Matrix (in)equalities for linear stochastic systems

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1 Introduction

We regard linear Itô differential equations of the type

$$\begin{aligned} dx(t) &= Ax(t)dt + Bv(t)dt \\ &\quad + \sum_{i=1}^{N} A_0^i x(t)dw_i(t) + \sum_{i=1}^{N} B_0^i(t)dw_i(t) \\ z(t) &= Cx(t) + \sum_{i=1}^{N} Dv(t), \end{aligned} \tag{1}$$

where $(A,\, C) \in R^{n\times n} \times R^{q\times n}$, and

$$(A_0^i, B_0^i,\, B,\, D) \in R^{n\times n} \times R^{n\times \ell} \times R^{n\times \ell} \times R^{q\times \ell}.$$

The $(w_i(t))_{t\in R_+}$ are independent zero mean real Wiener processes on a probability space $(\Omega, \mathcal{F}, \mu)$ w.r.t. an increasing family $(\mathcal{F}_t)_{t\in R_+}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$.
Let L_w^2 denote the corresponding space of non-anticipating R^ℓ-valued stochastic processes v with

$$|v(\cdot)|_{L_w^2}^2 := E\left(\int_0^\infty |v(t)|^2 dt\right) < \infty.$$

It is known from Itô-theory, that for all $(x_0, v) \in R^n \times L_w^2$ there exists a unique solution $x(\cdot, x_0, v)$ of (1) and thus also a unique output process $z(\cdot, x_0, v)$.
We write $z(\cdot, 0, v) = \boldsymbol{L}v(\cdot)$, and call $\boldsymbol{L}$ the *perturbation operator* of the system (1). It describes the effect of the input process v (viewed as a stochastic disturbance) on the output process z (interpreted as the vector of the to be controlled variables).

Definition 1.1 *The system (1) is said to be* internally (exponentially mean square) stable *if*

$$\exists M, \omega > 0 :\ \forall x_0 \in R^n,\, t \geq 0 :\ E\|x(t)\|^2 \leq Me^{-\omega t}\|x_0\|^2,$$

where $x(\cdot) = x(\cdot, x_0, 0)$ is the solution of the unperturbed system (with $v(\cdot) \equiv 0$).
The system (1) is called externally stable *if $\boldsymbol{L}$ is a bounded operator $\boldsymbol{L}$: $L_w^2 \to L_w^2$.*

In [3] it was shown, that internal stability of (1) implies external stability.
The norm $|\boldsymbol{L}|$ of the disturbance operator is of special interest. In the deterministic case (if all A_0^i, B_0^i vanish) it is equal to the H^∞-norm of the associated rational transfer matrix. Thus $|\boldsymbol{L}|$ can be seen as a generalized H^∞-type norm for the stochastic system (1). In [3] the problem of minimizing $|\boldsymbol{L}|$ by feedback control, i.e. the disturbance attenuation problem for (1), was addressed and a stochastic version of the bounded real lemma was proved. It gives necessary and sufficient conditions for (1) to be internally stable with $|\boldsymbol{L}| < \gamma$. These conditions comprise a parametrized linear matrix inequality (LMI) and, equivalently, a rational Riccati inequality which combines features of the continuous and the discrete Riccati inequalities in the deterministic case. Our aim is to find algebraic conditions for the solvability of this LMI-problem and to construct a solution if possible. As a first step we will analyze a lower dimensional matrix inequality, known as Khasminskij's criterion for internal stability of (1) which is a subproblem of our LMI-problem.
In the analysis of both the LMI and Khasminskij's criterion the notion of positive operators plays a crucial rôle.
We use the following notation:
Let $\mathcal{H}^n$ denote the real space of $n\times n$ Hermitian matrices, ordered by the cone $\mathcal{H}_+^n \subset \mathcal{H}^n$ of nonnegative definite matrices. We have $P \geq Q$ if $P - Q \in \mathcal{H}_+^n$. An operator $\mathcal{T} : \mathcal{H}^n \longrightarrow \mathcal{H}^m$ is called *positive* if $\mathcal{T}\left(\mathcal{H}_+^n\right) \subset \mathcal{H}_+^m$. By σ and ρ we denote the spectrum and the spectral radius of a linear operator.

