Positive groups on $\mathcal{H}^n$ are completely positive

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Abstract

We prove that an operator generates a positive group on the real space of real or complex Hermitian matrices, if and only if it is a Lyapunov operator. In particular it follows that every group of positive operators in fact is a group of completely positive operators.

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1 Notation and problem statement

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ let $\mathcal{H}^n \subset \mathbb{K}^{n\times n}$ denote the real space of $n \times n$ real or complex Hermitian matrices, and $\mathcal{H}^n_+ \subset \mathcal{H}^n$ the set of positive semidefinite matrices. It is well-known that $\mathcal{H}^n$ together with the inner product $\langle X, Y\rangle = \operatorname{trace} XY$ is a Hilbert space and $\mathcal{H}^n_+$ is a self-dual closed normal solid convex cone (e.g. [8]). We write A^* for the conjugate transpose of a matrix $A \in \mathbb{K}^{n\times n}$.

Definition 1.1 *A linear operator $T : \mathcal{H}^n \to \mathcal{H}^n$ is said to be*

(i) positive*, if $T(H^n_+) \subset \mathcal{H}^n_+$,*

(ii) completely positive*, if T has a representation of the form*

$$T : X \quad \mapsto \quad \sum_{j=1}^{N} A_j X A_j^*$$

with some $N \in \mathbb{N}$ and matrices $A_j \in \mathbb{K}^{n\times n}$,

(iii) a generator of a (completely) positive group*, if $e^{tT} : \mathcal{H}^n \to \mathcal{H}^n$ is (completely) positive for all $t \in \mathbb{R}$,*

(iv) a Lyapunov operator*, if there exists an $A \in \mathbb{K}^{n\times n}$ such that for all $X \in \mathcal{H}^n$*

$$T(X) \quad = \quad AX + XA^* \; .$$

In this case we write $T = L_A$.

Clearly, every completely positive operator is positive but not vice versa. Counterexamples can be found e.g. in [4, 3].

Our principal aim is to prove that Lyapunov operators and generators of positive groups on $\mathcal{H}^n$ are the same.

Theorem 1.2 *A linear operator $T : \mathcal{H}^n \to \mathcal{H}^n$ is a generator of a positive group if and only if T is a Lyapunov operator.*

A close relative of this result has been proven by Lindblad [9], see also [1]. In our terms, it states that a linear operator $T : \mathcal{H}^n \to \mathcal{H}^n$ generates a *completely* positive group if and only if T is a Lyapunov operator. Our Theorem 1.2 is therefore equivalent to the assertion in the title of this paper.

2 Exponentially positive operators

Both for the proof and further interpretation of Theorem 1.2, it is useful to recall the following definitions from [10, 5, 2].

Definition 2.1 *A linear operator $T : \mathcal{H}^n \to \mathcal{H}^n$ is said to be*

(i) exponentially positive*, if $e^{tT} : \mathcal{H}^n \to \mathcal{H}^n$ is positive for all $t \geq 0$.*

(ii) resolvent positive*, if $(\alpha I - T)^{-1}$ is positive for sufficiently large $\alpha > 0$,*

(iii) quasi-monotonic*, if for all $X \in \mathcal{H}^n_+$ there exists a $Y \in \mathcal{H}^n_+$ such that $\langle X, Y\rangle = 0$ and $\langle T(X), Y\rangle \geq 0$,*

(iv) cross-positive*, if $\langle X, Y\rangle = 0$ for $X, Y \in H^n_+$ implies $\langle T(X), Y\rangle \geq 0$,*

(v) essentially positive*, if $T \in \operatorname{cl}\{S - \alpha I \mid S : \mathcal{H}^n \to \mathcal{H}^n$ is positive$\}$,*

For general finite-dimensional vector spaces ordered by a closed normal solid cone, it was shown by Elsner in [5] that all these properties are equivalent.

Theorem 2.2 *For a linear operator $T : \mathcal{H}^n \to \mathcal{H}^n$ the properties (i)–(v) from Definition 2.1 are equivalent.*

It is obvious that T generates a positive group on $\mathcal{H}^n$, if and only if both T and $-T$ are exponentially positive. Hence we have the following criterion.

