On the parameter dependence of a class of rational matrix equations occuring in stochastic optimal control

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1 Introduction

This paper is concerned with rational matrix equations occuring in stochastic control that play an analogous role as the algebraic Riccati equation does in deterministic control. We will therefore sometimes refer to these equations as stochastic (algebraic) Riccati equations. A first rigorous treatment of a stochastic Riccati equation from LQ-control theory seems to have been undertaken by Wonham in [Won68]. Since then different versions of stochastic Riccati equations have been obtained in various control and stabilization problems (e.g. [Phi83], [Tes94], [DHS97], [HP98], [UP99], [YZ99]) and it is widely agreed that their analysis constitutes a challenging problem. *Algebraic* tools such as canonical transformations, Hamiltonians or factorization methods used for the deterministic Riccati equation, appear to be not available for the stochastic Riccati equation.

Our approach is of an *analytical* nature and based on a nonlocal convergence result for Newton's method applied to a certain class of nonlinear matrix equations. Resolvent positive and concave operators play a crucial role. We specify solvability conditions for stochastic Riccati equations of different kinds and analyze the dependence of their largest solution on parameters.

The paper is organized as follows. In Sec. 2 we introduce the stochastic Riccati operator and specify some notations. Then in Sec. 3 we discuss several problems of stochastic control that lead to stochastic Riccati inequalities with structurally different parameter matrices. These inequalities and the corresponding equations are discussed in Sec. 4, which contains our main results.

2 A class of rational matrix operators

We study matrix equations of the following form

$$\mathcal{R}(X) \quad = \quad P(X) - S(X)Q(X)^{-1}S(X)^* = 0. \qquad (1)$$

Here P, S and Q are affine linear operators on the real vector space $\mathcal{H}^n \subset \mathbb{K}^{n\times n}$ of $n \times n$ Hermitian matrices (with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$):

$$\begin{array}{rclcccc} P(X) & = & A^*X + XA & + & \Pi_1(X) & + & P_0 \\ S(X) & = & XB & + & \Sigma(X) & + & S_0 \\ Q(X) & = & & & \Pi_2(X) & + & Q_0, \end{array} \qquad (2)$$

where $A \in \mathbb{K}^{n\times n}$, $P_0 \in \mathcal{H}^n$, B, $S_0 \in \mathbb{K}^{n\times m}$, $Q_0 \in \mathcal{H}^m$. The spaces $\mathcal{H}^k$, $k \in \mathbb{N}$ are considered as ordered vector spaces; the order on $\mathcal{H}^k$ is induced by the cone $\mathcal{H}^n_+$ of nonnegative definite matrices (compare Section 4.1). We assume that the linear operators Π_1, Π_2, and Σ together form a positive linear operator (compare Definition 9)

$$\Pi : \mathcal{H}^n \to \mathcal{H}^{n+m}, \quad \Pi(X) = \begin{bmatrix} \Pi_1(X) & \Sigma(X) \\ \Sigma(X)^* & \Pi_2(X) \end{bmatrix}. \qquad (3)$$

In particular $\Pi_1 : \mathcal{H}^n \to \mathcal{H}^n$ and $\Pi_2 : \mathcal{H}^n \to \mathcal{H}^m$ must be positive.

The rational matrix operator $\mathcal{R}$ is a well-defined analytic operator on the domain

$$\operatorname{dom} \mathcal{R} = \{X \in \mathcal{H}^n \mid \det Q(X) \neq 0\}.$$

We further define $\operatorname{dom}_+ \mathcal{R} = \{X \in \mathcal{H}^n \mid Q(X) > 0\}$ and assume $\operatorname{dom}_+ \mathcal{R} \neq \emptyset$.

The dependence of the operator $\mathcal{R}$ on the matrices P_0, S_0 and Q_0 plays an important role in this paper. We collect these matrices in the 2×2 Hermitian block matrix

$$M \quad = \quad \begin{bmatrix} P_0 & S_0 \\ S_0^* & Q_0 \end{bmatrix} \in \mathcal{H}^{n+m}. \qquad (4)$$

If A, B and Π are fixed, we sometimes also write

$$\mathcal{R}^M(X) = P^M(X) - S^M(X)Q^M(X)^{-1}S^M(X)^* \qquad (5)$$

to highlight the dependence of the operators on M.

Matrix equations of type (1) contain as special cases the well known continuous-time and discrete-time algebraic Riccati equations that occur for example in the deterministic LQ-control problem or in the context of H^∞ optimal control. More complicated matrix equations of the form (1) occur in analogous control problems for linear systems involving state- and control-dependent multiplicative white noise.

3 Stochastic control systems

Regard the linear Itô differential equation

$$\begin{aligned} dx(t) &= Ax(t)dt + Bv(t)dt \qquad (6)\\ &\quad + \sum_{i=1}^{N} A_0^i x(t)dw_i(t) + \sum_{i=1}^{N} B_0^i(t)v(t)dw_i(t)\\ z(t) &= Cx(t) + Dv(t), \end{aligned}$$

where $(A, C, B, D) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{q\times m}$, and

$$(A_0^i, B_0^i) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}, \quad i = 1, \dots, N.$$

The $(w_i(t))_{t\in\mathbb{R}_+}$ are independent zero mean real Wiener processes on a probability space $(\Omega, \mathcal{F}, \mu)$ with respect to an increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$.
Let $L_w^2(\mathbb{R}_+, \mathbb{K}^m)$ denote the corresponding space of non-anticipating stochastic processes v with

$$\|v(\cdot)\|_{L_w^2}^2 := \mathcal{E}\left(\int_0^\infty \|v(t)\|^2 dt\right) < \infty,$$

where $\mathcal{E}$ denotes expectation.
The process v is regarded as the input of the given system and can either be viewed as a control or as a disturbance. It is known from Itô-theory, that for all $(x_0, v) \in \mathbb{K}^n \times L_w^2(\mathbb{R}_+, \mathbb{K}^m)$ there exists a unique solution $x(\cdot, x_0, v)$ of (6) with initial value $x(0, x_0, v) = x_0$ and thus also a unique output process $z(\cdot, x_0, v)$.
We write $z(\cdot, 0, v) = \boldsymbol{L}v(\cdot)$, and call $\boldsymbol{L}$ the *input/output operator* of the system (6). It describes the effect of the input process v on the output process z.

Definition 1 *The system (6) is said to be* internally (exponentially mean square) stable *if $x(\cdot, x_0, 0) \in L_w^2(\mathbb{R}_+, \mathbb{K}^m)$ for all $x_0 \in \mathbb{K}^n$, or equivalently if*

$$\exists M, \omega > 0 : \forall x_0 \in \mathbb{K}^n,\, t \geq 0 : \mathcal{E}|x(t)|^2 \leq Me^{-\omega t}|x_0|^2,$$

where $x(\cdot) = x(\cdot, x_0, 0)$ is the solution of the unperturbed system (with $v(\cdot) \equiv 0$).
We call the system (6) externally stable *if $\boldsymbol{L}$ is a bounded operator $\boldsymbol{L}$: $L_w^2(\mathbb{R}_+, \mathbb{K}^m) \to L_w^2(\mathbb{R}_+, \mathbb{K}^q)$.*
The system (6) is called stabilizable *(by static linear state-feedback) if there exists a matrix $F \in \mathbb{K}^{m\times n}$ such that the closed loop system*

$$dx(t) = (A + BF)x(t)dt + \sum_{i=1}^{N}(A_0^i + B_0^i F)x(t)dw_i(t)$$

is internally stable.

