State-feedback H^∞-type control of linear systems with time-varying parameter uncertainty

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Abstract

This paper is concerned with stabilization and disturbance attenuation problems for linear systems with uncertain parameters. We consider the case where the uncertainties are possibly unbounded but have a certain normal distribution and the case where the uncertainties are arbitrary but bounded. Criteria for the solvability of the disturbance attenuation problem are presented in terms of generalized Riccati-type matrix inequalities and equations. The main part of the paper is devoted to the analysis and solution of these equations. A numerical example is given to illustrate our results.

1 Linear models with uncertain parameters

We consider linear continuous-time control systems with parameter uncertainty. For motivation assume that we are given a time-invariant state-space model of some physical system and call this model the *nominal system*:

$$\dot{x} \;=\; Ax, \quad x \in \mathbb{R}^n \;.$$

It is clear that the dynamics of the nominal system only approximately describes the true dynamics of the physical system. Three important sources of error are the following: Firstly it is often impossible to determine the values of the system parameters exactly; secondly nonlinear effects have been neglected; thirdly the physical system is subject to exogenous disturbances, which might lead to changes in its dynamics.
In order to cope with these problems one can allow for parameter uncertainties in the nominal system. Corresponding to the different error sources, the nature of the

uncertainties can be different. If a linear time-invariant model is sufficient, but one is not able to determine the true values of some parameters exactly, then one can consider a whole class of systems

$$\dot{x} \;=\; (A + A_0)x, \quad A_0 \in \mathcal{A}_0 \subset \mathbb{R}^{n\times n} \;.$$

The set $\mathcal{A}_0$ represents all possible deviations of the true system parameters from those of the nominal system.
If, however, we wish to cope with nonlinear or exogenous effects, it is, in general, not sufficient to model the parameter uncertainties time-invariant. We are rather led to a model of the form

$$\dot{x} \;=\; \left(A + \sum_{i=1}^{N} \delta_i(t) A_0^{(i)} \right) x \;, \tag{1}$$

where the A_0^i represent the uncertain parameters and the δ_i are unknown scalar functions.
In the deterministic theory one usually assumes the δ_i to be bounded in norm, i.e. $\max_{t\in\mathbb{R}} |\delta_i(t)| \leq d_i$ for some given $d_i \geq 0$. Then a typical question is whether (1) is stable for all δ_i satisfying this constraint.
But one also might require the δ_i to have certain statistical properties. A common concept is to model the δ_i as zero-mean Gaussian stochastic processes with intensity $\sigma_i \geq 0$ and to consider the Itô-equation

$$dx \;=\; Ax\,dt + \sum_{i=1}^{N} \sigma_i A_0^{(i)} x\,dw \;. \tag{2}$$

This model allows for arbitrarily large deviations from the nominal system if these deviations occur not very often, i.e. with sufficiently small probability. Now a typical question is whether system (2) is stable in an adequate stochastic sense.
Systems of the form (2) were introduced into the control literature by Wonham [15] and have been dealt with extensively since. Robust H^∞-type control problems for stochastic systems have recently been studied e.g. in [8], [5] and [13].
On the other hand it has been observed in [1] that mean-square stability of (2) is closely related to the robust stability of (1): If (2) is mean-square asymptotically stable with a certain decay rate α then one can provide bounds d_i such that (1) is asymptotically stable for $|\delta_i| < d_i$.

The object of the present paper is threefold: Firstly we study a static state-feedback disturbance attenuation problem, where we allow for stochastic uncertainty in all parameters of the state equation. Note that [8] addresses the dynamic output feedback problem – which is more general – but does not allow for parameter uncertainties at the control input. Our problem leads to a very general indefinite Riccati-type matrix inequality, whereas the approach in [8] led to a coupled pair of Riccati-type matrix inequalities each of which is a special case of the one derived here. A thorough discussion of the inequality derived here seems to be indispensible to cope with the coupled pair of inequalities from [8].

Therefore, secondly, the main part of the paper is devoted to the analysis of this matrix inequality and the corresponding equation. We provide necessary and sufficient conditions for the existence of a stabilizing solution and suggest an iterative method to compute it.

Thirdly we show, how the stochastic disturbance attenuation problem relates to the corresponding deterministic problem with bounded parameter uncertainties; we give sufficient conditions for a disturbance attenuation problem to be solvable in this case. The results are illustrated by a simple example from the literature.

2 An H^∞-type control problem for a system with stochastic parameter uncertainty

We consider a linear stochastic system with the disturbance input v, the control input u, and the to be controlled output z:

$$\begin{aligned} dx(t) &= \quad Ax(t)dt + \sum_{i=1}^{N} A_0^i x(t)\, dw_i(t) \\ &\quad + B_1 v(t)dt + \sum_{i=1}^{N} B_{1,0}^i v(t)\, dw_i(t) \\ &\quad + B_2 u(t)dt + \sum_{i=1}^{N} B_{2,0}^i u(t)\, dw_i(t) \\ z(t) &= Cx(t) + D_1 v(t) + D_2 u(t). \end{aligned} \tag{3}$$

Here $x \in \mathbb{K}^n$, $u \in \mathbb{K}^m$, $v \in \mathbb{K}^\ell$, $z \in \mathbb{K}^q$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and all matrices are matrices of fitting sizes over $\mathbb{K}$. The processes $(w_i(t))_{t\in\mathbb{R}_+}$, $i = 1, \dots, N$ are independent uncorrelated zero mean real Wiener processes on a probability space $(\Omega, \mathcal{F}, \mu)$ with respect to an increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ (compare [6]).

Remark 2.1 *By our assumption the covariance matrix Q of the N-dimensional Wiener process $[w_1(t), \dots, w_N(t)]^T$ is the identity. This is not a restriction, for if $Q \neq I$, we can consider uncorrelated and normed linear combinations of the w_i, if we transform the matrices A_0^i and B_0^i as follows:*

$$\left[\begin{array}{cc} \tilde{A}_0^i & \tilde{B}_0^i \end{array}\right]_{i=1}^N = (\sqrt{Q} \otimes I_n) \left[\begin{array}{cc} A_0^i & B_0^i \end{array}\right]_{i=1}^N ,$$

where $\otimes$ denotes the Kronecker product, and $\left[\begin{array}{cc} A_0^i & B_0^i \end{array}\right]_{i=1}^N \in \mathbb{K}^{nN\times(n+m)}$.

Let $L^2_w(\mathbb{R}_+, \mathbb{K}^\ell)$ denote the corresponding space of non-anticipating stochastic processes v with values in $\mathbb{K}^\ell$ and norm

$$\|v(\cdot)\|^2_{L^2_w} \;:=\; \mathcal{E}\left(\int_0^\infty \|v(t)\|^2 dt\right) < \infty,$$

where $\mathcal{E}$ denotes expectation.
It is known from stochastic analysis, that for all $(x_0, v, u) \in \mathbb{K}^n \times L^2_w(\mathbb{R}_+, \mathbb{K}^\ell) \times L^2_w(\mathbb{R}_+, \mathbb{K}^m)$ there exists a unique solution $x(\cdot; x_0, v, u)$ of (3) and thus also a unique output process $z(\cdot; x_0, v, u)$.
Our aim is to construct a state-feedback control $u = Fx$, such that (3) is stabilized and the effect of v on z is reduced.
To be more precise, let us consider the closed-loop system

$$\begin{aligned} dx(t) &= \widehat{A}x(t)dt + \sum_{i=1}^N \widehat{A}_0^i x\, dw_i + B_1 v dt + \sum_{i=1}^N B_{1,0}^i v\, dw_i \\ z &= \widehat{C}x + D_1 v, \end{aligned} \tag{4}$$

with $\widehat{A} = A + B_2F$, $\widehat{A}_0^i = A_0^i + B_{2,0}^i F$, and $\widehat{C} = C + D_2 F$.
The solutions of (4) with initial value x_0 at $t = 0$ are denoted by $x(t; x_0, v)$; the corresponding output processes are denoted by $z(t; x_0, v)$.

Definition 2.2 *We call system (4)* internally (mean-square) stable *if for all $x_0 \in \mathbb{K}^n$ the unperturbed solution $x(\cdot, x_0, 0)$ is in $L^2_w(\mathbb{R}_+, \mathbb{K}^n)$, or eqivalently, if there exist numbers $M \geq 1$, $\alpha > 0$, such that*

$$\mathcal{E}\|x(t; x_0, 0)\|^2 \leq Me^{-2\alpha t}\|x_0\|^2 \quad \textit{for all} \;\; x_0 \in \mathbb{K}^n,\, t \geq 0.$$

If such an α is given we also say, that (4) is internally stable with decay rate greater than α*.*

If system (3) is internally stable then for all perturbations $v \in L^2_w(\mathbb{R}_+, \mathbb{K}^\ell)$ the solution $x(t; 0, v)$ (starting at $x_0 = 0$) is in $L^2_w(\mathbb{R}_+, \mathbb{K}^n)$ (see [8]).

Definition 2.3 *Let system (3) be internally stable. We define the* perturbation operator $\mathbb{L}^F : L^2_w(\mathbb{R}_+, \mathbb{K}^\ell) \to L^2_w(\mathbb{R}_+, \mathbb{K}^q)$ *by $\mathbb{L}^F(v) = z(\cdot; 0, v)$.*

In these terms we can give a precise definition of our design problem, which is a straight-forward generalization of the suboptimal H^∞-control design concept from deterministic to stochastic systems.

Definition 2.4 *Let $\gamma > 0$. The* γ-suboptimal stochastic H^∞ problem *consists in finding a feedback-gain matrix F, such that system (4) is internally (mean square) stable and the perturbation operator $\mathbb{L}^F : v \mapsto z$ has norm $\|\mathbb{L}\| < \gamma$.*

The stochastic bounded real lemma by Hinrichsen and Pritchard [8] gives a necessary and sufficient condition for F to solve the γ-suboptimal stochastic H^∞ problem.

Theorem 2.5 *System (4) is internally (mean-square) stable and the perturbation operator* $\mathbb{L}^F$ *has norm* $\|\mathbb{L}^F\| < \gamma$ *if and only if there exists a negative definite matrix* $0 > X = X^* \in \mathbb{K}^{n\times n}$ *such that*

$$Q_1^\gamma(X) > 0 \quad \text{and} \quad \mathcal{R}_F^\gamma(X) := P_F(X) - S_F(X) Q_1^\gamma(X) S_F^*(X) > 0\,. \tag{5}$$

Here the affine linear matrix operators P_F, S_F *and* Q_1^γ *are given by*

$$\begin{aligned} P_F(X) &= \widehat{A}^* X + X\widehat{A} + \sum_{i=1}^N \widehat{A}_0^{i*} X \widehat{A}_0^i - \widehat{C}^*\widehat{C} \\ S_F(X) &= XB_1 + \sum_{i=1}^N \widehat{A}_0^{i*} X B_{01}^i - \widehat{C}^* D_1 \\ Q_1^\gamma(X) &= \sum_{i=1}^N B_{01}^{i*} X B_{01}^i + \gamma^2 I - D^* D_1 \end{aligned} \tag{6}$$

with $\widehat{A} = A + B_2 F$, $\widehat{A}_0^i = A_0^i + B_{2,0}^i F$, *and* $\widehat{C} = C + D_2 F$, *as before.*

The inequality (5) and the equation $\mathcal{R}_F^\gamma(X) = 0$ have been studied in [3]:

Corollary 2.6 *If* (A, C) *is observable, then the following are equivalent:*

(i) $\exists X < 0$: $Q_1^\gamma(X) > 0$ *and* $\mathcal{R}_F^\gamma(X) > 0$,

(ii) $\exists X < 0$: $Q_1^\gamma(X) > 0$, $\mathcal{R}_F^\gamma(X) = 0$ *and* $\sigma\left((\mathcal{R}_F^\gamma)'_X\right) \subset \mathbb{C}_-$.

