LYAPUNOV ITERATIONS FOR COUPLED RICCATI DIFFERENTIAL EQUATIONS ARISING IN CONNECTION WITH NASH DIFFERENTIAL GAMES

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1 INTRODUCTION AND PROBLEM FORMULATION

In this note we study the existence of stabilizing solutions of two pairs of coupled matrix Riccati differential equations associated with linear-quadratic games of the form

$$\dot{x} = A(t)x(t) + B_1(t)u_1(t) + B_2(t)u_2(t)\ ; \quad x(0) = x_0,$$

$$\text{where} \quad x \in \mathbb{R}^n, \quad u_i \in \mathbb{R}^{r_i} \quad (i = 1, 2),$$

and where the cost functionals associated with each player are

$$J_1 = \frac{1}{2}x_f^T X_{1f} x_f + \frac{1}{2}\int_{t_0}^{t_f} (x^T Q_1(t)x + u_1^T R_{11}(t)u_1 + u_2^T R_{12}(t)u_2)\, dt,$$

$$x_f = x(t_f),$$

$$J_2 = \frac{1}{2}x_f^T X_{2f} x_f + \frac{1}{2}\int_{t_0}^{t_f} (x^T Q_2(t)x + u_1^T R_{21}(t)u_1 + u_2^T R_{22}(t)u_2)\, dt.$$

All weighting matrices are assumed to be real and symmetric with Q_i non-negative definite and $R_{ii} \quad (i = 1, 2)$ positive definite.

The Riccati equations examined in this paper are associated to two types of strategies of the two players, namely the feedback Nash strategy and the open-loop Nash strategy.
For $1 \leq i, j \leq 2$ let us use the abbreviations

$$\begin{aligned} S_i(t) &= B_i(t) R_{ii}^{-1}(t) B_i^T(t) , \\ S_{ij}(t) &= B_j(t) R_{jj}^{-1}(t) R_{ij}(t) R_{jj}^{-1}(t) B_j^T(t) . \end{aligned}$$

We consider two coupled pairs of Riccati differential equations for the two types of strategies.
For the *closed-loop* Nash strategy we have the system

$$\begin{aligned} &\frac{d}{dt} X_1 + A^T(t) X_1 + X_1 A(t) - X_1 S_1(t) X_1 - X_1 S_2(t) X_2 - X_2 S_2(t) X_1 + X_2 S_{12}(t) X_2 + Q_1(t) = 0 , \\ &\frac{d}{dt} X_2 + A^T(t) X_2 + X_2 A(t) - X_2 S_2(t) X_2 - X_2 S_1(t) X_1 - X_1 S_1(t) X_2 + X_1 S_{21}(t) X_1 + Q_2(t) = 0 , \end{aligned} \tag{1}$$

while for the *open-loop* Nash strategy the system is given by

$$\begin{aligned} &\frac{d}{dt} X_1 + A^T(t) X_1 + X_1 A(t) - X_1 S_1(t) X_1 - X_1 S_2(t) X_2 + Q_1(t) = 0 , \\ &\frac{d}{dt} X_2 + A^T(t) X_2 + X_2 A(t) - X_2 S_1(t) X_1 - X_2 S_2(t) X_2 + Q_2(t) = 0 . \end{aligned} \tag{2}$$

In both cases we are interested in the pair of solutions (X_1, X_2) satisfying

$$(X_1(t_f), X_2(t_f)) \;=\; (X_{1f}, X_{2f}) . \tag{3}$$

It is known from the literature (see [1, 3, 5] for precise definitions and further details on this topic) that the optimal feedback and open-loop Nash strategies have the form

$$\begin{aligned} u_1(t) &= -R_{11}^{-1}(t) B_1^T(t) X_1(t)\, x(t), \\ u_2(t) &= -R_{22}^{-1}(t) B_2^T(t) X_2(t)\, x(t), \end{aligned}$$

where $x(t)$ can be determined from the initial value problem

$$\dot{x} = \Big(A(t) - S_1(t) X_1(t) - S_2(t) X_2(t) \Big) x(t) , \quad x(0) = x_0 ,$$

provided the solution $(X_1(t), X_2(t))$ of the corresponding pair of Riccati differential equations (1) or (2), respectively, with terminal value (3) exists on the whole interval $[t_0, t_f]$.