In the following we will give characterizations of stability and stabilizability involving linear matrix equations of Lyapunov type and rational matrix equations of the form (1). The positive operator Π from (3) is given by

$$\Pi(X) = \sum_{i=1}^{N} \begin{bmatrix} A_0^{i*} X A_0^i & A_0^{i*} X B_0^i \\ B_0^{i*} X A_0^i & B_0^{i*} X B_0^i \end{bmatrix}, \qquad (7)$$

i.e. $\Pi_1(X) = \sum_{i=1}^N A_0^{i*} X A_0^i$, $\Sigma(X) = \sum_{i=1}^N A_0^{i*} X B_0^i$, $\Pi_2(X) = \sum_{i=1}^N B_0^{i*} X B_0^i$. Throughout this section we suppose that the operators $\Pi, \Pi_1, \Pi_2, \Sigma$ are defined by these formulae.

3.1 A stability criterion

Khasminskij's criterion for mean-square stability is a stochastic analog of Lyapunov's criterion for asymptotic stability of time-invariant linear differential equations. For convenience of notation we define the Lyapunov operator $\mathcal{L}_A : \mathcal{H}^n \to \mathcal{H}^n$ by $\mathcal{L}_A(X) := A^*X + XA$.

Theorem 2 ([Kha80]) *The following are equivalent:*

1. *System (6) is internally stable.*
2. *$\exists X < 0 : \quad \mathcal{L}_A(X) + \Pi_1(X) > 0$.*
3. *$\forall Y > 0 : \exists! X < 0 : \quad \mathcal{L}_A(X) + \Pi_1(X) = Y$.*
4. *$\sigma(\mathcal{L}_A + \Pi_1) \subset \mathbb{C}_-$.*

Remark 3 (i) Theorem 2 reduces to Lyapunov's stability theorem for deterministic linear differential equations if $\Pi_1 = 0$. Thus the inequality

$$\mathcal{L}_A(X) + \Pi_1(X) > 0, \qquad (8)$$

may be viewed as a generalization of Lyapunov's inequality. Given a solution $X < 0$ of (8), $V(\xi) := -\langle \xi, X\xi \rangle$ yields a Lyapunov function for the unperturbed system (6) (with $v \equiv 0$).
(ii) If $x(t)$ is a solution of the unperturbed system, then the operator $\mathcal{L}_A + \Pi_1$ describes the dynamics of the covariance matrix $Q(t) = \mathcal{E}x(t)x(t)^*$:

$$\dot{Q} = (\mathcal{L}_A + \Pi_1)(Q). \qquad (9)$$

By Theorem 2 system (6) is internally stable if and only if system (9) is asymptotically stable.

3.2 Stabilization and a rational matrix inequality

We try to find a feedback-gain matrix F such that the closed-loop system

$$dx = (A + BF)x\, dt + \sum_{j=1}^{N}(A_0^j + B_0^j F)x\, dw_j \qquad (10)$$

is mean-square stable. By Khasminskij's criterion the closed-loop system is mean-square stable if and only if for some matrix $Y \geq 0$ (and thus for all matrices $Y \geq 0$) there exists an $X < 0$, such that

$$\mathcal{L}_{A+BF}(X) + \begin{bmatrix} I \\ F \end{bmatrix}^* \Pi(X) \begin{bmatrix} I \\ F \end{bmatrix} > Y, \quad (11)$$

where $\Pi(X)$ is defined by (7). But if we want to characterize stabilizability, this condition is not really useful, since it contains two unknowns, the feedback-gain matrix F and the matrix X. In the following we will get rid of F by a completion of the square, where we make use of the fact, that the matrix $Y \geq 0$ can be chosen arbitrarily. In order to characterize the stabilizability of (6) in terms of a matrix inequality of the form $\mathcal{R}^M > 0$ (see 5) we suppose that $M \leq 0$ is any matrix of the form (4) with $Q_0 < 0$.

For $X < 0$ we define P, S, Q and $\mathcal{R} = \mathcal{R}^M$ by (2) and (1), respectively. Then $Q(X) \leq Q_0 < 0$ and hence $X \in \operatorname{dom} \mathcal{R}$. After adding and subtracting the term $P_0 - S(X)Q(X)^{-1}S(X)^*$ on the left-hand side of (11) and some reorganization we obtain

$$\begin{aligned} &\mathcal{L}_{A+BF}(X) + \begin{bmatrix} I \\ F \end{bmatrix}^* (\Pi(X) + M) \begin{bmatrix} I \\ F \end{bmatrix} \quad (12) \\ &= \mathcal{L}_A(X) + \Pi_1(X) + P_0 - S(X)Q(X)^{-1}S(X)^* \\ &+ \left(F + Q(X)^{-1}S(X)^*\right)^* Q(X) \left(F + Q(X)^{-1}S(X)^*\right). \end{aligned}$$

If F is stabilizing, then there exists an $X < 0$ such that (11) holds with $Y = -\begin{bmatrix} I \\ F \end{bmatrix}^* M \begin{bmatrix} I \\ F \end{bmatrix} \geq 0$. Since $Q(X) < 0$ it follows from (12) that for this X necessarily

$$\mathcal{R}^M(X) = P(X) - S(X)Q(X)^{-1}S(X)^* > 0. \quad (13)$$

If on the other hand some $X < 0$ satisfies (13), then (12) becomes positive for $F = -Q(X)^{-1}S(X)^*$. Thus we have proven the following result.

Theorem 4 *Let $M \leq 0$ be given like in (4) with $Q_0 < 0$. There exists a feedback-gain matrix F such that the closed-loop system (10) is mean-square stable if and only if the Riccati-type inequality (13) admits a negative definite solution X. In the latter case $F = -Q(X)^{-1}S(X)^*$ is a stabilizing feedback.*

In particular, if the system is stabilizable and $Q_0 < 0$ (S_0 arbitrary) then there exists an $X < 0$ such that $F = -(Q_0 + \Pi_2(X))^{-1}(B^*X + \Sigma(X)^* + S_0^*)$ is stabilizing.

Remark 5 The above solution $X < 0$ of the Riccati-type inequality (13) is not in $\operatorname{dom}_+ \mathcal{R}^M$ (since $Q(X) \leq Q_0 < 0$).

But if the system is stabilizable and $\tilde{M}$ is any matrix of the form (4) with $\tilde{Q}_0 > 0$, then there exists a matrix $X > 0$, such that

$$\begin{aligned} \tilde{F} &= -(-\tilde{Q}_0 + \Pi_2(-X))^{-1}(B^*(-X) + \Sigma(-X)^* - \tilde{S}_0^*) \\ &= -(\tilde{Q}_0 + \Pi_2(X))^{-1}(B^*X + \Sigma(X)^* + \tilde{S}_0^*) \end{aligned}$$

is stabilizing, where now $X \in \operatorname{dom}_+ \mathcal{R}^{\tilde{M}}$.

3.3 The stochastic LQ-control problem

As we have seen in the previous subsection, there is quite some freedom in choosing a stabilizing feedback matrix for a stabilizable system. In linear-quadratic (LQ) control one tries to choose an optimal stabilizing control $v \in L_w^2$ such that a given quadratic cost functional of the form

$$\mathcal{E} \int_0^\infty \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}^* M \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} dt \quad (14)$$

is minimized where M is of the form (4). If (6) is stabilizable and $M > 0$, then (see e.g. [Won70]) there exists an optimal stabilizing control of the form $v = Fx$, where $F = -Q(X)^{-1}S(X)^*$ and $X > 0$ is the largest solution of the equation

$$\mathcal{R}^M(X) = 0.$$

In fact the condition $M > 0$ can be weakened; for example like in the deterministic case the condition $P_0 > 0$ can be replaced by the conditions $P_0 \geq 0$ and (A, P_0) observable (compare [Won68]) But unlike in the deterministic case where the condition $Q_0 > 0$ is indispensible to ensure the existence of the minimum in (14) it has recently been observed (e.g. in [CLZ98]) that this is not necessarily so in the stochastic case where semidefinite or even indefinite control weights make sense and are of interest in applications.