Lemma 2.3 *A linear operator $T : \mathcal{H}^n \to \mathcal{H}^n$ generates a positive group, if and only if $\langle X, Y\rangle = 0$ for $X, Y \in H^n_+$ implies $\langle T(X), Y\rangle = 0$.*

Proof: By Theorem 2.2 the operator $T : \mathcal{H}^n \to \mathcal{H}^n$ generates a positive group, if and only if both T and $-T$ are cross-positive. The latter holds if and only if $\langle X, Y\rangle = 0$ for $X, Y \in H^n_+$ implies both $\langle T(X), Y\rangle \geq 0$ and $-\langle T(X), Y\rangle \geq 0$. □

Using the concept of resolvent positivity, one easily verifies that all Lyapunov operators generate positive groups.

Lemma 2.4 *Every Lyapunov operator generates a positive group on $\mathcal{H}^n$.*

Proof: Let $T = L_A$ for some $A \in \mathbb{K}^{n\times n}$. By Lyapunov's Theorem (e.g. [7]), L_A^{-1} is positive if $\sigma(L_A) \subset \mathbb{C}_+$. Thus $(\alpha I - L_A)^{-1} = (L_{-A+\frac{\alpha}{2}I})^{-1}$ is positive for sufficiently large $\alpha \in \mathbb{R}$. Hence T is resolvent positive and thus exponentially positive. Since $-T = L_{-A}$ is a Lyapunov operator, too, it is exponentially positive as well. By Lemma 2.3 it follows that T generates a positive group. □

Remark 2.5 *(i) It is immediate to see that the set of cross-positive operators $T : \mathcal{H}^n \to \mathcal{H}^n$ is a solid convex cone. Theorem 1.2 characterizes the maximal subspace contained in this cone as the set of Lyapunov operators.*

(ii) The role of Lyapunov operators in stability theory is well-known. If for instance we consider a homogeneous linear deterministic differential equation $\dot{X} = AX$, $X(0) = X_0$, with the solution $X(t)$ for $t \in \mathbb{R}$, then $P(t) = X(t)X(t)^*$ satisfies the equation $\dot{P} = L_A(P)$. Since $P(t) \geq 0$ for all $t \in \mathbb{R}$, we conclude again that L_A generates a positive group.

(iii) If we consider the stochastic differential equation of Itô type

$$dX = AX\,dt + \sum_{j=1}^{N} A_0^{(j)} X\,dw_j, \quad X(0) = X_0\,, \tag{1}$$

then for $t \geq 0$ the second moment $P(t) = E(X(t)X(t)^*) \geq 0$ satisfies

$$\dot{P} \;=\; L_A(P) + \sum_{j=1}^{N} A_0^{(j)} P A_0^{(j)*}\,. \tag{2}$$

Hence, the right-hand side of (2) generates a positive semigroup, but – in general – not a positive group, since (1) cannot be solved for $t < 0$.

Before proceeding with the proof of Theorem 1.2, we verify that the situation is different, if T is a *discrete-time Lyapunov operator* or *Stein operator*. This means that there exists an $A \in \mathbb{K}^{n\times n}$ such that

$$\forall X \in \mathcal{H}^n: \quad T(X) \;=\; A^*XA - X\,. \tag{3}$$

In this case we also write $T = S_A$. It follows from Theorem 2.2 (v) that S_A is resolvent positive. But this is not necessarily the case for $T = -S_A$ as the following example shows.

Example 1 Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $X_t = \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix}$ for $t > 0$.

For all $\alpha > 0$ we have $\alpha X_t + S_A(X_t) = \begin{bmatrix} t & \alpha - 1 \\ \alpha - 1 & \alpha t \end{bmatrix}$, which is positive for large t though X_t is always indefinite. Hence $(\alpha I + S_A)^{-1}$ is not positive for any α, and hence $-S_A$ is not resolvent positive.

