
Stability of linear systems and positive semigroups of symmetric matrices

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Abstract. The role of Lyapunov operators in stability theory is well-known. In this paper we present an interesting characterization of Lyapunov operators. We show that an operator generates a positive group on the real space of real or complex Hermitian matrices, if and only if it is a Lyapunov operator.

1 Introduction

There is an interesting relation between stability of linear dynamical systems in \mathbb{R}^n and positive semigroups on the space \mathcal{S}^n of symmetric matrices in $\mathbb{R}^{n \times n}$. A famous result due to Lyapunov states that the linear system $\dot{x} = Ax$ is asymptotically stable if and only if there exists a positive definite matrix $V \in \mathcal{S}^n$, such that $A^T V + V A \in \mathcal{S}^n$ is negative definite. In honour of Lyapunov, the matrix operator $\mathcal{L}_A : X \mapsto A^T X + X A$ is called a *Lyapunov operator*. It is easy to see that by the negativity of $A^T V + V A$ the positive definite matrix V defines a Euclidean norm $\|x\|_V = \sqrt{x^T V x}$ (an energy norm) on \mathbb{R}^n with respect to which all non-zero solutions of $\dot{x} = Ax$ are strictly decreasing. This is the usual interpretation of Lyapunov's result. But there is another useful point of view. Obviously, a solution $x(t)$ converges to zero, if and only if the family of rank-1 positive semidefinite matrices $X(t) = x(t)x(t)^T$ does. This family is subject to the differential equation $\dot{X} = \mathcal{L}_A^*(X)$, where $\mathcal{L}_A^* = \mathcal{L}_{A^T}$ is the operator adjoint to \mathcal{L}_A . Since $X(t) \geq 0$ for all $t \in \mathbb{R}_+$, it follows that for all $A \in \mathbb{R}^{n \times n}$ the Lyapunov operator \mathcal{L}_A^* generates a positive semigroup on \mathcal{S}^n . We note the simple fact that the given system is stable, if and only if the positive semigroup generated by \mathcal{L}_{A^T} is stable. It turns out (cf. Remark 1) that, in general, positive semigroups on \mathcal{S}^n can be used to characterize stability and robust stability properties of various types of linear dynamical systems in \mathbb{R}^n . This unifying way of characterization constitutes a useful connection between stability problems and spectral properties of generators of positive semigroups, which we discuss in Theorem 2. To classify different stability problems it would be useful to have a general representation result

for generators of positive *semigroups* on \mathcal{S}^n . To the author's best knowledge, however, this is an open problem. As a step in this direction, in Section 3 we characterize the set of generators of positive *groups* on \mathcal{S}^n . More precisely, we prove that this set coincides with the set of Lyapunov operators.

2 Exponentially positive operators on \mathcal{S}^n and stability of linear systems in \mathbb{R}^n

Let \mathcal{S}^n denote the set of $n \times n$ symmetric matrices and $\mathcal{S}_+^n \subset \mathcal{S}^n$ the cone of nonnegative definite matrices $X \geq 0$. It is well-known that \mathcal{S}^n together with the inner product $\langle X, Y \rangle = \text{trace } XY$ is a Hilbert space and \mathcal{S}_+^n is a self-dual closed normal solid convex cone (e.g. [14]). We write A^T for the conjugate transpose of a matrix $A \in \mathbb{R}^{n \times n}$.

Definition 1. A linear operator $T : \mathcal{S}^n \rightarrow \mathcal{S}^n$ is said to be

- (i) positive, if $T(\mathcal{S}_+^n) \subset \mathcal{S}_+^n$,
- (ii) a generator of a positive group, if $e^{tT} : \mathcal{S}^n \rightarrow \mathcal{S}^n$ is positive for all $t \in \mathbb{R}$,
- (iii) a Lyapunov operator, if there exists an $A \in \mathbb{R}^{n \times n}$, such that for all $X \in \mathcal{S}^n$

$$T(X) = AX + XA^T.$$

In this case we write $T = \mathcal{L}_A$.

If $e^{tT} : \mathcal{S}^n \rightarrow \mathcal{S}^n$ is positive only for all $t \in \mathbb{R}_+$, then T generates a positive semigroup. We recall a number of equivalent properties from [18, 7, 1].