3.4 A stochastic bounded-real lemma

A problem opposite to that of finding an optimal stabilizing control is to determine a worst case disturbance of a stable system. To be more precise, assume that system (6) is internally stable. Then, as proven in [HP98], the system is also externally stable. We wish to estimate the norm $\|\boldsymbol{L}\|$ of the input/output operator. In the deterministic case (if all A_0^i, B_0^i vanish) it is equal to the H^∞-norm of the associated rational transfer matrix. Thus $\|\boldsymbol{L}\|$ can be seen as a generalized H^∞-type norm for the stochastic system (6). Given $\gamma > 0$ we have $\|\boldsymbol{L}\| < \gamma$ if and only if the cost functional

$$\mathcal{E} \int_0^\infty \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}^* \underbrace{\begin{bmatrix} -C^*C & -C^*D \\ -D^*C & \gamma^2 I - D^*D \end{bmatrix}}_{=:M_\gamma} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} dt$$

is nonnegative for all $v \in L_w^2$, where $x(t) = x(t, 0, v)$ satisfies (6) with initial condition $x(0) = 0$. Moreover, if $\|\boldsymbol{L}\| < \gamma$, then the worst case disturbance that minimizes the above cost functional for all $x^0 \in \mathbb{K}^n$ is given by $v = Fx$, where $F = -Q(X)^{-1}S(X)^*$ and $X \in \operatorname{dom} \mathcal{R}^{M_\gamma}$ is the largest solution of the equation $\mathcal{R}^{M_\gamma}(X) = 0$. This nontrivial fact is the core of the following stochastic bounded real lemma from [HP98].

Theorem 6 *System (6) is internally stable and $\|\boldsymbol{L}\| < \gamma$ if and only if there exists an $X < 0$ such that $X \in \operatorname{dom}_+ \mathcal{R}^{M_\gamma}$ and $\mathcal{R}^{M_\gamma}(X) > 0$.*

3.5 A regular H^∞-type control problem

Now we regard a system with two inputs, one disturbance v and one control u.

$$\begin{aligned} dx(t) &= Ax(t)dt + \sum_{i=1}^N A_0^i x(t) dw_i(t) \\ &\quad + B_1 v(t)dt + \sum_{i=1}^N B_0^i v(t) dw_i(t) \qquad (15) \\ &\quad + B_2 u(t) dt \\ z(t) &= Cx(t) + D_2 u(t). \end{aligned}$$

We assume that the system is *regular* in the sense that $D_2^* D_2$ is nonsingular. Note that we have restricted ourselves to a simple model; e.g. one also might consider random vibrations of the parameter matrix B_2 or a direct feedthrough of the disturbance v to the output z. These extensions can be handled by similar methods as developed below, but they are technically more involved. If the control u in this system is given by static linear state feedback, i.e. $u = Fx$, then the resulting closed-loop system is of the form (6), with A and C replaced by $A + B_2F$ and $C + D_2F$, respectively. We then denote the input/output-operator of the closed-loop system with $\boldsymbol{L}^F$. With $B = B_1$, $D = D_2$ and $M = M_\gamma$ like in the Sec. 3.4 let P, S, Q and $\mathcal{R} = \mathcal{R}^{M_\gamma}$ be defined by (2) and (5), respectively, with Π given by (7).
Our aim is to find a feedback gain matrix F, such that the closed-loop system is internally stable, and $\|\boldsymbol{L}^F\| < \gamma$ for some given number $\gamma > 0$. By Theorem 6 we have to find a pair of matrices $F \in \mathbb{K}^{m\times n}$ and $X < 0$ such that $Q(X) = \sum_{i=1}^N B_0^{i*} X B_0^i + \gamma^2 I - D_2^* D_2 > 0$ (i.e. $X \in \operatorname{dom}_+ \mathcal{R}^{M_\gamma}$) and

$$\begin{aligned} 0 &< (A + B_2F)^* X + X(A + B_2F) + \Pi_1(X) \qquad (16) \\ &\quad - (C + D_2F)^*(C + D_2F) - S(X)Q(X)^{-1}S(X)^* \\ &= \mathcal{R}^{M_\gamma}(X) + (B_2^* X - D_2^* C)^*(D_2^* D_2)^{-1}(\ldots) \\ &\quad - \Big(F - (D_2^* D_2)^{-1}(B_2^* X - D_2^* C)\Big)^* D_2^* D_2 \Big(\ldots\Big). \end{aligned}$$

Here (and below) we use the short-hand notation $(\ldots)$ for the right factor which is the conjugate transpose of the left factor. Inequality (16) necessarily implies

$$0 \quad < \quad \hat{\mathcal{R}}^{M_\gamma}(X) \qquad (17)$$

with $\hat{\mathcal{R}}^{M_\gamma}(X) = \mathcal{R}^{M_\gamma}(X) + (B_2^* X - D_2^* C)^*(D_2^* D_2)^{-1}(\ldots)$. If on the other hand some $X < 0$ satisfies (17), then $F = (D_2^* D_2)^{-1}(B_2^* X - D_2^* C)$ and X satisfy (16). Thus we have proven the following result.

Theorem 7 *There exists a feedback gain matrix F, such that the closed-loop system (15) (with $u = Fx$) is internally stable and the corresponding disturbance operator $\boldsymbol{L}^F$ has norm $\|\boldsymbol{L}^F\| < \gamma$ if and only if there exists a matrix $X < 0$ in $\operatorname{dom}_+ \mathcal{R}^{M_\gamma}$ satisfying inequality (17). In this case a suitable F is given by $F = (D_2^* D_2)^{-1}(B_2^* X - D_2^* C)$.*

4 The equation $\mathcal{R}(X) = 0$

The problems described in the previous section have led us to matrix (in)equalities involving the operator $\mathcal{R} = \mathcal{R}^M$ corresponding to different weight matrices M. In this section we study solutions of the equation $\mathcal{R}(X) = 0$ in $\operatorname{dom}_+ \mathcal{R}$ where $\mathcal{R}$ is of the form (1). If such a solution exists and the underlying system is stabilizable, then there exists a greatest solution $X_+ = X_+^M$ (see Theorem 17). We will analyze the dependence of X_+^M on M.
The methods developed in this section turn out to be instrumental to address the above Riccati-type inequalities. First we will summarize some general facts about analytical properties of $\mathcal{R}$ as a nonlinear operator acting in a partially ordered vector space.

4.1 Analytical properties

Let $\mathcal{H}^n \subset \mathbb{K}^{n\times n}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) denote the real space of real or complex $n \times n$ Hermitian matrices, endowed with the Frobenius inner product $\langle X\,,\, Y\rangle = \operatorname{trace}(XY)$ and the corresponding (Frobenius) norm $\|\cdot\|$. By $\mathcal{H}_+^n := \{X \in \mathcal{H}^n \mid X \geq 0\}$ we denote the closed convex cone of nonnegative definite matrices and by $\operatorname{int}\left(\mathcal{H}_+^n\right)$ its inner, i.e. the open cone of positive definite matrices. The cone $\mathcal{H}_+^n$ induces a partial ordering on $\mathcal{H}^n$: we write $X \geq Y$, if $X - Y \in \mathcal{H}_+^n$.
The operator $\mathcal{R}$ defined in Sec. 2 possesses the following remarkable analytical properties (see [DH99]).