3 Proof of Theorem 1.2

It remains to show that a generator of a positive group on $\mathcal{H}^n$ is necessarily a Lyapunov operator. We distinguish between the real case $\mathbb{K} = \mathbb{R}$ and the complex case $\mathbb{K} = \mathbb{C}$. Though the idea in both cases is basically the same, the complex case is technically more involved and requires some extra considerations.

3.1 The real case

Let T generate a positive group. By Lemma 2.3 this is equivalent to

$$\Big(X, Y \geq 0 \quad \text{and} \quad \langle X, Y\rangle = 0\Big) \quad \Rightarrow \quad \langle TX, Y\rangle = 0\,. \tag{4}$$

If e_j denotes the j-th canonical unit vector in $\mathbb{R}^n$, then the set

$$B := \{e_j e_k^T + e_k e_j^T \mid 1 \leq j \leq k \leq n\} \subset \mathcal{H}^n \tag{5}$$

forms a basis of $\mathcal{H}^n \subset \mathbb{R}^{n\times n}$.
It suffices to find an $A \in \mathbb{R}^{n\times n}$ such that $T(X) = L_A(X)$ for all $X \in B$.

Let $X = e_j e_j^T$ (i.e. $2X \in B$). To apply criterion (4) we characterize all matrices

$$Y \in \mathcal{H}_+^n \quad \text{such that} \quad \langle X, Y\rangle = 0\,. \tag{6}$$

Let $Y \geq 0$ and $\langle X, Y\rangle = y_{jj} = 0$. Then necessarily the j-th row and column in Y vanish. Hence, (6) is true if and only if in $Y \geq 0$ the j-th row and column vanish. Criterion (4) in turn implies that in $T(X)$ everything vanishes except for the j-th row and column. Otherwise we could choose some Y satisfying (6) and $\langle T(X), Y\rangle \neq 0$.
For $j = 1, \ldots, n$ we thus have $T(e_j e_j^T) = a_j e_j^T + e_j a_j^T$ with vectors $a_1, \ldots, a_n \in \mathbb{R}^n$. If we build the matrix $A = (a_1, \ldots, a_n)$, then $T(X) = AX + XA^T$ for all $X = e_j e_j^T$. In other words, we have found a unique candidate for the Lyapunov operator.
It remains to show that also for $X_{jk} = e_j e_k^T + e_k e_j^T$ with $j < k$ we have

$$\begin{aligned} T(X_{jk}) &= AX_{jk} + X_{jk}A^T = a_j e_k^T + a_k e_j^T + e_j a_j^T + e_k a_k^T \qquad (7) \\ &= \begin{bmatrix} & & a_{1k} & \cdots & a_{1j} & & \\ & & \vdots & & \vdots & & \\ a_{1k} & \cdots & 2a_{jk} & \cdots & a_{jj}+a_{kk} & \cdots & a_{nk} \\ & & \vdots & & \vdots & & \\ a_{1j} & \cdots & a_{jj}+a_{kk} & \cdots & 2a_{kj} & \cdots & a_{nj} \\ & & \vdots & & \vdots & & \\ & & a_{nk} & \cdots & a_{nj} & & \end{bmatrix}. \end{aligned}$$

Let j and k be fixed. A matrix Y satisfies condition (6) with $X = X_{jk} + X_{jj} + X_{kk} \geq 0$ if in Y the j-th and k-th row and column vanish. As above we conclude from criterion (4) that in $T(X)$ and hence also in $T(X_{jk})$ everything vanishes except for the j-th and k-th row and column. Thus $T(X_{jk})$ is of the general form

$$\begin{aligned} T(X_{jk}) &= b_j e_j^T + e_j b_j^T + b_k e_k^T + e_k b_k^T, \quad \text{with } b_j, b_k \in \mathbb{R}^n \qquad (8) \\ &= \begin{bmatrix} & & b_{1j} & \cdots & b_{1k} & & \\ & & \vdots & & \vdots & & \\ b_{1j} & \cdots & 2b_{jj} & \cdots & b_{jk}+b_{kj} & \cdots & b_{nj} \\ & & \vdots & & \vdots & & \\ b_{1k} & \cdots & b_{jk}+b_{kj} & \cdots & 2b_{kk} & \cdots & b_{nk} \\ & & \vdots & & \vdots & & \\ & & b_{nj} & \cdots & b_{nk} & & \end{bmatrix} . \end{aligned}$$

