Differential equations with positive evolutions and some applications

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1 Introduction

Relations between algebraic Riccati equations and generators of positive semigroups on ordered vector spaces have proved to be useful in the analysis of control problems for linear time-invariant systems and the numerical solution of general matrix equations, compare e.g. the recent works [22, 12, 20, 11]. In particular, this approach allows a unified treatment – up to a certain extent – of symmetric and nonsymmetric Riccati equations arising in discrete or continuous time or in a stochastic framework. The notion of positivity facilitates the design of monotonically convergent solution schemes and has a clear interpretation with respect to stability properties of the system.

Similar ideas have also been developed for differential Riccati equations and corresponding time-varying control systems e.g. [21, 1]. However, to our knowledge, there is no concise treatment on the relation between positive evolutions and differential Riccati equations available in the literature. The present paper tries to fill this gap.

After reviewing some basic properties of finite dimensional ordered vector spaces in Section 2 we define positive evolutions in Section 3 and characterize their generators. Although this result is essentially a straight-forward extension from the time-invariant case, it seems that only special cases have been treated so far, e.g. [24]. Moreover, we provide some useful estimates. Section 4 is concerned with exponential stability of positive evolutions. Building upon some results in [13], we obtain a time-varying version of the generalized Lyapunov Theorem by Schneider [29], which – as mentioned above – has turned out to be a useful tool in the analysis of *algebraic* Riccati equations, (cf. [11]). Analogously, we apply our time-varying result to a certain *differential* Riccati equation in the last sections. Before that, however, we collect some interesting statements on uniform observability and exponential stability in Section 5 and on the robust stability of positive evolutions with respect to positive perturbations of their generator in Section 6. We conclude the paper with two substantial sections, where we apply our results (mainly from Sections 3 and 4) to coupled differential Riccati equation arising in game theory.

2 Preliminaries

In this section we recall some definitions which allow us to establish the framework of this paper.

Throughout this paper $\mathcal{X}$ is a finite dimensional real Hilbert space. We assume that $\mathcal{X}$ is ordered by a order relation " $\leq$ " induced by a regular, solid, closed, pointed and selfdual convex cone $\mathcal{X}^+$. For detailed definitions and other properties of convex cones we refer to [8, 9, 12, 11, 20, 27].

Here we recall that the dual cone with respect to the inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{X}$ is defined as $(\mathcal{X}^+)^* = \{y \in \mathcal{X} | \forall x \in \mathcal{X}^+ : \langle x, y\rangle \geq 0\}$; the cone $\mathcal{X}^+$ is called selfdual if $(\mathcal{X}^+)^* = \mathcal{X}^+$, By $|\cdot|_2$ we denote the norm on $\mathcal{X}$ induced by the inner product on $\mathcal{X}$, i.e. $|x|_2 = [\langle x, x\rangle]^{\frac{1}{2}}$. This norm is monotone in the sense that $0 \le x \le y$ implies $|x|_2 \le |y|_2$.

Fix some positive vector $\xi \in \operatorname{Int}\mathcal{X}^+$ and consider the set $B_\xi = \{x \in \mathcal{X} \mid -\xi < x < \xi\}$. By $|\cdot|_\xi$ we denote the norm with unit ball B_ξ, i.e. $B_\xi = \{x \,|\, |x|_\xi < 1\}$. In other words, $|\cdot|_\xi$ is the Minkowski-functional $|x|_\xi = \inf\{t > 0 : \frac{1}{t}x \in B_\xi\}$. The pair $(\xi, |\cdot|_\xi)$ has the following properties, which are easily verified:

***P*$_1$)** If $x, y, z \in \mathcal{X}$ are such that $y \le x \le z$ then

$$|x|_\xi \le \max(|y|_\xi; |z|_\xi)\ . \tag{2.1}$$

***P*$_2$)** $|\xi|_\xi = 1$ and for arbitrary $x \in \mathcal{X}$ with $|x|_\xi \le 1$ it holds that

$$-\xi \ \le\ x \ \le\ \xi\ . \tag{2.2}$$

***P*$_3$)** There exists a positive constant c_0 such that for all $x \in \mathcal{X}^+$:

$$\langle x, \xi\rangle \ \ \ge\ \ c_0|x|_\xi \tag{2.3}$$

If $T : \mathcal{X} \to \mathcal{X}$ is a linear operator and $|\cdot|$ is a norm on $\mathcal{X}$, then $\|T\| = \sup\limits_{|x|\le 1} |Tx|$ is the corresponding operator norm.

Remark 2.1 a) Since $\mathcal{X}$ is a finite dimensional space the norms $|\cdot|_\xi$ and $|\cdot|_2$ are equivalent. By definition it follows that $\|\cdot\|_\xi$ and $\|\cdot\|_2$ are also equivalent. This means that there are two positive constants c_1 and c_2 such that $c_1\|T\|_\xi \le \|T\|_2 \le c_2\|T\|_\xi$ for all linear operators $T : \mathcal{X} \to \mathcal{X}$.

b) If $T^* : \mathcal{X} \to \mathcal{X}$ is the adjoint operator of T with respect to the inner product on $\mathcal{X}$, then $\|T\|_2 = \|T^*\|_2$. In general the equality $\|T\|_\xi = \|T^*\|_\xi$ is not true. However, based on a) it follows that there are two positive constants $\tilde{c}_1, \tilde{c}_2$ such that

$$\tilde{c}_1\|T\|_\xi \le \|T^*\|_\xi \le \tilde{c}_2\|T\|_\xi. \tag{2.4}$$

Definition 2.2 *Let $(\mathcal{X}, \mathcal{X}^+)$ and $(\mathcal{Y}, \mathcal{Y}^+)$ be ordered vector spaces. An operator $T : \mathcal{X} \to \mathcal{Y}$ is called* positive, *if $T(\mathcal{X}^+) \subset \mathcal{Y}^+$. In this case we write $T \ge 0$. If $T(\operatorname{Int}\mathcal{X}^+) \subset \operatorname{Int}\mathcal{Y}^+$ we write $T > 0$.*

Proposition 2.3 *[14] If $T : \mathcal{X} \to \mathcal{X}$ is a linear operator then the following hold:*

(i) $T \ge 0$ if and only if $T^* \ge 0$.

(ii) If $T \ge 0$ then $\|T\|_\xi = |T\xi|_\xi$.

From (ii) of the previous Proposition we obtain:

Corollary 2.4 *Let $T_k : \mathcal{X} \to \mathcal{X}, k = 1, 2$ be linear positive operators. If $T_1 \le T_2$ then $\|T_1\|_\xi \le \|T_2\|_\xi$.*

Example 2.5 (i) Consider $\mathcal{X} = \mathbb{R}^n$ ordered by the regular, solid, closed, pointed and selfdual convex cone $\mathbb{R}^n_+ = \{x = (x_1, ..., x_n)^T \in \mathbb{R}^n | x_i \ge 0, 1 \le i \le n\}$. This is the so called *componentwise ordering*. For clarity, it will be denoted by " $\succeq$ ". If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator then $T \succeq 0$, if and only if the corresponding matrix A with respect to the canonical basis on $\mathbb{R}^n$ has nonnegative entries. For $\xi = (1\ \ 1\ ...\ 1)^T \in \operatorname{Int}(\mathbb{R}^n_+)$ the norm $|\cdot|_\xi$ is defined by

$$|x|_\xi = \max_{1\le i\le n} |x_i|\ . \tag{2.5}$$

The properties **P$_1$) – P$_3$)** hold with $c_0 = 1$.

(ii) Let $\mathcal{X} = \mathbb{R}^{m\times n}$ be the space of $m \times n$ matrices endowed with the inner product

$$\langle A, B\rangle = \mathrm{Tr}\,(B^T A) \tag{2.6}$$

$\forall A, B \in \mathbb{R}^{m\times n}$, $\mathrm{Tr}\,(M)$ denoting as usual the trace of a matrix M. On $\mathbb{R}^{m\times n}$ we consider the componentwise order relation induced by the regular solid closed selfdual pointed convex cone $\mathcal{X}^+ = \mathbb{R}^{m\times n}_+$ where

$$\mathbb{R}^{m\times n}_+ = \{A \in \mathbb{R}^{m\times n} | A = \{a_{ij}\}, a_{ij} \geq 0, 1 \leq i \leq m, 1 \leq j \leq n\} \tag{2.7}$$

If we define $\mathbf{1}_{m,n} \in \mathrm{Int}\,\mathbb{R}^{m\times n}_+$ as the matrix of all ones, the norm $|\cdot|_\xi$ will be given by

$$|A|_\xi = \max_{i,j} |a_{ij}|. \tag{2.8}$$

The properties $\mathbf{P_1}) - \mathbf{P_3})$ hold with $c_0 = 1$.

(iii) Let $\mathcal{X} = \mathcal{S}_n$ be the space of $n \times n$ symmetric matrices. On $\mathcal{S}_n$ we consider the inner product

$$\langle X, Y\rangle = \mathrm{Tr}\,(XY) \tag{2.9}$$

for arbitrary $X, Y \in \mathcal{S}_n$. The space $\mathcal{S}_n$ is ordered by the regular, solid, closed, pointed and selfdual convex cone $\mathcal{S}_n^+ = \{X \in \mathcal{S}_n, X \geq 0\}$ of nonnegative definite matrix. Its interior is the set of positive definite matrices $X > 0$.

For $\xi = I$ the norm $|\cdot|_\xi$ coincides with the Euclidian norm $\|\cdot\|_2$, since $|X|_\xi = \max_{\lambda\in\sigma(X)} |\lambda| = \rho(X) = \|X\|_2$,
where ρ denotes the spectral radius. The properties $\mathbf{P_1}) - \mathbf{P_3})$ are fulfilled with $c_0 = 1$.

3 Differential equations generating positive evolutions

Let $\mathcal{I} \subset \mathbb{R}$ be an interval and let $\mathcal{L} : \mathcal{I} \to \mathcal{B}(\mathcal{X})$ be a continuous operator valued function, $\mathcal{B}(\mathcal{X})$ is the space of linear operators defined on $\mathcal{X}$ taking values in $\mathcal{X}$.

Consider the linear differential equation:

$$\frac{d}{dt}x(t) = \mathcal{L}(t)x(t). \tag{3.1}$$

It is well known that for each $(t_0, x_0) \in \mathcal{I} \times \mathcal{X}$ equation (3.1) has a unique solution $x(\cdot, t_0, x_0) : \mathcal{I} \to \mathcal{X}$ which satisfies $x(t_0, t_0, x_0) = x_0$. Moreover the dependence $x_0 \mapsto x(\cdot, t_0, x_0)$ is linear.

Therefore for each $t, t_0 \in \mathcal{I}$ the linear operator $T(t, t_0) : \mathcal{X} \to \mathcal{X}$ is well defined by $T(t, t_0)x_0 = x(t, t_0, x_0)$. The operator $T(t, t_0)$ will be called *the linear evolution operator* defined on $\mathcal{X}$ by the operator $\mathcal{L}(\cdot)$. Here we recall some well known properties of an evolution operator:

Lemma 3.1 *The following hold:*

(i) $T(t, t) = I_{\mathcal{X}}$ *for all* $t \in \mathcal{I}$, $I_{\mathcal{X}}$ *being the identity operator on* $\mathcal{X}$.

(ii) $T(t, \tau)T(\tau, s) = T(t, s)$ *for all* $t, \tau, s \in \mathcal{I}$.

(iii) For all $t, s \in \mathcal{I}$, *the operator* $T(t, s)$ *is invertible and its inverse is* $T^{-1}(t, s) = T(s, t)$.

(iv) $t \mapsto T(t, s)$ *is differentiable and* $\frac{d}{dt}T(t, s) = \mathcal{L}(t)T(t, s)$.

(v) $s \mapsto T(t, s)$ *is differentiable and* $\frac{d}{ds}T(t, s) + T(t, s)\mathcal{L}(s) = 0$ ($\iff \frac{d}{ds}T^*(t, s) = -\mathcal{L}^*(s)T^*(t, s)$).

(vi) If $\mathcal{L}(t) = \mathcal{L}$ *for all* $t \in \mathcal{I}$ *then* $T(t, t_0) = e^{\mathcal{L}(t-t_0)} = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!}\mathcal{L}^k$.

Example 3.2 a) Let $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{L} : \mathcal{I} \to \mathbb{R}^{n\times n}$ be a continuous matrix valued function. In this case the evolution operator $T(t, t_0)$ defined by $\mathcal{L}(\cdot)$ is just the fundamental matrix solution $\Phi(t, t_0)$ of the linear differential equation on $\mathbb{R}^n$

$$\frac{d}{dt}x(t) = \mathcal{L}(t)x(t).$$

b) Let $A : \mathcal{I} \to \mathbb{R}^{n\times n}$ be a continuous matrix valued function. Define $\mathcal{L}_A : \mathcal{I} \to \mathcal{B}(\mathcal{S}_n)$ by

$$\mathcal{L}_A S = A(t)S + SA^T(t) \tag{3.2}$$

which is called Lyapunov operator defined by A. As usual the superscript T stands for the transpose of a matrix or a vector.

The adjoint of the Lyapunov operator is

$$\mathcal{L}_A^*(t)S = A^T(t)S + SA(t). \tag{3.3}$$

The linear evolution operator of the differential equation $\dot{X}(t) = \mathcal{L}_A(t)X(t)$ on $\mathcal{S}_n$ is given by

$$T(t, t_0)X_0 = \Phi_A(t, t_0)X_0\Phi_A^T(t, t_0) \tag{3.4}$$

where $\Phi_A(t, t_0)$ is the fundamental matrix solution of $\dot{x}(t) = A(t)x(t)$ with $\Phi_A(t_0, t_0) = I$.

