1} \\ &= \int_0^\infty \mathrm{e}^{2\frac{\operatorname{tr} A}{n}s} I\,\mathrm{d}s + \mathcal{O}(1/|k|) \qquad \text{as } |k| \to \infty.\end{aligned}$$

This completes the proof of the lemma. □

We also obtain an estimate for the growth of the condition number $\kappa_2(P_k) = \sigma_{\max}(P_k)/\sigma_{\min}(P_k)$ of P_k, which plays an important rôle in the proof of Theorem 2.3.

Corollary 3.5 *For any A with $\operatorname{tr} A < 0$ there exist numbers $a, k^* > 0$ such that*

$$\forall\, k \in \mathbb{R} \text{ with } |k| \geq k^* \ : \quad \kappa_2(P_k) \leq 1 + \frac{a}{|k|}\,. \tag{3.11}$$

Proof: Theorem 3.1, applied for $B = \frac{n}{-2\operatorname{tr} A} I$ and $\varepsilon = 1/|k|$, yields the existence of $\alpha, k_1 \geq 0$ such that

$$\forall\ k \in \mathbb{R} \text{ with } |k| \geq k_1,\ \forall\ j = 1, \ldots, n\ :\quad \frac{n}{-2\operatorname{tr} A} - \frac{\alpha}{|k|} \leq \lambda_j(P_k) \leq \frac{n}{-2\operatorname{tr} A} + \frac{\alpha}{|k|},$$

With $k^* := \max\left\{k_1, 1 + (-2\,\alpha \operatorname{tr} A)/n\right\}$ we thus obtain

$$\forall\ k \in \mathbb{R} \text{ with } |k| \geq k_1\ :\quad \kappa_2(P_k) = \frac{\sigma_{\max}(P_k)}{\sigma_{\min}(P_k)} \ \leq\ \frac{\frac{n}{-2\operatorname{tr} A} + \frac{\alpha}{|k|}}{\frac{n}{-2\operatorname{tr} A} - \frac{\alpha}{|k|}} \ \leq\ 1 + \frac{a}{|k|}\,. \qquad \square$$

Lemma 3.6 *Suppose that $A \in \mathbb{R}^{n\times n}$ has $\operatorname{tr} A < 0$. Then there exists $m^* > 0$ such that, with $M = 1 + \frac{2n}{-\operatorname{tr} A}\|A\|$,*

$$\forall\, m \in \mathbb{R} \text{ with } |m| \geq m^*\ \forall\, k \in \mathbb{R}\ :\ A_k^T P_m + P_m A_k \ \leq\ -\left(1 - \tfrac{|k-m|}{|m|} M\right) I\,. \tag{3.12}$$

Proof: By (3.8) and (1.5), we have, for all m with $|m| > k^*$,

$$-I \ = \ A^T P_m + P_m A + m\left(\Sigma_A^T P_m + P_m \Sigma_A\right),$$

or, equivalently,

$$\Sigma_A^T P_m + P_m \Sigma_A = -\frac{1}{m}\left(I + A^T P_m + P_m A\right).$$

Consequently, for all $k \geq 0$ and all $m > 0$,

$$\begin{aligned}\|A_k^T P_m + P_m A_k - (A_m^T P_m + P_m A_m)\| &= \|(A_k - A_m)^T P_m + P_m(A_k - A_m)\| \\ &= \|(k-m)\Sigma_A^T P_m + (k-m) P_m \Sigma_A\| \\ &= |k-m|\,\|\Sigma_A^T P_m + P_m \Sigma_A\| \\ &= \frac{|k-m|}{|m|}\,\|I + A^T P_m + P_m A\|\,.\end{aligned}$$

By Lemma 3.4 there exists $m^* > 0$ such that, for all $m \in \mathbb{R}$ with $|m| \geq m^*$, we have $\|I + A^T P_m + P_m A\| \leq M$, which proves (3.12). □

4 Time-varying linear systems

Suppose that $t \mapsto k(t)$ is some real-valued function. Then Lemma 3.6 implies that for m with $|m| \geq m^*$, where m^* is specified in the lemma, the matrix P_m given by (3.7) defines a Lyapunov function for the time-varying system $\dot{x} = \big(A + k(t)\Sigma_A\big)x$, provided that $k(t)$ is confined to a certain neighbourhood of m. The following lemma shows that the length of this neighbourhood can be chosen proportional to the size of $|m|$. This will be an important technical ingredient for the proofs of the results of Section 2.

Lemma 4.1 *Suppose that $A \in \mathbb{R}^{n\times n}$ has $\operatorname{tr} A < 0$, and let $k : [0,\infty) \to \mathbb{R}$ be measurable. Choose $m^* > 0$ so that* (3.12) *holds, where again $M = 1 + \frac{2\,n}{-\operatorname{tr} A}\|A\|$.*
If, for some $t_0 \geq 0$ and $t_1 \in (t_0, \infty]$, we have

$$\exists\, m \in \mathbb{R} \text{ with } |m| \geq m^* \;\forall\, t \in [t_0, t_1) \;:\quad |k(t) - m| \leq \frac{|m|}{2\,M}, \tag{4.1}$$

then every solution $x : [t_0, \infty) \to \mathbb{R}^n$ of

$$\dot{x} = \big(A + k(t)\Sigma_A\big)\, x \tag{4.2}$$

satisfies, for $\beta_m := \frac{1}{2\sigma_{\max}(P_m)}$,

$$\forall\, t \in [t_0, t_1) \;:\; \|x(t)\|^2 \leq \kappa_2(P_m)\, \mathrm{e}^{-\beta_m\,(t-t_0)}\, \|x(t_0)\|^2. \tag{4.3}$$

Proof: Differentiating $y \mapsto y^T P_m y$ along the solution of (4.2), invoking (3.12), (4.1), and

$$\forall\, y \in \mathbb{R}^n \;:\quad \sigma_{\min}(P_m)\, \|y\|^2 \;\leq\; y^T P_m y \;\leq\; \sigma_{\max}(P_m)\, \|y\|^2 \tag{4.4}$$

yields

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\, \big(x(t)^T P_m x(t)\big) &= x(t)^T \big(A_{k(t)}^T P_m + P_m A_{k(t)}\big) x(t) \\ &\leq -\big(1 - \tfrac{|k(t)-m|}{|m|} M\big) \|x(t)\|^2 \\ &\leq \frac{-1}{2\,\sigma_{\max}(P_m)}\, x(t)^T P_m\, x(t) \qquad \text{for all } t \in [t_0, t_1)\,. \end{aligned} \tag{4.5}$$

Integrating and applying (4.4) again, we obtain (4.3). □

It is interesting to note that the estimate (4.3) is robust with respect to arbitrary bounded skew-symmetric perturbations of A.