Now we consider matrices of the form $X = xx^T$ with $x = x_j e_j + x_k e_k$ where $x_j, x_k \in \mathbb{R}$ are arbitrary real numbers. Writing

$$X = x_j x_k X_{jk} + x_j^2 e_j e_j^T + x_k^2 e_k e_k^T ,$$

and exploiting the linearity of T we have the decomposition

$$\begin{aligned} T(X) &= x_j x_k T(X_{jk}) + x_j^2 T(e_j e_j^T) + x_k^2 T(e_k e_k^T) \qquad (9) \\ &= x_j x_k (b_j e_j^T + e_j b_j^T + b_k e_k^T + e_k b_k^T) \\ &\quad + x_j^2 (a_j e_j^T + e_j a_j^T) + x_k^2 (a_k e_k^T + e_k a_k^T) . \end{aligned}$$

Let $y \perp x$, for instance

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \quad \text{with} \quad \begin{cases} y_j = x_k , \\ y_k = -x_j , \\ y_\ell \text{ arbitrary for } \ell \notin \{j,k\} . \end{cases} \qquad (10)$$

Then $Y = yy^T$ satisfies condition (6), and by (4) we have $\langle T(X), Y \rangle = 0$.
If we write $T(X)$ like in equation (9) we obtain:

$$\begin{aligned} 0 &= \frac{1}{2} \langle T(X), Y \rangle = \frac{1}{2} \operatorname{trace}(T(X)Y) = \frac{1}{2} y^T T(X) y \\ &= x_j x_k \left(y_j \sum_{\ell=1}^n b_{\ell j} y_\ell + y_k \sum_{\ell=1}^n b_{\ell k} y_\ell \right) + x_j^2 y_j \sum_{\ell=1}^n a_{\ell j} y_\ell + x_k^2 y_k \sum_{\ell=1}^n a_{\ell k} y_\ell \\ &= x_j x_k^3 (b_{jj} - a_{jk}) + x_j^3 x_k (b_{kk} - a_{kj}) + x_j^2 x_k^2 (-b_{kj} - b_{jk} + a_{jj} + a_{kk}) \\ &\quad + x_j x_k^2 \sum_{\ell \notin \{j,k\}} y_\ell (b_{\ell j} - a_{\ell k}) + x_j^2 x_k \sum_{\ell \notin \{j,k\}} y_\ell (-b_{\ell k} + a_{\ell j}) . \end{aligned}$$

The right hand side is a homogeneous polynomial in the real unknowns x_j, x_k, and y_ℓ for $\ell \notin \{j,k\}$. Since these unknowns can be chosen arbitrarily, all the coefficients of the polynomial necessarily vanish, i.e.

$$\begin{aligned} b_{jj} &= a_{jk} \ , \\ b_{kk} &= a_{kj} \ , \\ b_{kj} + b_{jk} &= a_{jj} + a_{kk} \ , \\ b_{\ell j} &= a_{\ell k} \ , \\ b_{\ell k} &= a_{\ell j} \ . \end{aligned}$$

Inserting these data into (8), we see that (7) holds.
This concludes the proof for the real case.