Definition 2. A linear operator $T : \mathcal{S}^n \rightarrow \mathcal{S}^n$ is said to be

- (i) exponentially positive, if $e^{tT} : \mathcal{S}^n \rightarrow \mathcal{S}^n$ is positive for all $t \geq 0$ (i.e. T generates a positive semigroup).
- (ii) resolvent positive, if $(\alpha I - T)^{-1}$ is positive for sufficiently large $\alpha > 0$,
- (iii) quasi-monotonic, if for all $X \in \mathcal{S}_+^n$ there exists an $Y \in \mathcal{S}_+^n$, such that $\langle X, Y \rangle = 0$ and $\langle T(X), Y \rangle \geq 0$,
- (iv) cross-positive, if $\langle X, Y \rangle = 0$ for $X, Y \in \mathcal{S}_+^n$ implies $\langle T(X), Y \rangle \geq 0$,
- (v) essentially positive, if $T \in \text{cl}\{S - \alpha I \mid S : \mathcal{S}^n \rightarrow \mathcal{S}^n \text{ is positive, } \alpha \in \mathbb{R}\}$.

The equivalence of these properties was established in [7] (cf. also [1]) for general finite-dimensional vector spaces ordered by a closed normal solid cone.

Theorem 1. For a linear operator $T : \mathcal{S}^n \rightarrow \mathcal{S}^n$ the properties (i)–(v) from Definition 2 are equivalent.

It follows immediately from property (v), that every positive operator is exponentially positive, and so are arbitrary linear combinations of exponentially positive operators with nonnegative coefficients. In other words, the set of exponentially positive operators forms a closed solid wedge (cf. [14]) in the

space of endomorphisms on \mathcal{S}^n .

One easily verifies that all Lyapunov operators generate positive groups. In Section 3 we will show that, in fact, Lyapunov operators are the only generators of positive groups on \mathcal{S}^n .

Lemma 1. *Every Lyapunov operator generates a positive group on \mathcal{S}^n .*

Proof. Let $T = \mathcal{L}_A$ for some $A \in \mathbb{R}^{n \times n}$. By Lyapunov's Theorem (e.g. [11]), \mathcal{L}_A^{-1} is positive if all eigenvalues of \mathcal{L}_A have positive real part. Thus $(\alpha I - \mathcal{L}_A)^{-1} = (\mathcal{L}_{-A + \frac{\alpha}{2} I})^{-1}$ is positive for sufficiently large $\alpha \in \mathbb{R}$. Hence T is resolvent positive and thus exponentially positive. Since $-T = \mathcal{L}_{-A}$ is a Lyapunov operator, too, it is exponentially positive as well. Thus, $e^{\pm Tt}$ is positive for all $t \geq 0$.

Corollary 1. *Let $A, B_1, \dots, B_N \in \mathbb{R}^{n \times n}$. Then the operator*

$$X \mapsto T(X) = A^T X + X A + \sum_{j=1}^N B_j^T X B_j \quad (1)$$

is exponentially positive.

Proof. The operator is the sum of a Lyapunov operator and a positive operator and hence exponentially positive.

Remark 1. Operators of the form (1) occur in the stability analysis of different types of linear systems on \mathbb{R}^n :

1. The continuous-time deterministic system $\dot{x} = Ax$ is asymptotically stable, if and only if (cf. [9, 19])

$$\exists X > 0 : T_1(X) = A^T X + X A < 0 .$$

2. The discrete-time deterministic system $x_{k+1} = Ax_k$ is asymptotically stable, if and only if (cf. [19])

$$\exists X > 0 : T_2(X) = A^T X A - X < 0 .$$

3. The cont.-time stochastic differential equation of Itô-type $dx = Ax dt + \sum_{j=1}^N B_j x dw_j$ with independent normed Wiener processes w_j is asymptotically mean-square stable, if and only if (cf. [13])

$$\exists X > 0 : T_3(X) = A^T X + X A + \sum_{j=1}^N B_j^T X B_j < 0 .$$

4. The discr.-time stochastic system $x(k+1) = Ax(k) + \sum_{j=1}^N B_j x(k) w_j(k)$ is mean-square stable, if and only if (cf. [15])

$$\exists X > 0 : T_4(X) = A^T X A - X + \sum_{j=1}^N B_j^T X B_j < 0 .$$

5. The deterministic delay system $\dot{x}(t) = Ax(t) + Bx(t-h)$ is asymptotically stable for all delays $h > 0$, if (cf. [12])

$$\exists X > 0 : T_5(X) = A^T X + X A + X + B^T X B < 0 .$$

6. The uncertain linear system $\dot{x} = (A + \sum_{j=1}^N \delta_j(t, x) B_j) x$ is asymptotically stable for arbitrary measurable functions $\delta : \mathbb{R} \times \mathbb{R}^n \rightarrow [-1, 1]$, if (cf. [3])

$$\exists X > 0 : T_6(X) = A^T X + X A + nX + \sum_{j=1}^N B_j^T X B_j < 0 .$$

Based on these criteria, the following theorem (compiled from [17, 6, 16, 4, 8, 5]) constitutes a useful tool in the analysis of stability and stabilizability.