2 Khasminskij's criterion

Define $\mathcal{L}_A, \Pi_{A_0} : \mathcal{H}^n \longrightarrow \mathcal{H}^n$ by $\mathcal{L}_A(X) := A^*X + XA$ and $\Pi_{A_0}(X) := \sum_{i=1}^{N}\sum_{j=1}^{N}(A_0^i)^* X A_0^j$.

Theorem 2.1 (Khasminskij 1967, [4]) *The following are equivalent:*

1. *System (1) is internally stable.*

2. $\exists P < 0: \quad \mathcal{L}_A(P) + \Pi_{A_0}(P) > 0.$

3. $\sigma(\mathcal{L}_A + \Pi_{A_0}) \subset C_-$.

Remark 2.2 *a) Theorem 2.1 reduces to Lyapunov's stability theorem for deterministic linear differential equations if* $\Pi_{A_0} = 0$. *Thus the inequality*

$$\mathcal{L}_A(P) + \Pi_{A_0}(P) > 0, \tag{2}$$

may be viewed as a generalization of Lyapunov's inequality. Given a solution $P < 0$ *of (2),* $V(x) := -\langle x, Px \rangle$ *yields a Lyapunov function for the unperturbed system (1) (with* $v \equiv 0$*).*
b) The operator $\mathcal{L}_A + \Pi_{A_0}$ *describes the dynamics of the covariance matrix* Q *of the unperturbed system:*

$$\dot{Q} = (\mathcal{L}_A + \Pi_{A_0})(Q). \tag{3}$$

By Theorem 2.1 system (1) is stable iff system (3) is asymptotically stable. Note, that the system (3) is n^2*-dimensional. In [5] the problem was posed to find algebraic stability criteria for (3) in terms of the entries of* A *and* A_0*. For the special case of* n*-th order scalar linear stochastic equations Routh-Hurwitz-type criteria have been given in [4].*

In the proof of Theorem 2.1 the equivalence of 2. and 3. is shown by establishing their equivalence with 1. In the following we will derive further stability conditions and prove their equivalence without any stochastic considerations. Moreover we will show how to determine the threshold $\tau_{\min}$ at which the parametrized stochastic equation

$$dx(t) = Ax(t)dt + \tau \sum_{i=1}^{N} A_0^{(i)} x(t) dw_i(t) \tag{4}$$

becomes unstable assuming that $\dot{x} = Ax$ is stable. In other words we will answer the question, how far the diffusion term in (4) may be turned up without loss of internal stability:

$$\tau < \tau_{\min} \iff \text{(4) is mean square stable} \tag{5}$$

Our approach is based on Krein-Rutman's theory of positive operators and, more specifically, on the following special version of a result in [7]:

Theorem 2.3 (Schneider 1965, [7]) *Equivalent are:*

1. $\exists P < 0: \quad \mathcal{L}_A(P) + \Pi_{A_0}(P) > 0.$
2. $\forall Q > 0 : \exists^1 P < 0: \quad \mathcal{L}_A(P) + \Pi_{A_0}(P) = Q.$
3. $\sigma(\mathcal{L}_A) \subset C_-$ *and* $\rho(\mathcal{L}_A^{-1}\Pi_{A_0}) < 1.$

Condition 2. implies, that $-\mathcal{L}_A - \Pi_{A_0}$ is nonsingular and its inverse is a positive operator on $\mathcal{H}^n$. Note that $-\mathcal{L}_A^{-1}$ is positive iff $\sigma(A) \subset C_-$ ($\iff \sigma(\mathcal{L}_A) \subset C_-$).

Lemma 2.4 *Set* $\mathcal{T}_\tau := \mathcal{L}_A + \tau^2 \Pi_{A_0}$, $\tau \geq 0$, *and let* $b(\tau) := \max\{\operatorname{Re}\lambda;\ \lambda \in \sigma(\mathcal{T}_\tau)\}$ *be the spectral bound of* $\mathcal{T}_\tau$*. Then* $\mathcal{T}_\tau$ *is resolvent positive, i.e.* $R(\alpha, \mathcal{T}_\tau) := (\alpha I - \mathcal{T}_\tau)^{-1}$ *is positive for all* α *sufficiently large. Moreover* $b(\tau) \in \sigma(\mathcal{T}_\tau)$ *and* $b(\tau)$ *is monotonously increasing with* $\tau \geq 0$.