3.2 The complex case

In the complex case $\mathbb{K} = \mathbb{C}$ the computations are a little bit more involved, because we have to deal with real and imaginary parts (which are denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$ for $z \in \mathbb{C}$). It has to be noted that now $\dim \mathcal{H}^n = n^2$ while in the real case the dimension was $n(n+1)/2$. In particular B from (5) must be completed to a basis $B \cup B_i$ by

$$B_i = \{ie_je_k^* - ie_ke_j^* \mid 1 \le j < k \le n\} \subset \mathcal{H}^n \ . \tag{11}$$

Like in the real case we obtain a candidate for the Lyapunov operator L_A with $A = (a_1, \ldots, a_n)^* \in \mathbb{C}^{n\times n}$ from the relations $T(e_je_j^*) = a_je_j^* + e_ja_j^*$. But this candidate is not unique in the complex case; we can add an arbitrary diagonal matrix with purely imaginary entries to A without changing $(e_je_j^*)A + A^*(e_je_j^*)$. In particular, for arbitrarily given real numbers $\beta_1, \ldots, \beta_{n-1} \in \mathbb{R}$ we can choose the a_{jj} such that

$$\operatorname{Im}(a_{jj} - a_{nn}) \quad = \quad \beta_j, \quad j = 1, \ldots, n-1 \ . \tag{12}$$

This will be needed at the end of the proof. For the moment, we choose *some* matrix $A = (a_1, \ldots, a_n)^* \in \mathbb{C}^{n\times n}$ such that $T(e_je_j^*) = a_je_j^* + e_ja_j^*$ for $j = 1, \ldots, n$.
For $X_{jk} = e_je_k^* + e_ke_j^* \in B$ and $X_{jk}^i = ie_je_k - ie_ke_j^* \in B_i$ we have to verify that

$$T(X_{jk}) \quad = \quad AX_{jk} + X_{jk}A^* = a_je_k^* + a_ke_j^* + e_ja_j^* + e_ka_k^* \tag{13}$$

$$= \begin{bmatrix} & & a_{1k} & \cdots & a_{1j} & & \\ & & \vdots & & \vdots & & \\ \bar{a}_{1k} & \cdots & 2\operatorname{Re} a_{jk} & \cdots & a_{jj} + \bar{a}_{kk} & \cdots & \bar{a}_{nk} \\ & & \vdots & & \vdots & & \\ \bar{a}_{1j} & \cdots & \bar{a}_{jj} + a_{kk} & \cdots & 2\operatorname{Re} a_{kj} & \cdots & \bar{a}_{nj} \\ & & \vdots & & \vdots & & \\ & & a_{nk} & \cdots & a_{nj} & & \end{bmatrix}$$

and

$$
\begin{aligned}
T(X_{jk}^i) &= AX_{jk}^i + X_{jk}^i A^* = ia_j e_k^* - ia_k e_j^* + ie_j a_k^* - ie_k a_j^* \qquad (14)\\
&= \begin{bmatrix}
 & & -ia_{1k} & \cdots & ia_{1j} & & \\
 & & \vdots & & \vdots & & \\
i\bar{a}_{1k} & \cdots & -ia_{jk} + i\bar{a}_{jk} & \cdots & ia_{jj} + i\bar{a}_{kk} & \cdots & i\bar{a}_{nk} \\
 & & \vdots & & \vdots & & \\
-i\bar{a}_{1j} & \cdots & -i\bar{a}_{jj} - ia_{kk} & \cdots & -i\bar{a}_{kj} + ia_{kj} & \cdots & \bar{a}_{nj} \\
 & & \vdots & & \vdots & & \\
 & & -ia_{nk} & \cdots & ia_{nj} & &
\end{bmatrix} .
\end{aligned}
$$

Let j and k be fixed. Like in the real case we conclude that $T(X_{jk})$ is of the general form

$$
\begin{aligned}
T(X_{jk}) &= b_j e_j^* + e_j b_j^* + b_k e_k^* + e_k b_k^*, \quad \text{with } b_j, b_k \in \mathbb{R}^n \\
&= \begin{bmatrix}
 & & b_{1j} & \cdots & b_{1k} & & \\
 & & \vdots & & \vdots & & \\
\bar{b}_{1j} & \cdots & 2\operatorname{Re} b_{jj} & \cdots & b_{jk} + \bar{b}_{kj} & \cdots & \bar{b}_{nj} \\
 & & \vdots & & \vdots & & \\
\bar{b}_{1k} & \cdots & \bar{b}_{jk} + b_{kj} & \cdots & 2\operatorname{Re} b_{kk} & \cdots & \bar{b}_{nk} \\
 & & \vdots & & \vdots & & \\
 & & b_{nj} & \cdots & b_{nk} & &
\end{bmatrix} .
\end{aligned}
$$

Clearly $T(X_{jk}^i)$ has the same form with b_j, b_k replaced by some $c_j, c_k \in \mathbb{C}^n$.