Theorem 2. *Let $A, B_1, \dots, B_N \in \mathbb{R}^{n \times n}$ and set $\Pi_B : X \mapsto \sum_{j=1}^N B_j^T X B_j$. By σ and ρ we denote the spectrum and the spectral radius of a linear operator. The following are equivalent:*

- (a) $\sigma(\mathcal{L}_A + \Pi_B) \subset \mathbb{C}_- = \{z \in \mathbb{C} \mid \Re z < 0\}$.
- (b) $\max \sigma(\mathcal{L}_A + \Pi_B) \cap \mathbb{R} < 0$.
- (c) $\sigma(\mathcal{L}_A) \subset \mathbb{C}_-$ and $\forall \tau \in [0, 1] : \det(\mathcal{L}_A + \tau \Pi_B) \neq 0$.
- (d) $\sigma(\mathcal{L}_A) \subset \mathbb{C}_-$ and $\rho(\mathcal{L}_A^{-1} \Pi_B) < 1$.
- (e) $\forall Y < 0 : \exists X > 0 : \mathcal{L}_A(X) + \Pi_B(X) = Y$.
- (f) $\forall Y \leq 0$ with (A, Y) observable: $\exists X > 0 : \mathcal{L}_A(X) + \Pi_B(X) = Y$.
- (g) $\exists X > 0 : \mathcal{L}_A(X) + \Pi_B(X) < 0$.
- (h) $\exists X \geq 0 : \mathcal{L}_A(X) + \Pi_B(X) < 0$.
- (i) $\exists Y \leq 0$ with (A, Y) observable: $\exists X \geq 0 : \mathcal{L}_A(X) + \Pi_B(X) \leq Y$.
- (j) $\exists Y \leq 0$ s.t. $\mathcal{L}_A(X) + \Pi_B(X) \leq Y$ for some $X \in \mathcal{S}_+^n$, and $XY \neq 0$ for every pair $(X, \lambda) \in (\mathcal{S}_+^n \setminus \{0\}) \times \mathbb{C}_+$ satisfying $(\mathcal{L}_A + \Pi_B)^*(X) = \lambda X$.

In a sense, this theorem parallels [2, Theorem 2.3], which provides even fifty equivalent conditions for a Z -matrix to be an M -matrix. (Here, exponentially positive operators correspond to Z -matrices, while stable exponentially positive operators correspond to M -matrices.)

3 Generators of positive groups on \mathcal{S}^n

All exponentially positive operators considered in the previous section are of the form $\mathcal{L}_A + \Pi$, where $\Pi : \mathcal{S}^n \rightarrow \mathcal{S}^n$ is some positive operator. It is a natural question whether every exponentially positive operator on \mathcal{S}^n can be written in this form. As a partial result in this direction and the main result of this paper, we prove that every generator of a positive group on \mathcal{S}^n is a Lyapunov operator.

For general solid regular cones in a finite-dimensional real vector space, the question, whether every exponentially positive operator is the sum of a

positive operator and the generator of a positive group was posed in [20]; an affirmative answer was given for important classes of cones. On the other hand, it was shown in [10] that such a representation is impossible for almost all cones in a certain categorical sense. Nevertheless, the question seems to be still open for the cone \mathcal{S}_+^n in \mathcal{S}^n .

Theorem 3. *A linear operator $T : \mathcal{S}^n \rightarrow \mathcal{S}^n$ is a generator of a positive group, if and only if T is a Lyapunov operator.*

Before proceeding with the proof of Theorem 3, we verify that the situation is different for *discrete-time Lyapunov operators* $T_A : X \mapsto A^T X A - X$ (cf. Rem. 1). While T_A is exponentially positive, $-T_A$, in general, is not:

Example 1. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $X_t = \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix}$ for $t > 0$.

For all $\alpha > 0$ we have $\alpha X_t + T_A(X_t) = \begin{bmatrix} t & \alpha - 1 \\ \alpha - 1 & \alpha t \end{bmatrix}$, which is positive for large t , though X_t is always indefinite. Hence $(\alpha I + T_A)^{-1}$ is not positive for any α , and hence $-T_A$ is not resolvent positive.

