Detectability, Observability, and Asymptotic Reconstructability of Positive Systems

Tobias Damm and Cristina Ethington

Abstract. We give a survey of detectability, observability and reconstructability concepts for positive systems and sketch some applications to the analysis of stochastic equations.

1 Introduction

There have been a number of contributions to the definition of observability and detectability for different classes of linear control systems recently, see [21, 3, 24, 10, 4, 8, 14].

A common feature of these papers is the distinction between properties which are equivalent for deterministic linear time-invariant systems (compare also [20, 23]). In particular, the following properties are of interest.

Usually, one defines a system to be *detectable* (resp. *observable*), if all its unstable (resp. nontrivial) modes produce a non-zero output, i.e. if vanishing of the output $y(t) = 0$ for all t implies that the state $x(t)$ converges to zero (resp. is equal to zero).

In the deterministic case, it follows that a system is detectable if and only if the dual system is stabilizable, which again is equivalent to the existence of an asymptotically stable linear dynamic state observer. Here this property will be called *asymptotic reconstructability*. Moreover, there are equivalent algebraic criteria, the so called Hautus-test [15] (or Popov-Belevich-Hautus test), which play an important rôle in the discussion of algebraic Lyapunov and Riccati equations.

For many classes of stochastic systems, however, these properties fall apart and their usefulness differs. Therefore different concepts have been developed.

Department of Mathematics, TU Kaiserslautern.
D-67663 Kaiserslautern, Germany.
`damm@mathematik.uni-kl.de, cris@ethington.de`

Several authors (e.g. [5, 21, 9, 12, 13]) have chosen mean-square stabilizability of the dual system as a defining property for detectability. Some authors (e.g. [3]), call this property *MS-detectability.* However, as discussed in [8], this choice is not well-motivated. First no clear interpretation with respect to dynamical properties of the underlying stochastic control system has been given (as mentioned e.g. in [21]). Second, there is no simple equivalent algebraic criterion like the Hautus-test for deterministic systems. Third, in applications to generalized algebraic Lyapunov and Riccati equations only the generalized Hautus-test is used, which is weaker than stabilizability of the dual system (e.g. [13]).

In [8] a generalized version of the Hautus test was given and shown (for stochastic differential equations) to be equivalent to the system being detectable (in the sense that all unstable modes produce non-zero output). This property is called e.g. *W-detectability* in [4] or *β-detectability* in [7].

In the present note, we suggest abstract definitions and characterizations of *detectability*, *observability* and *asymptotic reconstructability* in terms of positive operators and clarify some relations between these notions. We then show how these definitions are related to the dynamical properties of different classes of systems.

2 Resolvent positive operators

We summarize some results on resolvent positive operators which have been collected e.g. in [1] or [6, 7]. Let $\mathcal{H}$ be some finite-dimensional real vector-space, ordered by a closed, solid, pointed convex cone $\mathcal{H}_+$. On $\mathcal{H}$ we consider a scalar product $\langle \cdot, \cdot \rangle$. By $\mathcal{H}_+^*$ we denote the dual cone, and for a linear operator $T : \mathcal{H} \to \mathcal{H}$ we denote the adjoint operator by T^*.

Definition 1. A linear operator $T : \mathcal{H} \to \mathcal{H}$ is called *positive*, $T \geq 0$, if $T(\mathcal{H}_+) \subset \mathcal{H}_+$. It is called *resolvent positive* if there exists an $\alpha_0 \in \mathbb{R}$ such that for all $\alpha \geq \alpha_0$ the resolvent $(\alpha I - T)^{-1}$ is positive, i.e.

$$(\alpha I - T)^{-1}(\mathcal{H}_+) \subset \mathcal{H}_+ \ .$$

There are many other equivalent characterizations of resolvent positivity (e.g. [11, 6]). Here, the following will be relevant.

Proposition 1. *A linear operator $T : \mathcal{H} \to \mathcal{H}$ is resolvent positive, if and only if it is exponentially positive, which means that $e^{Ts} \geq 0$ for all $s \geq 0$.*

Example 1. (i) Let $\mathcal{H} = \mathbb{R}^n$ be ordered by the cone $\mathcal{H}_+ = \mathbb{R}^n_+$. A matrix $A \in \mathbb{R}^{n \times n}$, regarded as a mapping $A : \mathcal{H} \to \mathcal{H}$, is positive, if and only if all its entries are nonnegative. It is resolvent positive, if and only if all off-diagonal entries of A are nonnegative, i.e. A is a Metzler matrix. This is, by definition, equivalent to saying that $-A$ is a *Z-matrix*. We call A *stable*, if $\sigma(A) \subset \mathbb{C}_-$. Hence, again by definition, A is resolvent positive and stable, if and only if $-A$ is an *M-matrix* (see [18, 16, 2]).