Here $(\mathcal{R}_F^\gamma)'_X$ *denotes the derivative of* $\mathcal{R}_F^\gamma$ *at* X.
The implication (ii)⇒(i) holds without the observability assumption.

By Theorem 2.5 the γ-suboptimal stochastic H^∞-problem is solvable if and only if there exists a pair of matrices (F, X) such that (5) is satisfied. To delete the matrix F from this condition we use a completion of the square argument, where we impose a regularity assumption. Let us first introduce the following notation. We set

$$M^\gamma = \begin{bmatrix} -C^*C & -C^*D_2 & -C^*D_1 \\ -D_2^*C & -D_2^*D_2 & -D_2^*D_1 \\ -D_1^*C & -D_1^*D_2 & \gamma^2 I - D_1^*D_1 \end{bmatrix} \tag{7}$$

$$\Pi(X) = \sum_{i=1}^N \begin{bmatrix} A_0^{i*}XA_0^i & A_0^{i*}XB_{02}^i & A_0^{i*}XB_{01}^i \\ B_{02}^{i*}XA_0^i & B_{02}^{i*}XB_{02}^i & B_{02}^{i*}XB_{01}^i \\ B_{01}^{i*}XA_0^i & B_{01}^{i*}XB_{02}^i & B_{01}^{i*}XB_{01}^i \end{bmatrix} \tag{8}$$

and with the same partition we define

$$\begin{bmatrix} P_0(X) & S_{02}(X) & S_{01}(X) \\ S_{20}(X) & Q_2(X) & S_{21}(X) \\ S_{10}(X) & S_{12}(X) & Q_1^\gamma(X) \end{bmatrix} = \begin{bmatrix} A^*X + XA & XB_2 & XB_1 \\ B_2^*X & 0 & 0 \\ B_1^*X & 0 & 0 \end{bmatrix} + \Pi(X) + M.$$

Note that Π is a positive operator in the sense of Definition 4.1 below.

Definition 2.7 *We call system (3)* regular *if $D_2^*D_2 > 0$.*

Note that regularity of (3) implies $Q_2(X) < 0$ for all $X \leq 0$. Often one also assumes $D_2^*C = 0$ for regular systems, since this can always be achieved by some transformation. In general, however, we do not make this assumption.

Theorem 2.8 *Let system (3) be regular. The γ-suboptimal stochastic H^∞ problem is solvable, if and only if there exists a negative definite matrix $X < 0$, such that*

$$Q_1^\gamma(X) \; > \; 0 \tag{9}$$

and the Riccati-type inequality

$$\mathcal{R}^\gamma(X) \; = \; P_0(X) - \begin{bmatrix} S_{20} \\ S_{10} \end{bmatrix}^* \begin{bmatrix} Q_2(X) & S_{21}(X) \\ S_{12}(X) & Q_1^\gamma(X) \end{bmatrix}^{-1} \begin{bmatrix} S_{20} \\ S_{10} \end{bmatrix} > 0 \tag{10}$$

is satisfied.
For any matrix $X < 0$ satisfying (9) and (10), the feedback-gain matrix

$$F = -(Q_2(X) - S_{21}(X)Q_1^\gamma(X)^{-1}S_{12}(X))^{-1}(S_{20}(X) - S_{01}(X)Q_1^\gamma(X)^{-1}S_{12}(X)) \tag{11}$$

solves the γ-suboptimal stochastic H^∞ problem.

Proof: Factoring out the matrix F we can write the second inequality in (5) as

$$\begin{aligned} 0 \; < \; \mathcal{R}_F^\gamma(X) \; &= \; P_0(X) - S_{01}(X)Q_1^\gamma(X)^{-1}S_{10}(X) \\ &\quad + (S_{02}(X) - S_{21}(X)Q_1^\gamma(X)^{-1}S_{10}(X))F \\ &\quad + F^*(S_{20}(X) - S_{01}(X)Q_1^\gamma(X)^{-1}S_{12}(X)) \\ &\quad + F^*(Q_2(X) - S_{21}(X)Q_1^\gamma(X)^{-1}S_{12}(X))F \\ &=: \; P(X) + S(X)^*F + F^*S(X) + F^*Q(X)F \end{aligned} \tag{12}$$

with an obvious correspondence of terms.
Note that $Q(X) = Q_2(X) - S_{21}(X)Q_1^\gamma(X)^{-1}S_{12}(X)$ is negative definite, since $Q_2 < 0$ and $Q_1^\gamma > 0$. By the relation

$$\begin{aligned} P + F^*S + S^*F + F^*QF \; &= \; P + (F^* + SQ^{-1})Q(F + Q^{-1}S^*) - SQ^{-1}S^* \qquad (13) \\ &\leq \; P - SQ^{-1}S^* \quad \text{(with equality for } F = -Q^{-1}S^*) \end{aligned}$$

it is clear, that $P + F^*S + S^*F + F^*QF > 0$ implies $P - SQ^{-1}S^* > 0$; if vice versa $P - SQ^{-1}S^* > 0$ then $P + F^*S + S^*F + F^*QF > 0$ holds with $F = -Q^{-1}S^*$.
In other words, if the γ-suboptimal H^∞-problem is solvable then there exists a matrix $X < 0$ such that $Q_1^\gamma(X) > 0$ and $\mathcal{R}^\gamma(X) := P(X) - S(X)Q(X)^{-1}S(X)^* > 0$; and if such an X exists, then the matrix F in (11) solves the γ-suboptimal H^∞-problem.
It remains to verify, that $\mathcal{R}^\gamma$ has the form specified in (10).
To this end we write $\mathcal{R}^\gamma(X)$ as the Schur-complement

$$\mathcal{R}^\gamma(X) \quad = \quad \mathrm{Schur}\left(\begin{bmatrix} P & S \\ S^* & Q \end{bmatrix} \Big/ Q\right) \qquad \text{of the matrix } \begin{bmatrix} P & S \\ S^* & Q \end{bmatrix}$$

with respect to the block Q. Furthermore we write

$$\begin{bmatrix} P & S \\ S^* & Q \end{bmatrix} \quad = \quad \mathrm{Schur}\left(\begin{bmatrix} P_0(X) & S_{02}(X) & S_{01}(X) \\ S_{20}(X) & Q_2(X) & S_{21}(X) \\ S_{10}(X) & S_{12}(X) & Q_1^\gamma(X) \end{bmatrix} \Big/ Q_1^\gamma \right) ,$$

i.e. we interpret $\mathcal{R}^\gamma(X)$ as a double Schur-complement.
Applying the quotient formula from Lemma A.2 we find

$$\begin{aligned} \mathcal{R}^\gamma(X) \quad &= \quad \mathrm{Schur}\left(\begin{bmatrix} P_0(X) & S_{02}(X) & S_{01}(X) \\ S_{20}(X) & Q_2(X) & S_{21}(X) \\ S_{10}(X) & S_{12}(X) & Q_1^\gamma(X) \end{bmatrix} \Big/ \begin{bmatrix} Q_2(X) & S_{21}(X) \\ S_{12}(X) & Q_1^\gamma(X) \end{bmatrix}\right) \\ &= \quad P_0(X) - \begin{bmatrix} S_{20} \\ S_{10} \end{bmatrix}^* \begin{bmatrix} Q_2(X) & S_{21}(X) \\ S_{12}(X) & Q_1^\gamma(X) \end{bmatrix}^{-1} \begin{bmatrix} S_{20} \\ S_{10} \end{bmatrix} . \end{aligned}$$

□

From our calculations we can also draw the following conclusions.

Corollary 2.9 *Let system (3) be regular. If $X \leq 0$ satisfies $Q_1^\gamma(X) > 0$ and F is defined according to (11), then $\mathcal{R}^\gamma(X) = \mathcal{R}_F^\gamma(X)$, and ${\mathcal{R}_X^\gamma}' = (\mathcal{R}_F^\gamma)'_X$.*

Proof: The first assertion follows from (13). With $F = -Q(X)^{-1}S(X)^*$ we have for the derivative of $\mathcal{R}^\gamma$ at X in direction H

$${\mathcal{R}_X^\gamma}'(H) \quad = \quad P'(H) + S'(H)F + F^*Q'(H)F + F^*S'(H)^* = (\mathcal{R}_F^\gamma)'_X(H) \; .$$

Here we have applied the product rule for $\mathcal{R}^\gamma$, whereas $\mathcal{R}_F^\gamma$ was evaluated according to (12). □

Remark 2.10 *Together with the constrained Riccati-type inequality (10) and (9) we study the constrained Riccati-type equation*

$$\mathcal{R}^\gamma(X) = 0 \qquad \text{with } X < 0 \text{ and } Q_1^\gamma(X) > 0 \; . \tag{14}$$

We can recover the Riccati equations from different types of H^∞-control problems in (14). For simplicity let $D_2^*[C, D_2] = [0, I]$ and $D_1 = 0$ now.

(i) If $\Pi = 0$, i.e. all stochastic terms vanish, then (10) specializes to the indefinite Riccati equation of deterministic continuous-time H^∞-control:

$$A^*X + XA - C^*C - X(-B_2^*B_2 + \gamma^{-2}B_1^*B_1)X \;=\; 0$$

(ii) If $A = -\frac{1}{2}I$, $B_1 = 0$, $B_2 = 0$, equation (14) turns into its counterpart from discrete-time stochastic control (compare [4]).

(iii) If $A = -\frac{1}{2}I$, $B_1 = 0$, $B_2 = 0$, and $\Pi(X) = \begin{bmatrix} A_0^*XA_0 & A_0^*XB_{01} & A_0^*XB_{02} \\ B_{02}^*XA_0 & B_{02}^*XB_{02} & 0 \\ B_{01}^*XA_0 & 0 & B_{01}^*XB_{01} \end{bmatrix}$, *we discover the constrained indefinite Riccati equation of deterministic discrete-time H^∞-control: $B_{01}^*XB_{01} + \gamma^2 I > 0$, and*

$$\begin{aligned} -X + A_0^*XA_0 - C^*C - A_0^*XB_{02}(B_{02}^*XB_{02} - I)^{-1}B_{02}^*XA_0 \\ - A_0^*XB_{01}(B_{01}^*XB_{01} + \gamma^2 I)^{-1}B_{01}^*XA_0 \;=\; 0\,. \end{aligned}$$

(iv) If we let $\gamma \to \infty$ in either of the Riccati-type equations from H^∞ control, we end up with the corresponding Riccati-type equation from LQ-control.