Proof: Given $\tau \geq 0$, for sufficiently large $\alpha > 0$ the shifted matrix $A - \alpha I_n$ is stable and $\rho\left(\mathcal{L}_{A-\alpha I_n}^{-1} \tau^2 \Pi_{A_0}\right) = \rho\left(\left(\mathcal{L}_A - 2\alpha I_{\mathcal{H}^n}\right)^{-1} \tau^2 \Pi_{A_0}\right) < 1$. Hence by Theorem 2.3

$$\begin{aligned} R(2\alpha, \mathcal{T}_\tau) &= \left(2\alpha I_{\mathcal{H}^n} - \left(\mathcal{L}_A + \tau^2 \Pi_{A_0}\right)\right)^{-1} \\ &= \left(-\mathcal{L}_{A-\alpha I_n} - \tau^2 \Pi_{A_0}\right)^{-1} \end{aligned}$$

is a positive operator on $\mathcal{H}^n$. The Theorem of Krein and Rutman implies $\rho_\alpha(\tau) := \rho\left(R(2\alpha, \mathcal{T}_\tau)\right) \in \sigma\left(R(2\alpha, \mathcal{T}_\tau)\right)$. Obviously $\lambda \in \sigma\left(R(2\alpha, \mathcal{T}_\tau)\right) \iff 2\alpha - 1/\lambda \in \sigma(\mathcal{T}_\tau)$ and an easy argument shows $b(\tau) = 2\alpha - \frac{1}{7}\rho_\alpha(\tau) \in \sigma(\mathcal{T}_\tau)$. Finally the monotonicity statement follows from $\tau' \geq \tau \Rightarrow \mathcal{T}_{\tau'} \geq \mathcal{T}_\tau \Rightarrow R(2\alpha, \mathcal{T}_{\tau'}) \geq R(2\alpha, \mathcal{T}_\tau) \Rightarrow \rho_\alpha(\tau') \geq \rho_\alpha(\tau)$. □

In the following proposition we list a series of stability criteria for the parametrized stochastic system (4).

Proposition 2.5 *Suppose* $\sigma(A) \subset C_-$*. System (4) is mean square stable iff one of the following equivalent conditions holds.*

1. $\exists P < 0: \quad \mathcal{L}_A(P) + \tau^2 \Pi_{A_0}(P) > 0.$
2. $\rho(\mathcal{L}_A^{-1}\Pi_{A_0}) < \tau^{-2}.$
3. $\sigma(\mathcal{L}_A + \tau^2 \Pi_{A_0}) \subset C_-$
4. $\forall \theta \in [0, \tau]: \ d(\theta) := \det(\mathcal{L}_A + \theta^2 \Pi_{A_0}) \neq 0$

Moreover the threshold $\tau_{\min}$ *defined in (5) is given by* $\tau_{\min} = \min\{\tau > 0 \mid \det(\mathcal{L}_A + \tau^2 \Pi_{A_0}) = 0\} = \rho\left(\mathcal{L}_A^{-1}\Pi_{A_0}\right)^{-\frac{1}{2}}$.

Proof: By Theorem 2.1 conditions 1. and 3. are equivalent to the mean square stability of system (4). We now prove the equivalence of the four conditions without referring to the stochastic equation.
By Theorem 2.3 we have 1. $\iff$ 2.
2. $\Rightarrow$ 4. follows, since $\rho(\mathcal{L}_A^{-1}\Pi_{A_0}) < \tau^{-2} \leq \theta^{-2}$ implies $\det\left(I + \mathcal{L}_A^{-1}\theta^2\Pi_{A_0}\right) \neq 0$. To prove the converse let $\rho := \rho(\mathcal{L}_A^{-1}\Pi_{A_0}) \geq \tau^{-2}$. As $-\mathcal{L}_A^{-1}\Pi_{A_0}$ is positive, we have by Krein-Rutman $\rho \in \sigma\left(-\mathcal{L}_A^{-1}\Pi_{A_0}\right)$, i.e. $\det\left(\mathcal{L}_A^{-1}\Pi_{A_0} + \rho I\right) = 0$. Thus the determinant condition in 4. fails for $\theta = \rho^{-\frac{1}{2}} \in [0, \tau]$.
The proof of 3. $\iff$ 4. is an application of Lemma 2.4:

$$\begin{aligned} \sigma(\mathcal{L}_A + \tau^2 \Pi_{A_0}) \subset C_- &\iff b(\tau) < 0 \\ \text{(as } b(\theta) \text{ increases!)} &\iff \forall \theta \in [0, \tau] : b(\theta) < 0 \\ &\iff \forall \theta \in [0, \tau] : d(\theta) \neq 0 \text{ and } \sigma(A) \subset C_- \end{aligned}$$ □