It is obvious by definition that T generates a positive group on \mathcal{S}^n , if and only if both T and $-T$ are exponentially positive. Hence we have the following criterion.

Lemma 2. *A linear operator $T : \mathcal{S}^n \rightarrow \mathcal{S}^n$ generates a positive group, if and only if $\langle X, Y \rangle = 0$ for $X, Y \in \mathcal{S}_+^n$ implies $\langle T(X), Y \rangle = 0$.*

Proof. By Theorem 1 the operator $T : \mathcal{S}^n \rightarrow \mathcal{S}^n$ generates a positive group, if and only if both T and $-T$ are cross-positive. The latter holds if and only if $\langle X, Y \rangle = 0$ for $X, Y \in \mathcal{S}_+^n$ implies both $\langle T(X), Y \rangle \geq 0$ and $-\langle T(X), Y \rangle \geq 0$.

We use this criterion in the proof Theorem 3.

Proof of Theorem 3. Let T generate a positive group. By Lemma 2 this is equivalent to

$$\left(X, Y \geq 0 \quad \text{and} \quad \langle X, Y \rangle = 0 \right) \Rightarrow \langle TX, Y \rangle = 0. \quad (2)$$

If e_j denotes the j -th canonical unit vector in \mathbb{R}^n , then the set

$$B := \{e_j e_k^T + e_k e_j^T \mid j, k = 1, \dots, n\} \subset \mathcal{S}_+^n \quad (3)$$

forms a basis of $\mathcal{S}^n \subset \mathbb{R}^{n \times n}$.

Thus it suffices to find an $A \in \mathbb{R}^{n \times n}$, such that $T(X) = \mathcal{L}_{A^T}(X)$ for all $X \in B$.

Let $X = e_j e_j^T$ (i.e. $2X \in B$). To apply criterion (2) we characterize all matrices

$$Y \in \mathcal{S}_+^n \quad \text{such that} \quad \langle X, Y \rangle = 0. \quad (4)$$

Let $Y \geq 0$ and $\langle X, Y \rangle = y_{jj} = 0$. Then necessarily the j -th row and column in Y vanish. Hence, (4) is true if and only if in $Y \geq 0$ the j -th row and column vanish. Criterion (2) in turn implies that in $T(X)$ everything vanishes except for the j -th row and column. Otherwise we could choose some Y satisfying (4) and $\langle T(X), Y \rangle \neq 0$.

For $j = 1, \dots, n$ we thus have $T(e_j e_j^T) = a_j e_j^T + e_j a_j^T$ with vectors $a_1, \dots, a_n \in \mathbb{R}^n$. If we build the matrix $A = (a_1, \dots, a_n)$, then $T(X) = AX + XA^T$ for all $X = e_j e_j^T$. In other words, we have found a unique candidate for the Lyapunov operator.

It remains to show, that also for $X_{jk} = e_j e_k^T + e_k e_j^T$ with $j < k$ we have

$$T(X_{jk}) = AX_{jk} + X_{jk}A^T = a_j e_k^T + a_k e_j^T + e_j a_j^T + e_k a_k^T \quad (5)$$

$$= \begin{bmatrix} & a_{1k} & \cdots & a_{1j} & & \\ & \vdots & & \vdots & & \\ a_{1k} & \cdots & 2a_{jk} & \cdots & a_{jj} + a_{kk} & \cdots & a_{nk} \\ & & \vdots & & \vdots & & \\ a_{1j} & \cdots & a_{jj} + a_{kk} & \cdots & 2a_{kj} & \cdots & a_{nj} \\ & & \vdots & & \vdots & & \\ & & a_{nk} & \cdots & a_{nj} & & \end{bmatrix}.$$

Let j and k be fixed. A matrix Y satisfies condition (4) with $X = X_{jk} + X_{jj} + X_{kk} \geq 0$ if in Y the j -th and k -th row and column vanish. As above we conclude from criterion (2), that in $T(X)$ and hence also in $T(X_{jk})$ everything vanishes except for the j -th and k -th row and column. Thus $T(X_{jk})$ is of the general form

$$T(X_{jk}) = b_j e_j^T + e_j b_j^T + b_k e_k^T + e_k b_k^T, \quad \text{with } b_j, b_k \in \mathbb{R}^n \quad (6)$$

$$= \begin{bmatrix} & b_{1j} & \cdots & b_{1k} & & \\ & \vdots & & \vdots & & \\ b_{1j} & \cdots & 2b_{jj} & \cdots & b_{jk} + b_{kj} & \cdots & b_{nj} \\ & & \vdots & & \vdots & & \\ b_{1k} & \cdots & b_{jk} + b_{kj} & \cdots & 2b_{kk} & \cdots & b_{nk} \\ & & \vdots & & \vdots & & \\ & & b_{nj} & \cdots & b_{nk} & & \end{bmatrix}.$$

