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A car-steering problem with random adhesion coefficient

Stabilization and disturbance attenuation problems for stochastic linear control systems have been studied recently e.g. in [3, 4, 5]. We recall some results and apply them to a modified version of the car-steering problem from [1].

1. Uncertain linear models, mean-square stability and disturbance attenuation

Consider a linear control system with parametric uncertainty of the form ($\star$):

$$\dot{x} = \Big(A+\sum_{j=1}^{N}\delta_j(t)A_0^{(j)}\Big)x+\Big(B_1+\sum_{j=1}^{N}\delta_j(t)B_{10}^{(j)}\Big)v+\Big(B_2+\sum_{j=1}^{N}\delta_j(t)B_{20}^{(j)}\Big)u, \quad z = Cx+D_1v+D_2u\,.$$

Here $x\in\mathbb{R}^n$ is the state vector, $u\in\mathbb{R}^m$ a control input, $v\in\mathbb{R}^q$ a disturbance input and $z\in\mathbb{R}^\ell$ an output vector. All matrices are of adequate sizes and the δ_j are real-valued uncertain functions.
Two different models for the uncertainties δ_j play a prominent role in the literature: *Bounded uncertainties*, where the $\delta_j:\mathbb{R}\to\mathbb{R}$ are assumed to be measurable and $\sup_{t\in\mathbb{R}}|\delta_j(t)|\le\Delta_j$ for some $\Delta_j\ge 0$, and *white noise*, where, formally, $\delta_j=\sigma_j\dot{w}_j$ with normed independent Wiener processes w_j and given intensities $\sigma_j\ge 0$. In the sequel, we consider stochastic differential equations of Itô type (i.e. $\delta_j=\sigma_j\dot{w}_j$) and use the notion of mean-square stability.

Definition 1. *The Itô equation $dx = Ax\,dt+\sum_{j=1}^N\sigma_jA_0^{(j)}x\,dw_j$ is called* mean-square stable *with decay rate (at least) $\alpha>0$ if $\exists M>0$: $\forall x_0=x(0)\in\mathbb{R}^n, t\ge 0$: $E\|x(t)\|^2\le Me^{-2\alpha t}\|x_0\|$ (where E denotes expectation).*

Lemma 2. *The Itô equation $dx=Ax\,dt+\sum_{j=1}^N\sigma_jA_0^{(j)}x\,dw_j$ is mean-square stable with decay rate (at least) $\alpha>0$, if and only if there exists a positive definite matrix X such that $A^TX+XA+2\alpha X+\sum_{j=1}^N\sigma_j^2A_0^{(j)T}XA_0^{(j)}<0$. In this event, the equation $\dot{x}=\left(A+\sum_{j=1}^N\delta_j(t)A_0^{(j)}\right)x$ is stable for arbitrary measurable functions $\delta_j:\mathbb{R}\to\mathbb{R}$ satisfying $\sup_{t\in\mathbb{R}}|\delta_j(t)|\le d_j\sigma_j$, where $\sum_{j=1}^N d_j^2\le 2\alpha$.*

Hence, the Itô model and the concept of mean-square stability offer some robustness margin with respect to bounded uncertainties (in contrast to Stratonovich models or stochastic stability); see [2, 3, 4, 6].
Our aim is to construct a feedback controller $u=Fx$ that stabilizes system ($\star$) and diminishes the effect of the disturbance v on the output z. A worst-case measure for this effect is the norm of the perturbation operator $\boldsymbol{L}^F: v\mapsto z$ (as an operator between L^2-spaces). We impose the regularity assumptions $D_2^T[C,D_1,D_2]=[0,0,\varepsilon^2I]$, $\varepsilon>0$, and $D_1^TC=0$ (only $D_2^TD_2>0$ is essential; the other assumptions simplify the notation). For $\gamma>0$, we define a rational matrix operator $\mathcal{R}^\gamma:\mathcal{S}^n\to\mathcal{S}^n$ (where $\mathcal{S}^n=\{X\in\mathbb{R}^{n\times n}\mid X=X^T\}$) and a target set $\mathrm{dom}_\pm\mathcal{R}^\gamma\subset\mathcal{S}^n$ via

$$\mathcal{R}^\gamma(X)=P(X)-S(X)Q^\gamma(X)^{-1}S(X)^T>0, \quad \mathrm{dom}_\pm\mathcal{R}^\gamma=\{X<0\mid\gamma^2I-D_1^TD_1+B_{01}^TXB_{01}>0\}$$

with $P(X)=A^TX+XA+A_0^TXA_0-C^TC$, $\quad S(X)=X[B_2,B_1]+A_0^TX[B_{20},B_{10}]$, $\quad$ and

$$Q^\gamma(X)=[B_{20},B_{10}]^TX[B_{20},B_{10}]-\mathrm{diag}(-\varepsilon^2I_m,\gamma^2I_q-D_1^TD_1)\,.$$