The adjoint operator of $T(t, t_0)$ will be given by

$$T^*(t, t_0)X_0 = \Phi_A^T(t, t_0)X_0\Phi_A(t, t_0). \tag{3.5}$$

c) Let $\mathcal{X} = \mathbb{R}^{m\times n}$ and $A : \mathcal{I} \to \mathbb{R}^{m\times m}$, $B : \mathcal{I} \to \mathbb{R}^{n\times n}$ continuous matrix valued functions. The nonsymmetric Lyapunov operator $\mathcal{L}_{AB} : \mathcal{I} \to \mathcal{B}(\mathbb{R}^{m\times n})$ is given by

$$\mathcal{L}_{AB}(t)X = A(t)X + XB^T(t). \tag{3.6}$$

Its adjoint operator is given by

$$\mathcal{L}_{AB}^*(t)X = A^T(t)X + XB(t). \tag{3.7}$$

The evolution operator of the linear differential equation $\dot{X}(t) = \mathcal{L}_{AB}(t)X(t)$ on $\mathbb{R}^{m\times n}$ is

$$T(t, t_0)X_0 = \Phi_A(t, t_0)X_0\Phi_B^T(t, t_0). \tag{3.8}$$

Its adjoint is $T^*(t, t_0)X_0 = \Phi_A^T(t, t_0)X_0\Phi_B(t, t_0)$.

Definition 3.3 *We say that $\mathcal{L}(\cdot)$ defines a* causal positive evolution *(or just* positive evolution *for shortness) if $T(t, t_0) \geq 0$ for all $t \geq t_0$, $t, t_0 \in \mathcal{I}$.*

From (3.4) we see that the Lyapunov operator $\mathcal{L}_A(\cdot)$ generates a positive evolution on $\mathcal{S}_n$. In the time-invariant case, where $\mathcal{L}(t) = \mathcal{L}$ for all t, it is well-known that $\mathcal{L}$ generates a positive evolution (or *semigroup* in this case), if and only if $\mathcal{L}$ is resolvent positive.
In [18] the following equivalent conditions were given for an operator to be resolvent positive (see also [30], [8]).

Theorem 3.4 *Let $\mathcal{X}$ be a finite-dimensional real Banach space ordered by a closed, solid, normal and convex cone $\mathcal{X}^+$. For linear operators $\mathcal{L} : \mathcal{X} \to \mathcal{X}$ the following are equivalent:*

(i) $\mathcal{L}$ is resolvent positive.

(ii) $\exp(t\mathcal{L})$ is positive for all $t \geq 0$.

(iii) $\forall x \in \partial\mathcal{X}^+ : \exists v \in \partial\mathcal{X}^{+*}, v \neq 0 : \langle x, v\rangle = 0$ *and* $\langle \mathcal{L}x, v\rangle \geq 0$.

(iv) $x \in \mathcal{X}^+$, $v \in \mathcal{X}^{+*}$, $\langle x, v\rangle = 0 \Rightarrow \langle \mathcal{L}x, v\rangle \geq 0$.

(v) $\mathcal{L} \in \operatorname{cl}\{S - \alpha I \,|\, S : X \to X \text{ positive}, \alpha \in \mathbb{R}\}$.

Remark 3.5 The term *resolvent positive* seems to have been coined by Arendt, e.g. [4]. A resolvent positive operator is also called *Metzler operator* e.g. in [19, 25]. Condition (ii) is often called *exponential positivity* or *exponential nonnegativity*, e.g. [8]. In [18] an operator is called *quasi-monotonic*, if it possesses property (iii). Operators satisfying (iv) are called *cross-positive* in [30] and $\mathcal{X}^+$*-subtangential* in [8], while e.g. in [3] property (iv) is termed *positive minimum principle*. An operator $\mathcal{L}$ is called *essentially nonnegative* in [8], if $\mathcal{L} + \alpha I$ is positive for sufficiently large $\alpha \in \mathbb{R}$. In the case $(\mathcal{X}, \mathcal{X}^+) = (\mathbb{R}^n, \mathbb{R}^n_+)$ the corresponding matrices are often called *Metzler matrices*. A matrix A is called Z-matrix, if $-A$ is a Metzler matrix; if additionally $\sigma(A) \subset \mathbb{C}_-$, then A is called M-matrix, e.g. [9].

Using the method of proof from [18], Theorem 3.4 can easily be extended to the time-varying case.

Proposition 3.6 *The operator family $\mathcal{L}(\cdot)$ generates a positive evolution on the interval $\mathcal{I}$, if and only if $\mathcal{L}(t)$ is resolvent positive for all $t \in \mathcal{I}$.*

Proof. '$\Leftarrow$': Assume that $\mathcal{L}(t)$ is resolvent positive for all $t \in \mathcal{I}$. For $\xi \in \operatorname{Int}\mathcal{X}^+$ and arbitrary $\delta \in \mathbb{R}$, $\delta > 0$, we consider the differential equation $\dot{x} = \mathcal{L}(t)x + \delta\xi$ with initial condition $x(t_0) = x_0 \in \operatorname{Int}\mathcal{X}^+$. We show that $x(t) = x(t, x_0, \delta) \in \operatorname{Int}\mathcal{X}^+$ for all $t \geq t_0$. If this was not the case, we could find some $\tau > t_0$ such that

$$x(\tau) \in \partial\mathcal{X}^+ \quad \text{and} \quad x(t) \in \operatorname{Int}\mathcal{X}^+ \quad \text{for all } t_0 \leq t < \tau. \tag{3.9}$$

By Theorem 3.4 (iii), there exists a $v \in (\mathcal{X}^+)^* = \mathcal{X}^+, v \neq 0$ with $\langle x(\tau), v\rangle = 0$ and $\langle \mathcal{L}(\tau)x(\tau), v\rangle \geq 0$. But from (3.9) we see that $\langle x(t), v\rangle > 0$ for $t_0 \leq t < \tau$ whence

$$0 \geq \frac{d}{dt}\langle x(t), v\rangle\Big|_{t=\tau} = \langle \mathcal{L}(\tau)x(\tau) + \delta\xi, v\rangle \geq \delta\langle \xi, v\rangle > 0\,,$$

which is a contradiction. Since the solution $x(t) = x(t, x_0, \delta)$ depends continuously on x_0 and δ, we conclude that $\mathcal{L}(\cdot)$ generates a positive evolution.
'$\Rightarrow$': Vice versa we assume that $\mathcal{L}(\tau)$ is not resolvent positive for some $\tau \in \mathcal{I}$. By Theorem 3.4 (iv) there exists a pair $x \in \mathcal{X}^+$, $v \in \mathcal{X}^{+*}$ with $\langle x, v\rangle = 0$ and $\langle \mathcal{L}(\tau)x, v\rangle < 0$.
We consider the initial value problem $\dot{x}(t) = \mathcal{L}(t)x(t)$ with $x(\tau) = x$. By construction $\langle x(\tau), v\rangle = 0$ and $\frac{d}{dt}\langle x(t), v\rangle\big|_{t=\tau} < 0$. Therefore $\langle x(\tau + \delta), v\rangle < 0$ for sufficiently small δ, implying $x(\tau + \delta) \notin \mathcal{X}^+$. Hence $\mathcal{L}(\cdot)$ does not generate a positive evolution. □

Remark 3.7 For the special case where $(\mathcal{X}, \mathcal{X}^+) = (\mathbb{R}^n, \mathbb{R}^n_+)$ it follows that a continuous matrix-valued function $A(t)$ defines a positive evolution, if and only if $A(t)$ is a Metzler matrix for all t. This result has been proved in [24].

Corollary 3.8 *Let $\mathcal{L}_k : \mathcal{I} \to \mathcal{B}(\mathcal{X}), k = 1, 2$ be continuous operator valued functions and let $T_k(t, s)$ be the corresponding linear evolution operators. Assume:*

a) $\mathcal{L}_1(t) \leq \mathcal{L}_2(t)$ *for all* $t \in \mathcal{I}$

b) $\mathcal{L}_1(\cdot)$ *generates a positive evolution on* $\mathcal{X}$.

Under these assumptions the following assertions hold:

(i) $\mathcal{L}_2(\cdot)$ *generates a positive evolution on* $\mathcal{X}$.

(ii) $T_2(t,s) \geq T_1(t,s)$ *for all* $t \geq s, t, s \in \mathcal{I}$.

Proof. (i) follows from the fact that $\mathcal{L}_2(t)$ is resolvent positive for all t, if $\mathcal{L}_1(t)$ is resolvent positive and $\mathcal{L}_2(t) \geq \mathcal{L}_1(t)$ for all t.

(ii) It is easy to see that for each $t \geq s$ we have $T_2(t,s) = T_1(t,s) + \int_s^t T_1(t,\sigma)(\mathcal{L}_2(\sigma) - \mathcal{L}_1(\sigma))T_2(\sigma,s)\, d\sigma$. The conclusion follows now from $T_1(t,\sigma) \geq 0,\ \mathcal{L}_2(\sigma) - \mathcal{L}_1(\sigma) \geq 0$ and $T_1(t,\sigma), T_2(\sigma,s) \geq 0$. □

Corollary 3.9 *Let* $\mathcal{L} : \mathcal{I} \to \mathcal{B}(\mathcal{X})$ *be a continuous operator valued function such that* $\mathcal{L}(t) \geq 0$ *for all* $t \in \mathcal{I}$. *Then* $\mathcal{L}(\cdot)$ *generates a positive evolution on* $\mathcal{X}$.

Proof. This follows from Corollary 3.8 with $\mathcal{L}_1(t) = 0$. □

Unlike in the discrete time case, where a sequence $\{\mathcal{L}_t\}_{t \geq t_0}$ generates a positive evolution if and only if $\mathcal{L}_t \geq 0$ for all $t \geq t_0$, in the continuous time case if an operator $\mathcal{L}(\cdot)$ generates a positive evolution on $\mathcal{X}$ then $\mathcal{L}(t)$ is not necessarily a positive operator. For example, the Lyapunov operators generate positive evolutions on $\mathcal{S}_n$ but they are not positive operators.

We give some more examples of non-positive operators which generate positive evolutions.

Example 3.10 a) Let $\mathcal{L} : \mathcal{I} \to \mathcal{B}(\mathbb{R}^n)$ be a continuous function. Let $A(t) = \{a_{ij}(t)\}$ be its corresponding matrix with respect to the canonical basis on $\mathbb{R}^n$. Assume that $a_{ij}(t) \geq 0$ for $i \neq j, t \in \mathcal{I}$. Then $\mathcal{L}(\cdot)$ generates a positive evolution on the ordered space $(\mathbb{R}^n, \mathbb{R}^n_+)$.

b) Let $\mathcal{L}_{A\Pi}(t) : \mathcal{S}_n \to \mathcal{S}_n$ defined by

$$\mathcal{L}_{A\Pi}(t)X = A(t)X + XA^T(t) + \Pi(t,X) \tag{3.10}$$

for all $X \in \mathcal{S}_n$ where $A : \mathcal{I} \to \mathbb{R}^{n\times n}$ and $t \mapsto \Pi(t,\cdot) : \mathcal{I} \to \mathcal{B}(\mathcal{S}_n)$ are continuous functions.

If for each $t \in \mathcal{I}, \Pi(t,\cdot) \geq 0$ then $\mathcal{L}_{A\Pi}$ generates a positive evolution on the ordered space $(\mathcal{S}_n, \mathcal{S}_n^+)$.

c) Let $A : \mathcal{I} \to \mathbb{R}^{m\times m}, B : \mathcal{I} \to \mathbb{R}^{n\times n}, t \mapsto \Pi(t,\cdot) : \mathcal{I} \to \mathcal{B}(\mathbb{R}^{m\times n})$ be continuous functions.

Assume that for all $t \in \mathcal{I}$ the elements of matrices $A(t), B(t)$ are such that $a_{ij}(t) \geq 0$ for all $i \neq j, b_{lk} \geq 0$ for all $l \neq k$ and $\Pi(t,\cdot)$ is a positive operator on the ordered space $(\mathbb{R}^{m\times n}, \mathbb{R}^{m\times n}_+)$.

Under these conditions the operator $\mathcal{L}_{AB} + \Pi$ generates a positive evolution on $(\mathbb{R}^{m\times n}, \mathbb{R}^{m\times n}_+)$.

We recall now that if $\mathcal{L} : \mathcal{I} \to \mathcal{B}(\mathcal{X})$ is a bounded and continuous operator valued function and if $T(t,s)$ is the corresponding linear evolution operator on $\mathcal{X}$ defined by the linear differential equation

$$\frac{d}{dt}x(t) = \mathcal{L}(t)x(t)$$

then we have

$$e^{-\gamma(t-s)}|x|_2 \leq |T(t,s)x|_2 \leq e^{\gamma(t-s)}|x|_2 \tag{3.11}$$

and

$$e^{-\gamma(t-s)}|x|_2 \leq |T^*(t,s)x|_2 \leq e^{\gamma(t-s)}|x|_2 \tag{3.12}$$

for all $t \geq s, t, s \in \mathcal{I}, x \in \mathcal{X}, \gamma > 0$ being a constant independent of t, s, x.

In the case of operators defining positive evolutions one obtains additionally:

Theorem 3.11 *Let $\mathcal{L} : \mathcal{I} \to \mathcal{B}(\mathcal{X})$ be a bounded and continuous operator valued function and $T(t, s)$ be the linear evolution operator defined by the linear differential equation*

$$\frac{d}{dt}x(t) = \mathcal{L}(t)x(t).$$

Fix some arbitrary $\xi \in \mathrm{Int}\,\mathcal{X}^+$. Then the following assertions hold:

$$\beta_1 e^{-\gamma(t-t_0)}\xi \le T(t, t_0)\xi \le \beta_2 e^{\gamma(t-t_0)}\xi \qquad (3.13)$$

$$\beta_1 e^{-\gamma(t-t_0)}\xi \le T^*(t, t_0)\xi \le \beta_2 e^{\gamma(t-t_0)}\xi \qquad (3.14)$$

for all $t \ge t_0, t, t_0 \in \mathcal{I}, \beta_1, \beta_2, \gamma$ being positive constants independent of t, t_0.