(ii) Let $\mathcal{H} = \mathcal{H}^n$ denote the space of $n \times n$ symmetric matrices ordered by the cone $\mathcal{H}_+ = \mathcal{H}_+^n$ of nonnegative definite matrices. Then for any $A \in \mathbb{R}^{n\times n}$, the operator $\Pi_A : \mathcal{H}^n \to \mathcal{H}^n$, $\Pi_A(X) = A^*XA$ is positive, whereas both the *continuous-time Lyapunov operator* $\mathcal{L}_A : \mathcal{H}^n \to \mathcal{H}^n$, $\mathcal{L}_A(X) = A^TX + XA$, and the *discrete-time Lyapunov operator* (also *Stein operator*) $\mathcal{S}_A : \mathcal{H}^n \to \mathcal{H}^n$, $\mathcal{S}_A(X) = A^*XA - X$, are resolvent positive but, in general, not positive.

Note that in both examples the cone $\mathcal{H}_+$ is self-dual, i.e. $\mathcal{H}_+ = \mathcal{H}_+^*$.
The following result goes back to [19].

Proposition 2. *Let $T : \mathcal{H} \to \mathcal{H}$ be resolvent positive and set $\beta = \max \operatorname{Re} \sigma(T)$. Then there exists an eigenvector $V \in \mathcal{H}_+$, $V \neq 0$, such that $TV = \beta V$. Moreover, the following are equivalent (where $X < 0$ means $X \in \operatorname{int} \mathcal{H}_+$):*

(a) $\beta(T) < 0$,
(b) $\exists X < 0 : \ T(X) < 0$,
(c) $\forall Y < 0 : \ \exists X < 0 : \ T(X) = Y$.

3 Detectability and observability

Definition 2. Let $T : \mathcal{H} \to \mathcal{H}$ be resolvent positive and $Y \in \mathcal{H}_+^*$. Consider the positive linear system $\dot{X} = T(X)$, $y(t) = \langle X(t), Y\rangle$. The solution of the initial value problem with $X(0) = X_0 \in \mathcal{H}_+$ will be denoted by $X(t, X_0)$.
We call the pair (T, Y)

(i) *detectable*, if $y(t, X_0) = 0$ for all $t \geq 0$ implies $X(t, X_0) \to 0$ for $t \to \infty$.
(ii) *observable*, if $y(t, X_0) = 0$ for all $t \geq 0$ implies $X_0 = 0$.

The following result is a positive analogue of the Hautus-criterion.

Proposition 3. *The pair (T, Y) as in the previous definition is*

(a) detectable, if and only if $\langle V, Y\rangle > 0$ for any eigenvector $V \in \mathcal{H}_+$ of T corresponding to an eigenvalue $\lambda \geq 0$,
(b) observable, if and only if $\langle V, Y\rangle > 0$ for any eigenvector $V \in \mathcal{H}_+$ of T corresponding to an arbitrary eigenvalue.

Proof. (a) Assume that the criterion in (a) does not hold, i.e. there exist $\lambda \geq 0$, $X_0 \in \mathcal{H}_+ \setminus \{0\}$, so that $T(X_0) = \lambda X_0$ and $\langle X_0, Y\rangle = 0$. Since $X(t, X_0) = e^{\lambda t} X_0$, we have $y(t, X_0) = 0$ for all t, but $X(t, X_0) \not\to 0$ as $t \to \infty$. Hence the system is not detectable.
Vice versa, assume that the criterion holds and for some $X_0 \in \mathcal{H}_+ \setminus \{0\}$ and all $t \geq 0$ we have $y(t, X_0) = 0$. Then we have to show that $X(t, X_0) \stackrel{t\to\infty}{\longrightarrow} 0$.