Now we consider matrices of the form $X = xx^*$ with $x = x_j e_j + x_k e_k$ where $x_j, x_k \in \mathbb{C}$ are arbitrary complex numbers. Writing

$$
X = \operatorname{Re}(\bar{x}_j x_k) X_{jk} - \operatorname{Im}(\bar{x}_j x_k) X_{jk}^i + |x_j|^2 e_j e_j^* + |x_k|^2 e_k e_k^* ,
$$

we have the decomposition

$$
\begin{aligned}
T(X) = \ & \operatorname{Re}(\bar{x}_j x_k)(b_j e_j^* + e_j b_j^* + b_k e_k^* + e_k b_k^*) \\
& - \operatorname{Im}(\bar{x}_j x_k)(c_j e_j^* + e_j c_j^* + c_k e_k^* + e_k c_k^*) \\
& + |x_j|^2 (a_j e_j^* + e_j a_j^*) + |x_k|^2 (a_k e_k^* + e_k a_k^*) . \qquad (15)
\end{aligned}
$$

Similarly as in (10) we choose $Y = yy^*$ with

$$
y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{C}^n \quad \text{and} \quad \begin{cases} y_j = \bar{x}_k , \\ y_k = -\bar{x}_j , \\ y_\ell \text{ arbitrary for } \ell \notin \{j, k\} , \end{cases}
$$

such that $y \perp x$. Then

$$\begin{aligned}
0 &= \frac{1}{2}\langle T(X), Y\rangle = \frac{1}{2}\,\mathrm{trace}(T(X)Y) = \frac{1}{2}\,y^*T(X)y \\
&= \mathrm{Re}(\bar{x}_j x_k)\,\mathrm{Re}\Big(y_j \sum_{\ell=1}^{n} b_{\ell j}\bar{y}_\ell + y_k \sum_{\ell=1}^{n} b_{\ell k}\bar{y}_\ell\Big) \\
&\quad - \mathrm{Im}(\bar{x}_j x_k)\,\mathrm{Re}\Big(y_j \sum_{\ell=1}^{n} c_{\ell j}\bar{y}_\ell + y_k \sum_{\ell=1}^{n} c_{\ell k}\bar{y}_\ell\Big) \\
&\quad + |x_j|^2\,\mathrm{Re}\Big(y_j \sum_{\ell=1}^{n} a_{\ell j}\bar{y}_\ell\Big) + |x_k|^2\,\mathrm{Re}\Big(y_k \sum_{\ell=1}^{n} a_{\ell k}\bar{y}_\ell\Big) \\
&= \mathrm{Re}(\bar{x}_j x_k)\,\mathrm{Re}\Big(|x_k|^2 b_{jj} - \bar{x}_k x_j b_{kj} + \bar{x}_k \sum_{\ell\notin\{j,k\}} b_{\ell j}\bar{y}_\ell \\
&\qquad\qquad - \bar{x}_j x_k b_{jk} + |x_j|^2 b_{kk} - \bar{x}_j \sum_{\ell\notin\{j,k\}} b_{\ell k}\bar{y}_\ell\Big) \\
&\quad - \mathrm{Im}(\bar{x}_j x_k)\,\mathrm{Re}\Big(|x_k|^2 c_{jj} - \bar{x}_k x_j c_{kj} + \bar{x}_k \sum_{\ell\notin\{j,k\}} c_{\ell j}\bar{y}_\ell \\
&\qquad\qquad - \bar{x}_j x_k c_{jk} + |x_j|^2 c_{kk} - \bar{x}_j \sum_{\ell\notin\{j,k\}} c_{\ell k}\bar{y}_\ell\Big) \\
&\quad + |x_j|^2\,\mathrm{Re}\Big(|x_k|^2 a_{jj} - \bar{x}_k x_j a_{kj} + \bar{x}_k \sum_{\ell\notin\{j,k\}} a_{\ell j}\bar{y}_\ell\Big) \\
&\quad + |x_k|^2\,\mathrm{Re}\Big(- \bar{x}_j x_k a_{jk} + |x_j|^2 a_{kk} + \bar{x}_j \sum_{\ell\notin\{j,k\}} a_{\ell k}\bar{y}_\ell\Big)\,.
\end{aligned}$$