Here we assume for convenience that $A : \mathbb{R} \to \mathbb{R}^{n \times n}$; $Q_i,\ S_i : \mathbb{R} \to \mathcal{S}_n, i = 1, 2$; $S_{ij} : \mathbb{R} \to \mathcal{S}_n, (ij) \in \{(1,2), (2,1)\}$ are bounded and continuous matrix valued functions; here, as usual, $\mathcal{S}_n \subset \mathbb{R}^{n \times n}$ is the linear subspace of all symmetric $n \times n$ matrices.

The equations (1) and (2) were investigated either as mathematical objects with interest in themselves in [1, Chapter 6], or in connection with several aspects of two players Nash differential games (see [2, 3, 5, 10, 11, 12] and references therein).
We mention that the system (2) can be rewritten as a non-symmetric (rectangular) matrix Riccati differential equation for the block matrix $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$. Therefore, we can use all results and methods known for this type of equations (see e.g. [1, Chapter 6], [7] and [9]); however, it is known that the global existence of the solutions of such differential equations is only guaranteed under rather restrictive conditions.
Existence results for the nonlinear coupled system (1) are also rare; although the solutions X_j,

$1 \leq j \leq 2$, of (1) are symmetric, if the terminal (or initial) values X_{jf}, $1 \leq j \leq 2$, are symmetric, the existence of the corresponding solutions can frequently only be guaranteed locally (see [8]). The situation becomes better if the differential equations (1) or (2) describe positive evolutions with respect to a suitable ordering; in particular, equation (1) and (2) were studied under these restrictions in [2], [4] and in [10], respectively.

In accordance with the assumptions from [10, 2, 4] we make the following hypothesis concerning the coefficients of (1) and (2):

Hypothesis $\boldsymbol{H_1}$. *(i) For each* $t \in \mathbb{R}$, $A(t) = (a_{ij}(t))$ *is a Metzler matrix, i.e.* $a_{ij}(t) \geq 0$ *for* $i \neq j$.

(ii) $S_i(t) \preceq 0, i = 1, 2, \ \forall\, t \in \mathbb{R}$.

(iii) $S_{ij}(t) \succeq 0, (i, j) \in \{(1, 2), (2, 1)\}, \ \forall\, t \in \mathbb{R}$.

(iv) $Q_l(t) \succeq 0, \ t \in \mathbb{R}, \ l = 1, 2$.

Here and below $\preceq$ and $\succeq$ denote the componentwise ordering.

Our aim is to construct sequences of iterates which converge towards the stabilizing solution of (1) and (2) respectively.

At each step we will have to solve two uncoupled symmetric Lyapunov differential equations or uncoupled nonsymmetric Lyapunov equations (Sylvester equations), respectively.

2 STABILIZING SOLUTIONS

Since (1) and (2) are nonstandard (coupled) Riccati differential equations, we consider that the obtained results could be useful to clarify the concept of stabilizing solutions of such equations.

To this end we regard these equations as nonlinear differential equations on a Hilbert space $\mathcal{X}$. For equation (1) we take $\mathcal{X} = \mathcal{S}_n \oplus \mathcal{S}_n$ while for equation (2) we take $\mathcal{X} = \mathbb{R}^{n \times n} \oplus \mathbb{R}^{n \times n}$. The usual inner product is given by

$$< X, Y >= \mathrm{Tr}\,(Y_1^T X_1) + \mathrm{Tr}\,(Y_2^T X_2) \tag{4}$$

for all $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ in $\mathcal{X}$.