Proof. It suffices to prove (3.13) since (3.14) follows by duality. Based on the property $\mathbf{P_3}$) we may write for each $x \in \mathcal{X}^+$:

$$\langle T(t, t_0)\xi, x\rangle = \langle \xi, T^*(t, t_0)x\rangle \ge c_0|T^*(t, t_0)x|_\xi.$$

From the equivalence of the norms $|\cdot|_\xi$ and $|\cdot|_2$ we deduce that there exists some $c > 0$ such that

$$\langle T(t, t_0)\xi, x\rangle \ge c_0 c|T^*(t, t_0)x|_2$$

for all $t \ge t_0,\ x \in \mathcal{X}^+,\ t, t_0 \in \mathcal{I}$. Using (3.12) we may write further

$$\langle T(t, t_0)\xi, x\rangle \ge c_0 c e^{-\gamma(t-t_0)}|x|_2.$$

Based on the Cauchy-Schwarz inequality we get

$$\langle T(t, t_0)\xi, x\rangle \ge \beta_1 e^{-\gamma(t-t_0)}\langle \xi, x\rangle \qquad (3.15)$$

for all $t \ge t_0,\ t, t_0 \in \mathcal{I},\ x \in \mathcal{X}^+$ where $\beta_1 = \frac{c_0 c}{|\xi|_2}$.

But (3.15) may be written as

$$\langle T(t, t_0)\xi - \beta_1 e^{-\gamma(t-t_0)}\xi, x\rangle \ge 0$$

for all $x \in \mathcal{X}^+$.

Since $\mathcal{X}^+$ is a self dual cone it follows that $T(t, t_0)\xi - \beta_1 e^{-\gamma(t-t_0)}\xi \in \mathcal{X}^+$, which is equivalent to

$$T(t, t_0)\xi \ge \beta_1 e^{-\gamma(t-t_0)}\xi;$$

hence the first part of (3.13) is valid.

Applying (2.2) for $x = \frac{1}{|T(t,t_0)\xi|_\xi}T(t, t_0)\xi$ we deduce that

$$T(t, t_0)\xi \le |T(t, t_0)\xi|_\xi \xi. \qquad (3.16)$$

Finally (3.16) together with (3.11) leads to $T(t, t_0)\xi \le \beta_2 e^{\gamma(t-t_0)}\xi$ where $\beta_2 = |\xi|_2$ and thus the second part of (3.13) is proved which completes our proof. □

4 Exponential stability

Throughout this section $\mathcal{I} \subset \mathbb{R}$ is a right unbounded interval. Here we investigate the exponential stability of the zero state equilibrium of the linear differential equation

$$\frac{d}{dt}x(t) = \mathcal{L}(t)x(t) \qquad (4.1)$$

in the case when $\mathcal{L}(\cdot)$ generates a positive evolution. We shall derive some necessary and sufficient conditions for exponential stability other then the ones based on Lyapunov functions.

We recall that the zero state equilibrium of (4.1) is exponentially stable or, equivalently, the operator $\mathcal{L}(\cdot)$ generates an exponentially stable evolution (E.S. evolution) if there exist $\beta \geq 1, \alpha > 0$ such that

$$\forall t, t_0 \in \mathcal{I}: \ \|T(t,t_0)\|_1 \leq \beta e^{-\alpha(t-t_0)} \ , \tag{4.2}$$

where $|\cdot|_1$ denotes an arbitrary norm.

In the following we fix some $\xi \in \operatorname{Int}\mathcal{X}^+$ and write $x(\cdot, t_0, \xi)$ for the solution of the initial value problem with $x(t_0) = \xi$.

Firstly from Proposition 2.3, Corollary 2.4 and Corollary 3.8 we obtain the following result specific to the case of operators which generate positive evolution:

Proposition 4.1 *Let $\mathcal{L}, \mathcal{L}_1 : \mathcal{I} \to \mathcal{B}(\mathcal{X})$ be a bounded and continuous operator valued functions generating positive evolutions on $\mathcal{X}$.*

(i) The following are equivalent:

a) $\mathcal{L}(\cdot)$ defines E.S. evolution.

b) There exist $\beta \geq 1, \alpha > 0$ such that $|x(t,t_0,\xi)| \leq \beta e^{-\alpha(t-t_0)}$ for all $t \geq t_0 \in \mathcal{I}$.

(ii) If $\mathcal{L}(t) \leq \mathcal{L}_1(t)$ for all $t \in \mathcal{I}$ and $\mathcal{L}_1(\cdot)$ generates an E.S. evolution, then $\mathcal{L}(\cdot)$ generates an E.S. evolution.

Theorem 4.2 *Let $\mathcal{L} : \mathbb{R}_+ \to \mathcal{B}(\mathcal{X})$ be a bounded and continuous operator valued function such that $\mathcal{L}(\cdot)$ generates a positive evolution on $\mathcal{X}$. Then the following are equivalent:*

(i) $\mathcal{L}(\cdot)$ defines an E.S. evolution.

(ii) $\int_{t_0}^t \|T(t,s)\|\, ds \leq \delta$ for all $t \geq t_0 \geq 0$ with $\delta > 0$ not depending upon t, t_0.

(iii) $\int_{t_0}^t T(t,s)\xi\, ds \leq \delta\xi$ for all $t \geq t_0 \geq 0, \delta > 0$ independent of t and t_0.

(iv) $\int_0^t T(t,s)\xi\, ds \leq \delta\xi$ for all $t \geq 0, \delta > 0$ independent of t.

(v) For arbitrary bounded and continuous functions $f : \mathbb{R}_+ \to \mathcal{X}$ the solution of the following initial value problem is bounded:

$$\frac{d}{dt}x(t) = \mathcal{L}(t)x(t) + f(t) \ , \quad x(0) = 0 \ . \tag{4.3}$$

Proof. The implication '(v)⇒(i)' is just Perron's theorem (see [23]).

It remains to prove the implications '(i)⇒(ii)⇒(iii)⇒(iv)⇒(v)'.

'(i)⇒(ii)' follows from (4.2) with $\delta = \frac{\beta}{\alpha}$.

From (2.2) we obtain $-\xi \leq \frac{1}{|T(t,s)\xi|_\xi} T(t,s)\xi \leq \xi$ which leads to

$$0 \leq T(t,s)\xi \leq \|T(t,s)\|_\xi \xi. \tag{4.4}$$

The implication '(ii)⇒(iii)' follows from (4.4).

'(iii)⇒(iv)' is obvious. We shall prove '(iv)⇒(v)'. Let $f : \mathbb{R}_+ \to \mathcal{X}$ be a bounded and continuous function. This means that there exist $\mu > 0$ such that $|f(t)|_\xi \leq \mu$.

Applying again (2.2) we obtain

$$-\xi \le \frac{1}{\mu} f(t) \le \xi,$$

hence $-\mu\xi \le f(t) \le \mu\xi$ for all $t \ge 0$.

Since $T(t,s) \ge 0$ we may write

$$-\mu T(t,s)\xi \le T(t,s)f(s) \le \mu T(t,s)\xi$$

for all $t \ge s \ge 0$. This leads to

$$-\mu \int_0^t T(t,s)\xi\, ds \le \int_0^t T(t,s)f(s)\, ds \le \mu \int_0^t T(t,s)\xi$$

for all $t \ge 0$.

Using (2.1) we have

$$|\int_0^t T(t,s)f(s)\, ds|_\xi \le \mu |\int_0^t T(t,s)\xi\, ds|_\xi.$$

If (iv) holds we conclude that $|\int_0^t T(t,s)f(s)\,ds|_\xi \le \mu\delta$, which shows that the solution with zero initial value of (4.3) is bounded. This completes the proof. □

Remark 4.3 From '(i)$\Longleftrightarrow$(iv)' in the preceding theorem it follows that in the case of linear differential equations defined by operators which generate positive evolutions exponential stability is equivalent to the boundedness of the solution with zero initial value of the affine equation

$$\frac{d}{dt}x(t) = \mathcal{L}(t)x(t) + \xi.$$

Let us now introduce the concept of uniform positivity.

Definition 4.4 *We say that the function $f : \mathcal{I} \to \mathcal{X}$ is* uniformly positive *if there exist $c_f > 0$ such that $f(t) \ge c_f\xi$ for all $t \in \mathcal{I}$. In this case we shall write $f(t) \gg 0, t \in \mathcal{I}$. Also we shall write $f(t) \ll 0$ if $-f(t) \gg 0, t \in \mathcal{I}$.*

The next result provides some necessary and sufficient conditions for exponential stability expressed in terms of the existence of bounded solutions for some backward affine equations.

Theorem 4.5 *Let $\mathcal{L} : \mathcal{I} \to \mathcal{B}(\mathcal{X})$ be a bounded and continuous operator valued function such that $\mathcal{L}(\cdot)$ generates a positive evolution on $\mathcal{X}$. Then the following are equivalent:*

(i) $\mathcal{L}(\cdot)$ generates an E.S. evolution.

(ii) There exist $\beta \ge 1, \alpha > 0$ such that

$$\|T^*(t,t_0)\|_\xi \le \beta e^{-\alpha(t-t_0)} \quad \textit{for all } t \ge t_0,\ t,t_0 \in \mathcal{I}.$$

(iii) There exist $\delta > 0$ independent of t such that

$$\int_t^\infty T^*(s,t)\xi\, ds \le \delta\xi \quad \textit{for all } t \in \mathcal{I}.$$

(iv) The backward affine equation

$$\frac{d}{dt}x(t) + \mathcal{L}^*(t)x(t) + \xi = 0 \tag{4.5}$$

has a bounded and uniformly positive solution $\tilde{x} : \mathcal{I} \to \mathrm{Int}\,\mathcal{X}^+$.

(v) For each bounded and continuous function $f : \mathcal{I} \to \mathrm{Int}\,\mathcal{X}^+, f(t) \gg 0, t \in \mathcal{I}$, *the backward affine differential equation*

$$\frac{d}{dt}x(t) + \mathcal{L}^*(t)x(t) + f(t) = 0 \tag{4.6}$$

has a bounded and uniformly positive solution $x : \mathcal{I} \to \mathrm{Int}\,\mathcal{X}^+$.

(vi) There exists a bounded and continuous function $f : \mathcal{I} \to \mathrm{Int}\,\mathcal{X}^+,\ f(t) \gg 0,\ t \in \mathcal{I}$ *such that the backward affine differential equation* (4.6) *has a bounded solution* $x : \mathcal{I} \to \mathcal{X}^+$.

(vii) There exists a C^1 *function bounded with bounded derivative* $y : \mathcal{I} \to \mathrm{Int}\,\mathcal{X}^+,\ y(t) \gg 0,\ t \in \mathcal{I}$, *such that for all* $t \in \mathcal{I}$:

$$\frac{d}{dt}y(t) + \mathcal{L}^*(t)y(t) \ll 0 \tag{4.7}$$

The proof of this theorem can be adapted from the proof of Theorem 2.11 in [13] and is omitted for shortness.

Example 4.6 Consider $(\mathcal{X}, \mathcal{X}^+) = (\mathbb{R}^2, \mathbb{R}^{2+})$. Let $\mathcal{L} : \mathbb{R} \to \mathcal{B}(\mathbb{R}^2)$ have the matrix with respect to the canonical basis on $\mathbb{R}^2$ given by

$$\mathcal{L}(t) = \begin{pmatrix} \frac{-2}{1+\sin^2 t} & \sin^2 t \\ \frac{1}{4+\cos^2 t} & -3+\cos^2 t \end{pmatrix}. \tag{4.8}$$

Based on Example 3.10 (a) it follows that $\mathcal{L}(\cdot)$ generates a positive evolution on $\mathbb{R}^2$.

It is easy to see that the corresponding inequality (4.7) from Theorem 4.5 is satisfied by $y(t) = (1,1)^T$ for all $t \in \mathbb{R}$. Therefore from the implication '(vii)⇒(i)' in the above Theorem, we deduce that the operator $\mathcal{L}$ defined by (4.8) generates an E.S. evolution.

The same conclusion may be obtained applying Proposition 4.1 (ii). So, if $\tilde{\mathcal{L}} = \begin{pmatrix} -1 & 1 \\ \frac{1}{4} & -2 \end{pmatrix}$ we have $\mathcal{L}(t) \le \tilde{\mathcal{L}}$ for all $t \in \mathbb{R}$. On the other hand the eigenvalues of $\tilde{\mathcal{L}}$ are $\lambda_1 = \frac{-3-\sqrt{2}}{2}$ and $\lambda_2 = \frac{-3+\sqrt{2}}{2}$ hence $\mathcal{L}(t)$ generates an E.S. evolution since $\tilde{\mathcal{L}}$ generates an E.S. evolution.

The next result provides more information concerning the bounded solutions of the backward affine differential equation of type (4.6).