We distinguish between several special cases, where the variables x_j, x_k and y_ℓ are of the form $x_j = \xi_j z_i$, $x_k = \xi_k z_k$, $y_\ell = \eta_\ell z_\ell$ with fixed complex numbers z_j, z_k, z_ℓ and real variables ξ_j, ξ_k, η_ℓ. In these cases the right hand side is a homogeneous, identically vanishing polynomial in ξ_j, ξ_k, η_ℓ. By inspecting the coefficients at different monomials we obtain relations between the entries of A, b_j, b_k, c_j and c_k. For each case considered, we provide a list of some monomials and the corresponding vanishing coefficients.

(i) Case $x_j = \xi_j, x_k = \xi_k$.

	$y_\ell = \eta_\ell$	$y_\ell = i\eta_\ell$
$\xi_j\xi_k^3$	$\mathrm{Re}(b_{jj} - a_{jk}) = 0$	
$\xi_j^3\xi_k$	$\mathrm{Re}(b_{kk} - a_{kj}) = 0$	
$\xi_j^2\xi_k^2$	$\mathrm{Re}(-b_{kj} - b_{jk} + a_{jj} + a_{kk}) = 0$	
$\xi_j\xi_k^2\eta_\ell$	$\mathrm{Re}(b_{\ell j} - a_{\ell k}) = 0$	$\mathrm{Im}(b_{\ell j} - a_{\ell k}) = 0$
$\xi_j^2\xi_k\eta_\ell$	$\mathrm{Re}(-b_{\ell k} + a_{\ell j}) = 0$	$\mathrm{Im}(-b_{\ell k} + a_{\ell j}) = 0$

Hence b_{jj}, b_{kk} and $b_{\ell j}$, $b_{\ell k}$ for all $\ell \notin \{j,k\}$ have the required form.

(ii) Case $x_j = \xi_j, x_k = i\xi_k$.

	$y_\ell = \eta_\ell$	$y_\ell = i\eta_\ell$
$\xi_j\xi_k^3$	$\mathrm{Re}(c_{jj}) - \mathrm{Im}(a_{jk}) = 0$	
$\xi_j^3\xi_k$	$\mathrm{Re}(c_{kk}) - \mathrm{Im}(a_{kj}) = 0$	
$\xi_j^2\xi_k^2$	$\mathrm{Im}(c_{jk} - c_{kj}) - \mathrm{Re}(a_{jj} + a_{kk}) = 0$	
$\xi_j\xi_k^2\eta_\ell$	$\mathrm{Re}(c_{\ell j}) - \mathrm{Im}(a_{\ell k}) = 0$	$\mathrm{Im}(c_{\ell j}) - \mathrm{Re}(a_{\ell k}) = 0$
$\xi_j^2\xi_k\eta_\ell$	$\mathrm{Im}(c_{\ell k}) + \mathrm{Re}(a_{\ell j}) = 0$	$\mathrm{Re}(c_{\ell k}) + \mathrm{Im}(a_{\ell j}) = 0$

Hence c_{jj}, c_{kk} and $c_{\ell j}$, $c_{\ell k}$ for all $\ell \notin \{j,k\}$ have the required form.