Corollary 4.2 *Assume the situation of Lemma 4.1, but instead of* (4.2) *consider*

$$\dot{x} = \big(A + \Sigma(t) + k\Sigma_A\big)x\;,$$

where $\Sigma : [0,\infty) \to \mathrm{so}(n,\mathbb{R})$ is measurable and bounded. Then inequality (4.3) *holds, for sufficiently large $m^* > 0$, for all $m \in \mathbb{R}$ with $|m| \geq m^*$ and $\beta_m = \frac{1}{4\sigma_{\max}(P_m)}$.*

Proof: By Lemma 3.4, there exist $\widetilde{m}, \alpha > 0$ such that, for all $t \geq 0$ and $m \in \mathbb{R}$ with $|m| \geq \widetilde{m}$,

$$\|\Sigma(t)^T P_m + P_m \Sigma(t)\| = \|\Sigma(t)^T \, \mathcal{O}(1/|m|) + \mathcal{O}(1/|m|)\, \Sigma(t)\| \leq \frac{\alpha}{|m|} \, .$$

Hence, differentiation along (4.5) gives, for all $t \geq 0$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \, \big(x(t)^T P_m x(t)\big) \leq \big(-1 + \tfrac{|k(t)-m|}{|m|} M + \tfrac{\alpha}{|m|}\big) \|x(t)\|^2 \, ,$$

and since, for sufficiently large $m^* \geq \widetilde{m}$, we have

$$\forall\, m \in \mathbb{R} \text{ with } |m| \geq m^* : \quad -1 + \tfrac{|k(t)-m|}{|m|} M + \tfrac{\alpha}{|m|} \leq -\frac{1}{4} \, ,$$

the claim follows as in the proof of Lemma 4.1. □

Remark 4.3 The following straightforward bound on the growth of $t \mapsto \|x(t)\|$ holds regardless of the values of $k(\cdot)$. It will be used below to obtain estimates for those times during the evolution of the system where $k(\cdot)$ is not (yet) good enough.
Let $A \in \mathbb{R}^{n \times n}$ and let $\Sigma : \mathbb{R} \to \mathrm{so}(n, \mathbb{R})$ be measurable and locally integrable. Then for any solution $t \mapsto x(t)$ of

$$\dot{x} = \big(A + \Sigma(t)\big) x$$

one has for Lebesgue almost all $t \in \mathbb{R}$, with $\lambda_{\max}(A) = \max \sigma(A + A^T)$,

$$\tfrac{d}{\mathrm{d}t} \|x(t)\|^2 = x(t)^T \big(A^T + \Sigma(t)^T + A + \Sigma(t)\big) x(t) = x(t)^T \big(A^T + A\big) x(t) \leq \lambda_{\max}(A) \, \|x(t)\|^2.$$

This implies, for $t_0 \leq t$,

$$\|x(t)\|^2 \leq \mathrm{e}^{\lambda_{\max}(A)\,(t-t_0)} \, \|x(t_0)\|^2 \, . \tag{4.6}$$

5 Numerical example

To illustrate the gain adaptation (2.6) in Theorem 2.5, consider a system of the form

$$\begin{aligned} \dot{x} &= \big(A + \delta\, \Sigma(t) + k(t)\, \Sigma_A\big) x \, , \\ \dot{k} &= \|x(t)\| \, , \end{aligned}$$

with

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}, \quad \text{which gives } \Sigma_A = \begin{bmatrix} 0 & 1.1401 & -1.2988 \\ -1.1401 & 0 & -0.1154 \\ 1.2988 & 0.1154 & 0 \end{bmatrix},$$

and

$$\Sigma(t) = \sin(t)\, \Sigma_0 - \cos(\sqrt{2}t)\, \Sigma_A, \quad t \geq 0.$$

We plot the norm of the solution and the size of the adaptation parameter, i.e. $\|x(t)\|$ and $k(t)$, for the values $\delta = 0, 10, 20$. Analogous numerical results have been obtained for matrices with different entries and for higher dimensions. One should note the fast oscillations in the solution as k increases.

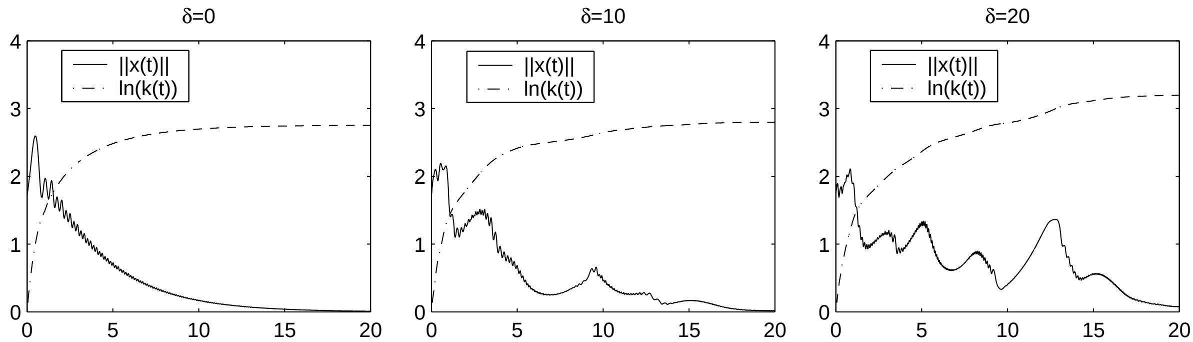

Fig. 2: Dynamic gain adaptation (2.6) for the time-varying system $\dot{x} = \big(A + \delta\Sigma(t) + k(t)\Sigma_A\big)x$ for different values of δ.