3 An H^∞-type control problem for a system with bounded parameter uncertainty

As we have pointed out in the first section, there are various ways of modelling parameter uncertainty, and in many cases one might argue, which one is appropriate. Therefore it is interesting to observe, that our approach via generalized Riccati inequalities captures different types of parameter uncertainties at once.

In analogy to the systems (3) and (4) we consider the open-loop system

$$\begin{aligned} \dot{x} &= \left(A + \sum_{i=1}^N \delta_i(t)A_0^i\right)x + \left(B_1 + \sum_{i=1}^N \delta_i(t)B_{1,0}^i\right)v + \left(B_2 + \sum_{i=1}^N \delta_i(t)B_{2,0}^i\right)u \\ z &= Cx + D_1 v + D_2 u\,, \end{aligned} \tag{15}$$

and the corresponding closed-loop system

$$\begin{aligned} \dot{x}(t) &= \left(\widehat{A} + \sum_{i=1}^N \delta_i(t)\widehat{A}_0^i\right)x + \left(B_1 + \sum_{i=1}^N \delta_i(t)B_{1,0}^i\right)v \\ z &= \widehat{C}x + D_1 v\,. \end{aligned} \tag{16}$$

The δ_i are arbitrary measurable real or complex functions, which are bounded by given numbers $d_i > 0$, i.e. $\forall t > 0 : |\delta_i(t)| < d_i$.
We denote the solutions of (16) by $x_\delta(t; x_0, v)$ and the output by $z_\delta(t; x_0, v)$.
System (16) is called internally stable if for all x_0 the unperturbed solution $x_\delta(t; x_0, 0)$ is exponentially stable, i.e. $\exists \omega > 0, M > 1 : \|x_\delta(t; x_0, 0)\| < Me^{\omega t}$.
For a given $\gamma > 0$ let the matrix operators $\mathcal{Q}_1^\gamma$ and $\mathcal{R}^\gamma$ be given like in (9) and (10). We will give a sufficient stabilization criterion with guaranteed disturbance attenuation bound γ for system (15) in terms of these matrix operators. It is based on the following very simple inequality, which has been used already in [1] to derive a sufficient robust stabilization result from a stochastic stabilization criterion.

Lemma 3.1 *Let $0 < X = X^* \in \mathbb{K}^{n\times n}$ and $V, W \in \mathbb{K}^{n\times k}$ be arbitrary. Then*

$$V^*XV + W^*XW \leq \bar{\delta}\, V^*XW + \delta\, W^*XV \quad \text{for all } \delta \in \mathbb{C} \text{ with } |\delta| \leq 1 .$$

Proof: The assertion follows from $V^*XV + W^*XW - (\bar{\delta}\, V^*XW + \delta\, W^*XV) = (\delta V + W)^*X(\delta V + W)(1 - |\delta|^2)V^*XV$ which is nonpositive if $|\delta| \leq 1$. □

Theorem 3.2 *Let $\gamma > 0$, $\alpha = \sum_{i=1}^N d_i^2$ and assume that there exist a matrix $X < 0$, such that*

$$\mathcal{Q}_1^\gamma(X) > 0 \quad \text{and} \quad \mathcal{R}^\gamma(X) > -\alpha X . \tag{17}$$

If F is given by (11) then system (16) is internally stable; moreover $z_\delta(\cdot; 0, v) \in L^2(\mathbb{R}_+, \mathbb{R}^\ell)$ if $v \in L^2(\mathbb{R}_+, \mathbb{R}^q)$ and $\|z_\delta(\cdot; 0, v)\|_{L^2} \leq \gamma\|v\|_{L^2}$.

Proof: By Lyapunov's second method, (16) is internally stable if there exist an $X < 0$, such that for all $t > 0$

$$\left(\widehat{A} + \sum_{i=1}^N \delta_i(t)\widehat{A}_0^i\right)^* X + X\left(\widehat{A} + \sum_{i=1}^N \delta_i(t)\widehat{A}_0^i\right) > 0 . \tag{18}$$

The inequality $\mathcal{R}^\gamma(X) > -\alpha X$ implies $P_0(X) > -\alpha X$, that is

$$A^*X + XA + \alpha X + \sum_{i=1}^N A_0^{i*}XA_0^i > 0.$$

Since by Lemma 3.1 for each summand $d_i^2 X + A_0^{i*}XA_0^i \leq \bar{\delta}_i A_0^{i*}X + \delta_i XA_0^i$, if $|\delta_i| < d_i$, we see that (18) holds, if $|\delta_i(t)| < d_i$ and $\alpha = \sum d_i^2$.
Now we consider the finite-horizon cost functional

$$\begin{aligned} J_T(v) &= \int_0^T \left(\gamma^2\|v(t)\| - \|z(t)\|\right)\, dt \\ &= \int_0^T \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}^* \begin{bmatrix} -C^*C & -C^*D \\ -D^*C & \gamma^2 I - D^*D \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} , \end{aligned}$$

where $x(\cdot) = x_\delta(\cdot; 0, v)$ and $z(\cdot) = z_\delta(\cdot; 0, v)$. We need to show, that $J_T(v) \geq 0$ for all $T > 0$. With $X < 0$ we have

$$J_T(v) \geq J_T(v) + x(T)^* X x(T) = J_T(v) + \int_0^T \frac{d}{dt} \left(x(t)^* X x(t) \right) dt .$$

Computing the derivative in the intgrand and using the previous expression for $J_T(v)$, we find that the right-hand side is nonnegative, if

$$0 < \Delta_0 + \begin{bmatrix} \widehat{A}^* \\ \widehat{B}_1^* \end{bmatrix}^* X \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} X \begin{bmatrix} \widehat{A}^* \\ \widehat{B}_1^* \end{bmatrix} + \begin{bmatrix} -C^*C & -C^*D \\ -D^*C & \gamma^2 I - D^*D \end{bmatrix} \tag{19}$$

where

$$\Delta_0 = \sum_{i=1}^N \left(\bar{\delta}_i(t) \begin{bmatrix} \widehat{A}_0^{i*} \\ \widehat{B}_{10}^{i*} \end{bmatrix}^* X \begin{bmatrix} I \\ 0 \end{bmatrix} + \delta_i(t) \begin{bmatrix} I \\ 0 \end{bmatrix} X \begin{bmatrix} \widehat{A}_0^{i*} \\ \widehat{B}_{10}^{i*} \end{bmatrix} \right) .$$

By Lemma A.2 condition (17) is equivalent to (19) if we replace Δ_0 by

$$\Delta_1 := \sum_{i=1}^N d_i^2 \begin{bmatrix} I \\ 0 \end{bmatrix}^* X \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} \widehat{A}_0^{i*} \\ \widehat{B}_{10}^{i*} \end{bmatrix}^* X \begin{bmatrix} \widehat{A}_0^{i*} \\ \widehat{B}_{10}^{i*} \end{bmatrix} .$$

Like above, we see by Lemma 3.1 that the i-th summand in Δ_0 exceeds the i-th summand in Δ_1, if $|\delta_i(t)| < d_i$, whence (19) holds.
Letting $T \to \infty$ we find $\gamma \|v\|_{L^2} \leq \gamma \|v\|_{L^2}$ □

Remark 3.3 *(i) If condition (17) holds for some value γ then it also for all $\tilde{\gamma}$ in a whole neighbourhood of γ. Hence we actually have $\|z_\delta(\cdot; 0, v)\|_{L^2} < \gamma \|v\|_{L^2}$ for $v \neq 0$.*

(ii) The same method can also be applied to deal with uncertainties that are combinations of the two types considered here.

4 Solution of the Riccati equation

For the analysis of the Riccati-type inequality (10) with constraint (9) it is useful to study the corresponding constrained Riccati-type equation

$$\mathcal{R}^\gamma(X) = 0 \quad \text{with } X < 0 \text{ and } Q_1^\gamma(X) > 0 . \tag{20}$$

For later use let us define

$$\begin{aligned}\operatorname{dom}\mathcal{R}^\gamma &= \left\{X \in \mathcal{H}^n : \det \begin{bmatrix} Q_2(X) & S_{21}(X) \\ S_{12}(X) & Q_1^\gamma(X) \end{bmatrix} \neq 0\right\}, &(21)\\ \operatorname{dom}_+\mathcal{R}^\gamma &= \{X \in \mathcal{H}^n : X < 0 \text{ and } Q_1^\gamma(X) > 0\}. &(22)\end{aligned}$$

Note that $\mathcal{R}^\gamma$ is well-defined on $\operatorname{dom}\mathcal{R}^\gamma$ and that $\operatorname{dom}_+\mathcal{R}^\gamma \subset \operatorname{dom}\mathcal{R}^\gamma$. We will express the constraint $X < 0$ and $Q_1^\gamma(X) > 0$ from now on as $X \in \operatorname{dom}_+\mathcal{R}^\gamma$.
One way to solve Riccati equations is Newton's method. In a series of papers including [9], [16], [7], [2], non-local convergence results were established for Newton's method applied to different types of Riccati equations (see also [10]). In [3] we have shown that these results rely only on a few properties of a certain class of concave operators in a partially ordered vector space. We briefly summarize the main facts and definitions.

4.1 A non-local convergence result for Newton's method

Let $\mathcal{H}^n \subset \mathbb{K}^{n\times n}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) denote the real space of real or complex $n \times n$ Hermitian matrices. By $\mathcal{H}^n_+ = \{X \in \mathcal{H}^n \mid X \geq 0\}$ we denote the closed convex cone of nonnegative definite matrices and by $\operatorname{int}\left(\mathcal{H}^n_+\right)$ its interior, i.e. the open cone of positive definite matrices. The cone $\mathcal{H}^n_+$ induces a partial ordering on $\mathcal{H}^n$: we write $X \geq Y$, if $X - Y \in \mathcal{H}^n_+$.

We need to introduce three notions for operators on $\mathcal{H}^n$: *Resolvent positivity*, *concavity*, and *stabilizability*.

Definition 4.1 *A linear operator $\mathcal{T} : \mathcal{H}^n \to \mathcal{H}^m$ is called* positive *($\mathcal{T} \geq 0$) if it maps $\mathcal{H}^n_+$ to $\mathcal{H}^m_+$. A linear operator $\mathcal{T} : \mathcal{H}^n \to \mathcal{H}^n$ is called* inverse positive *if it is invertible and $\mathcal{T}^{-1}$ is positive; it is called* resolvent positive, *if for all sufficiently large $\alpha > 0$ the resolvent operator $(\alpha I - \mathcal{T})^{-1}$ is positive. By $\sigma(\mathcal{T})$ we denote the spectrum of $\mathcal{T}$.*

Example 4.2 *(i) Let $A_0 \in \mathbb{K}^{n\times m}$, then the operator $\Pi_{A_0} : \mathcal{H}^n \to \mathcal{H}^m$ defined by $\Pi_{A_0}(X) = A_0^* X A_0$ is positive. In particular the operator Π from (8) is positive.*

(ii) All positive operators $\Pi : \mathcal{H}^n \to \mathcal{H}^n$ are resolvent positive, since for $\alpha > \rho(\Pi)$ the resolvent $(\alpha I - \Pi)^{-1} = \sum_{k=0}^\infty \alpha^{-(k+1)}\Pi^k$ is positive.