Proposition 8 *Let $\mathcal{R}$ be given by (1), (2) and suppose that Π defined by (3) is positive and $\operatorname{dom}_+ \mathcal{R} \neq \emptyset$. Then*

(i) The nonempty set $\operatorname{dom}_+ \mathcal{R} \subset \operatorname{dom} \mathcal{R}$ is saturated above, *i.e. $\operatorname{dom}_+ \mathcal{R} = \operatorname{dom}_+ \mathcal{R} + \mathcal{H}_+^n$.*

(ii) The operator $\mathcal{R}$ is concave *on $\operatorname{dom}_+ \mathcal{R}$ in the sense that for all $Y \in \operatorname{dom} \mathcal{R}$ and $Z \in \operatorname{dom}_+ \mathcal{R}$*

$$\mathcal{R}(Y) - \mathcal{R}(Z) + \mathcal{R}_Y'(Z - Y) \quad \geq \quad 0. \qquad (18)$$

(In the usual sense, concavity of $\mathcal{R}$ on $\operatorname{dom}_+ \mathcal{R}$ would only require that (18) holds for all $Y \in \operatorname{dom}_+ \mathcal{R}$; but in this paper concavity of an operator on a certain set is always understood in the stronger sense of (18)).

(iii) The operator $\mathcal{R}$ is Fréchet differentiable on $\operatorname{dom} \mathcal{R}$ and the Fréchet derivatives $\mathcal{R}_X'$ at points $X \in \operatorname{dom} \mathcal{R}$ are resolvent positive *operators in the sense of the next subsection.*

4.2 Resolvent positive operators

The main tool in our approach is the theory of positive operators in ordered vector spaces based on the work of Krein and Rutman [KR50]. In particular the class of resolvent positive operators plays an important role.

Definition 9 *A linear operator $T : \mathcal{H}^n \to \mathcal{H}^m$ is called* positive *($T \geq 0$) if it maps $\mathcal{H}^n_+$ to $\mathcal{H}^m_+$. A linear operator $T : \mathcal{H}^n \to \mathcal{H}^n$ is called* inverse positive *if it is invertible and T^{-1} is positive; it is called* resolvent positive, *if for all sufficiently large $\alpha > 0$ the resolvent operator $(\alpha I - T)^{-1}$ is positive.*

Remark 10 Resolvent positive operators [Are87] are also called *cross-positive*, *essentially positive*, *exponentially positive*, *$\mathcal{H}^n_+$-subtangential*, *quasimonotonic* or *Metzler operators* (compare [SV70], [Els74], [BNS89], [FHS98]). They can be seen as a generalization of M-matrices (e.g. [BP94]).

For a linear operator $\mathcal{T}$ let $\sigma(\mathcal{T})$ denote the spectrum, $\rho(\mathcal{T}) = \max\{|\lambda|; \lambda \in \sigma(\mathcal{T})\}$ the spectral radius, and $\beta(\mathcal{T}) = \max\{\Re(\lambda); \lambda \in \sigma(\mathcal{T})\}$ the spectral abscissa.

Examples 11 (i) Let $A_0 \in \mathbb{K}^{n\times n}$, then the operator $\Pi : \mathcal{H}^n \to \mathcal{H}^n$ defined by $\Pi(X) := A_0^* X A_0$ is positive. If A_0 is nonsingular, then it is also inverse positive.
(ii) All positive operators Π are also resolvent positive, since for $\alpha > \rho(\Pi)$ the resolvent $(\alpha I - \Pi)^{-1} = \sum_{k=0}^{\infty} \alpha^{-(k+1)} \Pi^k$ is positive.
(iii) Given $A \in \mathbb{K}^{n\times n}$, the associated *Lyapunov operator* $\mathcal{L}_A : \mathcal{H}^n \to \mathcal{H}^n$, $\mathcal{L}_A(X) := A^*X + XA$, is resolvent positive but, in general, not positive. It is well known that $\mathcal{L}_A$ is inverse positive if and only if $\sigma(A) \subset \mathbb{C}_+$. Since $\alpha I - \mathcal{L}_A = \mathcal{L}_{\frac{\alpha}{2}I - A}$, the resolvent is positive for $\alpha > 2\beta(A)$.

Theorem 12 [Els70] *Let $\mathcal{T} : \mathcal{H}^n \to \mathcal{H}^n$ be a resolvent positive linear operator. The following assertions hold:*

(i) $\beta(\mathcal{T}) \in \sigma(\mathcal{T})$ and there exists a matrix $V \in \mathcal{H}^n_+ \setminus \{0\}$ such that $\mathcal{T}(V) = \beta(\mathcal{T})V$.

(ii) If $\mathcal{S} : \mathcal{H}^n \to \mathcal{H}^n$ is a linear operator with $\mathcal{S} \geq \mathcal{T}$ then S is resolvent-positive and $\beta(\mathcal{S}) \geq \beta(\mathcal{T})$.

(iii) $\alpha I - \mathcal{T}$ is inverse positive $\iff$ $\alpha > \beta(\mathcal{T})$.

Corollary 13 ([Sch65]) *Let $\mathcal{L} : \mathcal{H}^n \to \mathcal{H}^n$ be resolvent positive and $\Pi : \mathcal{H}^n \to \mathcal{H}^n$ be positive. Then the following are equivalent:*

(i) $\mathcal{L} + \Pi$ is stable, i.e. $\sigma(\mathcal{L} + \Pi) \subset \mathbb{C}_-$.

(ii) $-(\mathcal{L} + \Pi)$ is inverse positive.

(iii) $\sigma(\mathcal{L}) \subset \mathbb{C}_-$ and $\rho\left(\mathcal{L}^{-1}\Pi\right) < 1$.

(iv) $\exists X > 0 : \quad (\mathcal{L} + \Pi)(X) < 0$.

The definiteness conditions in Corollary 13 (iv) can be weakened:

Theorem 14 *Let (A, G) be observable, $G \geq 0$, and assume*

$$\exists X \leq 0 : \mathcal{L}_A(X) + \Pi(X) \geq G.$$

Then $X < 0$ and $\mathcal{L}_A + \Pi$ is stable.

4.3 Stabilizability and Newton's method

We now define a concept of stabilization that only refers to the Riccati operator, not to the underlying system.

Definition 15 *We call $\mathcal{R}$* stabilizable, *if there exists a point $X \in \operatorname{dom}\mathcal{R}$, such that the Fréchet derivative of $\mathcal{R}$ at X is stable, i.e. $\sigma(\mathcal{R}'_X) \subset \mathbb{C}_-$. The matrix X is then called* stabilizing.

Remark 16 A calculation shows, that $\mathcal{R}'_X$ is given by the left side of (11) with F replaced by $Q(X)^{-1}S(X)^*$. Thus, if the operator $\mathcal{R}$ corresponds to system (6) (i.e. A, B are like in (6), Π is given by (7), and M is arbitrary) then the stabilizability of $\mathcal{R}$ implies the stabilizability of (6). By Theorem 4 the converse implication holds if $Q_0 < 0$ or $Q_0 > 0$. For indefinite or semidefinite Q_0 we do not know, whether this is true or not.