6 Proofs

Proof of Theorem 2.1:
The assertion $\sigma(A + k\Sigma_A) \subset \mathbb{C}_-$ for $k \geq k^*$ follows readily from Lemma 3.3 (i) and the assumption that $\operatorname{tr} A < 0$. For constant $k(\cdot) \equiv m$ with $|m| \geq k^*$, condition (4.1) of Lemma 4.1 is trivially satisfied. Hence the estimate (4.3) yields

$$\forall\, t \geq t_0 \;:\quad \big\|e^{A_m t}\big\| \leq \sqrt{\kappa_2(P_m)}\, \exp\Big(\tfrac{-1}{4\sigma_{\max}(P_m)}\, t\Big) \leq \sqrt{\kappa_2(P_m)}\,.$$

Now the claim follows from Corollary 3.5. □

Proof of assertion (i), Example 2.2:
Consider the matrices in (2.3). Then, for $k \geq 3$ and $\alpha_k := \sqrt{k^2 - 9}$, the eigenvalues of A_k are $-1 \pm i\alpha_k$ with corresponding eigenvectors $(-3 \pm i\alpha_k, k)^T$.
Setting

$$t_k = \frac{\pi}{2\alpha_k}$$

and invoking $e^{\pm i\alpha_k t_k} = \pm i$ gives

$$\begin{aligned} e^{A_k t_k} &= e^{-t_k} \begin{bmatrix} (-3 + i\alpha_k)e^{i\alpha_k t_k} & (-3 - i\alpha_k)e^{-i\alpha_k t_k} \\ k e^{i\alpha_k t_k} & k e^{-i\alpha_k t_k} \end{bmatrix} \begin{bmatrix} -3 + i\alpha_k & -3 - i\alpha_k \\ k & k \end{bmatrix}^{-1} \\ &= \frac{e^{-t_k}}{2k\alpha_k} \begin{bmatrix} -3 + i\alpha_k & 3 + i\alpha_k \\ k & -k \end{bmatrix} \begin{bmatrix} k & 3 + i\alpha_k \\ -k & -3 + i\alpha_k \end{bmatrix} = \frac{e^{-t_k}}{2\alpha_k} \begin{bmatrix} -6 & -2k \\ 2k & 6 \end{bmatrix}, \end{aligned}$$

and, for $\ell, m > 3$, we calculate

$$e^{A_\ell t} e^{A_m t} = -\frac{e^{-(t_\ell + t_m)}}{\alpha_\ell\, \alpha_m} \begin{bmatrix} \ell m - 9 & 3(\ell - m) \\ 3(\ell - m) & \ell m - 9 \end{bmatrix}, \tag{6.7}$$

with eigenvalue

$$\lambda(m, \ell) = -\frac{e^{-(t_\ell + t_m)}}{\alpha_\ell\, \alpha_m}[\ell m - 9 + 3(\ell - m)]$$

corresponding to the eigenvector $(1,1)^T$. Setting, in (2.4),

$$h = t_\ell, \quad \tilde{h} = t_m, \quad \text{and} \quad x^0 = (1,1)^T,$$

yields a solution of (2.2), (2.4) which satisfies

$$\forall\, j \in \mathbb{N}_0 \;:\; x(j(h+\tilde{h})) \;=\; \left(\mathrm{e}^{A_\ell t_\ell} \mathrm{e}^{A_m t_m}\right)^j \; x^0 \;=\; \lambda(m,\ell)^j\,(1,1)^T.$$

It remains to prove that, for suitable $m, \ell > 0$,

$$\mu := |\lambda(m,\ell)| > 1\,.$$

Invoking the convexity of the exponential function in the form

$$\mathrm{e}^{-(t_\ell + t_m)} \;\geq\; 1 - (t_\ell + t_m) = 1 - \frac{\pi}{2\alpha_\ell} - \frac{\pi}{2\alpha_m} = \frac{2\alpha_\ell\alpha_m - \pi\alpha_m - \pi\alpha_\ell}{2\alpha_\ell\alpha_m}$$

gives

$$\begin{aligned}
\mu \;&\geq\; \frac{2\alpha_\ell\,\alpha_m - \pi\alpha_m - \pi\alpha_\ell}{2\alpha_\ell^2\alpha_m^2}(\ell-3)(m+3) \\
&=\; \frac{2\sqrt{\ell^2-9}\sqrt{m^2-9} - \pi\sqrt{\ell^2-9} - \pi\sqrt{m^2-9}}{2(\ell+3)(m-3)} \\
&>\; \frac{2\sqrt{\ell^2-9}\sqrt{m^2-9} - \pi(\ell+m)}{2(\ell+3)(m-3)} \\
&=\; \sqrt{\frac{\ell-3}{\ell+3}\frac{m+3}{m-3}} - \frac{\pi(\ell+m)}{2(\ell+3)(m-3)} \\
&=\; \sqrt{1 + \frac{6(\ell-m)}{(\ell+3)(m-3)}} - \frac{\pi(\ell+m)}{2(\ell+3)(m-3)} \\
&\geq\; 1 + \frac{2(\ell-m)}{(\ell+3)(m-3)} - \frac{\pi(\ell+m)}{2(\ell+3)(m-3)}\;. \qquad (6.8) \\
&\geq\; 1 + \frac{1}{(\ell+3)(m-3)}\left[2(\ell-m) - \frac{\pi}{2}(\ell+m)\right], \qquad (6.9)
\end{aligned}$$

where the inequality in (6.8) follows, for $\ell > m > 9$, $\theta = \frac{6(\ell-m)}{(\ell+3)(m-3)} < 1$ and hence $\sqrt{1+\theta} \geq 1 + \theta/3$; the second term in (6.9) is positive if, e.g. $\ell = 9m$. This proves $|\lambda(10, 90)| > 1\,.$ □

Proof of assertion (ii), Example 2.2:
Choose a sequence (m_j) in $\mathbb{N}$ with $\sum_{j=1}^{\infty} m_j^{-1} = \infty$ and set, for $j \in \mathbb{N}$, $\ell_j = 9\,m_j$, and t_k, α_k as in the proof of assertion (i), and

$$\Phi_j \;=\; \mathrm{e}^{A_{\ell_j} t_{\ell_j}} \mathrm{e}^{A_{m_j} t_{m_j}}\,, \quad j \in \mathbb{N}.$$

Then the function $k : [0,\infty) \to [0,\infty)$, given by

$$k(t) = \begin{cases} \ell_j, & t \in [j(t_{\ell_j} + t_{m_j}), j(t_{\ell_j} + t_{m_j}) + t_{\ell_j}) \\ m_j, & t \in [j(t_{\ell_j} + t_{m_j}) + t_{\ell_j}, (j+1)(t_{\ell_j} + t_{m_j})) \end{cases}$$

for $j = 0, 1, 2, \ldots$, inserted into (2.2) yields, for the initial condition $x(0) = x^0 = \frac{1}{\sqrt{2}}(1,1)^T$, i.e. the normalized eigenvector of Φ_j, a solution x satisfying

$$\forall\, j \in \mathbb{N} :\ x(j(t_{\ell_j} + t_{m_j})) = \Phi_j \cdots \Phi_1\, x^0 .$$