On $\mathcal{X}$ the equations (1) and (2) may be written in a compact form as follows:

$$\frac{d}{dt} X + \mathcal{R}(t, X) + Q(t) = 0, \tag{5}$$

where $Q(t) = \big[Q_1(t), Q_2(t)\big]$ and $\mathcal{R}(t, X) = \big[\mathcal{R}_1(t, X), \mathcal{R}_2(t, X)\big]$ with

$$\mathcal{R}_1(t, X) = A^T(t)X_1 + X_1 A(t) - X_1 S_1(t) X_1 - X_1 S_2(t) X_2 - X_2 S_2(t) X_1 + X_2 S_{12}(t) X_2,$$

$$\mathcal{R}_2(t, X) = A^t(t)X_2 + X_2 A(t) - X_2 S_2(t) X_2 - X_2 S_1(t) X_1 - X_1 S_1(t) X_2 + X_1 S_{21}(t) X_1,$$

in case of equation (1), and

$$\mathcal{R}_1(t, X) = A^T(t)X_1 + X_1 A^T(t) - X_1 S_1(t) X_1 - X_1 S_2(t) X_2,$$

$$\mathcal{R}_2(t, X) = A^T(t)X_2 + X_2 A(t) - X_2 S_1(t) X_1 - X_2 S_2(t) X_2,$$

in case of equation (2).

For each solution $X(t) = (X_1(t), X_2(t))$ of equation (5) we construct the operator valued function $\mathcal{L}_X : \mathbb{R} \to \mathcal{B}[\mathcal{X}]$ by $\mathcal{L}_X(t)U = (\mathcal{L}_{1X}(t)U, \mathcal{L}_{2X}(t)U)$ where in the case of equation (1) we set

$$\begin{aligned}
\mathcal{L}_{1X}(t)U &= (A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))U_1 + U_1(A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))^T \\
&\quad - (S_1(t)X_2(t) - S_{21}(t)X_1(t))U_2 - U_2(X_2(t)S_1(t) - X_1(t)S_{21}(t))^T \ , \\
\mathcal{L}_{2X}(t)U &= -(S_2(t)X_1(t) - S_{12}(t)X_2(t))U_1 - U_1(X_1(t)S_2(t) - X_2(t)S_{12}(t)) \\
&\quad + (A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))U_2 + U_2(A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))^T \ ,
\end{aligned}$$

while in the case of equation (2) we have

$$\begin{aligned}
\mathcal{L}_{1X}(t)U &= (A(t) - S_1(t)X_1^T(t))U_1 + U_1(A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))^T - S_1(t)X_2^T(t)U_2 \ , \\
\mathcal{L}_{2X}(t)U &= (A(t) - S_2(t)X_2^T(t))U_2 + U_2(A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))^T - S_2(t)X_1^T(t)U_1 \ .
\end{aligned}$$

It is easy to see that

$$\mathcal{R}'(t, X(t)) = \mathcal{L}_X^*(t), \tag{6}$$

where $\mathcal{R}'(t, \cdot)$ is the Fréchet derivative of the function $X \to \mathcal{R}(t, X)$ and $\mathcal{L}_X^*(t)$ denotes the adjoint operator of $\mathcal{L}_X(t)$ with respect to the inner product (4).

Definition 2.1. We say that a solution $\tilde{X}(t) = (\tilde{X}_1(t), \tilde{X}_2(t))$ of (5) is

a) *a stabilizing solution* if the zero state equilibrium of the linear differential equation

$$\frac{d}{dt}Z = \mathcal{L}_{\tilde{X}}(t)Z \ , \quad Z \in \mathcal{X}$$

is exponentially stable.

b) *a strongly stabilizing solution* if the zero state equilibrium of the linear differential equation

$$\frac{d}{dt}x = A_{cl}(t)x \ , \quad x \in \mathbb{R}^n \tag{7}$$

is exponentially stable, where $A_{cl}(t) = A(t) - S_1(t)\tilde{X}_1(t) - S_2(t)\tilde{X}_2(t)$.