Theorem 4.7 *Let* $\mathcal{L} : \mathcal{I} \to \mathcal{B}(\mathcal{X})$ *be a bounded and continuous operator valued function such that* $\mathcal{L}(\cdot)$ *generates an E.S. evolution. Then the following assertions hold:*

(i) For an arbitrary bounded and continuous function $f : \mathcal{I} \to \mathcal{X}$ *the backward affine differential equation*

$$\frac{d}{dt}x(t) + \mathcal{L}^*(t)x(t) + f(t) = 0 \tag{4.9}$$

has a unique bounded solution $\tilde{x} : \mathcal{I} \to \mathcal{X}$ *which is given by*

$$\tilde{x}(t) = \int_t^\infty T^*(s,t)f(s)\,ds\,, \qquad t \in \mathcal{I}. \tag{4.10}$$

(ii) *If there exists $\theta > 0$ such that $\mathcal{L}(t+\theta) = \mathcal{L}(t)$ and $f(t+\theta) = f(t)$ for all $t \in \mathcal{I}$ then the unique bounded solution of* (4.9) *is a periodic function with period θ.*

(iii) *If $\mathcal{L}(t) = \mathcal{L}$ and $f(t) = f$, $t \in \mathcal{I}$ then the unique bounded solution of* (4.9) *is constant and it is given by $\tilde{x} = -(\mathcal{L}^*)^{-1} f$.*

(iv) *If $\mathcal{L}(\cdot)$ generates a positive evolution on $\mathcal{X}$, then for arbitrary bounded and continuous function $f : \mathcal{I} \to \mathcal{X}^+$ the unique bounded solution of* (4.9) *satisfies $\tilde{x}(t) \geq 0$ for all $t \in \mathcal{I}$. Moreover, if $f(t) \gg 0$, $t \in \mathcal{I}$ then $\tilde{x}(t) \gg 0$, $t \in \mathcal{I}$.*

Again, the proof of this theorem can be adapted from the proof of Theorem 2.13 in [13].

Remark 4.8 From the representation formula

$$x(t) = T(t,t_0)x(t_0) + \int_{t_0}^{t} T(t,s)f(s)\,ds$$

of the solutions of the forward affine equation

$$\frac{d}{dt}x(t) = \mathcal{L}(t)x(t) + f(t) \tag{4.11}$$

with $f : \mathcal{I} \to \mathcal{X}$ bounded and continuous function, it follows that if $\mathcal{L}(\cdot)$ generates an E.S. evolution then for each $(t_0, \infty) \subset \mathcal{I}$ all solutions of (4.11) with given initial values $x(t_0)$ are bounded on (t_0, ∞).

On the other hand from Theorem 4.7 (i) we see that for each subinterval $(t_0, \infty) \subset \mathcal{I}$ the backward affine equation (4.9) has a unique bounded solution on (t_0, ∞) namely the solution given by (4.10).

If $\mathcal{I} = \mathbb{R}$ then the equation (4.11) has also a unique bounded solution on $\mathbb{R}$.

Theorem 4.9 *Let $\mathcal{L} : \mathbb{R} \to \mathcal{B}(\mathcal{X})$ be a bounded and continuous operator valued function such that $\mathcal{L}(\cdot)$ generates an E.S. evolution. Then the following assertions hold:*

(i) *For arbitrary bounded and continuous functions $f : \mathbb{R} \to \mathcal{X}$ the forward affine equation* (4.11) *has a unique bounded on $\mathbb{R}$ solution given by:*

$$\hat{x}(t) = \int_{-\infty}^{t} T(t,s)f(s)\,ds \qquad \text{for } t \in \mathbb{R}\,. \tag{4.12}$$

(ii) *If there exists $\theta > 0$ such that $\mathcal{L}(t+\theta) = \mathcal{L}(t)$ and $f(t+\theta) = f(t)$, $t \in \mathbb{R}$ then the unique bounded solution of* (4.11) *is a periodic function with period θ.*

(iii) *If $\mathcal{L}(t) = \mathcal{L}, f(t) = f, t \in \mathbb{R}$ then the unique bounded solution of* (4.11) *is constant and it is given by $\hat{x} = -\mathcal{L}^{-1} f$.*

(iv) *If $\mathcal{L}(\cdot)$ generates a positive evolution on $\mathcal{X}$ and $f : \mathbb{R} \to \mathcal{X}^+$ is bounded and continuous function then the unique bounded solution of* (4.11) *satisfies $\hat{x}(t) \geq 0$, $t \in \mathbb{R}$. Moreover if $f(t) \gg 0$, $t \in \mathbf{R}$, then $\hat{x}(t) \gg 0$, $t \in \mathbf{R}$.*

The proof is standard and it is omitted.

If $\mathcal{L} : \mathbb{R} \to \mathcal{B}(\mathcal{X})$ we define $\mathcal{L}^\sharp : \mathbb{R} \to \mathcal{B}(\mathcal{X})$ by $\mathcal{L}^\sharp(t) = \mathcal{L}^*(-t)$. Let $T^\sharp(t,s)$ be the linear evolution operator defined by the equation $\dot{x}(t) = \mathcal{L}^\sharp(t)x(t)$. It can be verified that $T^\sharp(t,s) = T^*(-s,-t)$.

Lemma 4.10 *The following assertions hold:*

(i) *$\mathcal{L}(\cdot)$ defines a positive evolution if and only if $\mathcal{L}^\sharp(\cdot)$ defines a positive evolution.*

(ii) $\mathcal{L}(\cdot)$ *defines a E.S. evolution if and only if* $\mathcal{L}^\sharp(\cdot)$ *defines a E.S. evolution.*

Proof. (i) If $\mathcal{L}(t)$ is resolvent positive for all t, then so is $\mathcal{L}^\sharp(t)$.
(ii) follows from $T^\sharp(t,s) = T^*(-s,-t)$. □

Combining Lemma 4.10 and Theorem 4.5 we obtain the following result which provides criteria for an evolution to be exponentially stable in terms of some forward differential equations.

Theorem 4.11 *Let* $\mathcal{L} : \mathbb{R} \to \mathcal{B}(\mathcal{X})$ *be bounded and continuous operator valued function. Assume that* $\mathcal{L}(\cdot)$ *generates a positive evolution. Then the following statements are equivalent:*

(i) $\mathcal{L}(\cdot)$ *generates a E.S. evolution.*

(ii) There exists $\delta > 0$ *not depending upon* t*, such that* $\int_{-\infty}^t T(t,s)\xi\, ds \le \delta\xi$ *for all* $t \in \mathbb{R}$*.*

(iii) The forward affine differential equation

$$\frac{d}{dt}x(t) = \mathcal{L}(t)x(t) + \xi$$

has a bounded and uniformly positive solution.

(iv) For all bounded and continuous functions $f : \mathbb{R} \to \operatorname{Int}\mathcal{X}^+$*;* $f(t) \gg 0,\ t \in \mathbb{R}$ *the corresponding forward affine differential equation*

$$\frac{d}{dt}x(t) = \mathcal{L}(t)x(t) + f(t) \tag{4.13}$$

has a solution, which is bounded on $\mathbb{R}$ *and uniformly positive.*

(v) There exists a bounded and continuous function $f : \mathbb{R} \to \operatorname{Int}\mathcal{X}^+,\ f(t) \gg 0, t \in \mathbb{R}$ *such that the corresponding affine differential equation* (4.13) *has a bounded solution* $\tilde{x} : \mathbb{R} \to \mathcal{X}^+$*.*

(vi) There exists a C^1*-function* $y : \mathbb{R} \to \operatorname{Int}\mathcal{X}^+$ *bounded with bounded derivative* $y(t) \gg 0,\ t \in \mathbb{R}$ *such that*

$$\frac{d}{dt}y(t) - \mathcal{L}(t)y(t) \gg 0,\ t \in \mathbb{R}.$$

In the time invariant case part of the results in Theorem 4.5 and Theorem 4.11 recover criteria for exponential stability known for resolvent positive operators (see [12]).

5 Uniform observability and exponential stability

In this section we introduce the concept of *uniform observability* which extends the well-known concept of uniform observability from time-varying linear systems (see [2, 26, 28]) to our abstract framework.

Also we will show how the uniform observability can be used to derive some criteria for exponential stability. Thus we will obtain a version of Barbasin-Krasovski criteria for the case of linear differential equations generating positive evolution (see [7]).

Definition 5.1 *Let* $\mathcal{L} : \mathcal{I} \to \mathcal{B}(\mathcal{X})$ *be a bounded and continuous operator valued function such that* $\mathcal{L}(\cdot)$ *generates a positive evolution.*
Let $g : \mathcal{I} \to \mathcal{X}^+$ *be a bounded and continuous function. We say that the pair* $(g, \mathcal{L})$ *is* uniformly observable *if there exist* $\tau > 0, \gamma > 0$ *such that*

$$\int_t^{t+\tau} T^*(s,t)g(s) \ge \gamma\xi \quad \text{for all } t \in \mathcal{I}. \tag{5.1}$$

Remark 5.2 a) In the case when $\mathcal{X} = \mathcal{S}_n,\ \mathcal{L}(t) = \mathcal{L}_A(t),\ g(t) = C^T(t)C(t)$ it is easy to see that (5.1) becomes:

$$\int_t^{t+\tau} \Phi_A^T(s,t)C^T(s)C(s)\Phi_A(s,t)\,ds \ge \gamma I_n \tag{5.2}$$

for all $t \in \mathcal{I}$, which is just the condition which appears in definition of uniform observability in the case of continuous time, time-varying linear systems (see [2]).

b) Applying Theorem 3.11 we obtain that if $g(t) \gg 0,\ t \in \mathcal{I}$ then $(g, \mathcal{L})$ is uniformly observable.

If $\mathcal{L}(t) = \mathcal{L},\ g(t) = g,\ t \in \mathcal{I}$, then (5.1) takes the form

$$\gamma\xi \ \le\ \int_t^{t+\tau} e^{\mathcal{L}^*(s-t)} g\,ds \ =\ \int_0^{\tau} e^{\mathcal{L}^* s} g\,ds \ge \gamma\xi\ . \tag{5.3}$$

This shows that in the time invariant case (5.1) does not depend upon t. Hence, in this case we shall say that a pair $(g, \mathcal{L})$ is *observable* instead of *uniformly observable* if (5.1) or (5.3) is fulfilled.

Proposition 5.3 *If $\mathcal{L} : \mathcal{X} \to \mathcal{X}$ is a linear resolvent positive operator and $g \in \mathcal{X}^+$ the following are equivalent:*

(i) $(g, \mathcal{L})$ is observable.

(ii) There exist $\tau > 0, \gamma > 0$ such that (5.3) is satisfied.

(iii) There exist $\tau > 0, \gamma > 0$ such that $x_0(\tau) > \gamma\xi$ where $x_0(\cdot)$ is the solution of the problem with initial value

$$\frac{d}{dt}x(t) = \mathcal{L}^* x(t) + g\ , \quad x_0(0) = 0\ . \tag{5.4}$$

The next result will be needed in the proof of of Theorem 5.5, but it may be of interest in itself.

Lemma 5.4 *Let $\mathcal{L} : \mathcal{I} \to \mathcal{B}(\mathcal{X})$ be a bounded and continuous operator valued function such that $\mathcal{L}(\cdot)$ generates a positive evolution. Let $h : \mathcal{I} \to \mathcal{X}^+$ be a bounded and continuous function such that $h(t) \gg 0,\ t \in \mathcal{I}$, where $\mathcal{I} \subset \mathbb{R}$ is a right unbounded interval. If there exist $\tau > 0,\ q \in (0,1)$ such that*

$$T^*(t+\tau, t)h(t+\tau) \le q h(t) \tag{5.5}$$

for all $t \in \mathcal{I}$ then $\mathcal{L}(\cdot)$ defines an E.S. evolution.

Proof. Using the identity from Lemma 3.1 (ii) one obtains inductively from (5.4) that

$$T^*(s+m\tau, s)h(s+m\tau) \le q^m h(s) \tag{5.6}$$

for all $s \in \mathcal{I}$ and for all integer $m \ge 1$.

Since $\mu_1\xi \le h(s) \le \mu_2\xi$ for all $s \in \mathcal{I}$ where $\mu_k > 0,\ k = 1, 2$, we deduce from (5.5) that

$$0 \le T^*(s+m\tau, s)\xi \le \beta_1 q^m \xi$$

for all $s \in \mathcal{I},\ m \ge 1$ where $\beta_1 = \frac{\mu_2}{\mu_1}$.

Invoking (2.1) together with Proposition 2.3 (ii) we conclude that

$$\|T^*(s+m\tau, s)\|_\xi \le \beta_1 q^m \tag{5.7}$$

for all $s \in \mathcal{I}$ and $m \ge 1$.

On the other hand the boundedness of $\|\mathcal{L}(t)\|_\xi$ together with (3.12) leads to

$$\|T^*(t,s)\|_\xi \leq \beta_2 e^{\mu_3(t-s)} \tag{5.8}$$

for all $t, s \in \mathcal{I}$ with $\beta_3 > 0, \mu_3 > 0$.

If $t \geq s,\ t, s \in \mathcal{I}$ we have $t - s = m\tau + \tau_0$ with $0 \leq \tau_0 < \tau$. Using again Lemma 3.1 (ii) together with (5.6), (5.7) we obtain

$$\|T^*(t,s)\|_\xi \leq \beta_1\beta_2 e^{\mu_3\tau} q^m. \tag{5.9}$$

Since $m \geq \frac{(t-s)}{\tau}$ we obtain from (5.8) that

$$\|T^*(t,s)\|_\xi \leq \beta e^{-\alpha(t-s)} \tag{5.10}$$

for all $t \geq s,\ t, s \in \mathcal{I}$ where $\beta = \beta_1\beta_2 e^{\mu_3\tau}$ and $\alpha = -\frac{1}{\tau}\ln q$.