Finally, using (6.9) and $\sum_{j=1}^{\infty} m_j^{-1} = \infty$, we arrive at

$$\|\Phi_j \cdots \Phi_1\, x^0\| \geq \prod_{\lambda=1}^{j} \Big(1 + \frac{16 - 5\pi}{9} \frac{1}{m_\lambda}\Big) \to \infty \quad \text{for } j \to \infty. \qquad \square$$

Proof of Theorem 2.3:
Fix $A \in \mathbb{R}^{n\times n}$ with $\operatorname{tr} A < 0$. Choose k^*, $a > 0$ so that (3.11) and (3.12), with k^* taking the rôle of m^*, hold. Given $k : [0,\infty) \to [k^*,\infty)$ measurable and non-decreasing, let $x : [0,\infty) \to \mathbb{R}^n$ be a solution of (2.2).

If k is bounded, then $m := \lim_{t\to\infty} k(t)$ exists by monotonicity of k, so there exists $t_0 > 0$ such that (4.1) holds with $t \in [t_0,\infty)$. We thus may apply (4.3) to conclude that $\lim_{t\to\infty} x(t) = 0$. The case of unbounded k, i.e. $k(t) \to \infty$ as $t \to \infty$, is more subtle. Now we cannot find a common uniform static quadratic Lyapunov function for all $k(t) \geq k^*$. Therefore, we first define disjoint intervals of k-values, on each of which we are going to use one fixed Lyapunov function.
For $M = 1 + \frac{2n}{-\operatorname{tr} A}\|A\|$ put $t_0 = 0$ and, for $j \in \mathbb{N}$,

$$m_j := \Big(\tfrac{2M+1}{2M-1}\Big)^j k(0), \qquad k_j := m_j - \tfrac{m_j}{2M}, \qquad \text{and} \quad t_j := \sup\{t \geq 0 : k(t) \leq k_j\}\ .$$

By construction, $t \in [t_j, t_{j+1})$ implies $k(t) \in [k_j, k_{j+1})$, i.e.

$$\forall\, j \in \mathbb{N}_0\ \forall\, t \in [t_j, t_{j+1})\ :\ |k(t) - m_j| \leq \frac{m_j}{2\,M}. \tag{6.10}$$

Invoking Corollaries 4.1 and 3.5 gives, for all $t \in [t_j, t_{j+1})$,

$$\begin{aligned} \|x(t)\|^2 &\leq \kappa_2(P_{m_j}) \exp\Big(\frac{-1}{2\sigma_{\max}(P_{m_j})}(t - t_j)\Big)\|x(t_j)\|^2 \\ &\leq \Big\{1 + \frac{a}{k(0)}\big(\tfrac{2M-1}{2M+1}\big)^j\Big\} \exp\Big(\frac{-(t-t_j)}{2\sigma_{\max}(P_{m_j})}\Big)\|x(t_j)\|^2 . \end{aligned} \tag{6.11}$$

In view of the equivalence of $\sigma_{\max}(P_m) \leq \frac{n}{-4\operatorname{tr} A}$ and $\frac{-1}{2\sigma_{\max}(P_m)} \leq \frac{\operatorname{tr} A}{2n}$ Lemma 3.4 yields:

$$\exists\, j_0 \geq k^*\ \forall\, m \geq j_0\ :\quad \frac{-1}{2\sigma_{\max}(P_m)} \leq \frac{\operatorname{tr} A}{2n} < 0\ . \tag{6.12}$$

With $\gamma = \frac{2M-1}{2M+1} < 1$ and $\beta = -\frac{\operatorname{tr} A}{2n} > 0$ we obtain by inserting (6.12) into (6.11)

$$\forall\, j \geq j_0\ :\quad \|x(t_{j+1})\|^2 \leq \big(1 + \frac{a}{k(0)}\,\gamma^j\big)\mathrm{e}^{-\beta\,(t_{j+1}-t_j)}\,\|x(t_j)\|^2\ . \tag{6.13}$$

Since $t_j \to \infty$ for $j \to \infty$ by construction, the right hand side of the following chain of inequalities

$$\begin{aligned}\ln\Big(\prod_{\ell=j_0}^{j}\Big(1+\tfrac{a}{k(0)}\gamma^\ell\Big)\,\mathrm{e}^{-\beta\,(t_{j+1}-t_{j_0})}\Big) &= \sum_{\ell=j_0}^{j}\ln\Big(1+\tfrac{a}{k(0)}\gamma^\ell\Big)-\beta\,(t_{j+1}-t_{j_0})\\ &\le \sum_{\ell=j_0}^{j}\frac{a}{k(0)}\gamma^\ell-\beta\,(t_{j+1}-t_{j_0})\\ &\le \frac{a}{k(0)}\frac{\gamma}{1-\gamma}-\beta\,(t_{j+1}-t_{j_0})\end{aligned}$$

tends to $-\infty$ for $j \to \infty$, and so (6.13) yields $\lim_{j\to\infty} x(t_j) = 0$. Since for all $j \ge j_0$ and all $t \in [t_j, t_{j+1}]$ we have

$$\|x(t)\|^2 \le \big(1+\frac{a}{k(0)}\big)\|x(t_j)\|^2\,,$$

continuity of $t \mapsto x(t)$ yields $\lim_{t\to\infty} x(t) = 0$. □

Proof of Theorem 2.4:
By assumption (2.5) there exist $T > 0$, $K_0 > 0$, such that $|k(t+h)-k(t)| \le K_0\,h$ for all $t \ge T$ and all $h \ge 0$. Put $\beta := \frac{-\operatorname{tr} A}{2n}$ and $M := 1 + \frac{2\,n}{-\operatorname{tr} A}\|A\|$.
By Corollary 3.5 and Lemma 3.4 the inequalities $k \ge k^* \ge 4\ln 2\cdot M\,K_0/\beta$ imply $\kappa_2(P_k) \le 2$ and $-\frac{1}{2\sigma_{\max}(P_k)} \le -\beta$. Writing $h := \frac{2\ln 2}{\beta}$, we have

$$\forall\, t \ge T\ \forall\, \tau \in [t, t+h]\ :\quad |k(\tau)-k(t)| \le hK_0 \;=\; \frac{2\ln 2}{\beta}K_0 \;\le\; \frac{k^*}{2M} \;\le\; \frac{k(t)}{2M}\,.$$