(iii) Given $A \in \mathbb{K}^{n\times n}$, the associated Lyapunov operator *$\mathcal{L}_A : \mathcal{H}^n \to \mathcal{H}^n$, $\mathcal{L}_A(X) = A^*X + XA$, is resolvent positive but, in general, not positive.*

Theorem 4.3 [12] *Let $\mathcal{L} : \mathcal{H}^n \to \mathcal{H}^n$ be resolvent positive and $\Pi : \mathcal{H}^n \to \mathcal{H}^n$ be positive. Then $\mathcal{L} + \Pi$ is resolvent positive, and the following are equivalent:*

(i) $\mathcal{L}+\Pi$ *is stable, i.e.* $\sigma(\mathcal{L}+\Pi)\subset\mathbb{C}_-$.

(ii) $-(\mathcal{L}+\Pi)$ *is inverse positive.*

(iii) $\sigma(\mathcal{L})\subset\mathbb{C}_-$ *and* $\rho\left(\mathcal{L}^{-1}\Pi\right)<1$.

(iv) $\exists X<0:\quad(\mathcal{L}+\Pi)(X)>0$.

The definiteness conditions in (iv) can be weakened:

Corollary 4.4 *[3] Let* (A,G) *be observable,* $G\geq 0$, *and assume*

$$\exists X\leq 0:(\mathcal{L}+\Pi)(X)\geq G.$$

Then $X<0$ *and* $\mathcal{L}+\Pi$ *is stable.*

Let $\mathcal{G}$ be a nonlinear Fréchet-differentiable operator from some open domain $\operatorname{dom}\mathcal{G}\subset\mathcal{H}^n$ to $\mathcal{H}^n$. Let further $\operatorname{dom}_+\mathcal{G}$ be some nonempty open convex subset of $\operatorname{dom}\mathcal{G}$. By $\mathcal{G}'_X(H)$ we denote the derivative of $\mathcal{G}$ at X in direction H.

Definition 4.5 *The operator* $\mathcal{G}$ *is said to be* $\operatorname{dom}_+\mathcal{G}$-concave *on* $\operatorname{dom}\mathcal{G}$ *if for all* $Y\in\operatorname{dom}\mathcal{G}$ *and* $Z\in\operatorname{dom}_+\mathcal{G}$

$$\mathcal{G}(Y)-\mathcal{G}(Z)+\mathcal{G}'_Y(Z-Y)\geq 0.$$

In geometric terms $\mathcal{G}$ is $\operatorname{dom}_+\mathcal{G}$-concave on $\operatorname{dom}\mathcal{G}$, if the graph of $\mathcal{G}$ over $\operatorname{dom}_+\mathcal{G}$ lies below all tangents to the graph of $\mathcal{G}$ at arbitrary points in $\operatorname{dom}\mathcal{G}$. Thus $\operatorname{dom}_+\mathcal{G}$-concavity on $\operatorname{dom}\mathcal{G}$ implies concavity on $\operatorname{dom}_+\mathcal{G}$.

Example 4.6 *For a given matrix* F *the operator* $\mathcal{R}_F^\gamma$ *defined in (10) is* $\operatorname{dom}_+\mathcal{R}$-*concave on* $\operatorname{dom}\mathcal{R}_F^\gamma=\{X\in\mathcal{H}^n\mid\det Q_1^\gamma(X)\neq 0\}$, *see [3].*
The operator $\mathcal{R}^\gamma$, *however, is not concave.*

Definition 4.7 *The operator* $\mathcal{G}$ *is said to be* stabilizable *if there exists a matrix* $X\in\operatorname{dom}\mathcal{G}$, *such that* $\sigma(\mathcal{G}'_X)\subset\mathbb{C}_-$. *The matrix* X *is then called* stabilizing *(for* $\mathcal{G}$*).*

Now we can state the non-local convergence result for Newton's method.

Theorem 4.8 *Let* $\mathcal{G}$ *and the sets* $\operatorname{dom}\mathcal{G}$ *and* $\operatorname{dom}_+\mathcal{G}$ *be given as above and assume that the following conditions hold:*

(a) The set $\operatorname{dom}_+\mathcal{G}$ *is* saturated above, *i.e.* $\operatorname{dom}_+\mathcal{G}=\operatorname{dom}_+\mathcal{G}+\mathcal{H}_+^n$.

(b) The operator $\mathcal{G}$ *is* $\operatorname{dom}_+\mathcal{G}$*-concave on* $\operatorname{dom}\mathcal{G}$.

(c) For all $X \in \operatorname{dom} \mathcal{G}$ *the derivative* $\mathcal{G}'_X$ *is resolvent positive.*

Assume further that $\mathcal{G}$ *is stabilizable and let* X_0 *be stabilizing.*
If the inequality $\mathcal{G}(X) \geq 0$ *has a solution* $\hat{X}$ *in* $\operatorname{dom}_+ \mathcal{G}$*, then the iteration scheme*

$$X_{k+1} = X_k - (\mathcal{G}'_{X_k})^{-1}(\mathcal{G}(X_k)) \tag{23}$$

defines a sequence (X_k) *in* $\operatorname{dom} \mathcal{G}$ *with the following properties:*

(i) $\forall k = 1, 2, \ldots :\ X_k \in \operatorname{dom}_+ \mathcal{G},\ X_k \geq X_{k+1} \geq \hat{X}$*, and* $\sigma(\mathcal{G}'_{X_k}) \subset \mathbb{C}_-$.

(ii) (X_k) *converges to a limit matrix* $X_+ \in \operatorname{dom}_+ \mathcal{G}$ *that satisfies* $\mathcal{G}(X_+) = 0$ *and is the largest solution of* $\mathcal{G}(X) \geq 0$.

(iii) $\exists X \in \operatorname{dom}_+ \mathcal{G} :\ \mathcal{G}(X) > 0 \iff \sigma(\mathcal{G}'_{X_+}) \subset \mathbb{C}_-$.
In this case the Newton-iteration converges quadratically.

If the inequality $\mathcal{G}(X) \geq 0$ *is not solvable in* $\operatorname{dom}_+ \mathcal{G}$*, then either (i) fails, i.e. for some iterate* X_k *we have* $X_k \notin \operatorname{dom}_+ \mathcal{G}$ *or* $\sigma(\mathcal{G}'_{X_k}) \not\subset \mathbb{C}_-$*; or (ii) fails i.e. the* X_k *diverge to* ∞ *or the limit matrix* X_∞ *is a boundary point of* $\operatorname{dom}_+ \mathcal{G}$.

4.2 A transformation of the Riccati operator

We define another rational matrix operator $\mathcal{G}$ from some subset $\operatorname{dom} \mathcal{G}^\gamma \subset \mathcal{H}^n$ to $\mathcal{H}^n$ by

$$\mathcal{G}^\gamma(Y) \ := \ Y\mathcal{R}^\gamma(-Y^{-1})Y \ . \tag{24}$$

(with $\mathcal{R}^\gamma$ like in (10)) where we set

$$\begin{aligned} \operatorname{dom} \mathcal{G}^\gamma &= \{Y \mid \det Y \neq 0, \text{ and } X = -Y^{-1} \in \operatorname{dom} \mathcal{R}^\gamma\} \ , \\ \operatorname{dom}_+ \mathcal{G}^\gamma &= \{Y \mid X = -Y^{-1} \in \operatorname{dom}_+ \mathcal{R}^\gamma\} \ . \end{aligned} \tag{25}$$

Similar transformations have also been applied to solve the Riccati equation of the deterministic H^∞-control problem (e.g. [10]). The important point – in our terms – is, that $\mathcal{G}^\gamma$ is $\operatorname{dom}_+ \mathcal{G}^\gamma$-concave, whereas $\mathcal{R}^\gamma$ is not concave.

Lemma 4.9 *The following hold:*

(i) $\operatorname{dom}_+ \mathcal{G}^\gamma = \operatorname{dom}_+ \mathcal{G}^\gamma + \mathcal{H}^n_+$.

(ii) For all $Y \in \operatorname{dom} \mathcal{G}^\gamma$ *the derivative* ${\mathcal{G}^\gamma_Y}'$ *is* resolvent positive.

(iii) $\mathcal{G}^\gamma$ *is* $\operatorname{dom}_+ \mathcal{G}^\gamma$-concave *on* $\operatorname{dom} \mathcal{G}^\gamma$.

(iv) If $\det Y \neq 0$ *and* $X = -Y^{-1}$ *then* $\left(\mathcal{G}^\gamma(Y) \overset{(>)}{=} 0 \iff \mathcal{R}^\gamma(X) \overset{(>)}{=} 0\,.\right)$

(v) If $\det Y \neq 0$, $X = -Y^{-1}$, *and* $\mathcal{G}^\gamma(Y) = 0$ *then* $\sigma({\mathcal{G}_Y^\gamma}') = \sigma({\mathcal{R}_X^\gamma}')$.

(vi) $\lim_{\nu\to\infty} \frac{1}{\nu} {\mathcal{G}_{\nu I}^\gamma}' = \mathcal{L}_{\mathcal{R}^\gamma(0)}$ *and* $\mathcal{R}^\gamma(0) \leq 0$.

Proof: The statements (i) and (iv) are obvious. Assertion (v) follows from

$$\begin{aligned} {\mathcal{G}_Y^\gamma}'(H) &= H\mathcal{R}^\gamma(X)Y + Y\mathcal{R}^\gamma(X)H + Y{\mathcal{R}_X^\gamma}'(Y^{-1}HY^{-1})Y \\ &\overset{\text{by (iv)}}{=} Y{\mathcal{R}_X^\gamma}'(Y^{-1}HY^{-1})Y\,, \end{aligned}$$

because the operator $H \mapsto Y{\mathcal{R}_X^\gamma}'(Y^{-1}HY^{-1})Y$ is similar to $H \mapsto {\mathcal{R}_X^\gamma}'(H)$; in the notation of Kronecker products the similarity transformation is given by $Y \otimes Y^{-1}$. The proof of the remaining statements (especially (iii)) is very technical; we present it in the Appendix. □

Lemma 4.9 will be applied as follows: By (i)–(iii) we can apply Theorem 4.8 to solve the equation $\mathcal{G}^\gamma(Y) = 0$ which by (iv) is equivalent to solving the equation $\mathcal{R}^\gamma(X) = 0$; by (v) a solution Y is stabilizing for $\mathcal{G}^\gamma$ if and only if $X = -Y^{-1}$ is stabilizing for $\mathcal{R}^\gamma$. Assertion (vi) is used to guarantee the existence of stabilizing matrices for $\mathcal{G}^\gamma$.