Since $\mathcal{R}'_X$ is resolvent positive for all $X \in \operatorname{dom}\mathcal{R}$, we know by Theorem 12, that $-\mathcal{R}'_X$ is inverse positive if and only if X is stabilizing.
It has been observed in the literature (e.g. [Van67]) that for operator equations $\mathcal{G}(X) = 0$ with a convex operator $\mathcal{G}$ (on a partially ordered vector space) whose derivatives are *inverse positive everywhere* the Newton iteration converges monotonically from any starting point X_0 to a solution of the operator equation (provided a solution exists).
On the other hand it is known (e.g. [LR95]) that for standard Riccati equations Newton's method converges monotonically from any *stabilizing* starting point and the iterates never leave the set of stabilizing matrices.
These observations are unified and generalized in the following constructive existence result from [DH99] (compare also [Won68] or [DHS97] for special cases). At this point we forget about the special structure of $\mathcal{R}$ for a moment and assume only the properties from Proposition 8 to hold.

Theorem 17 *Let $\mathcal{R} : \operatorname{dom}\mathcal{R} \to \mathcal{H}^n$ be a Fréchet differentiable operator on an open set $\operatorname{dom}\mathcal{R} \subset \mathcal{H}^n$ and $\operatorname{dom}_+\mathcal{R}$ a nonempty open subset of $\operatorname{dom}\mathcal{R}$ such that the conditions (i), (ii) and (iii) from Proposition 8 are satisfied. Assume that $\mathcal{R}$ is stabilizable and that the inequality $\mathcal{R}(X) \geq 0$ has a solution $\hat{X} \in \operatorname{dom}_+\mathcal{R}$.*
Then there exists a solution $X_+ \in \operatorname{dom}_+\mathcal{R}$ of the equation $\mathcal{R}(X) = 0$ which can be computed by the standard Newton-iteration starting at an arbitrary stabilizing matrix $X_0 \in \operatorname{dom}\mathcal{R}$. This solution X_+ is the largest solution of the inequality $\mathcal{R}(X) \geq 0$ and hence uniquely determined. The matrix X_+ is stabilizing if and only if the strict inequality $\mathcal{R}(X) > 0$ is solvable in $\operatorname{dom}_+\mathcal{R}$; otherwise $0 \in \sigma\left(\mathcal{R}'_{X_+}\right) \subset \mathbb{C}_- \cup i\mathbb{R}$. If X_+ is stabilizing, then the convergence of the Newton-iteration is guaranteed to be quadratic.
If on the other hand a stabilizing solution $\tilde{X}_+ \in \operatorname{dom}_+\mathcal{R}$

of the equation $\mathcal{R}(X) = 0$ *is given, then necessarily* $\tilde{X}_+ = X_+$.

If a stabilizing matrix $X_0 \in \operatorname{dom}\mathcal{R}$ (such that $\sigma(\mathcal{R}_{X_0}) \subset \mathbb{C}_-$) has been found, Theorem 17 provides an iterative method to decide whether the inequality $\mathcal{R}(X) \geq 0$ is solvable in $\operatorname{dom}_+\mathcal{R}$ or not. In fact, if the inequality $\mathcal{R}(X) \geq 0$ is solvable in $\operatorname{dom}_+\mathcal{R}$ then the Newton algorithm starting at X_0 produces a decreasing sequence of stabilizing matrices $X_k \in \operatorname{dom}_+\mathcal{R}$ which converge to the largest solution X_+ of $\mathcal{R}(X) = 0$. If the inequality is not solvable in $\operatorname{dom}_+\mathcal{R}$, then either some iterate X_k is not stabilizing, or the iterates leave $\operatorname{dom}_+\mathcal{R}$, or converge to the boundary of $\operatorname{dom}_+\mathcal{R}$ or become arbitrarily large in norm.
Stabilizing solutions of $\mathcal{R}(X) = 0$ are particularly important in the context of the LQ-problem where they yield the optimal controls. According to the previous theorem a stabilizing solution $X_+ \in \operatorname{dom}_+\mathcal{R}$ of equation (1) exists, if and only if $\mathcal{R}$ is stabilizable and the inequality $\mathcal{R}(X) > 0$ is solvable in $\operatorname{dom}_+\mathcal{R}$. In this case it can be computed by Newton's method, if a stabilizing matrix for $\mathcal{R}$ is known. The problem of how to find such a stabilizing matrix will be discussed in Subsection 4.5 .

4.4 Dependence of X_+ on the data

We now suppose that $\mathcal{R}$ is given by (1), (2), Π defined by (3) is positive and $\operatorname{dom}_+\mathcal{R} \neq \emptyset$. In order to analyze the dependence of the largest solution X_+ of $\mathcal{R}(X) = 0$ on the parameter M (4) we make use of the close relationship between the rational matrix inequality $\mathcal{R}(X) \geq 0$ and a higher dimensional linear matrix inequality:

Proposition 18 *Let* $\Lambda(X) := \begin{bmatrix} A^*X + XA & XB \\ B^*X & 0 \end{bmatrix}$, Π *as in (3) and* M *as in (4), such that* $\operatorname{dom}_+\mathcal{R} \neq \emptyset$.

Then for $X \in \operatorname{dom}_+\mathcal{R}$ *the following equivalences hold:*

(i) $\mathcal{R}(X) \geq 0 \iff \Pi(X) + \Lambda(X) + M \geq 0.$

(ii) $\mathcal{R}(X) > 0 \iff \Pi(X) + \Lambda(X) + M > 0.$

Proof: It suffices to observe, that $\mathcal{R}(X)$ is the Schur complement of $\Pi(X) + \Lambda(X) + M$ with respect to the lower right block $Q(X)$. □

The matrix X_+ of Theorem 17 is thus also the largest solution of the inequality $\Pi(X) + \Lambda(X) + M \geq 0$. This observation can be used to prove the monotonic and concave dependence of X_+ on the weight matrix M:

Theorem 19 *Assume* $M_1 \geq M_0$. *If there exists a solution* $X_0 \in \operatorname{dom}_+\mathcal{R}^{M_0}$ *to* $\mathcal{R}^{M_0}(X) = 0$ *and* $\mathcal{R}^{M_1}$ *is stabilizable, then there exists a greatest solution* X_1 *to* $\mathcal{R}^{M_1}(X) = 0$ *and* $X_1 \geq X_0$. *If* X_0 *is stabilizing for* $\mathcal{R}^{M_0}$, *then* X_1 *is stabilizing for* $\mathcal{R}^{M_1}$.

Proof: By $M_1 \geq M_0$ and Proposition 18 we have $\operatorname{dom}_+\mathcal{R}^{M_0} \subset \operatorname{dom}_+\mathcal{R}^{M_1}$ and also $\mathcal{R}^{M_1}(X_0) \geq 0$. Thus by Theorem 17 there exists a greatest solution X_1 to $\mathcal{R}^{M_1}(X) = 0$ and $X_1 \geq X_0$.
If X_0 is stabilizing for $\mathcal{R}^{M_0}$, then by Theorem 17 there exists an $\tilde{X} \in \operatorname{dom}\mathcal{R}^{M_0}$ such that $\mathcal{R}^{M_0}(\tilde{X}) > 0$. Again by $M_1 \geq M_0$ and Proposition 18 we have $\mathcal{R}^{M_1}(\tilde{X}) > 0$. Thus again by Theorem 17 X_1 is stabilizing. □

An analogous argument shows that X_∞ depends on M in a concave fashion:

Theorem 20 *Let* $M_0, M_1 \in \mathcal{H}^{n+m}$ *be arbitrary, and set* $M_\tau := (1-\tau)M_0 + \tau M_1$ *for* $\tau \in [0,1]$. *Assume that for* $i = 0, 1$ *there exist solutions* $X_i \in \operatorname{dom}_+\mathcal{R}^{M_i}$ *to* $\mathcal{R}^{M_i}(X) = 0$ *and that* $\mathcal{R}^{M_{\tau_0}}$ *is stabilizable for some* $\tau_0 \in]0,1[$. *Then there exists a greatest solution* X_{τ_0} *to* $\mathcal{R}^{M_{\tau_0}}(X) = 0$ *and* $X_{\tau_0} \geq (1-\tau_0)X_0 + \tau_0 X_1$. *If* X_0 *or* X_1 *is stabilizing then so is* X_{τ_0}.