To get more information about $d(\tau)$ we regard special positive operators (see e.g. [1]):

Definition and Theorem 2.6 *An operator $\mathcal{T}:\mathcal{H}^n\to\mathcal{H}^m$ is called* completely positive *if for all $k\in N$ the operator $I_k\otimes\mathcal{T}:\mathcal{H}^{nk}\to\mathcal{H}^{mk}$ is positive.*
An equivalent condition is that $\mathcal{T}$ allows a representation of the form

$$\mathcal{T}(X)=\sum V_iXV_i^*,\qquad \text{where } V_i\in R^{m\times n}.$$

In this case $\operatorname{trace}\mathcal{T}=\sum|\operatorname{trace}V_i|^2\geq 0$.

Proposition 2.7 *If $R(\alpha,\mathcal{T}_\tau)$ is positive, then it is completely positive. Especially $-\mathcal{T}_\tau^{-1}$ is completely positive for $\tau\in[0,\tau_{\min}]$.*

Proof: Write $\alpha I-\mathcal{T}_\tau=\Pi_1-\Pi_2$, where $\Pi_1(X)=\frac{1}{2}\left(\frac{2\alpha+1}{2}I-2A\right)^*X\left(\frac{2\alpha+1}{2}I-2A\right)$ and $\Pi_2(X)=\frac{1}{2}\left(\frac{2\alpha-1}{2}I-2A\right)^*X\left(\frac{2\alpha-1}{2}I-2A\right)+\tau^2\Pi_{A_0}(X)$. By their representation Π_1^{-1}, Π_2 and $\Pi_1^{-1}\Pi_2$ are completely positive, and by Schneider's result in [7] we have $\rho(\Pi_1^{-1}\Pi_2)<1$ iff $R(\alpha,\mathcal{T}_\tau)$ is positive. In this case $R(\alpha,\mathcal{T}_\tau)=\sum_{k=0}^\infty(\Pi_1^{-1}\Pi_2)^k\Pi_1^{-1}$ is also completely positive. □

Theorem 2.8 *$d(\tau)$ is strictly monotonous on $[0,\tau_{\min}]$.*

Proof: As $\mathcal{T}_\tau:=\mathcal{L}_A+\tau\Pi_{A_0}$ is nonsingular on $[0,\tau_{\min}[$, by Liouville's formula

$$\dot d(\tau)=\operatorname{trace}\left(\Pi_{A_0}\mathcal{T}_\tau^{-1}\right)d(\tau).$$

By Proposition 2.7 $-\Pi_{A_0}\mathcal{T}_\tau^{-1}$ is completely positive and thus $\operatorname{trace}\left(\Pi_{A_0}\mathcal{T}_\tau^{-1}\right)\leq 0$. So $d(\tau)$ is monotonous on $[0,\tau_{\min}]$. Strict monotonicity follows since $d(\tau)$ is analytical and by $\Pi_{A_0}\neq 0$ not constant. □

3 A stochastic bounded real lemma

We turn back to the problem of estimating the norm $|\boldsymbol{L}|$ of the perturbation operator. For $X\in\mathcal{H}^n$ define

$$\begin{aligned}
\Lambda(X) &:= \begin{pmatrix} XA+A^*X & XB\\ B^*X & 0\end{pmatrix}\in\mathcal{H}^{2n},\\
\Pi_{B_0}(X) &:= \sum_{i=1}^N\sum_{j=1}^N(B_0^i)^*XB_0^j\in\mathcal{H}^n,\\
\Sigma(X) &:= \sum_{i=1}^N\sum_{j=1}^N(A_0^i)^*XB_0^j,\\
\Pi(X) &:= \begin{pmatrix}\Pi_{A_0}(X) & \Sigma(X)\\ \Sigma(X)^* & \Pi_{B_0}(X)\end{pmatrix}\in\mathcal{H}^{2n},\\
\Gamma_\gamma &:= \begin{pmatrix}-C^*C & -C^*D\\ -D^*C & \gamma^2I-D^*D\end{pmatrix}\in\mathcal{H}^{2n}.
\end{aligned}$$