4.3 Main result

Let $\mathcal{R}^\gamma$ and $\mathcal{G}^\gamma$ be defined according to (10) and (24). We study the relation between the Riccati-type inequality $\mathcal{R}^\gamma(X) \geq 0$ and the equation $\mathcal{R}^\gamma(X) = 0$. Our results parallel and generalize results from the deterministic theory: The existence of a *stabilizing* solution to the Riccati *equation* is equivalent to the solvability of the strict *inequality* if one either imposes an *observability* condition on the given system (3) or a *stabilizability* condition for the transformed operator $\mathcal{G}^\gamma$.
We set

$$\tilde{A} = A - B_2(D_2^*D_2)^{-1}D_2^*C, \quad \tilde{P}_0 = C^*C - C^*D_2(D_2^*D_2)^{-1}D_2^*C\,. \tag{26}$$

Note that $\tilde{A} = A$ and $\tilde{P}_0 = C^*C$ under the regularity assumption $D_2^*C = 0$.

Theorem 4.10 *Assume that (a)* $\mathcal{G}^\gamma$ *is stabilizable, or (b)* $(\tilde{A}, \tilde{P}_0)$ *is observable. Then the following are equivalent:*

(i) $\exists X \in \operatorname{dom}_+ \mathcal{R}^\gamma$, *such that* $\mathcal{R}(X) > 0$

(ii) $\exists X_+ \in \operatorname{dom}_+ \mathcal{R}^\gamma$, *such that* $\mathcal{R}(X_+) = 0$ *and* $\sigma\left(\mathcal{R}^{\gamma}_{X_+}{}'\right) \subset \mathbb{C}_-$.

The matrix F defined according to (11) with $X = X_+$ solves the γ-suboptimal H^∞-problem.

Proof: The equivalence of (i) and (ii) under the assumption that $\mathcal{G}^\gamma$ is stabilizable follows immediately from Theorem 4.8 and Lemma 4.9.
By Lemma 4.14 below, the observability of $(\tilde{A}, \tilde{P}_0)$ together with (i) implies the stabilizability of $\mathcal{G}^\gamma$. Hence (b) implies the equivalence of (i) and (ii).
It follows from Corollary 2.6, that F solves the γ-suboptimal H^∞-problem. □

By Theorem 4.8 we can also suggest an iterative method to solve the inequality $\mathcal{R}^\gamma(X) > 0$ with an almost canonical choice of the initial matrix. The idea is to perturb $\mathcal{R}^\gamma$ slightly, such that $\mathcal{G}^\gamma$ becomes stabilizable and the strict inequality remains solvable. We define $\mathcal{R}^{\gamma,\varepsilon} : X \mapsto \mathcal{R}^\gamma(X) - \varepsilon I$ and $\mathcal{G}^{\gamma,\varepsilon} : Y \mapsto \mathcal{G}^\gamma(Y) - \varepsilon Y^2$.

Proposition 4.11 *Assume that for some $\varepsilon > 0$ there exists an $X \in \operatorname{dom}_+ \mathcal{R}^\gamma$, such that $\mathcal{R}^\gamma(X) > \varepsilon I$. Consider the sequence (Y_k) produced by Newton's method applied to the equation $\mathcal{G}^{\gamma,\varepsilon}(Y) = 0$ starting at $Y_0 = \nu I$.*
If $\nu > 0$ is chosen sufficiently large, then the Y_k converge quadratically to a stabilizing solution $Y_+^\varepsilon \in \operatorname{dom}_+ \mathcal{G}^\gamma$ of this equation. Hence $X_+^\varepsilon = (Y_+^\varepsilon)^{-1} \in \operatorname{dom}_+ \mathcal{R}^\gamma$ is a stabilizing solution of the equation $\mathcal{R}^\gamma(X) = \varepsilon I$; moreover it is the largest solution of the inequality $\mathcal{R}^\gamma(X) \geq \varepsilon I$.

Proof: The operator $\mathcal{G}^{\gamma,\varepsilon}$ is well-defined on $\operatorname{dom} \mathcal{G}^\gamma$ and the properties (ii) – (v) of Lemma 4.9 carry over to $\mathcal{G}^{\gamma,\varepsilon}$, By property (vi) of the same Lemma we have $\lim_{\nu\to\infty} \frac{1}{\nu} \mathcal{G}^{\gamma,\varepsilon}_{\nu I}{}' = \mathcal{L}_{\mathcal{R}^\gamma(0)-\varepsilon I}$. Since $\mathcal{R}^\gamma(0) - \varepsilon I < 0$ the operator $\mathcal{G}^{\gamma,\varepsilon}_{\nu I}{}'$ is stable for sufficiently large $\nu > 0$. By assumption the inequality $\mathcal{G}^{\gamma,\varepsilon}(Y) > 0$ is solvable in $\operatorname{dom}_+ \mathcal{G}^\gamma$. Therefore the result follows from Theorem 4.8. □

Corollary 4.12 *Assume that there exists an $X \in \operatorname{dom}_+ \mathcal{R}^\gamma$, such that $\mathcal{R}^\gamma(X) > 0$. Then there exists an X_+ in the closure of $\operatorname{dom}_+ \mathcal{R}^\gamma$, such that $\mathcal{R}^\gamma(X_+) = 0$ and $\sigma\left(\mathcal{R}^{\gamma}_{X_+}{}'\right) \subset \mathbb{C}_- \cup i\mathbb{R}$. The matrix X_+ is the largest solution of $\mathcal{R}^\gamma(X) \geq 0$.*

Proof: For sufficiently small ε the assumptions of Proposition 4.11 are satisfied. Since the X_+^ε are the largest solutions of $\mathcal{R}^\gamma(X) \geq \varepsilon I$, they increase as ε decreases. As elements of $\operatorname{dom}_+ \mathcal{R}^\gamma$, however, the X_+^ε are bounded above by 0. Hence they converge to some $X_+ \in \operatorname{cl} \operatorname{dom}_+ \mathcal{R}^\gamma$ and our assertions hold by continuity. □

If we consider the matrix X_+ from Corollary 4.12 we can fill the gap in the proof of

Theorem 4.10 by showing that the observability of $(\tilde{A}, \tilde{P}_0)$ implies $X_+ \in \operatorname{dom}_+ \mathcal{R}^\gamma$ and $\sigma\left({\mathcal{R}^\gamma_{X_+}}'\right) \subset \mathbb{C}_-$. We need another technical lemma, partly from [16].

Lemma 4.13 *Let $\tilde{C}$, $\tilde{D}$, $\tilde{E}$, $\tilde{G}$ be matrices of adequate sizes, such that*

$$\tilde{C}^*\tilde{C} + \tilde{D}^*\tilde{D} \;=\; \tilde{E}^*\tilde{E}\;. \tag{27}$$

(i) If the pair $(\tilde{A}, \tilde{C})$ is observable, then also $(\tilde{A} + \tilde{G}\tilde{D}, \tilde{E})$ is observable.
(ii) In particular, if $(\tilde{A}, \tilde{P}_0)$ is observable, then $(A + B_2F, C + D_2F)$ is observable for arbitrary $F \in \mathbb{K}^{m\times n}$.

Proof: (i) If $(\tilde{A} + \tilde{G}\tilde{D}, \tilde{E})$ is not observable, then there exist a vector $x \neq 0$ and a number $\lambda \in \mathbb{C}$, such that $\tilde{E}x = 0$ and $(\tilde{A} + \tilde{G}\tilde{D})x = \lambda x$. By (27) $\tilde{E}x = 0$ implies $\tilde{C}x = 0$ and $\tilde{G}x = 0$, whence also $\tilde{A}x = \lambda x$. Therefore $(\tilde{A}, \tilde{C})$ is not observable.
(ii) If we set $\tilde{C} = \tilde{P}_0^{\frac{1}{2}}$, $\tilde{D} = (D_2^*D_2)^{\frac{1}{2}}F + (D_2^*D_2)^{-\frac{1}{2}}D_2^*C$, $\tilde{E} = C + D_2F$ we find that (27) holds. Moreover, with $\tilde{G} = B_2(D_2^*D_2)^{-\frac{1}{2}}$, we have $\tilde{A}+\tilde{G}\tilde{D} = A+B_2F$. □

Lemma 4.14 *Assume that there exists an $X \in \operatorname{dom}_+ \mathcal{R}^\gamma$, such that $\mathcal{R}^\gamma(X) > 0$. If $(\tilde{A}, \tilde{P}_0)$ is observable, then there exists an $X_+ \in \operatorname{dom}_+ \mathcal{R}^\gamma$, such that $\mathcal{R}^\gamma(X_+) = 0$ and $\sigma\left({\mathcal{R}^\gamma_{X_+}}'\right) \subset \mathbb{C}_-$.*

Proof: It remains to show, that the matrix $X_+ \leq 0$ from Corollary 4.12 is negative definite and stabilizing. By Corollary 2.9 and the definition of P_F in (6) it follows that

$$(A + B_2F)^*X_+ + X_+(A + B_2F) \;\geq\; (C + D_2F)^*(C + D_2F)\;,$$

whence by Lemma 4.13 (ii) and Corollary 4.4 we have $X_+ < 0$, i.e. $X_+ \in \operatorname{dom}_+ \mathcal{R}^\gamma$. Thus also $Y_+ = -X_+^{-1} \in \operatorname{dom}_+ \mathcal{G}^\gamma$ and $\mathcal{G}^\gamma(Y_+) = 0$. Moreover Y_+ is the largest solution of the inequality $\mathcal{G}^\gamma(Y) \geq 0$. By our first assumtion there exists an $Y \in \operatorname{dom}_+ \mathcal{G}^\gamma$, such that $\mathcal{G}(Y) > 0$. We conclude, that $Y \leq Y_+$. Since $\mathcal{G}$ is concave on $\operatorname{dom}_+ \mathcal{G}$, we have

$${\mathcal{G}^\gamma_{Y_+}}'(Y - Y_+) \;\geq\; \mathcal{G}^\gamma(Y) - \mathcal{G}^\gamma(Y_+) = \mathcal{G}^\gamma(Y) > 0\;.$$

As a continuous operator ${\mathcal{G}^\gamma_{Y_+}}'$ maps a whole neighbourhood of $Y - Y_+ \leq 0$ to $\operatorname{int} \mathcal{H}^n_+$. Since ${\mathcal{G}^\gamma_{Y_+}}'$ is resolvent positive, Theorem 4.3 yields $\sigma\left({\mathcal{G}^\gamma_{Y_+}}'\right) \subset \mathbb{C}_-$. □

5 Example

In [13] and [14] a linear model of the two-mass spring system with uncertain stiffness depicted in Fig. 1 was considered.