Now we consider matrices of the form $X = xx^T$ with $x = x_j e_j + x_k e_k$ where $x_j, x_k \in \mathbb{R}$ are arbitrary real numbers. Writing

$$X = x_j x_k X_{jk} + x_j^2 e_j e_j^T + x_k^2 e_k e_k^T,$$

and exploiting the linearity of T we have the decomposition

$$\begin{aligned}
 T(X) &= x_j x_k T(X_{jk}) + x_j^2 T(e_j e_j^T) + x_k^2 T(e_k e_k^T) \\
 &= x_j x_k (b_j e_j^T + e_j b_j^T + b_k e_k^T + e_k b_k^T) \\
 &\quad + x_j^2 (a_j e_j^T + e_j a_j^T) + x_k^2 (a_k e_k^T + e_k a_k^T).
 \end{aligned} \tag{7}$$

Let $y \perp x$, for instance

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \quad \text{with} \quad \begin{cases} y_j = x_k, \\ y_k = -x_j, \\ y_\ell \text{ arbitrary for } \ell \notin \{j, k\}. \end{cases} \tag{8}$$

Then $Y = yy^T$ satisfies condition (4), and by (2) we have $\langle T(X), Y \rangle = 0$.

If we write $T(X)$ like in equation (7) we obtain:

$$\begin{aligned}
 0 &= \frac{1}{2} \langle T(X), Y \rangle = \frac{1}{2} \text{trace}(T(X)Y) = \frac{1}{2} y^T T(X) y \\
 &= x_j x_k \left(y_j \sum_{\ell=1}^n b_{\ell j} y_\ell + y_k \sum_{\ell=1}^n b_{\ell k} y_\ell \right) + x_j^2 y_j \sum_{\ell=1}^n a_{\ell j} y_\ell + x_k^2 y_k \sum_{\ell=1}^n a_{\ell k} y_\ell \\
 &= x_j x_k^3 (b_{jj} - a_{jk}) + x_j^3 x_k (b_{kk} - a_{kj}) + x_j^2 x_k^2 (-b_{kj} - b_{jk} + a_{jj} + a_{kk}) \\
 &\quad + x_j x_k^2 \sum_{\ell \notin \{j, k\}} y_\ell (b_{\ell j} - a_{\ell k}) + x_j^2 x_k \sum_{\ell \notin \{j, k\}} y_\ell (-b_{\ell k} + a_{\ell j}).
 \end{aligned}$$

The right hand side is a homogeneous polynomial in the real unknowns x_j , x_k , and y_ℓ for $\ell \notin \{j, k\}$. Since these unknowns can be chosen arbitrarily, all the coefficients of the polynomial necessarily vanish, i.e.

$$b_{jj} = a_{jk}, \quad b_{kk} = a_{kj}, \quad b_{kj} + b_{jk} = a_{jj} + a_{kk}, \quad b_{\ell j} = a_{\ell k}, \quad b_{\ell k} = a_{\ell j}.$$

Inserting these data into (6), we see that (5) holds.

4 Open questions

We conclude this paper with two open questions:

1. Can every exponentially positive operator T on \mathcal{S}^n be written in the form $T = \mathcal{L}_A + \Pi$ with some Lyapunov operator \mathcal{L}_A and some positive operator Π ?
2. A positive operator $\Pi : \mathcal{S}^n \rightarrow \mathcal{S}^n$ is called *completely positive*, if for some $N \in \mathbb{N}$ there exist matrices B_1, \dots, B_N , such that $T(X) = \sum_{j=1}^N B_j^T X B_j$ for all $X \in \mathcal{S}^n$. Let us call T *completely resolvent positive* if $\alpha I - T$ is completely positive for $\alpha \gg 1$. Then we ask:
Can every completely resolvent positive operator T on \mathcal{S}^n be written in the form $T = \mathcal{L}_A + \Pi$ with some Lyapunov operator \mathcal{L}_A and some completely positive operator Π ?

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