For convenience we introduce the following subsets of $\mathcal{H}^{n+m}$ (depending on Λ and Π but these are fixed):

$$\begin{aligned}
\mathcal{M}_+ &:= \{M \text{ partitioned as in (4) with } Q_0 > 0\},\\
\mathcal{M}_0 &:= \{M : \mathcal{R}^M \text{ is stabilizable}\},\\
\mathcal{M}_1 &:= \{M : \mathcal{R}^M(X) \geq 0 \text{ is solvable in } \operatorname{dom}_+\mathcal{R}^M\},\\
\mathcal{M}_2 &:= \{M : \mathcal{R}^M(X) = 0 \text{ has a stabilizing solution in } \operatorname{dom}_+\mathcal{R}^M\}.
\end{aligned}$$

Proposition 21 *(i)* $\mathcal{M}_0 \neq \emptyset \Rightarrow \mathcal{M}_0 \supset \pm\mathcal{M}_+$

(ii) $\mathcal{M}_1 = \mathcal{M}_1 + \mathcal{H}_+^{n+m}$.

(iii) $\mathcal{M}_2 = \mathcal{M}_0 \cap (\mathcal{M}_2 + \mathcal{H}_+^{n+m}) \subset \mathcal{M}_0 \cap \mathcal{M}_1$.

Proof: (i) follows from the Remarks 5 and 16, (ii) is obvious, and (iii) follows from Theorem 17. □

Theorem 22 *The sets* $\mathcal{M}_0$, $\mathcal{M}_2$ *are open in* $\mathcal{H}^{n+m}$ *and there exists a (real) analytic order-preserving function* $X_+ : \mathcal{M}_2 \to \mathcal{H}^n$ *such that* $X_+(M)$ *is the stabilizing solution of* $\mathcal{R}^M(X) = 0$ *for all* $M \in \mathcal{M}_2$.

Proof: Clearly $\mathcal{D} = \{(M,X) \in \mathcal{H}^{n+m} \times \mathcal{H}^n \mid Q^M(X) > 0\}$ is non-empty and open in the real vector space $\mathcal{H}^{n+m} \times \mathcal{H}^n$ and the map $G : \mathcal{D} \to \mathcal{H}^n$ defined by

$$G : (M,X) \mapsto G(M,X) := \mathcal{R}(X)^M$$

is (real) analytic. As a consequence also the derivative

$$\frac{\partial G}{\partial X} : (M,X) \mapsto \frac{\partial G}{\partial X}(M,X) = \left(\mathcal{R}^M\right)'_X$$

is an analytic map from $\mathcal{D}$ to $\mathcal{L}(\mathcal{H}^n)$.
Now let $M_0 \in \mathcal{M}_0$, then there exists $X_0 \in \operatorname{dom}\mathcal{R}^{M_0}$ such that $\sigma\left(\left(\mathcal{R}^{M_0}\right)'_{X_0}\right) \subset \mathbb{C}_-$. Since the spectrum depends continuously on the operator, $(M_0, X_0) \in \mathcal{D}$

and $\frac{\partial G}{\partial X}$ is analytic on $\mathcal{D}$ there is an open ball $\mathcal{B}(M_0,\varepsilon) := \{M \in \mathcal{H}^{n+m} \mid \|M - M_0\| < \varepsilon\}$ in $\mathcal{H}^{n+m}$ such that $X_0 \in \operatorname{dom}\mathcal{R}^M$ and $\sigma((\mathcal{R}^{\mathcal{M}})'_{X_0}) \subset \mathbb{C}_-$ for all $M \in \mathcal{B}(M_0,\varepsilon)$. Thus X_0 is stabilizing for all $\mathcal{R}^M$ with $M \in \mathcal{B}(M_0,\varepsilon)$, and this proves that $\mathcal{M}_0$ is open.
Now assume that $M_0 \in \mathcal{M}_2$ and let $X_0 \in \operatorname{dom}_+\mathcal{R}^{M_0}$ be the stabilizing solution of $\mathcal{R}^{M_0}(X) = 0$. Then $(M_0,X_0) \in \mathcal{D}$ and $\frac{\partial G}{\partial X}(M_0,X_0) = (\mathcal{R}^{M_0})'_{X_0}$ is stable, in particular invertible. As a consequence of the implicit function theorem for analytic functions [Die69] there is an open ball $\mathcal{B}(M_0,\varepsilon_0)$ in $\mathcal{H}^{n+m}$ such that for all $M \in \mathcal{B}(M_0,\varepsilon_0)$ there exists a unique solution $X(M) \in \mathcal{H}^n$ of $\mathcal{R}^M(X) = 0$ which depends analytically on $M \in \mathcal{B}(M_0,\varepsilon_0)$. But then $M \mapsto \frac{\partial G}{\partial X}(M,X(M)) = (\mathcal{R}^M)'_{X(M)}$ is continuous (even analytic) on $\mathcal{B}(M_0,\varepsilon_0)$ and since $\sigma((\mathcal{R}^{M_0})'_{X(M_0)}) \subset \mathbb{C}_-$ there exists $\varepsilon \in]0,\varepsilon_0[$ such that $\sigma((\mathcal{R}^M)'_{X(M)}) \subset \mathbb{C}_-$ for all $M \in \mathcal{B}(M_0,\varepsilon)$. Hence $X(M)$ is a stabilizing solution of $\mathcal{R}^M(X) = 0$ for all $M \in \mathcal{B}(M_0,\varepsilon)$ and so $\mathcal{B}(M_0,\varepsilon) \subset \mathcal{M}_2$. This shows that $\mathcal{M}_2$ is open in $\mathcal{H}^{n+m}$, and the restriction of $X_+(\cdot)$ to $\mathcal{B}(M_0,\varepsilon)$ coincides with $X(\cdot)$ on $\mathcal{B}(M_0,\varepsilon)$. Therefore $X_+(\cdot) : \mathcal{M}_2 \to \mathcal{H}^n$ is analytic and by Theorem 19 order-preserving. □

Corollary 23 $\mathcal{M}_2 \subset \mathcal{M}_0 \cap \mathcal{M}_1 \subset \overline{\mathcal{M}_2}$.