Remark 2.2. a) Based on (6) it follows that in the time invariant case, the concept for a stabilizing solution introduced above can be characterized by the fact that the eigenvalues of the operator $\mathcal{R}'(X)$ are located in the open left half-plane Re $\lambda < 0$.

b) In [4] it was shown that if $\tilde{X}(t) \succeq 0$ is a stabilizing solution of (1) then it is also a strongly stabilizing solution of the same equation.

Reasoning as in Lemma 8.1 (ii), (iii) in [4] one obtains that if $\tilde{X}(t) \succeq 0$ is a stabilizing solution of (2) then the solution $Z_k = 0$ of the linear differential equations

$$\frac{d}{dt}Z_k = \Lambda_{k,\tilde{X}}(t)Z_k, \ k = 1, 2,$$

is exponentially stable, where

$$\Lambda_{k,\tilde{X}}(t)Z_k = \Big(A(t) - S_k(t)\tilde{X}_k^T(t)\Big)Z_k + Z_k\Big(A(t) - S_1(t)\tilde{X}_1(t) - S_2(t)\tilde{X}_2(t)\Big)^T \tag{8}$$

is a nonsymmetric Lyapunov operator (i.e. a Sylvester operator).

It is, however, not true that the exponential stability of the evolution generated by the Sylvester operator (8) implies the exponential stability of the corresponding closed-loop matrix $A_{cl}(t)$ defined by (7). As a simple example we can consider a time-invariant situation where $A = -5I$ and $S_k(t)\tilde{X}^T = 3I$. Then $\Lambda_{k,\tilde{X}}(t) = -\mathrm{id}$ is stable, while $A_{cl}(t) = I$ is unstable.

c) Necessary and sufficient conditions under which a strongly stabilizing solution of (5) is also a stabilizing solution can be derived using the developments from Section 6 in [4].

In [10, 2, 4] the sequences of iterates $X^j = (X_1^j, X_2^j)$ converging towards the stabilizing solution were provided. In each step $X^j \in \mathcal{X}$ is obtained either as the solution of the linear differential equations

$$\frac{d}{dt}X^j + \mathcal{L}^*_{X^{j-1}}(t)X^j + Q^j(t) = 0 \tag{9}$$

in the time-varying case or as the solution of the algebraic linear equations

$$\mathcal{L}^*_{X^{j-1}}X^j + Q^j = 0 \tag{10}$$

in the time invariant case.

In this paper we replace equations (9) and (10) respectively, by uncoupled Lyapunov differential equations or uncoupled algebraic Lyapunov equations, respectively.

We wil need the existence of a solution to the corresponding strict Riccati inequality. To formalize this we introduce the following set of functions related to equation (5)

$$\Omega(\mathcal{R}, Q) = \{P : \mathbb{R} \to \mathcal{X} | P(t) \succeq 0 \quad \text{and} \quad \frac{d}{dt}P(t) + \mathcal{R}(t, P(t)) + Q(t) \ll 0\} \,. \tag{11}$$

Recall that for $H : \mathbb{R} \to \mathcal{X}$ we write $H(t) \gg 0$, if there exists a positive constant δ such that $H(t) \succeq \delta\, \mathbf{1} \succ 0$, where $\mathbf{1}$ is the $n \times 2n$ matrix with all ones (cf. Ex. 2.5 (ii) in [4]).

We shall write $H(t) \ll 0$ if $-H(t) \gg 0$.

Remark 2.3. In (11) equation $\mathcal{R}(\cdot, \cdot)$ takes different forms according to the fact that the set $\Omega(\mathcal{R}, Q)$ is associated either to equation (1) or to equation (2).