Let $\mathcal{X}_+ = \operatorname{cl}\operatorname{conv}\{X(t, X_0) \mid t \geq 0\}$ denote the closed convex hull of the positive orbit of $X(t, X_0)$. Let further $\mathcal{X} = \mathcal{X}_+ - \mathcal{X}_+$ be the minimal subspace of $\mathcal{H}$ containing $\mathcal{X}_+$. Then $\mathcal{X}_+$ is a closed solid pointed convex cone in $\mathcal{X}$. By construction, both $\mathcal{X}_+$ and $\mathcal{X}$ are invariant with respect to $\dot{X} = T(X)$. That means $T(\mathcal{X}) \subset \mathcal{X}$, and the restriction $T\big|_{\mathcal{X}}$ is resolvent positive with respect to $\mathcal{X}_+$. Let $\beta_{\mathcal{X}}$ be the spectral bound of $T\big|_{\mathcal{X}}$. By Proposition 2 there exists an eigenvector $V_{\mathcal{X}} \in \mathcal{X}_+ \subset \mathcal{H}_+$, such that $T(V_{\mathcal{X}}) = \beta_{\mathcal{X}} V_{\mathcal{X}}$. Since $\langle X(t, X_0), Y\rangle = 0$ for all $t \geq 0$ we conclude $\langle V, Y\rangle = 0$ for all $V \in \mathcal{X}$. In particular $\langle V_{\mathcal{X}}, Y\rangle = 0$. It follows now from the detectability criterion that $\beta_{\mathcal{X}} < 0$, which implies asymptotic stability of $X(t, X_0)$ for all $X_0 \in \mathcal{X}$.

(b) If the criterion does not hold, then – as in (a) – we have a nontrivial solution $X(t, X_0) = e^{\lambda t} \in \mathcal{H}_+$ with $y(t, X_0) = 0$.
Conversely, if $y(t, X_0) = 0$ for all $t \geq 0$, then – as in (a) – we have the T-invariant subspace $\mathcal{X}$ with $\langle \mathcal{X}, Y\rangle = \{0\}$. If $X_0 \neq 0$ then $\mathcal{X}$ contains an eigenvector, i.e. the criterion is violated. □

4 Asymptotic reconstructability

Note that detectability and observability in the sense of Definition 2 do not imply any means to reconstruct the state of the system. In fact, it is obvious that the measurement $y = \langle X, Y\rangle$ in general will not be sufficient to distinguish two different solutions of the system.

Example 2. Consider $\mathcal{H} = \mathbb{R}^2$ ordered by $\mathbb{R}^2_+$. For $T(X) = X$, the differential equation $\dot{X} = T(X)$ defines a positive system, and for any $Y > 0$ the pair (T, Y) is observable. But $y(t, X_0) = e^t\langle X_0, Y\rangle$ just depends on $\langle X_0, Y\rangle$. If for instance $Y = [1, 1]^T$, then $y(t, [0, 1]^T) = y(t, [1, 0]^T)$ for all t.

Note that $Y > 0$ always implies observability. In this case $Y^\perp \cap \mathcal{H}_+ = \{0\}$. The smaller the dimension of span $\left(Y^\perp \cap \mathcal{H}_+\right)$ the smaller also the number of nonobservable modes is likely to be. When using positive systems e.g. on $\mathcal{H}^n$ to analyze stochastic systems on $\mathbb{R}^n$, this is a natural requirement. We therefore aim at a concept of asymptotic reconstructability which also has this property. To this end we consider a different condition on the pair (T, Y), which can easily be formulated for arbitrary ordered vector spaces. For the important special cones introduced in Example 1, we show how it leads to positive asymptotic observer equations.

Definition 3. Let $T : \mathcal{H} \to \mathcal{H}$ be resolvent positive and $Y \in \mathcal{H}^*_+$. We call the pair (T, Y) asymptotically reconstructable if $T^*(Z) - Y < 0$ for some $Z > 0$.

Let us note a simple general implication.

Lemma 1. *If (T, Y) is asymptotically reconstructable then it is detectable.*

Proof. If $T^*(Z)-Y<0$ for some $Z\in\operatorname{int}\mathcal{H}_+^*$. If $TV=\lambda V$ for some eigenpair $(\lambda,V)\in\mathbb{R}\times\mathcal{H}_+\setminus\{0\}$ satisfying $\langle V,Y\rangle=0$ then

$$0\;>\;\langle V,T^*(Z)-Y\rangle\;=\;\langle T(V),Z\rangle\;=\;\lambda\langle V,Z\rangle\;.$$

Since $\langle V,Z\rangle\geq 0$, we have $\lambda<0$. □

The converse implication does not hold in general as was demonstrated in Example 2. Now let us consider two special cases.