Proof: The first inclusion follows directly from the definition. Now suppose $M_0 \in \mathcal{M}_0 \cap \mathcal{M}_1$ and $X_0 \in \operatorname{dom}\mathcal{R}^{M_0}$ is a solution of $\mathcal{R}^{M_0}(X) = 0$. By Theorem 22 we have $\mathcal{B}(M_0,\varepsilon_0) \subset \mathcal{M}_0$ for some $\varepsilon_0 > 0$. If we set $M_\varepsilon = M_0 + \varepsilon I$ for arbitrary $\varepsilon \in]0,\varepsilon_0[$ we get

$$R^{M_\varepsilon}(X_0) := \Lambda(X_0) + \Pi(X_0) + M_0 + \varepsilon I > 0.$$

It follows from Proposition 18(ii) and Theorem 17 that there exists a stabilizing solution of $R^{M_\varepsilon}(X) = 0$ and so $M_\varepsilon \in \mathcal{M}_2$. Since M_ε comes arbitrarily close to M_0 as $\varepsilon \to 0$, the corollary is proved. □

4.5 Stabilization

In this subsection we discuss the question how to find a stabilizing matrix X_0 for $\mathcal{R}$ where $\mathcal{R}$ is defined by (1), (2) with Π given by (7). In this case $\mathcal{R} = \mathcal{R}^M$ is determined by the parameters of the system (6) and the weight matrix M as in (4). We suppose that $\operatorname{dom}_+\mathcal{R} = \{X \in \mathcal{H}^n \mid Q(X) > 0\} \neq \emptyset$ and that $\mathcal{R}$ (hence the system (6)) is stabilizable. Additionally we assume $M \in \pm\mathcal{M}_+$, i.e. Q_0 is either positive or negative definite. We need this definiteness assumption in order to construct a stabilizing matrix X_0 for $\mathcal{R}$. Note that under this additional assumption the stabilizability of the operator $\mathcal{R}$ and of the system (6) are equivalent by Remark 16. Moreover, by Theorem 4, the system (6) is stabilizable if there exists a solution $X < 0$ to the modified equation

$$\mathcal{R}^{\tilde{M}}(X) \quad = \quad 0. \tag{19}$$

Here

$$\tilde{M} = \begin{bmatrix} \tilde{P}_0 & \tilde{S}_0 \\ \tilde{S}_0^* & \tilde{Q}_0 \end{bmatrix} < 0, \quad (\tilde{S}_0^*, \tilde{Q}_0) = \pm(S_0^*, Q_0) \tag{20}$$

with $(\tilde{S}_0^*, \tilde{Q}_0) = (S_0^*, Q_0)$ if $Q_0 < 0$ and $(\tilde{S}_0^*, \tilde{Q}_0) = -(S_0^*, Q_0)$ if $Q_0 > 0$. The block $\tilde{P}_0$ is only constrained by the condition $\tilde{M} < 0$, i.e. $\tilde{P}_0 < \tilde{S}_0\tilde{Q}_0^{-1}\tilde{S}_0^*$. In fact, the equation (19) implies the corresponding strict inequality (required in Theorem 4) where $\tilde{P}_0$ is slightly increased.
We wish to solve (19) with the help of Theorem 17. But as pointed out before, $X < 0$ implies

$$Q^{\tilde{M}}(X) = \sum_{i=1}^{N} B_0^{i*} X B_0^i + \tilde{Q}_0 < 0$$

and thus $X \notin \operatorname{dom}_+\mathcal{R}^{\tilde{M}}$, whence Theorem 17 is not directly applicable.
Therefore we use the following transformation: We replace X by $-Y^{-1}$ and multiply the equation from both sides with Y; thus we have to solve the equation

$$\mathcal{G}(Y) := Y\mathcal{R}^{\tilde{M}}(-Y^{-1})Y = 0, \tag{21}$$

in $\operatorname{dom}_+\mathcal{G} := \operatorname{int}\mathcal{H}_+^n$.
Let

$$\operatorname{dom}\mathcal{G} \quad := \quad \{Y \in \mathcal{H}^n \mid \det Y \neq 0, \det Q^{\tilde{M}}(-Y^{-1}) \neq 0\}$$

Lengthy calculations show that the operator $\mathcal{G} : \operatorname{dom}\mathcal{G} \to \mathcal{H}^n$ satisfies the conditions of Proposition 8:

(i) $\operatorname{dom}_+\mathcal{G}$ is saturated above.

(ii) $\mathcal{G}$ is concave on $\operatorname{dom}_+\mathcal{G}$.

(iii) $\mathcal{G}$ is Fréchet differentiable on $\operatorname{dom}\mathcal{G}$ with resolvent positive derivatives.

Thus we can apply Theorem 17 to solve (21). At the first glance this does not really seem to be a progress, because now we need to find a stabilizing matrix for the operator $\mathcal{G}$. But it turns out that such a matrix can easily be found in standard form:

Lemma 24 *For sufficiently large $\alpha \in]0,\infty[$, the matrix $Y_0 = \alpha I$ is stabilizing for $\mathcal{G}$, i.e. $\sigma(\mathcal{G}'_{\alpha I}) \subset \mathbb{C}_-$.*

Proof: We set $F(Y) = -Q^{\tilde{M}}(-Y^{-1})^{-1}\tilde{S}^{\tilde{M}}(-Y^{-1})^*Y$; a calculation shows, that

$$\begin{aligned}\mathcal{G}'_Y(H) \quad &= \quad \mathcal{L}_{-A^*+(\tilde{P}_0-\Pi_1(-Y^{-1}))Y+(\tilde{S}_0-\Sigma_0(Y^{-1}))F(Y)}(H) \\ &\quad + \begin{bmatrix} Y \\ F(Y) \end{bmatrix}^* \Pi(Y^{-1}HY^{-1}) \begin{bmatrix} Y \\ F(Y) \end{bmatrix},\end{aligned}$$

where $\mathcal{L}$ denotes the Lyapunov operator as in Sec. 3.1. By the above definition of F and the definitions of S and Q in Sec. 2 we have $\lim_{\alpha\to+\infty}\frac{1}{\alpha}\,F(\alpha I) = -\tilde{Q}_0^{-1}\tilde{S}_0^*$, and thus

$$\lim_{\alpha\to+\infty}\frac{1}{\alpha}\,\mathcal{G}'_{\alpha I} = \mathcal{L}_{\tilde{P}_0-\tilde{S}_0\tilde{Q}_0^{-1}\tilde{S}_0^*}\,. \tag{22}$$

Since $\tilde{P}_0 - \tilde{S}_0\tilde{Q}_0^{-1}\tilde{S}_0^* < 0$, it follows that the Lyapunov operator in (22) is stable. By the continuous dependence of $\mathcal{G}'_{\alpha I}$ on α the latter operator is stable for large enough $\alpha > 0$. In fact α can be chosen arbitrarily small, if $-\tilde{P}_0 > 0$ is appropriately large. □

By similar considerations as in the previous subsection this leads to the following result.

Theorem 25 *Let the system (6) be stabilizable. For all $M \in \pm\mathcal{M}_+$ there exists a matrix $X_0 \in \operatorname{int}\mathcal{H}_+^n$ such that $\sigma\Big((\mathcal{R}^M)'_{X_0})\Big) \subset \mathbb{C}_-$.*
If $\tilde{M}$ is specified as in (20), then a stabilizing matrix X_0 for $\mathcal{R}^M$ is given by $\pm Y_+^{-1}$, where $Y_+ = Y_+(\tilde{M}) > 0$ is the largest solution of (21). The matrix Y_+ is given as the limit of the Newton iteration applied to (21) starting from $Y_0 = \alpha I$ for sufficiently large $\alpha > 0$.