Theorem 3. *[3, 4, 5] Let $\gamma>0$. There exists an $X\in\mathrm{dom}_\pm\mathcal{R}^\gamma$, satisfying $\mathcal{R}^\gamma(X)>0$, if and only if there exists an $F\in\mathbb{R}^{m\times n}$, such that $\|\boldsymbol{L}_w^F\|<\gamma$ (where $\boldsymbol{L}_w^F$ is the perturbation operator of the (Itô-)system ($\star$) with feedback-control $u=Fx$ and stochastic parameter uncertainty $\delta_j=\dot{w}_j$).*
If $\mathcal{R}^\gamma(X)>2\alpha X$ for some $X\in\mathrm{dom}_\pm\mathcal{R}^\gamma$, $\alpha>0$, then $\|\boldsymbol{L}_\Delta^F\|<\gamma$ (where $\boldsymbol{L}_\Delta^F$ is the perturbation operator of system ($\star$) with feedback-control $u=Fx$ and bounded parameter uncertainty $\sup|\delta_j(t)|<\Delta_j$, $\sum_{j=1}^N\Delta_j^2\le 2\alpha$).

Rational matrix equations and inequalities have been studied in [3, 4]. If the pair (A,C) is observable and there exists an $X\in\mathrm{dom}_\pm\mathcal{R}^\gamma$, satisfying $\mathcal{R}^\gamma(X)>0$, then also the equation $\mathcal{R}^\gamma(X)=0$ has a solution $X_+\in\mathrm{dom}_\pm\mathcal{R}^\gamma$. In view of the example below, let us simplify the notation further and assume $B_{10}^{(j)}=0$ for all j. Then the feedback gain matrix $F = -\left(\sum B_{20}^{(j)T} X_+ B_{20}^{(j)} - \varepsilon^2 I\right)^{-1}\left(X_+ B_2 + \sum A_{20}^{(j)T} X_+ B_{20}^{(j)}\right)^T$ solves the disturbance attenuation problem. To compute X_+, we can apply Newton's method to the *dual* equation $\mathcal{G}^\gamma(Y) = Y\mathcal{R}^\gamma(-Y^{-1})Y = 0$ starting at an arbitrary point $Y_0 \in \mathcal{S}^n$, where the eigenvalues of the derivative $\mathcal{G}'_{Y_0}$ lie in the open left half plane; then $X_+ = -Y_+^{-1}$, if Y_+ denotes the limit of the Newton iteration. We have $\sigma(\mathcal{G}'_{Y_0}) \subset \mathbb{C}_-$, e.g. if $Y_0 = \nu X_0^{-1}$, where X_0 is the stabilizing solution of the standard Riccati equation $A^T X + XA - C^T C + X^2 = 0$, and $\nu > 0$ is sufficiently large (see [3]).

2. A car-steering problem

We consider the four-wheel car-steering model discussed in [1]. An important feature of this model is the uncertainty of the adhesion coefficient μ between tyre and road-surface. Here, we model the coefficient μ as a stochastic process with mean value 0.5 and intensity σ. Thus the extended state-space system [1, p.18] takes the stochastic form

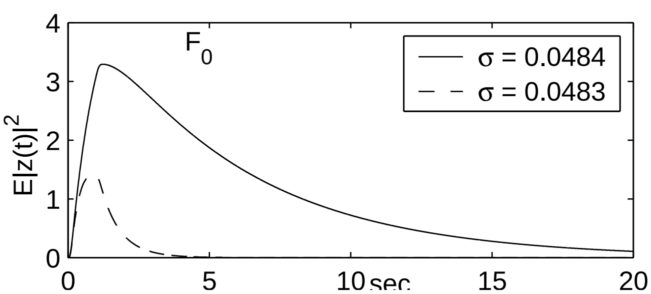
$$dx = (Ax + B_1 v + B_2 u)\,dt + (A_0^{(1)}x + B_{20}^{(1)}u)\sigma\,dw_1 + (A_0^{(2)}x + B_{20}^{(2)}u)\sigma\,dw_2\,, \quad z = Cx\,,$$