The conclusion follows now from the implication '(ii)⇒(i)' in Theorem 4.5. □

The main result of this section is:

Theorem 5.5 *Let $\mathcal{L} : \mathcal{I} \to \mathcal{B}(\mathcal{X})$ be a bounded and continuous operator valued function such that $\mathcal{L}(\cdot)$ generates a positive evolution. Let $g : \mathcal{I} \to \mathcal{X}^+$ be a bounded and continuous function. Assume that:*

a) $(g, \mathcal{L})$ is uniformly observable.

b) The backward affine equation

$$\frac{d}{dt}x(t) + \mathcal{L}^*(t)x(t) + g(t) = 0 \tag{5.11}$$

has a bounded solution $\tilde{x} : \mathcal{I} \to \mathcal{X}^+$.

Under these assumptions $\mathcal{L}(\cdot)$ generates an E.S. evolution.

Proof. For each $t \leq \tau,\ t, \tau \in \mathcal{I}$ we may write $\tilde{x}(t) = T^*(\tau, t)\tilde{x}(\tau) + \int_t^\tau T^*(s,t)g(s)\,ds$.
Since $T^*(s,t) \geq 0$ for all $s \geq t,\ s, t \in \mathcal{I}$ and $\tilde{x} \geq 0$ is a bounded function we deduce:

$$0 \leq \int_t^\tau T^*(s,t)g(s)\,ds \leq \tilde{x}(t) \leq \lambda\xi \quad \text{for all } t \leq \tau,\ t, \tau \in \mathcal{I}. \tag{5.12}$$

Since $\mathcal{X}^+$ is closed we conclude from (5.11) that $h(t)$ is well defined by

$$h(t) = \int_t^\infty T^*(s,t)g(s)\,ds,\ t \in \mathcal{I}.$$

For $t \in \mathcal{I}$ it follows from (5.11) that $h(t) \leq \lambda\xi$ while assumption (a) implies

$$h(t) \geq \int_t^{t+\tau} T^*(s,t)g(s)\,ds \geq \gamma\xi\ . \tag{5.13}$$

Applying Lemma 3.1 (ii) we may write

$$T^*(t+\tau, t)h(t+\tau) = \int_{t+\tau}^\infty T^*(s,t)g(s)\,ds = h(t) - \int_t^{t+\tau} T^*(s,t)g(s)\,ds \leq qh(t)$$

with $q = 1 - \frac{\gamma}{\lambda} \in (0,1)$. Exponential stability follows now from Lemma 5.4. □

Remark 5.6 a) In the particular case $\mathcal{X} = \mathcal{S}_n \oplus \mathcal{S}_n \oplus \ldots \oplus \mathcal{S}_n$, $\mathcal{X}^+ = \mathcal{S}_n^+ \oplus \mathcal{S}_n^+ \oplus \ldots \oplus \mathcal{S}_n^+$ the result of the above theorem was proved in [15]. The discrete time version of the result in Theorem 5.5 may be found in [16].

b) The result proved in Theorem 5.5 may be viewed as an alternative of the implication '(vi)⇒(i)' of Theorem 4.5 in the case when the free term of the affine equation (4.6) is not uniformly positive. The absence of the uniform positivity is compensated here by the uniform observability.

c) From Theorem 5.5 and Theorem 4.7 it follows that under the assumption of uniform observability if the equation (5.10) has a bounded solution $\tilde{x}(t) \in \mathcal{X}^+$ then it is the unique bounded solution of equation (5.10) and additionally $\tilde{x}(t) \gg 0,\ t \in \mathcal{I}$.

6 On the robustness of exponential stability

In this section we want to investigate the problem of the preservation of exponential stability of equation (4.1) in the case when the operator $\mathcal{L}(t)$ is replaced by a perturbed one $\hat{\mathcal{L}}(t) = \mathcal{L}(t) + P(t)$.

So, combining the result of the above Theorem 3.4 with Theorem 2.11 from [12] (cf. also [29]), one obtains that if $\mathcal{L} \in \mathcal{B}(\mathcal{X})$ is a linear operator which defines a positive evolution and $P \in \mathcal{B}(\mathcal{X}),\ P \geq 0$, then the operator $\mathcal{L} + P$ defines an E.S. evolution if and only if the operator $\mathcal{L}$ has eigenvalues in the open left half plane, i.e. $\operatorname{Re}\lambda < 0$, and $\rho(\mathcal{L}^{-1}P) < 1$, where $\rho(\cdot)$ is the spectral radius.

In this section we shall prove a result which extends the above mentioned result to the periodic case.

Let us assume that there exists $\theta > 0$ such that $\mathcal{L}(t+\theta) = \mathcal{L}(t)$, $P(t+\theta) = P(t)$ for all $t \in \mathcal{I}$.

If $\mathcal{L}(t)$ generates an E.S. evolution then $\lambda = 1$ is not an eigenvalue of the operator $T(\theta, 0)$. Hence we can define $G : [0,\theta] \times [0,\theta] \to \mathcal{B}(\mathcal{X})$ by

$$G(t,s) = T(t,0)(I_{\mathcal{X}} - T(\theta,0))^{-1} T(\theta,s) + \chi_t(s) T(t,s), \tag{6.1}$$

where $\chi_t(s)$ is the indicator function of the interval $[0,t]$, that is $\chi_t(s) = \begin{cases} 1, & s \in [0,t] \\ 0, & s \notin [0,t] \end{cases}$.

Note that $G(0,s) = G(\theta,s)$ for all $s \in [0,\theta]$.

Let $\mathbf{C}_0[0,\theta]$ denote the space of continuous functions $f : [0,\theta] \to \mathcal{X}$ with $f(0) = f(\theta)$. This is an ordered Banach space with the norm $|\cdot|_\infty$, defined by $|f|_\infty = \max_{t\in[0,\theta]} |f(t)|$, and the order relation induced by the closed, solid, normal and pointed convex cone $\mathbf{C}_0^+[0,\theta] = \{f \in \mathbf{C}_0[0,\theta] \mid f(t) \in \mathcal{X}^+, t \in [0,\theta]\}$. We define the linear bounded operator $\Pi : \mathbf{C}_0[0,\theta] \to \mathbf{C}_0[0,\theta]$, by $f \mapsto \Pi f = g$ where

$$(\Pi f)(t) = g(t) = \int_0^t G(t,s) P(s) f(s)\, ds. \tag{6.2}$$

Lemma 6.1 *If $\mathcal{L}(\cdot), P(\cdot)$ are periodic functions with period θ and $\mathcal{L}(\cdot)$ generates a positive and E.S. evolution and $P(t) \geq 0$, then the operator Π defined in (6.2) is positive.*

Proof. Let $f \in \mathbf{C}_0[0,\theta]$ and $\tilde{x} = \Pi f$ as defined in (6.2). Let $\hat{x} : \mathbb{R} \to \mathcal{X}, \hat{f} : \mathbb{R} \to \mathcal{X}$ be periodic functions such that $\hat{x}|_{[0,\theta]} = \tilde{x}$ and $\hat{f}|_{[0,\theta]} = f$. The periodicity of $P(\cdot, s)$ implies that $\hat{x}(t)$ is a periodic solution with period θ of the affine equation

$$\frac{d}{dt}x(t) = \mathcal{L}(t)x(t) + g(t) \quad \text{with} \quad g(t) = P(t)\hat{f}(t),\ t \in \mathbb{R}. \tag{6.3}$$

Since $P(t) \geq 0,\ f(t) \geq 0$ it follows that $g(t) \geq 0,\ t \in \mathbb{R}$. Theorem 4.9 now yields $\hat{x}(t) \geq 0$. □

Now we are ready to state the main result of this section.

Theorem 6.2 *Let $\mathcal{L} : \mathbb{R} \to \mathcal{B}(\mathcal{X}), P : \mathbb{R} \to \mathcal{B}(\mathcal{X})$ be bounded and continuous periodic functions such that $\mathcal{L}(\cdot)$ generates a positive evolution and let $P(t) \geq 0,\ t \in \mathbb{R}$. Then the following statements are equivalent:*

(i) The sum operator $\hat{\mathcal{L}} = \mathcal{L} + P$ defines an E.S. evolution on $\mathcal{X}$.

(ii) The operator $\mathcal{L}(\cdot)$ defines an E.S. evolution on $\mathcal{X}$ and $\rho(\Pi) < 1$, where Π is the linear operator defined by (6.2).

Proof. '(i)⇒ (ii)': If (i) holds then based on the implication '(i)⇒ (iii)' in Theorem 4.11 one deduces that – for an arbitrary $\xi \in \operatorname{Int} \mathcal{X}^+$ – the forward equation

$$\frac{d}{dt}x(t) = [\mathcal{L}(t) + P(t)]x(t) + \xi \tag{6.4}$$

has a bounded and uniformly positive solution $\tilde{x} : \mathbb{R} \to \operatorname{Int} \mathcal{X}_+$.

It can be seen that $\tilde{x}(\cdot)$ satisfies (4.13) and thus one concludes that $\mathcal{L}(\cdot)$ defines an E.S. evolution.

Theorem 4.9 (ii) yields that $\tilde{x}(\cdot)$ is a periodic function with period θ. Hence from (6.4) we have that

$$\tilde{x}(t) = \int_0^t G(t,s)P(s)\tilde{x}(s)\, ds + \int_0^t G(t,s)\xi\, ds \qquad t \in [0,\theta]. \tag{6.5}$$

If $\hat{x}(t) = \tilde{x}|_{[0,\theta]}$ we obtain from (6.5) that $\hat{x}$ solves the equation

$$(-\mathcal{I}_{\mathbf{C}_0} + \Pi)\hat{x} + \tilde{g} = 0, \tag{6.6}$$

where $\mathcal{I}_{\mathbf{C}_0}$ is the identity operator on $\mathbf{C}_0[0,\theta]$, and $\tilde{g}(t) = \int_0^t G(t,s)\xi\, ds$ is the bounded solution on $\mathbb{R}$ of the affine equation

$$\frac{d}{dt}x(t) = \mathcal{L}(t)x(t) + \xi. \tag{6.7}$$

Applying Theorem 4.9 (iv) we deduce that $\tilde{g}(t) \gg 0,\ t \in \mathbb{R}$. That means that $\tilde{g} \in \operatorname{Int} \mathbf{C}_0^+[0,\theta]$. Using the implication '(v)⇒(vi)' of Theorem 2.11 in [12] for $R = -I_{\mathbf{C}_0}$ and $P = \Pi$ one obtains that $\rho(\Pi) < 1$ and so (ii) is valid.

'(ii)⇒(i)': Let $\xi \in \operatorname{Int} \mathcal{X}^+$. If (ii) holds then $\tilde{g}$ is well defined by

$$\tilde{g}(t) = \int_0^t G(t,s)\xi\, ds\ , \quad t \in [0,\theta].$$

Let $\hat{g} : \mathbb{R} \to \mathcal{X}$ be a periodic function with period θ such that $\hat{g} = \tilde{g},\ t \in [0,\theta]$. Then $\hat{g}(\cdot)$ is the unique bounded on $\mathbb{R}$ solution of equation (6.7). By Theorem 4.9 (iv) we deduce that $\hat{g}(t) \gg 0,\ t \in \mathbb{R}$. Hence $\tilde{g} \in \operatorname{Int} \mathbf{C}_0^+[0,\theta]$. Applying Theorem 2.11 from [12] we conclude that the equation

$$[-\mathcal{I}_{\mathbf{C}_0} + \Pi]x + \tilde{g} = 0$$

has a solution $\hat{x} \in \mathbf{C}_0^+[0,\theta]$. Let $\tilde{x} : \mathbb{R} \to \mathcal{X}$ be the periodic function with period θ such that $\tilde{x}(t) = \hat{x}(t)$ for all $t \in [0,\theta]$. It follows that $\tilde{x}(\cdot) \geq 0$ is a bounded solution of the equation

$$\frac{d}{dt}x(t) = [\mathcal{L}(t) + P(t)]x(t) + \xi.$$

The implication '(v)⇒(i)' of Theorem 4.11 yields that $\mathcal{L} + P$ generates an E.S. evolution. □

As we can see from the proof, the result of Theorem 6.2 is mainly based on the result in Theorem 2.11 in [12] and the above Theorem 4.11. Combining Theorem 2.11 in [12] with Theorem 4.5 (from above) we may obtain:

Theorem 6.3 *Under the assumptions of Theorem 6.2 the following statements are equivalent:*

(i) The operator $\mathcal{L}+P$ defines an E.S. evolution.

(ii) The operator $\mathcal{L}$ defines an E.S. evolution and $\rho(\tilde{\Pi})<1$ where $\tilde{\Pi}:\mathbf{C}_0[0,\theta])\to\mathbf{C}_0[0,\theta]$ is defined by $y=\tilde{\Pi}(f)$, where

$$y(t)=\int_0^t G^*(s,t)P^*(s)f(s)\,ds\ .$$

Similar results as in Theorems 6.2 and 6.3 may be obtained also in the case when $\mathcal{L}(\cdot),P(\cdot)$ are bounded and continuous functions without any periodicity assumption. In this case we define

$$\begin{aligned}[\Pi(f)](t) &= \int_{-\infty}^t T(t,s)P(s)f(s)\,ds, \quad t\in\mathbb{R}\ ,\\ [\tilde{\Pi}(f)](t) &= \int_t^{\infty} T^*(s,t)P^*(s)f(s)\,ds, \quad t\in\mathbb{R}\ .\end{aligned}$$

7 Nash differential games for time-varying positive systems

In this section, as well in the next one, we illustrate the applicability of the results derived in the previous sections to obtain time-varying counterparts of some results in [5]. More precisely we shall prove the existence of a stabilizing solution of a system of coupled nonlinear matrix differential equations arising in connection with the computation of the deterministic feedback Nash equilibrium strategy in the case of linear quadratic (two player) Nash game associated to a time-varying positive linear system. For further details on this topic, we refer to [6] and [17].