4.1 Asymptotic reconstructability on $\mathbb{R}^n$

This case may look a bit artificial, but it illustrates the concept. Let $\mathcal{H}=\mathbb{R}^n$ and $\mathcal{H}_+=\mathbb{R}^n_+$. Then T is a Metzler-matrix and $Y\in\mathbb{R}^n_+$. Let $\operatorname{diag}(Y)$ denote the diagonal matrix whose diagonal contains the entries of Y and assume that $y=\operatorname{diag}(Y)X$. Consider the extended system

$$\begin{aligned}\dot X&=TX\;,\quad y=\operatorname{diag}(Y)X\\ \dot{\hat X}&=T\hat X+K\operatorname{diag}(Y)\hat X-y\;,\end{aligned}$$

where $\hat X$ is the state of the observer parametrized by the diagonal matrix K. For the error $E=\hat X-X$ we have

$$\dot E=(T+K\operatorname{diag}(Y))E=:T_KE\;. \tag{1}$$

Lemma 2. *There exists a diagonal matrix K, so that $\sigma(T_K)\subset\mathbb{C}_-$, if and only if (T,Y) is asymptotically reconstructable.*

Proof. Note that T_K is Metzler and $\beta(T_K)=\beta(T_K^*)$
If $T^*Z-Y<0$ for some $Z\in\operatorname{int}\mathcal{H}_+^*$, then we set $K=-\operatorname{diag}(Z)^{-1}$, so that $T_K^*Z=T^*Z-Y<0$, i.e. $\beta(T_K)<0$ by Proposition 2.
Conversely, if $\beta(T_K)<0$ $T_K^*\tilde Z-Y=T^*\tilde Z+\operatorname{diag}(Y)K\tilde Z<0$ for some $\tilde Z\in\operatorname{int}\mathcal{H}_+^*$. It is easy to see that $\operatorname{diag}(Y)K\tilde Z>-\alpha Y$ for some $\alpha>0$. Hence $T^*Z-Y<0$ for $Z=\tilde Z/\alpha$. □

Remark 1. For general systems with $A\in\mathbb{R}^{n\times n}$, $C\in\mathbb{R}^{n\times n}$ the existence of a matrix K so that $\sigma(A+KC)\subset\mathbb{C}_-$ is equivalent to detectability in the usual sense, i.e. $\operatorname{rank}(sI-A^T,C^T)=n$ for all $s\in\mathbb{C}_+$. But the characterization in Lemma 2 requires the assumptions that A is Metzler and that $C=\operatorname{diag}(Y)$ is diagonal and nonnegative. For instance, the pair $(A,C)=\left(\begin{bmatrix}1&-2\\0&-1\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix}\right)$, with $Y=0$ is not detectable, but $AZ-Y<0$ for $Z=[1,1]^T$. Similarly, the pair $(A,C)=\left(\begin{bmatrix}0&0\\0&-1\end{bmatrix},\begin{bmatrix}-1&0\\0&0\end{bmatrix}\right)$ with $Y=\begin{bmatrix}-1\\0\end{bmatrix}$ is detectable, but $AZ-Y\not<0$ for any $Z>0$. Definition 3 thus is specific for positive systems.

4.2 Asymptotic reconstructability on $\mathcal{H}^n$

Let $\mathcal{H} = \mathcal{H}^n$ and $\mathcal{H}_+ = \mathcal{H}^n_+$. Let $y = YX$ and consider the extended system

$$\begin{aligned} \dot{X} &= T(X)\ , \quad y = YX \\ \dot{\hat{X}} &= T(\hat{X}) + KY\hat{X} + \hat{X}YK^T - y - y^T\ , \end{aligned} \tag{2}$$

where $K \in \mathbb{R}^{n\times n}$. For the error $E = \hat{X} - X$ we have

$$\dot{E} = T(E) + KYE + EYK^T =: T_K(E)\ . \tag{3}$$

Lemma 3. *There exists a matrix $K \in \mathbb{R}^{n\times n}$, so that $\sigma(T_K) \subset \mathbb{C}_-$, if and only if (T, Y) is asymptotically reconstructable.*

Proof. Note that T_K is resolvent positive for all K and thus $\sigma(T_K) \subset \mathbb{C}_-$ if and only if $T_K^*(Z) = T^*(Z) + ZKY + YK^TZ < 0$ for some $Z > 0$.
If $T^*(Z) - Y < 0$ then we may just set $K = -\frac{1}{2}Z^{-1}$ to get $T_K^*(Z) < 0$.
Vice versa, assume that $T_K^*(Z) < 0$ with $Z > 0$. Let $U = [U_1, U_2] \in \mathbb{R}^{n\times n}$ be orthogonal with the columns of U_1 spanning $\operatorname{Ker} Y$. Then $U_1^TYU_1 = 0$ together with $U_1^TT_K^*(Z)U_1 > 0$ implies $U_1^TT^*(Z)U_1 < 0$. For $\alpha > 0$ we have