3 LYAPUNOV TYPE ITERATIONS FOR EQUATION (1)

Let $\{X^j(t)\}_{j\geq 0}$ be the sequence of functions $X^j : \mathbb{R} \to \mathcal{X}, X^j(t) = (X_1^j(t), X_2^j(t))$ with $X_l^j(t)$ being the unique bounded on $\mathbb{R}$ solution of the Lyapunov differential equation:

$$\begin{aligned} &\frac{d}{dt}X_l^j(t) + [A(t) - S_1(t)X_1^{j-1}(t) - S_2(t)X_2^{j-1}(t)]^T X_l^j(t) \\ &+X_l^j(t)[A(t) - S_1(t)X_1^{j-1}(t) - S_2(t)X_2^{j-1}(t)] + Q_l^{j-1}(t) = 0, \end{aligned} \tag{12}$$

$l = 1, 2, X_l^0(t) = 0, t \in \mathbb{R}$, where

$$\begin{aligned} Q_1^{j-1}(t) &= Q_1(t) + X_1^{j-1}(t)S_1(t)X_1^{j-1}(t) + X_2^{j-1}(t)S_{12}(t)X_2^{j-1}(t) \,, & (13) \\ Q_2^{j-1}(t) &= Q_2(t) + X_2^{j-1}(t)S_2(t)X_2^{j-1}(t) + X_1^{j-1}(t)S_{21}(t)X_1^{j-1}(t) \,. & (14) \end{aligned}$$

Before stating the main result of this section we make the following assumption:

Hypothesis $\boldsymbol{H_2}$. *(i) The zero state equilibrium of the linear differential equation on $\mathbb{R}^n$:*

$$\frac{d}{dt}x(t) = A(t)x(t)$$

is exponentially stable.

(ii) The set $\Omega(\mathcal{R}, Q)$ is not empty.

Theorem 3.1. *Under the assumptions $\mathbf{H_1}$ and $\mathbf{H_2}$ the sequence $\{X^j(t)\}_{j\geq 0}$ defined by* (12) – (14) *is well defined and convergent.*

If $\tilde{X}(t) := \lim_{j\to\infty} X^j(t)$ then $\tilde{X}(t)$ is the stabilizing solution of (1). *Moreover $\tilde{X}(t)$ is the minimal solution of* (1) *with respect to the class of globally bounded nonnegative solutions of* (1).

Proof: By induction, we prove for $j = 0, 1, 2, \ldots$ that

$a_j)$ $0 \preceq X^j(t) \preceq P(t)$ for all $P(t) \in \Omega(\mathcal{R}, Q)$,

$b_j)$ the zero state equilibrium of the linear differential equation $\frac{d}{dt}x(t) = A_j(t)x(t)$ is exponentially stable, where

$$A_j(t) = A(t) - S_1(t)X_1^j(t) - S_2(t)X_2^j(t)\ , \tag{15}$$

$c_j)$ $X^j(t) \preceq X^{j+1}(t)$ for all $t \in \mathbb{R}$.

By assumption $\mathbf{H_2}$ together with $X_l^0(t) = 0$ the items $a_0)$ and $b_0)$ are fulfilled.

To check that $c_0)$ is also true we note that $X_l^1(t)$ is the unique bounded solution of the Lyapunov differential equation

$$\frac{d}{dt}X_l^1(t) + A^T(t)X_l^1(t) + X_l^1(t)A(t) + Q_l(t) = 0.$$

Since $Q_l(t) \succeq 0$ Theorem 4.7 (iv) of [4] yields $X_l^1(t) \succeq 0 = X_l^0(t), t \in \mathbb{R}$, which is just $c_0)$. Let us assume next that $a_i)$, $b_i)$, $c_i)$ are fulfilled for $0 \leq i \leq j-1$ and prove that then they also hold for $i = j$.

If $b_{j-1})$ is fulfilled then from Theorem 4.7 (i) of [4] it follows that equation (12) has a unique bounded on $\mathbb{R}$ solution and thus $X^j(t)$ is well defined.