Hence we may apply Lemma 4.1 on the interval $[t, t+h]$ to obtain, for any solution x of (2.2) and all $t \ge T$,

$$\|x(t+h)\|^2 \le 2\mathrm{e}^{-\beta h}\|x(t)\|^2 = 2\mathrm{e}^{-2\ln 2}\|x(t)\|^2 = \tfrac{1}{2}\|x(t)\|^2\,,$$

whence

$$\forall\, j \in \mathbb{N}\ \forall\, t \in [T, T+h)\ :\ \|x(t+jh)\|^2 \le \big(\tfrac{1}{2}\big)^j\ \|x(t)\|^2\,,$$

or, equivalently,

$$\forall\, j \in \mathbb{N}\ \forall\, t \in [T+jh, T+(j+1)h)\ :\ \|x(t)\|^2 \le \big(\tfrac{1}{2}\big)^j\ \|x(t-jh)\|^2\,.$$

For $t \in [T+jh, T+(j+1)h)$ and $t_0 \in [0, T)$ we have, by Remark 4.3,

$$\begin{aligned}\|x(t)\|^2 &\le \big(\tfrac{1}{2}\big)^j\ \|x(t-jh)\|^2\\ &\le \big(\tfrac{1}{2}\big)^{\frac{t-T}{h}-1}\max_{s\in[T,T+h)}\|x(s)\|^2\\ &\le \big(\tfrac{1}{2}\big)^{\frac{t-T}{h}-1}\,\mathrm{e}^{\lambda_{\max}(A)\,(T+h-t_0)}\|x(t_0)\|^2\\ &\le 2\,\mathrm{e}^{\lambda_{\max}(A)\,(T+h-t_0)+\ln 2\,\frac{T-t_0}{h}}\ \mathrm{e}^{-\ln 2\,\frac{T-t_0}{h}}\ \|x(t_0)\|^2\,.\end{aligned}$$

It remains to consider the case $T \in [0, t_0)$. Invoking Remark 4.3 again gives, for $t \in [t_0 + jh, t_0 + (j+1)h)$,

$$\|x(t)\|^2 \leq \left(\tfrac{1}{2}\right)^j \|x(t-jh)\|^2 \leq \left(\tfrac{1}{2}\right)^j \mathrm{e}^{\lambda_{\max}(A)\,h} \|x(0)\|^2 \leq \left(\tfrac{1}{2}\right)^{\frac{t-t_0}{h}-1} \mathrm{e}^{\lambda_{\max}(A)\,h} \|x(t_0)\|^2 \,.$$

This completes the proof of the theorem. □

It is quite instructive to see how both Theorems 2.3 and 2.4 are crucial in Step 2 of the following proof.

Proof of Theorem 2.5:
Consider, for $r \in (0, \infty]$, $p \geq 1$, $x^0 \in \mathbb{R}^n$, $k^0 > 0$, the system (2.6), (2.7). Note that $\dot{k} = \|x(t)\|^p$ if $r = \infty$.

Step 1: Since the right hand side of (2.6), (2.7) is locally Lipschitz, the initial value problem has a unique solution $(x, k) : [0, \omega) \to \mathbb{R}^n \times \mathbb{R}$ for some $\omega \in (0, \infty]$, the latter is assumed to be maximal. By Remark 4.3, $\|x(\cdot)\|$ grows at most exponentially and therefore $x(\cdot)$ cannot escape in finite time. Hence $k(t) \leq k^0 + \int_0^t \|x(\tau)\|^p \, d\tau < \infty$ for all $t < \infty$, whence $\omega = \infty$.

Step 2: We show that k is bounded, whence assertion (i).
Seeking a contradiction, suppose that k is unbounded, i.e., by (2.6), $k(t)$ tends monotonically to ∞ as $t \to \infty$.
Suppose that r is finite. Theorem 2.3 ensures that $x(t)$ tends to 0 for $t \to \infty$. By the gain-adaptation law (2.6), there exists $t_0 \geq 0$ such that $0 \leq \dot{k}(t) \leq r$ for all $t \geq t_0$. Therefore, Theorem 2.4 yields that $x(t)$ tends to 0 *exponentially*, and, invoking (2.6) again, we obtain that k, being the integral of an exponentially decaying function, is bounded. This contradicts the assumption that k is unbounded.
It remains to consider the case $r = \infty$. Since $x(t)$ tends to 0 as $t \to \infty$, $\dot{k}$ is bounded and so k satisfies (2.5), which gives exponential decay of $x(t)$ for $t \to \infty$. However, the latter entails that $\|x\|^p$ is integrable, and so k has to be bounded, which again contradicts the assumption.

Step 3: We show that x is bounded.
Seeking a contradiction, suppose that x is unbounded. Observe that by boundedness of k and (2.7), there exists $c_1 > 0$ so that

$$\forall t > 0 : \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\|x(t)\| \;\leq\; c_1 \|x(t)\| \,.$$

Choose $t_0 \geq 0$ such that

$$\|x(t_0)\| \geq \|x^0\| \,,$$

and set, for arbitrary $R > 0$,

$$\tau_R := \inf\left\{ t > t_0 : \|x(t)\| = \mathrm{e}^R \|x(t_0)\| \right\}, \qquad \sigma_R := \sup\left\{ t \in [t_0, \tau_R) : \|x(t)\| = \|x(t_0)\| \right\}.$$

Then

$$\forall\, t \in [\sigma_R, \tau_R] : \|x(t_0)\| \leq \|x(t)\| \leq \mathrm{e}^R \|x(t_0)\| = \|x(\tau_r)\| \leq \mathrm{e}^{c_1(\tau_R - \sigma_R)} \|x(t_0)\| \,,$$

whence, by monotonicty of k,

$$\begin{aligned} k(\tau_R) &= k(\sigma_R) + \int_{\sigma_R}^{\tau_R} \min\{r, \|x(t)\|^p\} \, \mathrm{d}t \\ &\geq k^0 + (\tau_R - \sigma_R) \, \min\big\{r, \|x(t_0)\|^p\big\} \\ &\geq k^0 + \frac{R}{c_1} \, \min\big\{r, \|x^0\|^p\big\} \,. \end{aligned}$$

Since R is arbitrary, the latter contradicts boundedness of k. Therefore, x is bounded.