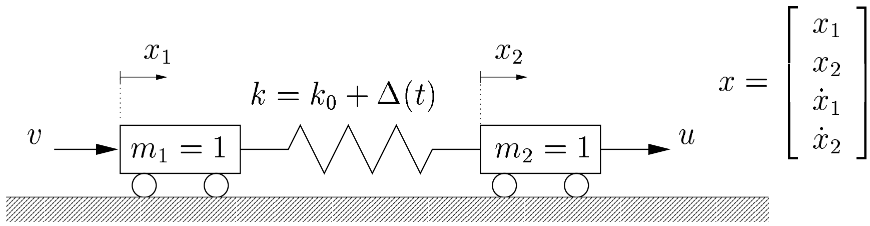

Figure 1: Two carts on a rail, connected by a spring

The system consists of two carts connected by a spring; there is a disturbance v acting on the left cart and a control force u driving the right cart. The control is to be chosen such that the distance between the carts is always nearly the same, i.e. the effects of v have to be annihilated. This is a typical disturbance attenuation problem.
It is assumed that the spring constant $k = k(t)$ has the nominal value $k_0 = 5/4$, but it can vary (e.g. due to nonlinear effects of the spring) and is considered uncertain. In a series of experiments under different conditions it was found out, that at each instant of time the values approximately have a Gaussian distribution centered at k_0. The experiments also showed, that k ranges over $[0.5, 2]$. Therefore the difference $\Delta(t) = k(t) - k_0$ was modelled as a Gaussian white noise process with intensity $\sigma = 1/4$ ensuring that $|\Delta(t)| < 0.75$ and hence $k \in [0.5, 2]$ with sufficiently high probability.
As pointed out before, one might argue, whether this model is adequate. It is possible, that the Gaussian distribution of the values k was not produced by one individual spring at different instances of time, but by different springs, with time-invariant stiffness each. Or the stiffness of each spring is in fact time-varying but not a Gaussian process. In this case the concept of bounded uncertainties might be preferable. To see the difference, we follow the approach of [13] and discuss, what it gives for the case of bounded uncertainties.

The state-space equations of the stochastic spring system take the general form

$$\begin{aligned} dx &= (Ax + B_1 v + B_2 u)dt + A_0 x \, dw \, , \\ z &= Cx + Du \, . \end{aligned}$$

Hence the Riccati equation of the stochastic γ-subobtimal H^∞ problem is given by

$$\mathcal{R}^\gamma(X) = A^*X + XA + A_0^*XA_0 - C^*C + X\left(B_2(D^*D)^{-1}B_2^* - \gamma^{-2}B_1B_1^*\right)X = 0 \, .$$

For the transformed operator

$$\mathcal{G}^\gamma(Y) = -YA^* - AY - YA_0^*Y^{-1}A_0Y - YC^*CY + B_2(D^*D)^{-1}B_2^* - \gamma^{-2}B_1B_1^*$$

with $\operatorname{dom}\mathcal{G}^\gamma = \{Y \in \mathcal{H}^n \mid \det Y \neq 0\}$, and $\operatorname{dom}_+\mathcal{G}^\gamma = \operatorname{int}\mathcal{H}_+^n$ we have

$$\begin{aligned}\mathcal{G}^{\gamma}_{Y_0}{}'(H) \;&=\; (-A^* - C^*CY_0 - A_0^*Y_0^{-1}A_0Y_0)^*H + H(-A^* - C^*CY_0 - A_0^*Y_0^{-1}A_0Y_0)\\ &\quad + (Y_0^{-1}A_0Y_0)^*H(Y_0^{-1}A_0Y_0)\;.\end{aligned}$$

If (A, C) is observable, a stabilizing Y_0 exists and can be chosen independently of γ.

With the data in [13]:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5/4 & 5/4 & 0 & 0 \\ 5/4 & -5/4 & 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/4 & 1/4 & 0 & 0 \\ 1/4 & -1/4 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

we found $Y_0 = 2I - (e_1^Te_4 + e_4^Te_1)$ (with canonical unit vectors e_i) to be stabilizing. For $\gamma = 2$ we could reproduce (by Newton's method in 10 steps) the solution obtained by Ugrinovskii; by a bisection search we found the optimal attenuation value to be $\gamma_* \approx 1.8293$. For $\gamma = 1.8293$ the solution

$$X = Y_\infty^{-1} \;\approx\; \begin{bmatrix} 4.6440 & -6.4837 & -4.7299 & -3.3828 \\ -6.4837 & 19.1359 & 17.4201 & 10.4627 \\ -4.299 & 17.4201 & 18.9092 & 9.8687 \\ -3.3828 & 10.4627 & 9.8687 & 7.0731 \end{bmatrix} > 0$$

of $\mathcal{R}^\gamma(X) = 0$ is obtained in 13 steps, whereas for $\gamma = 1.8292$ the 11th iterate Y_{11} is not stabilizing (i.e. by Thm. 4.8 in this case $\mathcal{G}_\gamma(Y) = 0$ is not solvable in $\operatorname{int}\mathcal{H}_+^n$).

For comparison we study the system

$$\begin{aligned} \dot{x} &= (A + \delta(t)A_0)x + B_1v + B_2u\;, \\ z &= Cx + Du\;. \end{aligned} \tag{28}$$

In view of Theorem 3.2 we wish to determine the largest value $\alpha = \alpha_{\max}$, such that

$$\mathcal{G}^\gamma(Y) - \alpha Y \;=\; 0$$

has a stabilizing solution in $\operatorname{dom}_+\mathcal{G}^\gamma$.
For $\gamma = 2$ we found $\alpha_{\max} = 0.0756$, whence for all $|\delta(t)| \leq d =: \sqrt{\alpha_{\max}} = 0.275$ system (28) is stabilizable with attenuation level $\gamma = 2$. This bound correponds to

$|\Delta(t)| = |k(t) - k_0| < 0.0687 =: \Delta$, which is rather far away from 0.75.
Considering different attenuation levels γ we found (by a bisection search) the following maximal admissible uncertainty bounds Δ:

γ	1.893	2	4	10	100	1000	∞
Δ	0	0.0687	0.1994	0.2826	0.4146	0.5590	0.8803
Δ/k_0	0	0.05	0.15	0.23	0.33	0.45	0.70

Apparently we can guarantee stability for an arbitrary time-varying parameter $k : t \mapsto k(t) \in [0.5, 2]$ only if we abandon disturbance attenuation. But if we aim at an attenuation value of $\gamma = 2$ or $\gamma = 4$, we can still allow deviations of $k(t)$ from k_0 of about 5 or 15 percent, respectively.
We repeat, that our criteria for the case of bounded uncertainties are sufficient only. One might, for instance, get better results if one chooses a different σ, i.e. another scaling of A_0.

A Schur complements

We give a brief account of Schur complements, as far as they are needed in this paper. Our standard reference is [11]. The notation is adapted to our framework, such that the formulae can be applied easily. All variables, however, are local; they do not necessarily have the same meaning as in the rest of the paper.

Definition A.1 *For a Hermitian 2×2-block matrix $M = \begin{bmatrix} P & S \\ S^* & Q \end{bmatrix}$ we define the Schur complement* $\mathrm{Schur}(M/Q) = P - SQ^{-1}S^*$ *with respect to Q, if Q is invertible, and the Schur complement* $\mathrm{Schur}(M/P) = Q - S^*P^{-1}S$ *with respect to P, if P is invertible.*
The inertia $\mathrm{in}\, M$ *of M is defined to be the triple $(\pi, \nu, \delta) \in \mathbb{N}^3$, where π, ν, δ denotes the number of positive, negative and zero eigenvalues of M, respectively.*

Whenever the inverse of a matrix occurs in this section, it is tacitly assumed, that the matrix is nonsingular.

Lemma A.2 *(i) Inertia formula:*

$$\mathrm{in}\, M = \mathrm{in}\, Q + \mathrm{in}\,\mathrm{Schur}(M/Q) = \mathrm{in}\, P + \mathrm{in}\,\mathrm{Schur}(M/P)\,.$$

In particular $M > 0 \iff Q > 0$ *and* $P - SQ^{-1}S^* > 0.$

(ii) Inversion formula:

$$M^{-1} = \begin{bmatrix} P^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -P^{-1}S \\ I \end{bmatrix} \mathrm{Schur}(M/P) \begin{bmatrix} -P^{-1}S \\ I \end{bmatrix}^*$$

(iii) Quotient formula: If $Q = \begin{bmatrix} Q_2 & S_{21} \\ S_{12} & Q_1 \end{bmatrix}$ *then*

$$\mathrm{Schur}(M/Q) \;=\; \mathrm{Schur}\left(\mathrm{Schur}(M/Q)\Big/\,\mathrm{Schur}(Q/Q_1)\right) .$$

B Analysis of the operator $\mathcal{G}^\gamma$

The proof of the statements (ii), (iii) and (vi) in Lemma 4.9 proceeds by explicit calculations, involving large expressions with block matrices. Therefore we have to use an efficient notation, that is short enough for us to retain the overview and explicit enough to make the calculation transparent.
The subblocks of the matrix M^γ from (7) and the positive operator Π from (8) are denoted as follows:

$$\begin{aligned} M^\gamma &= \begin{bmatrix} -C^*C & -C^*D_2 & -C^*D_1 \\ -D_2^*C & -D_2^*D_2 & -D_2^*D_1 \\ -D_1^*C & -D_1^*D_2 & \gamma^2 I - D_1^*D_1 \end{bmatrix} = \begin{bmatrix} P_0 & S_{02} & S_{01} \\ S_{20} & P_2 & S_{21} \\ S_{10} & S_{12} & P_1 \end{bmatrix} \\ \Pi(X) &=: \begin{bmatrix} \Pi_0(X) & \Sigma_{02}(X) & \Sigma_{01}(X) \\ \Sigma_{20}(X) & \Pi_2(X) & \Sigma_{21}(X) \\ \Sigma_{10}(X) & \Sigma_{12}(X) & \Pi_1(X) \end{bmatrix} . \end{aligned}$$

That is, we use the shorthand notation $P_0 = P_0(0)$, $S_{02} = S_{02}(0)$, etc., and set $\Pi_0(X) = \sum_{i=1}^N A_0^{i*} X A_0^i$, $\Sigma_{02}(X) = \sum_{i=1}^N A_0^{i*} X B_{02}^i$, ...
Moreover we set

$$\begin{aligned} P_Y &= -YA^* - AY - Y\Pi_0(Y^{-1})Y + YP_0Y \\ B &= \begin{bmatrix} B_2 & B_1 \end{bmatrix} \\ \Sigma_0(\cdot) &= \begin{bmatrix} \Sigma_{02}(\cdot) & \Sigma_{01}(\cdot) \end{bmatrix} \\ S_0 &= \begin{bmatrix} S_{02}(0) & S_{01}(0) \end{bmatrix} \\ \tilde\Sigma_0(\cdot) &= \Sigma_0(\cdot) - S_0 \\ S_Y &= B + Y\Sigma_0(Y^{-1}) - YS_0 = B + Y\tilde\Sigma_0(Y^{-1}) \\ \Pi_{21}(\cdot) &= \begin{bmatrix} \Pi_2(\cdot) & \Sigma_{21}(\cdot) \\ \Sigma_{12}(\cdot) & \Pi_1(\cdot) \end{bmatrix} \\ Q_0 &= \begin{bmatrix} Q_2(0) & S_{21}(0) \\ S_{12}(0) & Q_1(0) \end{bmatrix} \\ Q_Y &= Q_0 - \Pi_{21}(Y^{-1}), \end{aligned}$$

such that

$$\mathcal{G}(Y) \;=\; P_Y - S_Y Q_Y^{-1} S_Y$$

In products of the form V^*WV we sometimes write $[\ldots]$ for the right factor, if it is the conjugate transpose of the left factor.