If $P(t) = (P_1(t), P_2(t)) \in \Omega(\mathcal{R}, Q)$ one can see that it solves the differential equation

$$\frac{d}{dt}P(t) + \tilde{\mathcal{R}}(t, P(t)) + Q(t) + \hat{Q}(t) = 0\ ,$$

where $\hat{Q}(t) = (\hat{Q}_1(t), \hat{Q}_2(t)) \gg 0$.

Each block $P_l(t)$ satisfies the Lyapunov equation

$$\frac{d}{dt}P_l(t) + A_{j-1}^T(t)P_l(t) + P_l(t)A_{j-1}(t) + H_l^{j-1}(t) = 0\ ,\ l = 1, 2, \tag{16}$$

where $A_{j-1}(t)$ is as in (15) with $X_l^j(t)$ replaced by $X_l^{j-1}(t)$ and where $H^{j-1}(t) = (H_1^{j-1}(t), H_2^{j-1}(t))$ with

$$\begin{aligned} H_1^{j-1}(t) = -[P_1(t) - X_1^{j-1}(t)]S_1(t)[P_1(t) - X_1^{j-1}(t)] \\ -[P_2(t) - X_2^{j-1}(t)]S_2(t)P_1(t) - P_1(t)S_2(t)[P_2(t) - X_2^{j-1}(t)] \\ +P_2(t)S_{12}(t)P_2(t) + X_1^{j-1}(t)S_1(t)X_1^{j-1}(t) + Q_1(t) + \hat{Q}_1(t), \end{aligned} \tag{17}$$

$$\begin{aligned} H_2^{j-1}(t) = -[P_2(t) - X_2^{j-1}(t)]S_2(t)[P_2(t) - X_2^{j-1}(t)] \\ -[P_1(t) - X_1^{j-1}(t)]S_1(t)P_2(t) - P_2(t)S_1(t)[P_1(t) - X_1^{j-1}(t)] \\ +P_1(t)S_{21}(t)P_1(t) + X_2^{j-1}(t)S_2(t)X_2^{j-1}(t) + Q_2(t) + \hat{Q}_2(t). \end{aligned} \tag{18}$$

From (12) and (16) one obtains

$$\frac{d}{dt}(P_l(t) - X_l^j(t)) + A_{j-1}^T(t)(P_l(t) - X_l^j(t)) + (P_l(t) - X_l^j(t))A_{j-1}(t) + M_l^{j-1}(t) = 0, \tag{19}$$

where $M_l^{j-1}(t) = H_l^{j-1}(t) - Q_l^{j-1}(t)$.

Since $a_{j-1})$ is fulfilled one obtains from (12), (14) and (17) – (18) that $M_l^{j-1}(t) \succ 0, t \in \mathbb{R}$.

Applying Theorem 4.7 (iv) in [4] to the equation (19) one concludes that for $t \in \mathbb{R}$

$$P_l(t) - X_l^j(t) \succ c\,\mathbf{1}_n, \tag{20}$$

where c is a positive constant. Thus we deduce that $a_j)$ is fulfilled.

To check that $b_j)$ is fulfilled we rewrite equation (16) in the form:

$$\frac{d}{dt}P_l(t) + A_j^T(t)P_l(t) + P_l(t)A_j(t) + H_l^j(t) = 0, \tag{21}$$

where $A_j(t)$ is as in (15) and the matrices $H_l^j(t)$ are as in (17) – (18) with $X_l^{j-1}(t)$ replaced by $X_l^j(t)$.