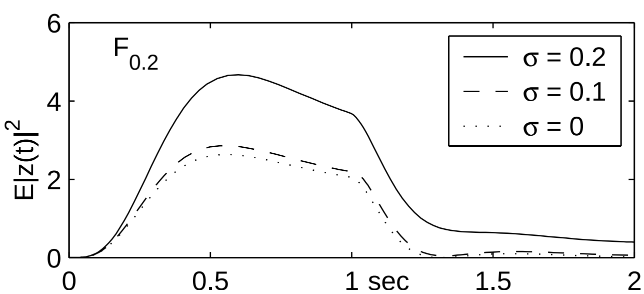
with $A = \begin{bmatrix} -\frac{c_r+c_f}{2mv} & \frac{c_r\ell_r - c_f\ell_f}{2mv^2} - 1 & 0 & 0 \\ \frac{c_r\ell_r + c_f\ell_f}{2J} & -\frac{c_r\ell_r^2 + c_f\ell_f^2}{2Jv} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ v & \ell_0 & v & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ 0 \\ v \\ 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} \frac{c_f}{2mv} & \frac{c_r}{2mv} \\ \frac{c_f\ell_f}{2J} & \frac{-c_r\ell_r}{2J} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T$,

$A_0^{(1)} = \begin{bmatrix} \frac{-c_f}{mv} & \frac{-c_f\ell_f}{mv^2} & 0 & 0 \\ \frac{c_f\ell_f}{J} & \frac{-c_f\ell_f^2}{Jv} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A_0^{(2)} = \begin{bmatrix} \frac{-c_r}{mv} & \frac{c_r\ell_r}{mv^2} & 0 & 0 \\ \frac{c_r\ell_r}{J} & \frac{-c_r\ell_r^2}{Jv} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $B_{20}^{(1)} = \begin{bmatrix} \frac{c_f}{mv} & 0 \\ \frac{c_f\ell_f}{J} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B_{20}^{(2)} = \begin{bmatrix} 0 & \frac{c_r}{mv} \\ 0 & \frac{-c_r\ell_r}{J} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

The state vector $x = [\beta, r, \Delta, \mu]^T$ contains the sideslip angle β, the yaw rate r, the displacement angle Δ and the measured displacement μ. The front and rear steering angles $[\delta_f, \delta_r]^T = u$ are the control variables. We assume that the car is to follow a straight line ($\rho_{\text{ref}} = 0$), such that all deviations v from the nominal value ρ_{ref} are regarded as an exogenous disturbance, whose effect on the output z has to be diminished. We regularize the system by introducing the additional output εu with $\varepsilon = 0.1$. The parameter values are $\ell_f = 3.67\,[\text{m}]$, $\ell_r = 1.93\,[\text{m}]$, $\ell_0 = 6.12\,[\text{m}]$, $c_f = 198000\,[\text{N/rad}]$, $c_r = 470000\,[\text{N/rad}]$, $v = 10[\text{m/s}]$, $m = 10000[\text{kg}]$, $J = i^2\mu m\,, i^2 = 10.85[\text{m}^2]$.
For $\sigma = 0$, $\sigma = 0.2$ we found the optimal attenuation values $\gamma_0 = 1.62$, $\gamma_{0.2} = 4.58$ and the feedback-gain matrices

$$F_0 = \begin{bmatrix} -64.8261 & -509.0833 & -365.0595 & 12.2410 \\ 7.4033 & 118.1471 & 78.9999 & -13.2715 \end{bmatrix}, \qquad F_{0.2} = \begin{bmatrix} -4.1149 & -3.4930 & -5.0236 & -2.0495 \\ -1.9438 & -1.7667 & -3.0303 & -1.3690 \end{bmatrix}.$$





In the diagrams, we see the response of the controlled systems to the input signal $v(t) = 1$ for $t \in [0, 1]$, $v(t) = 0$ otherwise. While in the absence of noise, F_0 has better attenuation properties than $F_{0.2}$, it deteriorates rapidly as the noise increases and even fails to stabilize the system for 0.0485. In contrast, $F_{0.2}$ stabilizes the system and still has acceptable attenuation properties even for $\sigma = 0.2$. (Notice the different scalings.)

3. References

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