7.1 Time-varying positive linear systems

Consider the controlled system described by

$$\frac{d}{dt}x(t)=A(t)x(t)+B(t)u(t) \tag{7.1}$$

where $A:\mathcal{I}\to\mathbb{R}^{n\times n}, B:\mathcal{I}\to\mathbb{R}^{n\times m}$ are bounded and continuous matrix valued functions, $\mathcal{I}\subseteq\mathbb{R}$ is a right unbounded interval.

The class of admissible controls consist of the set of piecewise continuous functions $u:\mathcal{I}\to\mathbb{R}^m$.

We define positive systems with respect to the componentwise ordering " $\succeq$ " of $\mathbb{R}^n$ (see Ex. 2.5).

Definition 7.1 *We say that the system* (7.1) *is a* positive system, *if for all $t_0\in\mathcal{I}$, for all initial states $x_0\in\mathbb{R}^n_+$ and for all admissible controls u, such that $u(t)\in\mathbb{R}^m_+$ for $t\geq t_0$ we have $x_u(t,t_0,x_0)\in\mathbb{R}^n_+$ for all $t\geq t_0$, where $x_u(\cdot,t_0,x_0)$ is the solution of* (7.1) *corresponding to the input u and taking the initial value $x_u(t_0,t_0,x_0)=x_0$.*

We have the following criterion:

Theorem 7.2 *System* (7.1) *is a positive system if and only if the following two conditions hold:*

a) For all $t\in\mathcal{I}$, $A(t)$ is a Metzler matrix.

b) All elements of the matrix $B(t)$ are nonnegative.

Proof. *Sufficiency*: If a) is fulfilled then by Example 3.10 a) one obtains that $\Phi_A(t,s) \succeq 0$, where $\Phi_A(t,s)$ is the fundamental matrix solution of the differential equation $\dot{x}(t) = A(t)x(t)$.
The conclusion follows from the representation formula

$$x_u(t,t_0,x_0) = \Phi_A(t,t_0)x_0 + \int_{t_0}^{t} \Phi_A(t,s)B(s)u(s)\,ds. \tag{7.2}$$

Necessity: For $u(t) = 0, t \in \mathcal{I}$, the condition $x_u(t,t_0,x_0) \in \mathbf{R}_+^n, t \geq t_0, t, t_0 \in \mathcal{I}$ reduces to the condition $\Phi_A(t,t_0)x_0 \geq 0$, for all $x_0 \in \mathbb{R}_+^n$, $t \geq t_0, t, t_0 \in \mathcal{I}$. Hence $A(\cdot)$ defines a positive evolution on $(\mathbb{R}^n, \mathbb{R}_+^n)$ and a) follows from Remark 3.7.

To prove b) we assume the contrary, i.e. that for some $\tau > t_0$ there is an entry $b_{ij}(\tau) < 0$. We consider the admissible control $u(t)$ defined by $u(t) = 0$ for $t_0 \leq t < \tau$, $u(t) = e_j$ for $t \geq \tau$, where e_j is the j-th unit vector. For $x_0 = 0$ and the corresponding solution $x_u(\cdot,t_0,x_0)$, we have $x_u(\tau,t_0,x_0) = 0$. Let $x_u^{(i)}(\cdot)$ denote the i-th component of $x_u(\cdot,t_0,x_0)$. In $t = \tau$ we have $\dot{x}_u^{(i)}(\tau) = b_{ij}(\tau) < 0$. By continuity we have $\dot{x}_u^{(i)}(t) < 0$ for all t in some interval $[\tau, \tau+\Delta]$. Hence $x_u^{(i)}(\tau+\Delta) < 0$ and $x(\tau+\Delta,t_0,x_0) \notin \mathbb{R}_+^n$. We conclude that b) is necessary. □

7.2 Deterministic feedback Nash equilibria for a time-varying positive systems

Throughout the remainder of this section we assume that in the system (7.1) the matrix $B(t)$ is partitioned as $B(t) = \begin{pmatrix} B_1(t) & B_2(t) \end{pmatrix}$ where $B_i(t) \in \mathbb{R}^{n\times m_i}$, $m_i > 0, i = 1,2$, $m_1 + m_2 = m$. Thus the system may be written as

$$\frac{d}{dt}x(t) = A(t)x(t) + B_1(t)u_1(t) + B_2(t)u_2(t)\ , \tag{7.3}$$

where u_i is interpreted as the control action executed by the i-th player. The set
$\mathcal{F} = \{(F_1,F_2)|F_j : \mathcal{I} \to \mathbb{R}^{m_j\times n}$ *are piecewise continuous and bounded such that* $A + B_1F_1 + B_2F_2$ *defines a positive exponentially stable evolution on* $\mathbb{R}^n\}$
defines the admissible feedback controls. We associate the gain functionals

$$J_i(F_1,F_2,t_0,x_0) = \int_{t_0}^{\infty} x^T(t)\left(Q_i(t) + \sum_{j=1}^{2} F_j^T(t)R_{ij}(t)F_j(t)\right)x(t)\,dt\ , \qquad \text{i=1,2}\ , \tag{7.4}$$

for all $(F_1,F_2) \in \mathcal{F}$, where $Q_i : (t_0,\infty) \to \mathcal{S}_n$, $R_{ij} : [t_0,\infty) \to \mathcal{S}_{m_j}$ are piecewise continuous and bounded matrix valued functions and $R_{ii}(t)$ is negative definite for all $t \geq t_0$ $(i = 1,2)$.

According to [10], [5] and [17] we define a deterministic feedback Nash equilibrium as a pair of admissible feedback control strategies, which yields optimal gain for each player under the assumption that the other player does not change her strategy.

Definition 7.3 *A pair* $(\tilde{F}_1,\tilde{F}_2) \in \mathcal{F}$ *is called a (two player)* deterministic feedback Nash equilibrium *on the positive system* (7.3) *if the following inequalities hold:*

$$\begin{aligned} J_1(\tilde{F}_1,\tilde{F}_2,t_0,x_0) &\geq J_1(F_1,\tilde{F}_2,t_0,x_0)\ , \\ J_2(\tilde{F}_1,\tilde{F}_2,t_0,x_0) &\geq J_2(\tilde{F}_1,F_2,t_0,x_0)\ , \end{aligned}$$

for all initial states $x_0 \in \mathbb{R}_+$ *and for all* $(F_1,\tilde{F}_2),(\tilde{F}_1,F_2) \in \mathcal{F}$.

Notice that, consistently with the interpretation of the functionals (7.4) as *gain* functionals and the negative definite weight term $R_{ii}(t) < 0$, we are looking for *maximizing* control strategies. This may deviate slightly from other publications on our topic, but should not cause any difficulties.

Let us consider the following system of coupled nonlinear matrix differential equations:

$$
\begin{aligned}
\frac{d}{dt}K_1(t) &= -A^T(t)K_1(t) - K_1(t)A(t) + K_1(t)S_2(t)K_2(t) + K_2(t)S_2(t)K_1(t) \\
&\quad + K_1(t)S_1(t)K_1(t) - K_2(t)S_{12}(t)K_2(t) - Q_1(t)\ , \\
\frac{d}{dt}K_2(t) &= -A^T(t)K_2(t) - K_2(t)A(t) + K_2(t)S_1(t)K_1(t) + K_1(t)S_1(t)K_2(t) \\
&\quad + K_2(t)S_2(t)K_2(t) - K_1(t)S_{21}(t)K_1(t) - Q_2(t)\ ,
\end{aligned}
\tag{7.5}
$$

where $S_i(t) = B_i(t)R_{ii}^{-1}(t)B_i^T(t)$, $S_{ij}(t) = B_j(t)R_{jj}^{-1}(t)R_{ij}(t)R_{jj}^{-1}(t)B_j^T(t)$, $i, j = 1, 2$.

Let $\tilde{K} : \mathcal{I} \to \mathcal{S}_n \oplus \mathcal{S}_n$, $\tilde{K}(t) = \begin{pmatrix} \tilde{K}_1(t) \\ \tilde{K}_2(t) \end{pmatrix}$ be a global bounded solution of (7.5) such that $A_{cl}(t) = A(t) - S_1(t)\tilde{K}_1(t) - S_2(t)\tilde{K}_2(t)$ generates an exponentially stable evolution on $\mathbb{R}^n$.

By direct calculation one obtains the equations

$$
\begin{aligned}
J_1(F_1, F_2, t_0, x_0) &= x_0^T\tilde{K}_1(t_0)x_0 + \int_{t_0}^{\infty} x^T(t)[F_1(t) - \tilde{F}_1(t)]^T R_{11}(t)[F_1(t) - \tilde{F}_1(t)]x(t)\,dt \\
&\quad + \int_{t_0}^{\infty} x^T(t)\Big([F_2(t) - \tilde{F}_2(t)]^T B_2^T(t)\tilde{K}_1(t) + \tilde{K}_1(t)B_2(t)[F_2(t) - \tilde{F}_2(t)] \\
&\qquad + F_2^T(t)R_{12}(t)F_2(t) - \tilde{F}_2^T(t)R_{12}(t)\tilde{F}_2(t)\Big)x(t)\,dt
\end{aligned}
$$

$$
\begin{aligned}
J_2(F_1, F_2, t_0, x_0) &= x_0^T\tilde{K}_2(t_0)x_0 + \int_{t_0}^{\infty} x^T(t)[F_2(t) - \tilde{F}_2(t)]^T R_{22}(t)[F_2(t) - \tilde{F}_2(t)]x(t)\,dt \\
&\quad + \int_{t_0}^{\infty} x^T(t)\Big([F_1(t) - \tilde{F}_1(t)]^T B_1^T(t)\tilde{K}_2(t) + \tilde{K}_2(t)B_1(t)[F_1(t) - \tilde{F}_1(t)] \\
&\qquad + F_1^T(t)R_{21}(t)F_1(t) - \tilde{F}_1^T(t)R_{21}(t)\tilde{F}_1(t)\Big)x(t)\,dt
\end{aligned}
$$

for all $(F_1, F_2) \in \mathcal{F}$, where $\tilde{F}_j(t) = -R_{jj}^{-1}(t)B_j^T(t)\tilde{K}_j(t), j = 1, 2$. These equalities show that – under the assumption $R_{ii}(t) < 0$ – the pair $(\tilde{F}_1, \tilde{F}_2)$ is a feedback Nash strategy: If the first player chooses strategy $\tilde{F}_1$, then $\tilde{F}_2$ is the optimal strategy for the second player and vice versa (compare [5] and [17]); therefore we are interested in the global solvability of the system (7.5).

Since – in contrast to standard matrix Riccati equations – the coupled nonlinear system (7.5) of differential equations cannot be transformed to an equivalent linear system of differential equations, it is a nontrivial task to find sufficient conditions for the existence of the above-mentioned solutions $\tilde{K}_1$, $\tilde{K}_2$, compare [21] and [1, Ch. 6].

8 Stabilizing solutions of the feedback Nash differential equations

In this section we introduce the concept of a stabilizing solution of the coupled feedback Nash differential equations (7.5) and provide conditions which guarantee the existence of such a solution. To prove the main result of this section we make use of the results developed in the previous sections of the paper.

We consider the space $\mathcal{X} = \mathcal{S}_n \otimes \mathcal{S}_n$ endowed with the inner product

$$
\langle X, Y\rangle = \mathrm{Tr}\,[Y_1X_1] + \mathrm{Tr}\,[Y_2X_2] = \mathrm{Tr}\,(Y^TX)\ . \tag{8.1}
$$

Here and in the sequel we tacitly assume for every element $X \in \mathcal{X}$ that we are given a partitioning $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ with $X_{1,2} \in \mathcal{S}^n$.

Together with the norm $|\cdot|_2$ induced by the inner product (8.1) we define the norm $|\cdot|_1$ via

$$|X|_1 = \max\{|X_1|_\xi, |X_2|_\xi\} \tag{8.2}$$

for all $X \in \mathcal{X}$, where $|X_i|_\xi$ is defined as in (2.8) and $\xi = \mathbf{1}_{2n,n}$ (cf. Ex 2.5 (ii)).

On $\mathcal{X}$ we consider the order relation induced by the cone $\mathcal{X}^+ = \{X \in \mathcal{X} \mid X_k \succeq 0, k = 1,2\}$.
The system of feedback Nash differential equations (7.5) may be written in a compact form as a nonlinear equation on $\mathcal{X}$

$$\frac{d}{dt}K(t) + \mathcal{R}(t, K(t)) + Q(t) = 0 \tag{8.3}$$

with

$$\begin{aligned}
\mathcal{R}_1(t,K) &= A^T(t)K_1 + K_1A(t) - K_1S_2(t)K_2 - K_2S_2(t)K_1 - K_1S_1(t)K_1 + K_2S_{12}(t)K_2 \ , \\
\mathcal{R}_2(t,K) &= A^T(t)K_2 + K_2A(t) - K_2S_1(t)K_1 - K_1S_1(t)K_2 - K_2S_2(t)K_2 + K_1S_{21}(t)K_1 \ .
\end{aligned}$$

First we shall prove an auxiliary result which is of interest in itself. Set $\tilde{\mathcal{L}}(t)K = \begin{pmatrix} \tilde{\mathcal{L}}_1(t)K \\ \tilde{\mathcal{L}}_2(t)K \end{pmatrix}$ with

$$\tilde{\mathcal{L}}_1(t)K = \tilde{A}(t)K_1 + K_1\tilde{A}^T(t) + \tilde{\mathcal{N}}(t)K_2 + K_2\tilde{\mathcal{N}}^T(t) \ , \tag{8.4}$$

$$\tilde{\mathcal{L}}_2(t)K = \tilde{\mathcal{M}}(t)K_1 + K_1\tilde{\mathcal{M}}^T(t) + \tilde{A}(t)K_2 + K_2\tilde{A}^T(t) \ , \tag{8.5}$$

where $\tilde{A} : \mathcal{I} \to \mathbb{R}^{n\times n}$, $\tilde{\mathcal{N}} : \mathcal{I} \to \mathbb{R}^{n\times n}$, $\tilde{\mathcal{M}} : \mathcal{I} \to \mathbb{R}^{n\times n}$ are piecewise continuous matrix valued functions. Then we have the decomposition

$$\tilde{\mathcal{L}}(t)K = \Lambda(t)K + \Pi(t)K \tag{8.6}$$

with $\Lambda(t)K = \begin{pmatrix} \tilde{A}(t)K_1 + K_1\tilde{A}^T(t) \\ \tilde{A}(t)K_2 + K_2\tilde{A}^T(t) \end{pmatrix}$, $\Pi(t)K = \begin{pmatrix} \tilde{\mathcal{N}}(t)K_2 + K_2\tilde{\mathcal{N}}^T(t) \\ \tilde{\mathcal{M}}(t)K_1 + K_1\tilde{\mathcal{M}}^T(t) \end{pmatrix}$ for all $K \in \mathcal{X}$.