Equation (16) can be rewritten as

$$\frac{d}{dt}X_l^j(t) + A_j^T(t)X_l^j(t) + X_l^j(t)A_j(t) + G_l^j(t) = 0, \tag{22}$$

where $A_j(t)$ is given by (15) and

$$\begin{aligned} G_1^j(t) = Q_1(t) + X_1^{j-1}(t)S_1(t)X_1^{j-1}(t) + X_2^{j-1}(t)S_{12}(t)X_2^{j-1}(t) \\ -(X_1^{j-1}(t) - X_1^j(t))S_1(t)X_1^j(t) - X_1^j(t)S_1(t)(X_1^{j-1}(t) - X_1^j(t)) \\ -X_2^j(t)X_2(t)(X_2^{j-1}(t)X_2^j(t)) - (X_2^{j-1}(t) - X_2^j(t))S_2(t)X_2^j(t), \end{aligned} \tag{23}$$

$$\begin{aligned} G_2^j(t) = Q_2(t) + X_1^{j-1}(t)S_{21}(t)X_1^{j-1}(t) + X_2^{j-1}(t)S_2(t)X_2^{j-1}(t) \\ -X_2^j(t)S_1(t)(X_1^{j-1}(t) - X_1^j(t)) - (X_1^{j-1}(t) - X_1^j(t))S_1(t)X_2^j(t) \\ -X_2^j(t)S_2(t)(X_2^{j-1}(t) - X_2^j(t)) - (X_2^{j-1}(t) - X_2^j(t))S_2(t)X_2^j(t). \end{aligned} \tag{24}$$

Subtracting (22) from (21) and taking into account (20) one obtains that the function $t \to P_l(t) - X_l^j(t)$ is a bounded and uniform positive solution of the Lyapunov equation

$$\frac{d}{dt}Y_l(t) + A_j^T(t)Y_l(t) + Y_l(t)A_j(t) + \Theta_l^j(t) = 0 \tag{25}$$

with $\Theta_l^j(t) = H_l^j(t) - G_l^j(t)$.

It is easy to see that $\Theta_l^j(t) \gg 0$.

Applying the implication '(vi)⇒(i)' of Theorem 4.5 in [4] to equation (25) one concludes that the zero state equilibrium of the equation

$$\frac{d}{dt}x(t) = A_j(t)x(t)$$

is exponentially stable. Thus we obtained that item $b_j)$ is fulfilled.

To check the validity of item c_j) one subtracts equation (22) from equation (12) written for $X_l^{j+1}(t)$ and obtains:

$$\frac{d}{dt}(X_l^j(t) - X_l^{j+1}(t)) = A_j^T(t)(X_l^{j+1}(t) - X_l^j(t)) + (X_l^{j+1}(t) - X_l^j(t))A_j(t) + \Delta_l^j(t), \tag{26}$$

where $\Delta_l^j(t) = Q_l^j(t) - G_l^j(t),\ l = 1, 2.$

Combining (12) – (14) written for $j+1$ instead of j with (23) – (24) one can see that $\Delta_l^j(t) \gg 0$.

Applying Theorem 4.7 (iv) of [4] to equation (26) we conclude that

$$X_l^{j+1}(t) - X_l^j(t) \geq 0,\ t \in \mathbb{R}.$$

This shows that c_j) is fulfilled.

From a_j) and c_j), $j \geq 0$ it follows that the sequences $\{X_l^j(t)\}_{j\geq 0}, l = 1, 2, t \in \mathbb{R}$ are convergent. Set $\tilde{X}_l(t) = \lim_{j\to\infty} X_l^j(t), l = 1, 2, t \in \mathbb{R}$. By standard arguments one obtains that $t \to \tilde{X}(t) = (\tilde{X}_1(t), \tilde{X}_2(t))$ is a solution of (1). As in [4] one proves that $\tilde{X}(t)$ is just the stabilizing solution of (1).