$$U^T(T^*(\alpha Z) - Y)U = \begin{bmatrix} \alpha U_1^TT(Z)U_1 & \alpha U_1^TT^*(Z)U_2 \\ \alpha U_2^TT^*(Z)U_1 & \alpha U_2^TT^*(Z)U_2 - U_2^TYU_2 \end{bmatrix}\ ,$$

where $U_2^TYU_2 > 0$. If α is sufficiently small the Schur-complement

$$-U_2^TYU_2 + \alpha\Big(U_2^TT^*(Z)U_2 - U_2^TT^*(Z)U_1(U_1^TT^*(Z)U_1)^{-1}U_1^TT^*(Z)U_2\Big)\ ,$$

is negative, proving $T(\alpha Z) - Y < 0$. Since $\alpha Z > 0$, this proves that (T, Y) is asymptotically reconstructable. □

As a consequence we obtain a simple characterization of detectability.

Corollary 1. *Let $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$. The pair (A, C) is detectable in the usual sense, i.e.* $\operatorname{rank}(sI - A^T, C^T) = n$*, if and only if there exists a positive definite matrix Z, so that $AZ + ZA^T - C^TC < 0$.*

5 Some applications

We sketch two set-ups where our concepts can be applied directly. An in-depth analysis of these examples is beyond the scope of this contribution.

5.1 Stochastic differential equations

Consider the Itô-type stochastic differential equation

$$d\xi = A\xi\,dt + A_0\xi\,dw\,, \quad \eta \;=\; C\xi\;.$$

Then $X = E(\xi\xi^T)$ satisfies $\dot{X} = AX + XA^T + A_0XA_0^T = T(X)$, which is a positive system on $\mathcal{H}^n$. If from the measurements η it is possible to estimate $E(\eta\eta^T) = CXC^T$, then also the output $y = \operatorname{trace} XC^TC = \langle X, Y\rangle$ is available. Then the system is detectable, if and only if the pair (T, Y) with $Y = C^TC$ is detectable.
If further we assume that even the output $YX = C^TCX$ is available, then we can set up the observer equation (2) for the second moments X. Note, however, that this requires different measurements than just $\eta = C\xi$ and may be unrealistic, but it is exactly the underlying assumption in the concept of MS-detectability (compare [8]).

5.2 Markov jump linear systems

Here we consider n-dimensional systems of the form (e.g. [22])

$$\dot{\xi}(t) = A(\theta(t))\xi(t)\;, \quad \eta(t) \;=\; C(\theta(t))\xi(t)\;, \tag{4}$$

where $\theta(t)$ is a Markov process in continuous time on a finite sample space $S = \{1, 2, \ldots, N\}$. The distributions $p \in [0, 1]^N$ with $p_j = P(\theta = j)$ of the process are subject to the transition equation $\dot{p}(t) = \Lambda p(t)$ where e^{Λ} is a stochastic matrix. For $X_j = E(x(t)x(t)^T\delta_{\theta(t),j})$, $j = 1, \ldots, N$, where $\delta_{i,j}$ is Kronecker's delta, we have the coupled set of Lyapunov equations (cf. [17])

$$\dot{X}_j = A_i^TX_i + X_iA_i + \sum_{j=1}^{N}\lambda_{ij}X_j = T_j(X)\;, \quad j = 1, \ldots, N\;,\; (\lambda_{ij}) = \Lambda\,. \tag{5}$$

Here we may consider the space $\mathcal{H} = (\mathcal{H}^n)^N$ ordered by $\mathcal{H}_+ = (\mathcal{H}^n_+)^N$. Then $T = (T_j, \ldots, T_N)$ is resolvent positive, and $(C_1^TC_1, \ldots, C_N^TC_N) = Y \in \mathcal{H}_+^*$. Detectability, observability and reconstructability properties can now immediately be transferred to the system (4) like in the previous subsection.

6 Summary and outlook

Positive linear systems on matrix algebras are used as auxiliary systems for various linear control problems. The notions of detectability, observability and asymptotic reconstructability can be formulated conveniently in terms of resolvent positive operators. While the interpretation of the first two is clear for arbitrary spaces $\mathcal{H}$, we have clarified the meaning of asymptotic reconstructability only for $(\mathcal{H},\mathcal{H}_+) = (\mathbb{R}^n,\mathbb{R}^n_+)$ and for $(\mathcal{H},\mathcal{H}_+) = (\mathcal{H}^n,\mathcal{H}^n_+)$. A general analysis will be an issue for further research. Similarly, a unified treatment of different classes of stochastic systems (e.g. from networked control) shall be carried out in more detail.

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