(iii) Case $x_j = \xi_j$, $x_k = (1+i)\xi_k$, $y_\ell = 0$.
Considering the coefficient at $\xi_j^2\xi_k^2$ we obtain

$$\begin{aligned} 0 &= \mathrm{Re}\Big(-(1-i)b_{kj} - (1+i)b_{jk} + 2a_{jj} \\ &\quad + (1-i)c_{kj} + (1+i)c_{jk} + 2a_{kk}\Big) \\ &= \mathrm{Re}(-b_{kj} - b_{jk} + a_{jj} + a_{kk}) + \mathrm{Im}(-b_{kj} + b_{jk}) \\ &\quad - \mathrm{Re}(-c_{kj} - c_{jk}) - \mathrm{Im}(-c_{kj} + c_{jk}) + \mathrm{Re}(a_{jj} - a_{kk}) \\ &= \mathrm{Im}(-b_{kj} + b_{jk}) - \mathrm{Re}(c_{kj} + c_{jk}). \end{aligned}$$

Here we have made use of the corresponding relations in the cases (i) and (ii).

The proof would be complete, if we could show that

$$\mathrm{Im}(-b_{kj} + b_{jk}) = \mathrm{Im}(a_{jj} - a_{kk}) = -\mathrm{Re}(c_{kj} + c_{jk}) .$$

But without any further specification of the $\mathrm{Im}\, a_{jj}$ this need not be true. Recall from (12) that there was some freedom in their choice. Notice further that the $\mathrm{Im}\, a_{jj}$ have played no role in our considerations so far.
Hence, to finish the proof, let us consider the difference $S = T - L_A$, which by assumption and Lemma 2.4 also satisfies (4). For $s_{jk} := \mathrm{Im}(-b_{kj} + b_{jk} - a_{jj} + a_{kk})$ we know from the cases (i), (ii), and (iii) that

$$S(X_{jk}) = s_{jk} X_{jk}^i \quad \text{and} \quad S(X_{jk}^i) = s_{jk} X_{jk} . \tag{16}$$

Since j and k were arbitrary, we conclude that for all $j < k$ there exist real numbers s_{jk} such that (16) holds. By the construction of A we have $S(X_{jj}) = 0$ for all $j = 1, \ldots, n$. As noted in (12) we can assume that $\mathrm{Im}(a_{jj} - a_{nn}) = \mathrm{Im}(-b_{nj} + b_{jn})$ for $1 \le j < n$, i.e.

$$s_{1n} = s_{2n} = \ldots = s_{n-1,n} = 0 .$$

This determines A up to an imaginary multiple of the identity matrix, which can always be added to A without changing L_A.
Now let $j < k < n$ be fixed again and set $x = e_j + ie_k + e_n$ and $y = e_j + e_k -$

$(1-i)e_n$ such that $x^*y = 0$. Taking into account that $S(X_{jj}) = S(X_{kk}) = S(X_{nn}) = 0$ and $s_{jn} = s_{kn} = 0$ we find

$$0 \quad = \quad y^* S(xx^*) y \;=\; y^* \Big(s_{jk} X_{jk} \Big) y \;=\; 2 s_{jk} \; .$$

Hence $S = 0$ and therefore $T = L_A$.
This concludes the proof for the complex case as well.

4 An open question

Following [6], we use the term *lineality space* for the maximal subspace contained in the cone of resolvent positive operators on some ordered vector space. In [11] the interesting question was posed whether every resolvent positive operator T can be represented as the sum

$$T \quad = \quad L + P \tag{17}$$

of a positive operator P and an element L from the lineality space. Although in [11] an affirmative answer could be given for important classes of cones, it was shown in [6] that this representation is impossible for almost all cones in a certain categorial sense.
Nevertheless, the question seems to be still open for the ordered vector space of Hermitian matrices. In view of our main result, we ask: Is it true that every exponentially positive operator on $\mathcal{H}^n$ can be decomposed in the sum of a Lyapunov operator and a positive operator?

Again, the analogous result for completely positive operators is already available in [9]: If $T : \mathcal{H}^n \to \mathcal{H}^n$ generates a semigroup of *completely* positive operators, then T possesses a so-called *Lindblad-decomposition* $T = L + P$, where L is a Lyapunov operator and P is *completely* positive.

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