Step 4: We show assertion (ii).
Since x and k are bounded, it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\|x\|^p$ is bounded, and so $\|x\|^p$ is uniformly continuous. Consequently, also $t \mapsto \min\{r, \|x(t)\|^p\}$ is uniformly continuous. Thus we may apply Barbălat's Lemma [4] to conclude that $k_\infty - k^0 = \int_0^\infty \min\{r, \|x(t)\|^p\}\,\mathrm{d}t \in \mathbb{R}$ yields $\min\{r, \|x(t)\|^p\} \to 0$ as $t \to \infty$, which is assertion (ii). $\square$

Proof of Corollary 2.7:
The key tool in proving the Theorems 2.3, 2.4 and 2.5 is Lemma 4.1. To derive Corollary 2.7, repeat the arguments exploiting the assertion of Corollary 4.2. We omit the details for brevity. $\square$

Proof of Theorem 2.9:
Consider a piecewise monotone periodic function p with period $\omega > 0$, and with discrete zeros. By piecewise monotonicity $t \mapsto p(t)$ is measurable, hence the initial value problem

$$\dot{x} = \big(A + kp(t)\Sigma_A\big)x, \qquad x(0) = x^0 \tag{6.14}$$

has, for any $x^0 \in \mathbb{R}^n$, a unique solution on $\mathbb{R}$. Linearity of (6.14) implies that the zero solution is asymptotically stable if it is attractive. It therefore remains to determine some $k^* > 0$ such that, for every k with $k \geq k^*$, the zero solution of $\dot{x} = \big(A + kp(t)\Sigma_A\big)x$ is globally attractive. This follows, by invoking (4.6) again, if there exists some $\rho \in (0,1)$ such that, for all k with $k \geq k^*$ and all $x^0 \in \mathbb{R}^n \setminus \{0\}$, the solution of (6.14) satisfies

$$\|x(\omega)\|^2 \;\leq\; \rho\,\|x^0\|^2\,. \tag{6.15}$$

With $M = 1 + \frac{2n\|A\|}{-\mathrm{tr}\,(A)}$ and $\gamma = \frac{2M-1}{2M+1}$ put, for $j \in \mathbb{N}_0$,

$$p_0 = \sup\{|p(t)| : t \in [0,\omega)\}, \qquad p_j = \gamma^j\, p_0, \qquad \text{and} \quad m_j = \tfrac{2M}{2M+1}\, p_j$$

as well as

$$T_j = \{t \in [0,\omega] : |p(t)| \in (p_{j+1}, p_j]\} \qquad \text{and} \quad T_\infty = \{t \in [0,\omega] : p(t) = 0\}.$$

Note that $p_0 < \infty$ by assumption, so that $\lim_{j\to\infty} p_j = \lim_{j\to\infty} m_j = 0$ by virtue of $\gamma < 1$. Furthermore, T_∞ is a finite set by assumption, and the time interval $[0,\omega]$ can be written as the disjoint union $[0,\omega] = \bigcup_{j=0}^\infty T_j \cup T_\infty$. We partition each T_j further as follows. Since p is piecewise monotone, for each $j \in \mathbb{N}_0$ there exist $L_j \in \mathbb{N}_0$ (set $L_j = 0$ if $T_j = \emptyset$) disjoint intervals $T_{j\ell} = (\underline{t}_{j\ell}, \overline{t}_{j\ell}) \subset T_j$ such that $p(\cdot)$ is monotone and has constant sign on $T_{j\ell}$, $\ell = 1, \ldots, L_j$, and such that $T_j = \bigcup_{\ell=1}^{L_j} (\underline{t}_{j\ell}, \overline{t}_{j\ell})$ except for finitely many points (in fact, consisting of $\underline{t}_{j\ell}$). Writing

$$\forall\, j \in \mathbb{N}_0 \;\; \forall\, \ell = 1, \ldots, L_j \;:\quad h_j := |T_j|, \qquad h_{j\ell} := |T_{j\ell}|\,,$$

gives

$$\forall\, j \in \mathbb{N}_0 \;:\quad |T_j| = \sum_{\ell=1}^{L_j} |T_{j\ell}| = \sum_{\ell=1}^{L_j} (\overline{t}_{j\ell} - \underline{t}_{j\ell}) = \sum_{\ell=1}^{L_j} h_{j\ell}\,.$$

By Lemma 3.6 and Lemma 3.4 together with Theorem 3.1, we may choose $m^*, \beta > 0$ such that

$$\forall\, m \in \mathbb{R} \text{ with } |m| \geq m^* \ \forall\, k \in \mathbb{R} \ : \quad A_k^T P_m + P_m A_k \ \leq \ -\Big(1 - \tfrac{|k-m|}{|m|}\, M\Big)\, I \tag{6.16}$$

and

$$\forall\, m \in \mathbb{R} \text{ with } |m| \geq m^* \ : \quad \beta \ < \ \frac{1}{\sigma_{\max}(P_m)}\,. \tag{6.17}$$

Define the strictly increasing and unbounded $\mathbb{N}$-valued sequences $(\psi(j))_{j\in\mathbb{N}_0}$ and $(j(\psi))_{\psi\in\mathbb{N}_{\psi(0)}}$ by

$$\psi(j) = \min\{p \in \mathbb{N} : p\, m_j > m^*\}, \quad j \in \mathbb{N}_0, \qquad j(\psi) = \max\{j \in \mathbb{N} : \psi(j) \leq \psi\}, \quad \psi \in \mathbb{N}_{\psi(0)}.$$

Since $\sum_{j=1}^{\infty} h_j = \omega$ implies

$$\mu(\psi) := \sum_{j=j(\psi)+1}^{\infty} h_j \to 0 \quad \text{ as } \psi \to \infty\,, \tag{6.18}$$

we may choose $\psi \in \mathbb{N}$ so large that

$$\mathrm{e}^{\lambda_{\max}(A)\,\mu(\psi) - \frac{\beta}{2}(\omega - \mu(\psi))} =: \rho < 1\,. \tag{6.19}$$

Put

$$m_{j\ell}(k) := \mathrm{sgn}\Big(p\big(\tfrac{\bar{t}_{j\ell} + \underline{t}_{j\ell}}{2}\big)\Big) k\, m_j, \quad j \in \mathbb{N}_0, \quad \ell \in \{1, \ldots, L_j\}, \quad k > 0.$$