In the same way as in the proof of item a_j) one shows that $X^j(t) \preceq Y(t)$ for arbitrary $Y(t) = [Y_1(t), Y_2(t)]$ satisfying

$$\frac{d}{dt}Y(t) + \mathcal{R}(t, Y(t)) + Q(t) \preceq 0, Y_l(t) \succeq 0.$$

This allows us to conclude that $\tilde{X}(t)$ is the minimal solution of (1). □

Remark 3.2. a) Applying Theorem 4.7 (iii) in [4] one deduces that in the time invariant case the unique bounded solution of (12) is constant. Therefore in the time invariant case, for each iteration we have to solve two algebraic Lyapunov equations

$$A_{j-1}^T X_l^j + X_l^j A_{j-1} + Q_l^j = 0,\ l = 1, 2,$$

with $A_{j-1} = A - S_1 X_1^{j-1} - S_2 X_2^{j-1}$ and Q_l^{j-1} as in (12) – (14).

b) If $A(\cdot), S_j(\cdot), S_{kl}(\cdot), Q_l(\cdot)$ are periodic functions with period $\theta > 0$, then the unique bounded solution of (12) is a periodic function with the same period θ.

The initial value $X_l^j(0)$ is obtained as solution of the Stein equation:

$$X_l^j(0) = \Phi_{j-1}^T(\theta, 0) X_l^j(0) \Phi_{j-1}(\theta, 0) + \int_0^\theta \Phi_{j-1}^T(s, 0) Q_l^{j-1}(s) \Phi_{j-1}(s, 0) ds,$$

where $\Phi_{j-1}(t, \tau)$ is the fundamental matrix solution of $\frac{d}{dt}x(t) = A_{j-1}(t)x(t)$.

4 LYAPUNOV TYPE ITERATIONS FOR EQUATION (2)

We construct the following sequence of functions $\{X^j(t)\}_{j\geq 0}$, $X^j(t) = (X_1^j(t), X_2^j(t))$, where $X_l^j : \mathbb{R} \to \mathbb{R}^{n\times n}$ is the unique bounded solution of the following nonsymmetric Lyapunov differential equations:

$$\begin{aligned}\frac{d}{dt}X_l^j(t) + (A^T(t) - X_l^{j-1}(t)S_l(t))X_l^j(t) + X_l^j(t)(A(t) - S_1(t)X_1^{j-1}(t) - S_2(t)X_2^{j-1}(t))\\ +Q_l(t) + X_l^{j-1}(t)S_l(t)X_l^{j-1}(t) = 0, \quad l = 1, 2,\ j \geq 1,\ X_l^0(t) = 0,\ l = 1, 2.\end{aligned} \tag{27}$$

The main result of this section is:

Theorem 4.1. *Under the assumptions* $\mathbf{H_1}$) *and* $\mathbf{H_2}$) *the sequence* $\{X^j(t)\}_{j\geq 0}$ *is well defined and convergent. If* $\tilde{X}(t) = \lim_{j\to\infty} X^j(t), t \in \mathbb{R}$ *then* $\tilde{X}(t)$ *is the stabilizing and minimal solution of* (2).

The proof follows the same line as in the case of Theorem 3.1 and it is omitted for shortness. However we remark that instead of the item b_j) one proofs the following new item

b_j^*) The zero state equilibrium of the linear differential equation on $\mathbb{R}^{n\times n}$:

$$\frac{d}{dt}Z_l(t) = [A(t) - S_l(t)(X_l^j(t))^T]Z_l(t) + Z_l(t)[A(t) - S_1(t)X_1^j(t) - S_2(t)X_2^j(t)]^T$$

is exponentially stable.

Remark 4.2. If in (2), $A(\cdot), S_j(\cdot), Q_l(\cdot)$ are constants, then one obtains inductively that the unique solution of (27) is constant. Therefore in the time invariant case at each iteration we solve the following nonsymmetric algebraic Lyapunov equations:

$$(A^T - X_l^{j-1}(t))X_l^j + X_l^j(A - S - 1X_1^{j-1} - S_2X_2^j) + Q_l + X_l^{j-1}S_lX_l^{j-1} = 0.$$

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