4.6 The inequality $\hat{\mathcal{R}}(X) > 0$

Finally we treat the matrix inequality (17) of the regular H^∞-type control problem presented in Sec. 3.5. Using Proposition 8 one verifies immediately that on $\operatorname{dom}_+\mathcal{R}^{M_\gamma}$ the operator

$$\hat{\mathcal{R}}^{M_\gamma}(X) = \mathcal{R}^{M_\gamma}(X) + (B_2^*X - D_2^*C)^*(D_2^*D_2)^{-1}(\ldots)$$

is the sum of a concave and a convex operator.
We apply the same transformation as in the previous subsection, i.e. we substitute $-Y^{-1}$ for X and multiply (17) from both sides with Y. Thus we wish to solve the inequality

$$\hat{\mathcal{G}}(Y) := Y\hat{\mathcal{R}}^{M_\gamma}(-Y^{-1})Y > 0 \tag{23}$$

in $\operatorname{dom}_+\hat{\mathcal{G}} := \{Y \in \operatorname{int}\mathcal{H}_+^n \mid Q^{M_\gamma}(-Y^{-1}) > 0\}$.
We set

$$\operatorname{dom}\hat{\mathcal{G}} := \{Y \in \mathcal{H}^n \mid \det Y \neq 0, \det Q^{\tilde{M}}(-Y^{-1}) \neq 0\}.$$

The operator $\hat{\mathcal{G}} : \operatorname{dom}\hat{\mathcal{G}} \to \mathcal{H}_+^n$ arises from $\mathcal{G}$ in (21) if to the latter we add the constant term $C^*D_2(D_2^*D_2)^{-1}D_2^*C$ and substitute $\hat{A} := A - B_2(D_2^*D_2)^{-1}D_2^*C$ for A and

$$\hat{P}_0 \;:=\; -C^*C + C^*D_2(D_2^*D_2)^{-1}D_2^*C \leq 0 \tag{24}$$

for $\tilde{P}_0$. Thus structurally the operators $\mathcal{G}$ and $\hat{\mathcal{G}}$ differ only in the constant term, but note, that in the previous subsection we had $\tilde{M} < 0$, whereas now

$$M_\gamma = \begin{bmatrix} P_0 & S_0 \\ S_0^* & Q_0 \end{bmatrix} = \begin{bmatrix} -C^*C & -C^*D \\ -D^*C & \gamma^2 I - D^*D \end{bmatrix}$$

is indefinite with positive block Q_0. Above we had automatically $Q^{\tilde{M}}(-Y^{-1}) < 0$ for all $Y \in \operatorname{int}\mathcal{H}_+^n$, whereas now $Q^{M_\gamma}(-Y^{-1}) > 0$ is a further constraint for $Y \in \operatorname{int}\mathcal{H}_+^n$. Nevertheless the operator $\hat{\mathcal{G}} : \operatorname{dom}\hat{\mathcal{G}} \to \mathcal{H}^n$ just as above satisfies the conditions of Proposition 8, that we have listed under (i), (ii), (iii) in the previous subsection. Therefore we infer from Theorem 17:

Lemma 26 *Assume that inequality (17) is solvable in $\operatorname{dom}_+\mathcal{R}^{M_\gamma} \cap -\operatorname{int}\mathcal{H}_+^n$ and that $\hat{\mathcal{G}}$ from (23) is stabilizable. Then there exists a matrix $\hat{X} \in \operatorname{dom}_+\mathcal{R}^{M_\gamma} \cap -\operatorname{int}\mathcal{H}_+^n$ such that $\hat{\mathcal{G}}(-\hat{X}^{-1}) = \hat{\mathcal{R}}^{M_\gamma}(\hat{X}) = 0$; moreover one can prove that $\sigma(\hat{\mathcal{G}}'_{-\hat{X}^{-1}}) = \sigma({\mathcal{R}_{\hat{X}}^{M_\gamma}}') \subset \mathbb{C}_-$.*
The matrix $\hat{X}$ yields a stabilizing feedback-gain matrix $F = -Q(\hat{X})^{-1}S(\hat{X})^*$ such that the corresponding disturbance operator $\boldsymbol{L}^F$ has norm $\|\boldsymbol{L}^F\| \leq \gamma$.

As in (22) we have $\lim_{\alpha\to+\infty}\frac{1}{\alpha}\,\hat{\mathcal{G}}'_{\alpha I} = \mathcal{L}_{P_0-S_0Q_0^{-1}S_0^*}$, which is stable if $P_0 = -C^*C$ is nonsingular. But if inequality (17) holds for some $X \in \operatorname{dom}_+\mathcal{R} \cap -\operatorname{int}\mathcal{H}_+^n$ then we may replace P_0 by $P_0 - \varepsilon I$ for small enough $\varepsilon > 0$ without affecting the inequality. The corresponding modified operator $\hat{\mathcal{G}}_\varepsilon$ is then stabilizable by αI for large α.
So a solution $\hat{X}_\varepsilon = -\hat{Y}^{-1} \in \operatorname{dom}_+\mathcal{R}^{M_\gamma} \cap -\operatorname{int}\mathcal{H}_+^n$ of the equation $\hat{\mathcal{R}}^{M_\gamma}(X) - \varepsilon I = 0$ (and thus of (17)) can be found by applying Newton's method to the equation $\hat{\mathcal{G}}_\varepsilon(Y) = 0$ with initial matrix $Y_0 = \alpha I$, where $\varepsilon > 0$ is sufficiently small and $\alpha > 0$ is sufficiently large. Moreover $\sigma(\hat{\mathcal{R}}_{\hat{X}_\varepsilon}^{M_\gamma}) \subset \mathbb{C}_-$ and $\hat{X}_\varepsilon$ is the largest solution of the inequality $\hat{\mathcal{R}}(X) - \varepsilon I \geq 0$. Thus $\hat{X}_\varepsilon$ increases montonically as ε decreases to 0. Since $\hat{X}_\varepsilon < 0$ for all ε and $\operatorname{dom}_+\mathcal{R}^{M_\gamma}$ is saturated above, the $\hat{X}_\varepsilon$ converge to a solution $\hat{X} \in \operatorname{dom}_+\mathcal{R}^{M_\gamma} \cap -\mathcal{H}_+$ of $\hat{\mathcal{R}}^{M_\gamma}(X) = 0$. In general, however, $\hat{X}$ is only negative semidefinite and almost stabilizing, i.e. $\sigma(\hat{\mathcal{R}}_{\hat{X}}^{M_\gamma}) \subset \mathbb{C}_- \cup i\mathbb{R}$.
But if the pair $(A, \hat{P}_0)$ (see (24)) is observable, then it follows (by Lemma 4.1 in [Won68]) that also the pair $(A + B_2F, C + D_2F)$ is observable for $F = (D_2^*D_2)^{-1}(B_2^*X - D_2^*C)$. Since the equation $\hat{\mathcal{R}}(\hat{X}) = 0$ implies $(\mathcal{L}_{A+B_2F} + \Pi_1)(\hat{X}) \geq (C + D_2F)^*(C + D_2F)$ Theorem 14 yields $\hat{X} < 0$ and $\sigma(\mathcal{L}_{A+B_2F} + \Pi_1) \subset \mathbb{C}_-$. Thus the stabilizability assumption in Lemma 26 can be replaced by the above observability assumption.

Theorem 27 *If the inequality (17) is solvable in $\operatorname{dom}_+\mathcal{R} \cap -\operatorname{int}\mathcal{H}_+^n$ and if $(A, C^*(I - D_2(D_2^*D_2)^{-1}D_2^*)C)$ is observable, then the equation $\hat{\mathcal{R}}^{M_\gamma}(X) = 0$ has a solution $X \in \operatorname{dom}_+\mathcal{R}^{M_\gamma} \cap -\operatorname{int}\mathcal{H}_+^n$. Each such solution X yields a stabilizing feedback-gain matrix $F = -Q(X)^{-1}S(X)^*$ such that the corresponding disturbance operator $\boldsymbol{L}^F$ has norm $\|\boldsymbol{L}^F\| \leq \gamma$.*

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