Then

$$\begin{aligned}
&\forall\, j \in \mathbb{N}_0 \ \ \forall\, \ell \in \{1, \ldots, L_j\} \ \ \forall\, t \in [\underline{t}_{j\ell}, \bar{t}_{j\ell}) \ \ \forall\, k \geq \psi(j) \ : \\
&\quad |kp(t) - m_{j\ell}(k)| = km_j \left| \frac{|p(t)|}{m_j} - 1 \right| = km_j \left[1 - \frac{|p(t)|}{m_j} \right] \\
&\qquad\qquad \leq km_j \left[1 - \frac{p_{j+1}}{m_j} \right] = \frac{k\, m_j}{2M} = \frac{|m_{j\ell}(k)|}{2M}.
\end{aligned} \tag{6.20}$$

Now (6.16) and (6.20) ensure that the assumptions of Lemma 4.1 are fulfilled, and so applying (4.3) to the solution x of the initial value problem (6.14) gives, by invoking (6.17),

$$\forall\, j \in \mathbb{N}_0 \ \ \forall\, \ell \in \{1, \ldots, L_j\} \ \ \forall\, k \geq \psi(j) \ : \quad \|x(\bar{t}_{j\ell})\|^2 \ \leq \ \kappa_2(P_{m_{j\ell}(k)})\, \mathrm{e}^{-\beta\, h_{j\ell}}\, \|x(\underline{t}_{j\ell})\|^2. \tag{6.21}$$

In view of (4.6) and (6.21), we have, for any $x^0 \neq 0$,

$$\begin{aligned}
\frac{\|x(\omega)\|^2}{\|x^0\|^2} \ &= \ \prod_{j=1}^{\infty} \prod_{\ell=1}^{L_j} \frac{\|x(\bar{t}_{j\ell})\|^2}{\|x(\underline{t}_{j\ell})\|^2} \\
&= \ \Big(\prod_{j=j(k)+1}^{\infty} \prod_{\ell=1}^{L_j} \frac{\|x(\bar{t}_{j\ell})\|^2}{\|x(\underline{t}_{j\ell})\|^2} \Big) \Big(\prod_{j=1}^{j(k)} \prod_{\ell=1}^{L_j} \frac{\|x(\bar{t}_{j\ell})\|^2}{\|x(\underline{t}_{j\ell})\|^2} \Big) \\
&\leq \ \mathrm{e}^{\lambda_{\max}(A)\,\mu(\psi)} \prod_{j=1}^{j(\psi)} \prod_{\ell=1}^{L_j} \kappa_2(P_{m_{j\ell}(k)})\, \mathrm{e}^{-\beta\, h_{j\ell}} \\
&\leq \ \mathrm{e}^{\lambda_{\max}(A)\,\mu(\psi)} \prod_{j=1}^{j(\psi)} \Big(\prod_{\ell=1}^{L_j} \kappa_2(P_{m_{j\ell}(k)}) \Big) \mathrm{e}^{-\beta\, h_j}\,.
\end{aligned} \tag{6.22}$$

By Corollary 3.5, we may choose $a, \widetilde{k} = \widetilde{k}(\psi) > 0$ such that

$$\forall\, k \geq \widetilde{k} \quad \forall\, j \in \{1, \ldots, j(\psi)\} \quad \forall\, \ell \in \{1, \ldots, L_j\} \ : \quad \kappa_2(P_{m_{j\ell}(k)}) \leq 1 + \tfrac{a}{|m_{j\ell}(k)|} = 1 + \tfrac{a}{k m_j}\,,$$

and hence

$$\forall\, k \geq \widetilde{k} \quad \forall\, j \in \{1, \ldots, j(\psi)\} \ : \quad \prod_{\ell=1}^{L_j} \kappa_2(P_{m_{j\ell}(k)}) \leq \Big(1 + \tfrac{a}{k m_j}\Big)^{L_j}\,,$$

and furthermore, we may choose $k^* = k^*(\psi) \geq \widetilde{k}$ such that

$$\forall\, k \geq k^* \quad \forall\, j \in \{1, \ldots, j(\psi)\} \ : \quad \prod_{\ell=1}^{L_j} \kappa_2(P_{m_{j\ell}(k)}) \leq \mathrm{e}^{\beta\, h_j/2}\,,$$

which, when inserted into (6.22) and invoking (6.15), yields

$$\forall\, k \geq k^* \ : \ \frac{\|x(\omega)\|^2}{\|x^0\|^2} \ \leq \ \mathrm{e}^{\lambda_{\max}(A)\,\mu(\psi)} \prod_{j=1}^{j(\psi)} \mathrm{e}^{-\beta\, h_j/2} \ \leq \ \mathrm{e}^{\lambda_{\max}(A)\,\mu(\psi) - \frac{\beta}{2}(\omega - \mu(\psi))} \ < \rho\,.$$

This shows (6.15). □

Proof of Theorem 2.11:
Let $r > 0$, $p \geq 1$, $x^0 \in \mathbb{R}^n$, $k^0 > 0$. Existence and uniqueness of the solution $x : [0, \infty) \to \mathbb{R}^n$ to the initial value problem (2.9), (2.9) follows as in Step 1 of the proof of Theorem 2.5.
If k is bounded, then it follows as in Step 3 and 4 of the proof of Theorem 2.5 that x is bounded, and that assertion (ii) holds.
Therefore, it remains to show boundedness of k, whence assertion (i). Seeking a contradiction, suppose that k is unbounded, i.e. $k(t)$ tends monotonically to ∞ as t tends to ∞. If $\Phi(\cdot, \cdot)$ denotes the transition matrix of (2.9), then in view of (4.6) it remains to show that

$$\exists\, \rho \in (0, 1)\ \exists\, i \in \mathbb{N}\ \forall\, j \in \mathbb{N}_i \ : \quad \|\Phi(t_{j+1}, t_j)\, x(t_j)\| \leq \rho\, \|x(t_j)\|\,. \tag{6.23}$$

For $N \in \mathbb{N}_3$ put

$$\begin{array}{rlll}
t_j & = j\pi, & t_{j\ell} = j\pi + \frac{\ell}{N}\pi, & j \in \mathbb{N}_0,\ \ell \in \{0, \ldots, N\} \\
\bar{t}_{j\ell} & = \frac{t_{j,\ell+1} + t_{j\ell}}{2}\,, & & j \in \mathbb{N}_0,\ \ell \in \{0, \ldots, N-1\} \\
h_j & = t_{j+1} - t_j, & h_{j\ell} = t_{j,\ell+1} - t_{j\ell} = \pi/N & j \in \mathbb{N}_0,\ \ell \in \{1, \ldots, N-1\} \\
m_j & = k(t_{j1})\, |\sin(\pi/N)| & & j \in \mathbb{N}_0 \\
m_{j\ell} & = k(\bar{t}_{j\ell})\, \sin(\bar{t}_{j\ell}), & & j \in \mathbb{N}_0,\ \ell \in \{1, \ldots, N-2\}\,.
\end{array}$$