Proof of Lemma 4.9

(ii) We have to show that ${\mathcal{G}_Y^{\gamma}}'$ is resolvent positive for all $Y \in \operatorname{dom} \mathcal{G}^\gamma$.
For the factors we have

$$\begin{aligned} P_Y'(H) &= -HA^* - AH + H(P_0 - \Pi_0(Y^{-1}))Y + Y(P_0\Pi_0(Y^{-1}))H \\ &\quad + Y\Pi_0(Y^{-1}HY^{-1})Y \\ S_Y'(H) &= H(\Sigma_0(Y^{-1}) - S_0) - Y\Sigma_0(Y^{-1}HY^{-1}) \\ Q_Y'(H) &= \Pi_{21}(Y^{-1}HY^{-1}) \;. \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{G}_Y'(H) &= P_Y'(H) - S_Y'(H)Q_Y^{-1}S_Y^* - S_YQ_Y^{-1}S_Y'(H)^* + S_YQ_Y^{-1}Q_Y'(H)Q_Y^{-1}S_Y^* \\ &= -AH - HA^* + H(P_0 - \Pi_0(Y^{-1}))Y + Y(P_0 - \Pi_0(Y^{-1}))H \\ &\quad + H(S_0 - \Sigma_0(Y^{-1}))Q_Y^{-1}S_Y^* + Y\Sigma_0(Y^{-1}HY^{-1})Q_Y^{-1}S_Y^* \\ &\quad + S_YQ_Y^{-1}(S_0 - \Sigma_0(Y^{-1}))^*H + S_YQ_Y^{-1}\Sigma_0(Y^{-1}HY^{-1})^*Y \\ &\quad + Y\Pi_0(Y^{-1}HY^{-1})Y + S_YQ_Y^{-1}\Pi_{21}(Y^{-1}HY^{-1})Q_Y^{-1}S_Y^* \\ &= (-A + S_YQ_Y^{-1}(S_0 - \Sigma_0(Y^{-1}))^* + Y(P_0 - \Pi_0(Y^{-1})))H \qquad (29) \\ &\quad + H(-A^* + (S_0 - \Sigma_0(Y^{-1}))Q_Y^{-1}S_Y^* + (P_0 - \Pi_0(Y^{-1}))Y) \\ &\quad + \begin{bmatrix} Y \\ Q_Y^{-1}S_Y^* \end{bmatrix}^* \begin{bmatrix} \Pi_0(Y^{-1}HY^{-1}) & \Sigma_0(Y^{-1}HY^{-1}) \\ \Sigma_0(Y^{-1}HY^{-1})^* & \Pi_{21}(Y^{-1}HY^{-1}) \end{bmatrix} \begin{bmatrix} Y \\ Q_Y^{-1}S_Y^* \end{bmatrix} . \end{aligned}$$

which is the sum of a Lyapunov operator and a positive operator and thus resolvent positive by Theorem 4.3 together with Example 4.2.

(iii) We have to show that $\mathcal{G}^\gamma$ is $\operatorname{dom}_+ \mathcal{G}^\gamma$-concave on $\operatorname{dom} \mathcal{G}^\gamma$, i.e.

$$\forall Y \in \operatorname{dom} \mathcal{G}, Z \in \operatorname{dom}_+ \mathcal{G} : \quad \mathcal{G}(Y) - \mathcal{G}(Z) + \mathcal{G}_Y'(Z - Y) \geq 0.$$

Letting $H = Z - Y$ in the previous expression for $\mathcal{G}_Y'(H)$ we obtain after some cancellation and reordering of terms

$$\begin{aligned} \mathcal{G}_Y'(Z - Y) &= A(Y - Z) + (Y - Z)A^* \\ &+ \begin{bmatrix} Y \\ Z \\ Q_Y^{-1}S_Y^* \end{bmatrix}^* \begin{bmatrix} \Pi_0(Y^{-1} + Y^{-1}ZY^{-1}) - 2P_0 & P_0 - \Pi_0(Y^{-1}) & \Sigma_0(Y^{-1}ZY^{-1}) - S_0 \\ P_0 - \Pi_0(Y^{-1}) & 0 & S_0 - \Sigma_0(Y^{-1}) \\ (\Sigma_0(Y^{-1}ZY^{-1}) - S_0)^* & (S_0 - \Sigma_0(Y^{-1}))^* & \Pi_{21}(Y^{-1}ZY^{-1} - Y^{-1}) \end{bmatrix} \begin{bmatrix} \ldots \end{bmatrix}, \end{aligned}$$

whence

$$\begin{aligned} &\mathcal{G}(Y) - \mathcal{G}(Z) + \mathcal{G}_Y'(Z - Y) = \\ &= -AY - YA^* + Y(P_0 - \Pi_0(Y^{-1}))Y - S_YQ_Y^{-1}(Q_0 - \Pi_{21}(Y^{-1}))Q_Y^{-1}S_Y^* \qquad (30) \end{aligned}$$

$$+ AZ + ZA^* + Z(\Pi_0(Z^{-1}) - P_0)Z + S_Z Q_Z^{-1} S_Z^* + \mathcal{G}'_Y(Z-Y)$$

$$= \begin{bmatrix} Y \\ Z \\ Q_Y^{-1}S_Y^* \end{bmatrix}^* \begin{bmatrix} \Pi_0(Y^{-1}ZY^{-1}) & -\Pi_0(Y^{-1}) & \Sigma_0(Y^{-1}ZY^{-1}) \\ -\Pi_0(Y^{-1}) & \Pi_0(Z^{-1}) & -\Sigma_0(Y^{-1}) \\ \Sigma_0(Y^{-1}ZY^{-1})^* & -\Sigma_0(Y^{-1})^* & \Pi_{21}(Y^{-1}ZY^{-1}) \end{bmatrix} \begin{bmatrix} Y \\ Z \\ Q_Y^{-1}S_Y^* \end{bmatrix}$$
$$+ \begin{bmatrix} Y \\ Z \\ Q_Y^{-1}S_Y^* \end{bmatrix}^* \begin{bmatrix} -P_0 & P_0 & -S_0 \\ P_0 & -P_0 & S_0 \\ -S_0^* & S_0^* & -Q_0 \end{bmatrix} \begin{bmatrix} Y \\ Z \\ Q_Y^{-1}S_Y^* \end{bmatrix} + S_Z Q_Z^{-1} S_Z^*. \qquad (31)$$

Now we factorize $S_Z Q_Z^{-1} S_Z^*$ in a similar fashion like the other summands. We replace S_Z by $S_Y + Z\tilde{\Sigma}_0(Z^{-1}) - Y\tilde{\Sigma}_0(Y^{-1})$ in the first step and Q_Y by $Q_Z + \Pi_{21}(Z^{-1}) - \Pi_{21}(Y^{-1}) = Q_Z + \Pi_{21}(Z^{-1} - Y^{-1})$ in the last step:

$$\begin{aligned} S_Z Q_Z^{-1} S_Z^* &= S_Y Q_Z^{-1} S_Y^* + S_Y Q_Z^{-1}\tilde{\Sigma}_0(Z^{-1})^* Z - S_Y Q_Z^{-1}\tilde{\Sigma}_0(Y^{-1})^* Y \\ &\quad + Z\tilde{\Sigma}_0(Z^{-1})Q_Z^{-1}S_Y^* + Z\tilde{\Sigma}_0(Z^{-1})Q_Z^{-1}\tilde{\Sigma}_0(Z^{-1})^* Z - Z\tilde{\Sigma}_0(Z^{-1})Q_Z^{-1}\tilde{\Sigma}_0(Y^{-1})^* Y \\ &\quad - Y\tilde{\Sigma}_0(Y^{-1})Q_Z^{-1}S_Y^* - Y\tilde{\Sigma}_0(Y^{-1})Q_Z^{-1}\tilde{\Sigma}_0(Z^{-1})^* Z + Y\tilde{\Sigma}_0(Y^{-1})Q_Z^{-1}\tilde{\Sigma}_0(Y^{-1})^* Y \end{aligned}$$

$$= \begin{bmatrix} Y \\ Z \\ Q_Y^{-1}S_Y^* \end{bmatrix}^* \begin{bmatrix} \tilde{\Sigma}_0(Y^{-1})Q_Z^{-1}\tilde{\Sigma}_0(Y^{-1})^* & -\tilde{\Sigma}_0(Y^{-1})Q_Z^{-1}\tilde{\Sigma}_0(Z^{-1})^* & -\tilde{\Sigma}_0(Y^{-1})Q_Z^{-1}Q_Y \\ -\tilde{\Sigma}_0(Z^{-1})Q_Z^{-1}\tilde{\Sigma}_0(Y^{-1})^* & \tilde{\Sigma}_0(Z^{-1})Q_Z^{-1}\tilde{\Sigma}_0(Z^{-1})^* & \tilde{\Sigma}_0(Z^{-1})Q_Z^{-1}Q_Y \\ -Q_Y Q_Z^{-1}\tilde{\Sigma}_0(Y^{-1})^* & Q_Y Q_Z^{-1}\tilde{\Sigma}_0(Z^{-1})^* & Q_Y Q_Z^{-1} Q_Y \end{bmatrix} \begin{bmatrix} \dots \end{bmatrix}$$

$$= \begin{bmatrix} Y \\ Z \\ Q_Y^{-1}S_Y^* \end{bmatrix}^* \begin{bmatrix} -\tilde{\Sigma}_0(Y^{-1}) \\ \tilde{\Sigma}_0(Z^{-1}) \\ \Pi_{21}(Z^{-1} - Y^{-1}) \end{bmatrix} Q_Z^{-1} \begin{bmatrix} -\tilde{\Sigma}_0(Y^{-1}) \\ \tilde{\Sigma}_0(Z^{-1}) \\ \Pi_{21}(Z^{-1} - Y^{-1}) \end{bmatrix}^* \begin{bmatrix} Y \\ Z \\ Q_Y^{-1}S_Y^* \end{bmatrix}$$
$$+ \begin{bmatrix} Y \\ Z \\ Q_Y^{-1}S_Y^* \end{bmatrix}^* \begin{bmatrix} 0 & 0 & S_0 - \Sigma_0(Y^{-1}) \\ 0 & 0 & \Sigma_0(Z^{-1}) - S_0 \\ S_0^* - \Sigma_0(Y^{-1})^* & \tilde{\Sigma}_0(Z^{-1})^* - S_0^* & Q_0 + \Pi_{21}(Z^{-1} - 2Y^{-1}) \end{bmatrix} \begin{bmatrix} Y \\ Z \\ Q_Y^{-1}S_Y^* \end{bmatrix} . (32)$$

Substituting $S_Z Q_Z^{-1} S_Z^*$ in formula (31) by (32) we obtain

$$\mathcal{G}(Y) - \mathcal{G}(Z) + \mathcal{G}'_Y(Z-Y) = \begin{bmatrix} Y \\ Z \\ Q_Y^{-1}S_Y^* \end{bmatrix}^* \Theta \begin{bmatrix} Y \\ Z \\ Q_Y^{-1}S_Y^* \end{bmatrix}$$

where

$$\begin{aligned} \Theta &= \begin{bmatrix} \Pi_0(Y^{-1}ZY^{-1}) & -\Pi_0(Y^{-1}) & \Sigma_0(Y^{-1}ZY^{-1} - Y^{-1}) \\ -\Pi_0(Y^{-1}) & \Pi_0(Z^{-1}) & \Sigma_0(Z^{-1} - Y^{-1}) \\ \Sigma_0(Y^{-1}ZY^{-1} - Y^{-1})^* & \Sigma_0(Z^{-1} - Y^{-1})^* & \Pi_{21}(Y^{-1}ZY^{-1} - 2Y^{-1} + Z^{-1}) \end{bmatrix} \\ &\quad + \begin{bmatrix} \tilde{\Sigma}_0(Y^{-1}) \\ -\tilde{\Sigma}_0(Z^{-1}) \\ \Pi_{21}(Z^{-1} - Y^{-1}) \end{bmatrix} Q_Z^{-1} \begin{bmatrix} \tilde{\Sigma}_0(Y^{-1}) \\ -\tilde{\Sigma}_0(Z^{-1}) \\ \Pi_{21}(Z^{-1} - Y^{-1}) \end{bmatrix}^* + \begin{bmatrix} -P_0 & P_0 & 0 \\ P_0 & -P_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (33) \\ &= \Theta_1 + \Theta_2 + \Theta_0 \end{aligned}$$

The remaining part of the proof amounts to verifying that $\Theta \geq 0$ for $Z \in \mathrm{dom}_+ \mathcal{G}^\gamma$.