Lemma 8.1 *Assume:* *a) For each $t \in \mathcal{I}$, $\tilde{A}(t)$ is a Metzler matrix.*

b) $\tilde{\mathcal{M}}(t) \succeq 0, \tilde{\mathcal{N}} \succeq 0$ for all $t \in \mathcal{I}$.

Under these conditions the following assertions hold:

(i) The adjoint operator of $\tilde{\mathcal{L}}(t)$ with respect to the inner product (8.1) is given by
$$\tilde{\mathcal{L}}^*(t)Z = \begin{pmatrix} \tilde{A}^T(t)Z_1 + Z_1\tilde{A}(t) + \tilde{\mathcal{M}}^T(t)Z_2 + Z_2\tilde{\mathcal{M}}(t) \\ \tilde{\mathcal{N}}^T(t)Z_1 + Z_1\tilde{\mathcal{N}}(t) + \tilde{A}^T(t)Z_2 + Z_2\tilde{A}(t) \end{pmatrix} \text{ for all } Z \in \mathcal{X}.$$

(ii) $\tilde{\mathcal{L}}(\cdot)$ generates a positive evolution on $\mathcal{X}$.

(iii) If $\tilde{\mathcal{L}}(\cdot)$ generates an E.S. evolution on $\mathcal{X}$ then $\tilde{A}(\cdot)$ generates an E.S. evolution on $\mathbb{R}^n$.

Proof. (i) follows immediately from the definition of the adjoint operator.

(ii) If $T_\Lambda(t,t_0)$ denotes the linear evolution operator defined by $\Lambda(\cdot)$ from (8.6) then

$$T_\Lambda(t,t_0)K = \begin{pmatrix} \Phi_{\tilde{A}}(t,t_0)K_1\Phi_{\tilde{A}}^T(t,t_0) \\ \Phi_{\tilde{A}}(t,t_0)K_2\Phi_{\tilde{A}}^T(t,t_0) \end{pmatrix} \quad \text{for all} \quad K \in \mathcal{X} \ , \tag{8.7}$$

where $\Phi_{\tilde{A}}(t,t_0)$ is the fundamental matrix solution of the linear differential equation

$$\frac{d}{dt}x(t) = \tilde{A}(t)x(t) \ , \quad x \in \mathbb{R}^n \ . \tag{8.8}$$

From Remark 3.7, it follows that $\Phi_{\tilde{A}}(t,t_0) \succeq 0$ for all $t \geq t_0$, $t, t_0 \in \mathcal{I}$, this leads to $T_\Lambda(t,t_0)K \succeq 0$ if $K \succeq 0$. Therefore $\Lambda(\cdot)$ generates a positive evolution on $\mathcal{X}$. On the other hand from assumption b) one deduces that $\Pi(t)K \succeq 0$ if $K \succeq 0$. This shows that $\Pi(t)$ is a linear and positive operator on $\mathcal{X}$. Therefore, the conclusion is obtained by applying Corollary 3.8.

(iii) The inequality $\Lambda(t) \leq \tilde{\mathcal{L}}(t)$ together with Proposition 4.1 (ii) allows us to deduce that $\Lambda(\cdot)$ generates an E.S. evolution if $\tilde{\mathcal{L}}(t)$ generates an E.S. evolution.

If $x(t)$ is an arbitrary solution of (8.8), then $X(t) = \begin{pmatrix} x(t)x(t)^T \\ x(t)x(t)^T \end{pmatrix}$ satisfies $\dot{X}(t) = \Lambda(t)X(t)$. Hence there exist α, β independent of $x(\cdot)$ and t_0, such that

$$\beta e^{\alpha(t-t_0)} \geq \langle X(t), X(t)\rangle^{\frac{1}{2}} = \sqrt{\mathrm{Tr}\,(x(t)x(t)^T x(t)x(t)^T)} = |x(t)|_2^2 \,.$$

This proves exponential stability of (8.8). □

In the following, we consider linear operators on $\mathcal{X}$ of the form $Z \mapsto \mathcal{L}_K(t)Z = \begin{pmatrix} \mathcal{L}_{1K}(t)Z \\ \mathcal{L}_{2K}(t)Z \end{pmatrix}$, with

$$\begin{aligned} \mathcal{L}_{1K}(t)Z &= (A(t) - S_1(t)K_1(t) - S_2(t)K_2(t))Z_1 + Z_1(A(t) - S_1(t)K_1(t) - S_2(t)K_2(t))^T \\ &\quad -(S_1(t)K_2(t) - S_{21}(t)K_1(t))Z_2 - Z_2(K_2(t)S_1(t) - K_1(t)S_{21}(t)) \qquad (8.9) \\ \mathcal{L}_{2K}(t)Z &= (A(t) - S_1(t)K_1(t) - S_2(t)K_2(t))Z_2 + Z_2(A(t) - S_1(t)K_1(t) - S_2(t)K_2(t)) \\ &\quad - (S_2(t)K_1(t) - S_{12}(t)K_2(t))Z_1 - Z_1(K_1(t)S_2(t) - K_2(t)S_{12}(t)) \qquad (8.10) \end{aligned}$$

for all $Z \in \mathcal{X}$, with $K(t)$ being a fixed solution of (8.3).

Setting

$$\begin{aligned} \tilde{A}(t) &= A(t) - S_1(t)K_1(t) - S_2(t)K_2(t)\,, \\ \tilde{\mathcal{M}}(t) &= -(S_2(t)K_1(t) - S_{12}(t)K_2(t))\,, \\ \tilde{\mathcal{N}}(t) &= -(S_1(t)K_2(t) - S_{21}(t)K_1(t))\,, \end{aligned}$$

one can see that the operators $\mathcal{L}_K(t)$ defined by (8.9) – (8.10) are of type (8.4) – (8.5).

If $K(t)$ is a solution of (8.3) then the Fréchet derivative of the operator $Z \mapsto \mathcal{R}(t,Z)$ along $K(t)$ is given by

$$\mathcal{R}'(t)K(t) = \mathcal{L}_K^*(t) \qquad (8.11)$$

where $\mathcal{L}_K^*(t)$ is the adjoint operator of the operator defined by (8.9) – (8.10) for the solution $K(t)$.

Definition 8.2 *A global solution $\tilde{K} : \mathcal{I} \to \mathcal{X}$ of* (8.3) *is a* stabilizing solution *if the operator $\mathcal{L}_{\tilde{K}}$ generates an E.S. evolution, where $\mathcal{L}_{\tilde{K}}(t)$ is associated to the solution $\tilde{K}(t)$ by* (8.9) – (8.10).

Remark 8.3 It is easy to see that this definition of a stabilizing solution is a natural extension of the concept of a stabilizing solution of Riccati equations (see e.g. [13]).

Following [5] we make the assumptions:

Assumption $\mathbf{H_1}$ a) For each $t \in \mathcal{I}$, $A(t)$ is a Metzler matrix.

b) $S_j(t) \preceq 0, j = 1,2$, $S_{21}(t) \succeq 0$, $S_{12}(t) \succeq 0$ for all $t \in \mathcal{I}$.

We draw an immediate consequence from Lemma 8.1.

Corollary 8.4 *Assume* $\mathbf{H_1}$. *If $\tilde{K}(t)$ is a stabilizing solution of* (8.3) *such that $\tilde{K}(t) \succeq 0$ then the closed loop matrix $A_{cl}(t) = A(t) - S_1(t)\tilde{K}_1(t) - S_2(t)\tilde{K}_2(t)$ generates an E.S. evolution on $\mathbb{R}^n$.*

Remark 8.5 It is possible that for a solution $\tilde{K}(t)$ of (8.3) the corresponding closed-loop matrix

$$\tilde{A}_{cl}(t) = A(t) - S_1(t)\tilde{K}_1(t) - S_2(t)\tilde{K}_2(t) \tag{8.12}$$

generates an E.S. evolution but $\tilde{K}(t)$ is not a stabilizing solution of (8.3) in the sense of Def. 8.2.

To see if a solution $\tilde{K}(t) \succeq 0$ of (8.3) with the property that the corresponding closed-loop matrix (8.12) generates an E.S. evolution, is just a stabilizing solution of (8.3) one may use the results developed in Section 6.

For $W \in \mathcal{X}$ we define $\Gamma(t, W) = \begin{pmatrix} W_2 S_{12}(t) W_2 - W_2 S_2(t) W_1 - W_1 S_2(t) W_2 - W_1 S_1(t) W_1 \\ W_1 S_{21}(t) W_1 - W_1 S_1(t) W_2 - W_2 S_1(t) W_1 - W_2 S_2(t) W_2 \end{pmatrix}$.

Assumption $\mathbf{H_1}$ b) implies that $\Gamma(t, W) \succeq 0$ if $W \succeq 0$.

By standard algebraic manipulations we obtain the following result, which will be used repeatedly.

Lemma 8.6 *The following assertions hold:*

(i) If $K(t)$ is a solution of (8.3) *then it also solves the equation*

$$\dot{K}(t) + \mathcal{L}_W^*(t)K(t) + \Gamma(t, K(t) - W(t)) + Q(t) - \Gamma(t, W(t)) = 0$$

where $W : \mathcal{I} \to \mathcal{X}$ is an arbitrary continuous function.

(ii) If $K(t)$ is a solution of the linear equation

$$\dot{K}(t) + \mathcal{L}_W^*(t)K(t) + Q(t) - \Gamma(t, W(t)) = 0$$

with $W : \mathcal{I} \to \mathcal{X}$ fixed then $K(t)$ solves the differential equation

$$\dot{K}(t) + \mathcal{L}_K^*(t)K(t) - \Gamma(t, K(t) - W(t)) + Q(t) - \Gamma(t, K(t)) = 0.$$

Note that system (7.5) and equivalently equation (8.3) are completely determined by the pair $(\mathcal{R}, Q)$. It is useful to introduce two sets of functions associated to this pair.

$$\Omega(\mathcal{R}, Q) = \{P \in C_b^1(\mathcal{I}, \mathcal{X}) | \dot{P}(t) + \mathcal{R}(t, P(t)) + Q(t) \preceq 0, P(t) \succeq 0, t \in \mathcal{I}\}$$

$$\tilde{\Omega}(\mathcal{R}, Q) = \{P \in C_b^1(\mathcal{I}, \mathcal{X}) | \dot{P}(t) + \mathcal{R}(t, P(t)) + Q(t) \prec\!\prec 0, P(t) \succeq 0, t \in \mathcal{I}\}$$

where $C_b^1(\mathcal{I}, \mathcal{X})$ is the set of C^1-functions $P : \mathcal{I} \to \mathcal{X}$ being bounded with bounded derivative.

We recall that for a function $H : \mathcal{I} \to \mathcal{X}$, $H(x) \prec\!\prec 0$ means that there exist a positive number h such that $H(t) \preceq -h\xi$ for all $t \in \mathcal{I}$.

Remark 8.7 a) $\tilde{\Omega}(\mathcal{R}, Q) \subseteq \Omega(\mathcal{R}, Q)$.

b) $\Omega(\mathcal{R}, Q)$ contains all bounded and global solutions $K(t) \succeq 0$ of the equation (8.3).

Assumption $\mathbf{H_2}$ a) $A(\cdot)$ generates an E.S. evolution on $\mathbb{R}^n$.

b) The set $\tilde{\Omega}(\mathcal{R}, Q)$ is not empty.

Under our Assumptions $\mathbf{H_1}$ and $\mathbf{H_2}$ we prove the existence of a stabilizing solution to the Nash differential equations by constructing iteratively a sequence $\{K^j(t)\}_{j \geq 0}$ as follows: For each $j \geq 1$ let $K^j(t)$ be the unique bounded solution of the linear equation on $\mathcal{X}$

$$\frac{d}{dt}K^j(t) + \mathcal{L}_{K^{j-1}}^*(t)K^j + Q(t) - \Gamma(t, K^{j-1}(t)) = 0 \quad \text{with} \quad K^0(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathcal{X} \,. \tag{8.13}$$

Convergence of this sequence is established in the following theorem.

Theorem 8.8 *Under the Assumptions* $\mathbf{H_1}$, $\mathbf{H_2}$ *the sequence* $K^j(t)$ *given by* (8.13) *is well defined and convergent. If* $\tilde{K}(t) = \lim_{j\to\infty} K^j(t)$ *then* $\tilde{K}(t)$ *is a stabilizing solution of* (7.5). *Moreover it satisfies* $0 \preceq \tilde{K}(t) \preceq P(t)$ *for all* $P(t) \in \Omega(\mathcal{R}, Q)$.