In passing, note that $t_{j0} = t_j$ and $t_{jN} = t_{j+1}$ for all $j \in \mathbb{N}_0$ and, since k is assumed to be unbounded,

$$\exists\, i \in \mathbb{N}\ \forall\, j \in \mathbb{N}_i\ \ell \in \{1, \ldots, N-2\} \ : \quad m_j \leq |m_{j\ell}|\,.$$

By Lemma 3.6 and Lemma 3.4 we may choose $m^*, \beta > 0$ such that for every real m with $|m| \geq m^*$ and for every $k \in \mathbb{R}$

$$A_k^T P_m + P_m A_k \ \leq \ -\Big(1 - \tfrac{|k-m|}{|m|}\, M\Big)\, I \qquad \text{and} \quad \beta \ < \ \frac{1}{\sigma_{\max}(P_m)}, \tag{6.24}$$

where again $M = 1 + \frac{2n\|A\|}{-\mathrm{tr}\,(A)}$. Next choose $N \in \mathbb{N}$ and $i \in \mathbb{N}$, both sufficiently large, and, in view of Corollary 3.5, $a > 0$ such that

$$\exp\big(2\,\lambda_{\max}(A) - \beta(N-2)\big) =: \rho_N < 1\,, \tag{6.25}$$

$$\forall\, j \in \mathbb{N}_i\ \forall\, \ell \in \{1,\ldots,N-2\}\ \ \forall\, t \in (t_{j\ell}, t_{j,\ell+1})\ :\ \frac{k(\bar t_{j\ell}) + \frac{r\pi}{2N}}{k(\bar t_{j\ell})}\ \frac{|\sin t|}{|\sin(\bar t_{j\ell})|} \le 1 + \tfrac{1}{2M}, \tag{6.26}$$

$$\exists\, \rho \in (0,1)\ \forall\, j \in \mathbb{N}_i\ :\quad \rho_N\ \Big(1 + \tfrac{a}{|m_j|}\Big)^{N-2} \le \rho\,, \tag{6.27}$$

$$\forall\, j \in \mathbb{N}_i\ \forall\, \ell \in \{1,\ldots,N-2\}\ :\quad \kappa_2(P_{m_{j\ell}}) \le 1 + \tfrac{a}{|m_{j\ell}|} \le 1 + \tfrac{a}{m_j}\,. \tag{6.28}$$

In view of $\dot k(t) \le r$ and (6.26), we have

$$\forall\, j \in \mathbb{N}_i\ \forall\, \ell \in \{1,\ldots,N-2\}\ \forall\, t \in (t_{j\ell}, t_{j,\ell+1})\ :$$
$$\frac{|k(t)\,\sin t - m_{j\ell}|}{|m_{j\ell}|} \;=\; \frac{k(t)\,|\sin t|}{k(\bar t_{j\ell})|\sin(\bar t_{j\ell})|} - 1 \;\le\; \frac{k(\bar t_{j\ell}) + \frac{r\pi}{2N}}{k(\bar t_{j\ell})}\,\frac{|\sin t|}{|\sin(\bar t_{j\ell})|} - 1 \;\le\; \frac{1}{2M}\,,$$

and together with (6.24) we may apply Lemma 4.1 to conclude, by invoking (4.6) again,

$$\begin{aligned}
&\forall\, j \in \mathbb{N}_i\ \forall\, \ell \in \{1,\ldots,N-2\}\ :\\
&\qquad \|\Phi(t_{j+1}, t_j)\, x(t_j)\| \;\le\; \|\Phi(t_{jN}, t_{j,N-1}) \cdots \Phi(t_{j1}, t_{j0}) x(t_j)\|\\
&\qquad\qquad \le\; \mathrm{e}^{\lambda_{\max}(A) 2\pi/N} \prod_{\ell=1}^{N-2} \kappa_2(P_{m_{j\ell}})\, \mathrm{e}^{-\beta\, h_{j\ell}}\ \|x(t_j)\|\\
&\qquad\qquad \overset{(6.28)}{\le}\; \mathrm{e}^{\lambda_{\max}(A) 2\pi/N} \big(1 + \tfrac{a}{|m_{j\ell}|}\big)^{N-2} \mathrm{e}^{-\beta\,\pi(N-2)/N}\ \|x(t_j)\|\\
&\qquad\qquad \overset{(6.25),(6.28)}{\le}\; \rho_N\ \big(1 + \tfrac{a}{|m_j|}\big)^{N-2}\ \|x(t_j)\|\\
&\qquad\qquad \overset{(6.27)}{\le}\; \rho\, \|x(t_j)\|\,.
\end{aligned}$$

This proves (6.23). □

7 Conclusions

We have derived several stabilization results of linear systems by rotation:

(a) For any A with $\mathrm{tr}\, A < 0$, there exists a skew-symmetric matrix Σ_A, such that $A + k\Sigma_A$ is stable for k with $|k|$ large enough. The transient bound of the system $\dot x = (A + k\Sigma)x$ approaches the optimal value 1 as $|k| \to \infty$. The matrix Σ_A depends only on the symmetric part $A + A^T$ of A. This, in particular, implies that $A + A^T$ alone does not yield any information on the transient bound.

(b) The system $\dot x = \big(A + k(t)\Sigma_A\big)\, x$ is stable, if $t \mapsto k(t)$ becomes sufficiently large and k grows monotonically. If k is not monotone, then the system may be unstable, even if k tends to ∞.

(c) A stabilizing controller gain function k can be determined by a servo mechanism so that $u(t) = k(t)\Sigma_A\, x(t)$ is stabilizing.

(d) The dynamic state feedback controller is robust with respect to bounded skew-symmetric perturbations.

(e) A system $\dot{x} = Ax$ is vibrationally stabilizable in the sense of Meerkov if, and only if, $\operatorname{tr} A < 0$.

(f) A stabilizing controller gain function k can be determined by a servo mechanism so that $u(t) = k(t)p(t)\Sigma_A\, x(t)$ is vibrationally stabilizing.

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