To estimate the second summand Θ_2 we remember that $Q_Z = \begin{bmatrix} Q_2(Z^{-1}) & S_{21}(Z^{-1}) \\ S_{12}(Z^{-1}) & Q_1(Z^{-1}) \end{bmatrix}$ with $Q_2(Z^{-1}) < 0$ and $Q_1(Z^{-1}) > 0$ for $Z \in \mathrm{dom}_+ \mathcal{G}^\gamma$. Therefore also the Schur complement

$$\hat{Q} := \text{Schur}\left(Q_Z/Q_2(Z^{-1})\right) \quad = \quad Q_1(Z^{-1}) - S_{12}(Z^{-1})Q_2(Z^{-1})^{-1}S_{21}(Z^{-1})$$

of Q_Z with respect to the left-upper block is positive definite. By the matrix inversion formula of Lemma A.2 we have

$$Q_Z^{-1} \quad = \quad \begin{bmatrix} Q_2(Z^{-1})^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Q_2(Z^{-1})^{-1}S_{21}(Z^{-1}) \\ I \end{bmatrix} \hat{Q} \begin{bmatrix} Q_2(Z^{-1})^{-1}S_{21}(Z^{-1}) \\ I \end{bmatrix},$$

which is the sum of a negative and a positive semidefinite matrix. Resubstituting $\tilde{\Sigma}_0$ and Π_{21} by their defining expressions we get

$$\begin{aligned}
\Theta_2 \quad &= \quad \begin{bmatrix} \tilde{\Sigma}_0(Y^{-1}) \\ -\tilde{\Sigma}_0(Z^{-1}) \\ \Pi_{21}(Z^{-1} - Y^{-1}) \end{bmatrix} Q_Z^{-1} \begin{bmatrix} \tilde{\Sigma}_0(Y^{-1}) \\ -\tilde{\Sigma}_0(Z^{-1}) \\ \Pi_{21}(Z^{-1} - Y^{-1}) \end{bmatrix}^* \\
&\geq \quad \begin{bmatrix} \Sigma_{02}(Y^{-1}) - S_{02} & \Sigma_{01}(Y^{-1}) - S_{01} \\ -\Sigma_{02}(Z^{-1}) + S_{02} & -\Sigma_{01}(Z^{-1}) + S_{01} \\ \Pi_2(Z^{-1} - Y^{-1}) & \Sigma_{21}(Z^{-1} - Y^{-1}) \\ \Sigma_{12}(Z^{-1} - Y^{-1}) & \Pi_1(Z^{-1} - Y^{-1}) \end{bmatrix} \begin{bmatrix} Q_2(Z^{-1})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cdots \end{bmatrix} \\
&= \quad \begin{bmatrix} \Sigma_{02}(Y^{-1}) - S_{02} \\ -\Sigma_{02}(Z^{-1}) + S_{02} \\ \Pi_2(Z^{-1} - Y^{-1}) \\ \Sigma_{12}(Z^{-1} - Y^{-1}) \end{bmatrix} Q_2(Z^{-1})^{-1} \begin{bmatrix} \Sigma_{02}(Y^{-1}) - S_{02} \\ -\Sigma_{02}(Z^{-1}) + S_{02} \\ \Pi_2(Z^{-1} - Y^{-1}) \\ \Sigma_{12}(Z^{-1} - Y^{-1}) \end{bmatrix}^*
\end{aligned}$$

Substituting the last expression for Θ_2 in (33) we can only make Θ smaller. The arising expression can be written as the Schur complement

$$\text{Schur}\left(\begin{bmatrix} \Theta_1 + \Theta_0 & \tilde{\Theta}_2 + \tilde{\Theta}_0 \\ \tilde{\Theta}_2^* + \tilde{\Theta}_0 & -Q_2(Z^{-1}) \end{bmatrix} \Big/ - Q_2(Z^{-1}\right),$$

where

$$\tilde{\Theta}_2 = \begin{bmatrix} \Sigma_{02}(Y^{-1}) \\ -\Sigma_{02}(Z^{-1}) \\ \Pi_2(Z^{-1} - Y^{-1}) \\ \Sigma_{12}(Z^{-1} - Y^{-1}) \end{bmatrix}, \quad \tilde{\Theta}_0 = \begin{bmatrix} -S_{02} \\ S_{02} \\ 0 \\ 0 \end{bmatrix}, \text{ and } - Q_2(Z^{-1}) = \Pi_2(Z^{-1}) - Q_2 .$$

We show now, that $\begin{bmatrix} \Theta_1 + \Theta_0 & \tilde{\Theta}_2 + \tilde{\Theta}_0 \\ \tilde{\Theta}_2^* + \tilde{\Theta}_0 & \Pi_2(Z^{-1}) - Q_2 \end{bmatrix} \geq 0$, which implies that $\Theta \geq 0$ for $Z \in \mathrm{dom}_+ \mathcal{G}^\gamma$ and therefore proves (iii).

For the constant term we have

$$\begin{bmatrix} \Theta_0 & \tilde\Theta_0 \\ \tilde\Theta_0 & -Q_2 \end{bmatrix} = \begin{bmatrix} -P_0 & P_0 & 0 & 0 & -S_{02} \\ P_0 & -P_0 & 0 & 0 & S_{02} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -S_{20} & S_{20} & 0 & 0 & -Q_2 \end{bmatrix} = \begin{bmatrix} C \\ -C \\ 0 \\ 0 \\ D_2 \end{bmatrix}^* \begin{bmatrix} C \\ -C \\ 0 \\ 0 \\ D_2 \end{bmatrix} \geq 0.$$

Setting $\mathcal{D}_i := \operatorname{diag}(A_0^i, A_0^i, B_{2,0}^i, B_{1,0}^i, B_{2,0}^i)$ the remaining term can be written as

$$\begin{bmatrix} \Theta_1 & \tilde\Theta_2 \\ \tilde\Theta_2^* & \Pi_2(Z^{-1}) \end{bmatrix} = \sum_{i=1}^{N} \mathcal{D}_i^* \Upsilon \mathcal{D}_i$$

with Υ =

$$\begin{bmatrix} Y^{-1}ZY^{-1} & -Y^{-1} & Y^{-1}ZY^{-1}-Y^{-1} & Y^{-1}ZY^{-1}-Y^{-1} & Y^{-1} \\ -Y^{-1} & Z^{-1} & Z^{-1}-Y^{-1} & Z^{-1}-Y^{-1} & -Z^{-1} \\ Y^{-1}ZY^{-1}-Y^{-1} & Z^{-1}-Y^{-1} & Y^{-1}ZY^{-1}-2Y^{-1}+Z^{-1} & Y^{-1}ZY^{-1}-2Y^{-1}+Z^{-1} & Z^{-1}-Y^{-1} \\ Y^{-1}ZY^{-1}-Y^{-1} & Z^{-1}-Y^{-1} & Y^{-1}ZY^{-1}-2Y^{-1}+Z^{-1} & Y^{-1}ZY^{-1}-2Y^{-1}+Z^{-1} & Z^{-1}-Y^{-1} \\ Y^{-1} & -Z^{-1} & Z^{-1}-Y^{-1} & Z^{-1}-Y^{-1} & Z^{-1} \end{bmatrix}.$$

For $Z \in \operatorname{dom}_+ \mathcal{G}^\gamma$ we have $Z^{-1} > 0$, and a straight-forward calculation yields $\operatorname{Schur}(\Upsilon/Z^{-1}) = 0$. Hence $\Upsilon \geq 0$, which completes our proof of (iii).

(vi) Since $\lim_{\nu\to\infty} \frac{1}{\nu} Q_{\nu I} S_{\nu I}^* = Q_0^{-1} S_0^*$, we get from (29) that

$$\lim_{\nu\to\infty} \frac{1}{\nu} \mathcal{G}'_{\nu I}(H) = \mathcal{L}_{P_0 - S_0 Q_0^{-1} S_0^*}(H) = \mathcal{L}_{\mathcal{R}^\gamma(0)}(H) .$$

It remains to show, that $P_0 - S_0 Q_0^{-1} S_0^* \leq 0$.
We write

$$P_0 - S_0 Q_0^{-1} S_0^* = -C^* \operatorname{Schur}\left(\tilde M^\gamma / Q_0\right) C$$

with $\tilde M^\gamma = \begin{bmatrix} I & D_2 & D_1 \\ D_2^* & D_2^* D_2 & D_2^* D_1 \\ D_1^* & D_1^* D_2 & D_1^* D_1 - \gamma^2 I \end{bmatrix}$ and $Q_0 = \begin{bmatrix} D_2^* D_2 & D_2^* D_1 \\ D_1^* D_2 & D_1^* D_1 - \gamma^2 I \end{bmatrix}$.

By Lemma A.2 the inertia of $\tilde M^\gamma$ can be expressed through the Schur complement with respect either to Q_0 or to I:

$$\begin{aligned} \operatorname{in} \tilde M^\gamma &= \operatorname{in} Q_0 + \operatorname{in} \operatorname{Schur}\left(\tilde M^\gamma / Q_0\right) \\ &= \operatorname{in} I + \operatorname{in} \begin{bmatrix} 0 & 0 \\ 0 & -\gamma^2 I \end{bmatrix} = (n, -\ell, 0) . \end{aligned}$$

Since $\operatorname{in} Q_0 = (m, \ell, 0)$, we have $\operatorname{in} \operatorname{Schur}\left(\tilde M^\gamma / Q_0\right) = (n-m, 0, 0)$, that is $\operatorname{Schur}\left(\tilde M^\gamma / Q_0\right) \geq 0$ (with rank $n-m$), which we needed to show.

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