If the coefficients of (7.5) *are periodic functions with period* θ, *then* $\tilde{K}(t)$ *is also a periodic function with period* θ. *If the coefficients of* (7.5) *is independent of* t *then* $\tilde{K}$ *is constant and solves the algebraic equation* $\mathcal{R}(\tilde{K}) + Q = 0$ *associated to* (8.3).

Proof. For $j = 1$, $K^1(t)$ is obtained as the unique bounded solution of the equation

$$\frac{d}{dt}K_i^1(t) + A^T(t)K_i^1(t) + K_i^1(t)A(t) + Q_i(t) = 0\ , \qquad i = 1, 2\ . \tag{8.14}$$

For each $i \in \{1, 2\}$ the equation (8.14) has a unique bounded solution $K^i : \mathcal{I} \to \mathcal{S}_n$. Based on Theorem 4.7 (iv) one obtains that $K_i^1(t) \succeq 0, t \in \mathcal{I}$. Therefore the first term $K^1(t)$ of the sequence of equations is well defined and $0 \preceq K^1(t), t \in \mathcal{I}$.

The equations (8.14) may be written in a compact form as:

$$\frac{d}{dt}K^1(t) + \Lambda^*(t)K^1(t) + Q(t) = 0, \tag{8.15}$$

where $\Lambda(t)$ is the operator introduced in connection with the decomposition (8.6), where $\tilde{A}(t)$ is replaced by $A(t)$.

For $P(t) \in C_b^1(\mathcal{I}, \mathcal{X})$ we define $\hat{Q}(t) = -\frac{d}{dt}P(t) - \mathcal{R}(t, P(t)) - Q(t)$.

By definition, we have $\hat{Q}(\cdot) \succeq 0$, if $P \in \Omega(\mathcal{R}, Q)$ and $\hat{Q}(\cdot) \succ\succ 0$, if $P \in \tilde{\Omega}(\mathcal{R}, Q)$.

Also we have

$$\frac{d}{dt}P(t) + \mathcal{R}(t, P(t)) + Q(t) + \hat{Q}(t) = 0. \tag{8.16}$$

Subtracting (8.15) from (8.16) and exploiting $\Gamma(t, P) = \mathcal{R}(t, P) - \Lambda^* P$, one obtains that

$$\frac{d}{dt}(P(t) - K^1(t)) + \Lambda^*(t)(P(t) - K^1(t)) + H^1(t) = 0,$$

where $H^1(t) = \Gamma(t, P(t)) + \hat{Q}(t)$ for $t \in \mathcal{I}$. It is easy to check that $H^1(t) \succeq 0$ if $P(t) \in \Omega(\mathcal{R}, Q)$ and $H^1 \succ\succ 0$ if $P(t) \in \tilde{\Omega}(\mathcal{R}, Q)$.

By Theorem 4.7 (iv) we thus have for $t \in \mathcal{I}$

$$\begin{array}{rcl} P(t) - K^1(t) & \succeq & 0 \quad \text{if} \quad P(\cdot) \in \Omega(\mathcal{R}, Q)\ , \\ P(t) - K^1(t) & \succ\succ & 0 \quad \text{if} \quad P(\cdot) \in \tilde{\Omega}(\mathcal{R}, Q)\ . \end{array} \tag{8.17}$$

Now we show that $\mathcal{L}_{K^1}(t)$ generates an E.S. evolution on $\mathcal{X}$. To this end we choose $\hat{P}(t) \in \tilde{\Omega}(\mathcal{R}, Q)$. Applying Lemma 8.6 (i) with $W(t) = K^1(t)$ and $K(t) = \hat{P}(t)$ we obtain from (8.16)

$$\frac{d}{dt}P(t) + \mathcal{L}_{K^1}^*(t)P(t) + \Gamma(t, P(t) - K^1(t)) + Q(t) + \hat{Q}(t) - \Gamma(t, K^1(t)) = 0\ . \tag{8.18}$$

Applying Lemma 8.6 (ii) for $W(t) = 0$, $K(t) = K^1(t)$ equation (8.15) yields

$$\frac{d}{dt}K^1(t) + \mathcal{L}_{K^1}^*(t)K^1(t) - 2\Gamma(t, K^1(t)) + Q(t) = 0\ . \tag{8.19}$$

Subtracting (8.19) from (8.18) and invoking (8.17) one obtains that $t \mapsto P(t) - K^1(t)$ is a bounded and uniformly positive solution of the backward affine equation

$$\frac{d}{dt}Z(t) + \mathcal{L}_{K^1}^*(t)Z(t) + \Delta^1(t) = 0,$$

where $\Delta^1(t) = \Gamma(t, P(t) - K^1(t)) + \Gamma(t, K^1(t)) + \hat{Q}(t) \succ\succ 0$, $t \in \mathcal{I}$.

Invoking implication '(vi)⇒(i)' of Theorem 4.5 we conclude that $\mathcal{L}_{K^1}(t)$ generates an E.S. evolution on $\mathcal{X}$.

By induction, we prove the following items:

$\mathbf{a_j}$) $K^j(t) \preceq P(t)$ for all functions $P(t) \in \Omega(\mathcal{R}, Q)$ and $K^j(t) \prec\prec P(t)$ if $P(t) \in \tilde{\Omega}(\mathcal{R}, Q)$.

$\mathbf{b_j}$) $K^{j-1} \preceq K^j(t)$.

$\mathbf{c_j}$) $\mathcal{L}_{K^j}(\cdot)$ generates an E.S. evolution on $\mathcal{X}$.

For $j = 1$ the assertions $a_1), b_1)$ and $c_1)$ were proved before.

Assuming that $a_i), b_i), c_i)$ hold for $i \in \{1, 2, \ldots, j-1\}$, we show that they also hold for $i = j$.

If $P(t) \in \Omega(\mathcal{R}, Q)$ then by Lemma 8.6 (i) with $W(t) = K^{j-1}(t)$ and $K(t) = P(t)$ we obtain that equation (8.16) may be written as follows

$$\frac{d}{dt}P(t) + \mathcal{L}^*_{K^{j-1}}(t)P(t) + \Gamma(t, P(t) - K^{j-1}(t)) + Q(t) + \hat{Q}(t) - \Gamma(t, K^{j-1}(t)) = 0. \tag{8.20}$$

Subtracting (8.13) from (8.20) we obtain that $t \mapsto P(t) - K^j(t)$ is a bounded solution of the equation

$$\frac{d}{dt}Z(t) + \mathcal{L}^*_{K^{j-1}}(t)Z(t) + H^j(t) = 0, \tag{8.21}$$

where $H^j(t) = \Gamma(t, P(t) - K^{j-1}(t)) + \hat{Q}(t)$.

Since $a_{j-1})$ is fulfilled it follows that $\Gamma(t, P(t) - K^{j-1}(t)) \succeq 0$. Hence $H^j(t) \succeq 0$ if $P(t) \in \Omega(\mathcal{R}, Q)$ and $H^j(t) \succ\succ 0$ if $P(t) \in \tilde{\Omega}(\mathcal{R}, Q)$. Since $\mathcal{L}_{K^{j-1}}(\cdot)$ generates an E.S. evolution we conclude via Theorem 4.7 (iv) that $P(t) - K^j(t) \succeq 0$ if $P(t) \in \Omega(\mathcal{R}, Q)$ and

$$P(t) - K^j(t) \succ\succ 0 \tag{8.22}$$

if $P(t) \in \tilde{\Omega}(\mathcal{R}, Q)$ and thus $a_j)$ is valid.

To prove that $b_j)$ is fulfilled we rewrite equation (8.13) for j replaced by $j-1$ in the form

$$\frac{d}{dt}K^{j-1}(t) + \mathcal{L}^*_{K^{j-1}}(t)K^{j-1}(t) - \Gamma(t, K^{j-1}(t) - K^{j-2}(t)) + Q(t) - \Gamma(t, K^{j-1}(t)) = 0. \tag{8.23}$$

Subtracting (8.22) from (8.12) one obtains that $t \mapsto K^j(t) - K^{j-1}(t)$ is a bounded on $\mathcal{I}$ solution of the equation

$$\frac{d}{dt}Y(t) + \mathcal{L}^*_{K^{j-1}}Y(t) + \Gamma(t, K^{j-1}(t) - K^{j-2}(t)) = 0. \tag{8.24}$$

Since $K^{j-1}(t) - K^{j-2}(t) \succeq 0, t \in \mathcal{I}$ it follows that $\Gamma(t, K^{j-1}(t) - K^{j-2}(t)) \succeq 0, t \in \mathcal{I}$.

Applying again Theorem 4.7 (iv) we conclude that $K^j(t) - K^{j-1}(t) \succeq 0$. This means that $b_j)$ is fulfilled.

To prove $c_j)$ we rewrite equation (8.13) in the form

$$\frac{d}{dt}K^j(t) + \mathcal{L}^*_{K^j}(t)K^j(t) - \Gamma(t, K^j(t) - K^{j-1}(t)) + Q(t) - \Gamma(t, K^j(t)) = 0. \tag{8.25}$$

Also, if $P(t) \in \tilde{\Omega}(\mathcal{R}, Q)$ then equation (8.16) may be rewritten as

$$\frac{d}{dt}P(t) + \mathcal{L}^*_{K^j}(t)P(t) + \Gamma(t, P(t) - K^j(t)) + Q(t) + \hat{Q}(t) - \Gamma(t, K^j(t)) = 0. \tag{8.26}$$

From (8.22), (8.25) and (8.26) we conclude that $t \mapsto P(t) - K^j(t)$ is a bounded and uniformly positive solution of the backward affine equation

$$\frac{d}{dt}Y(t) + \mathcal{L}^*_{K^j}(t)Y(t) + \Delta^j(t) = 0, \tag{8.27}$$

where $\Delta^j(t) = \Gamma(t, P(t) - K^j(t)) + \Gamma(t, K^j(t) - K^{j-1}(t)) + \hat{Q}(t)$.

Since $\hat{Q}(t) \succ\succ 0$ and $a_j), b_j)$ hold, we conclude that $\Delta^j(t) \succ\succ 0, t \in \mathcal{I}$.

Applying implication '(vi)$\Rightarrow$ (i)' in Theorem 4.5 we deduce that $\mathcal{L}_{K^j}(\cdot)$ generates an E.S. evolution and thus $c_j)$ is fulfilled.

From $a_j)$ and $b_j)$ one deduces that the sequence $\{K^j\}_{j\geq 1}$ is convergent. Set $\tilde{K}(t) = \lim_{j\to\infty} K^j(t), t \in \mathcal{I}$. By a standard reasoning one obtains that $t \mapsto \tilde{K}(t)$ is a solution of (8.3). Also from $a_j)$ we obtain that

$$\tilde{K}(t) \preceq P(t) \tag{8.28}$$

for any $P(t) \in \Omega(\mathcal{R}, Q)$. This shows that $\tilde{K}(t)$ is a bounded and minimal positive solution of (8.3). To show that $\tilde{K}(t)$ is just a stabilizing solution of (8.3) we take $P(t) \in \tilde{\Omega}(\mathcal{R}, Q)$. Applying Lemma 8.6 (i) with $W(t) = \tilde{K}(t)$ we obtain that $t \mapsto P(t) - \tilde{K}(t)$ is the bounded solution of the equation

$$\frac{d}{dt}(P(t) - \tilde{K}(t)) + \mathcal{L}^*_{\tilde{K}}(t)(P(t) - \tilde{K}(t)) + \tilde{H}(t) = 0 \tag{8.29}$$

where $\tilde{H}(t) = \Gamma(t, P(t) - \tilde{K}(t)) + \hat{Q}(t)$.

Since $P(t) - \tilde{K}(t) \succeq 0$ and $\hat{Q}(t) \succ\succ 0, t \in \mathcal{I}$ we have $\tilde{H}(t) \succ\succ 0, t \in \mathcal{I}$.

Using again implication '(vi)$\Rightarrow$(i)' of Theorem 4.5 we conclude that $\mathcal{L}_{\tilde{K}}(\cdot)$ generates an E.S. evolution which means that $\tilde{K}(t)$ is a stabilizing solution of (8.3) or equivalently, of the system (7.5).

Finally we remark that if the coefficients of (7.5) are periodic functions with period θ then based on Theorem 4.7 (ii) the unique bounded solution of (8.13) is periodic function with the same period θ. In this case $\tilde{K}(t)$ will be a periodic function with period θ. On the other hand if the coefficients of (7.5) are constant functions then from Theorem 4.7 (iii) the unique bounded solution of (8.13) is a constant function. In this case the stabilizing solution $\tilde{K}(t)$ is a constant function, too. □

Remark 8.9 The previous result extends Theorem 8 in [5] to the time-varying case. From our proof it follows that $A - S_1P_1 - S_2P_2$ needs not necessarily be an M-matrix if $A_{cl} = A - S_1\tilde{K} - S_2\tilde{K}$ is required to be an M-matrix.

Remark 8.10 It may be verified that under Assumption $\mathbf{H_1}$ if there exist a solution $\tilde{K}(t) \succeq 0$ of (7.5) such that $A_{cl}(t) = A(t) - S_1(t)\tilde{K}_1(t) - S_2(t)\tilde{K}_2(t)$ generates an E.S. evolution on $\mathbb{R}^n$, then from $A(t) \preceq A_{cl}(t)$ together with (ii) from Proposition 4.1 one obtains that $A(t)$ generates an E.S. evolution on $\mathbb{R}^n$. This shows that Assumption $\mathbf{H_2}$, a) is also a necessary condition for the existence of a stabilizing solution $\tilde{K}(t) \succeq 0$ of (7.5) in the presence of the assumption $\mathbf{H_1}$.

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