Theorem 3.1 (Bounded Real Lemma, [3]) *Let $\gamma>0$. Then system (1) is internally stable and $|\boldsymbol{L}|<\gamma$ if and only if*

$$\exists P<0:\quad \Lambda(P)+\Pi(P)+\Gamma_\gamma>0. \tag{6}$$

Note that in the case when in system (1) all A_0^i,B_0^i vanish, condition (6) reduces to: $\exists P<0$:

$$\begin{pmatrix}PA+A^*P & PB\\ B^*P & 0\end{pmatrix}-\begin{pmatrix}C^*C & C^*D\\ D^*C & D^*D-\gamma^2I\end{pmatrix}>0, \tag{7}$$

which is the well known LMI from the bounded real lemma for deterministic systems. It is feasible iff the H^∞-norm of the corresponding transfer matrix is less then γ. By calculating the Schur complement, this LMI can be transformed into an algebraic Riccati inequality (ARI) (compare (11) below), but for (6) we obtain a rational matrix inequality (set $R(P):=\gamma^2I-D^*D+\Pi_{B_0}(P)$):

$$\begin{aligned}&\mathcal{L}_A(P)-C^*C+\Pi_{A_0}(P)\\ &\quad-\Big(PB-C^*D+\Sigma(P)\Big)R(P)^{-1}\Big(\cdots\Big)^*>0.\end{aligned} \tag{8}$$

Note further that (6) arises from (7) through the addition of the *positive operator* Π. The special case where $\Pi_{B_0}=0,\Sigma=0$ was studied first in [8] by Wonham and more recently e.g. in [2].

In Theorem 3.4 below we present an iteration scheme to solve the *nonstrict* problem

$$\exists P<0:\quad \Lambda(P)+\Pi(P)+\Gamma_\gamma\geq 0, \tag{9}$$

which corresponds to the *nonstrict* version of (8); if (A,C) is observable the solvability of (9) is sufficient for $|\boldsymbol{L}|\leq\gamma$. In each step the iteration uses the greatest solution (w.r.t. the order in $\mathcal{H}^n$) of an LMI

$$\Lambda(X)+\Gamma\geq 0\quad\text{with }\Gamma=\begin{pmatrix}T & S\\ S^* & R\end{pmatrix}, \tag{10}$$

or, equivalently, of the corresponding ARI:

$$\mathrm{Ric(X)}:=\mathcal{L}_\mathrm{A}(\mathrm{X})+\mathrm{T}-(\mathrm{XB}+\mathrm{S})\mathrm{R}^{-1}(\mathrm{XB}+\mathrm{S})^*\geq 0. \tag{11}$$

In practice it might be too costly to solve an LMI in every step, but still the result gives important theoretical insights in the solution set of (9). Again the positivity of Π in connection with the following well-known theorem (e.g. [6]) is the decisive tool:

Theorem 3.2 *Assume (A,B) to be stabilizable and regard the LMI (10). The following are equivalent:*

1. $\exists X=X^*:\quad \Lambda(X)+\Gamma\geq 0$.
2. *There exists a greatest solution $X_+=X_+^*$ of $\Lambda(X_+)+\Gamma\geq 0$, and, what is more,* $\mathrm{Ric}(\mathrm{X}_+)=0$.
3. $\begin{pmatrix}(i\omega I-A)^{-1}B\\ I\end{pmatrix}^*\Gamma\begin{pmatrix}(i\omega I-A)^{-1}B\\ I\end{pmatrix}\overset{\forall\omega\in R}{\geq}0.$

Furthermore, if $\sigma(A)\subset C_-$ and $T\leq -C^*C$ with (A,C) observable, then $X_+<0$.

Comparing the solution sets of LMIs with different data yields a monotonicity result:

Corollary 3.3 *Let* $\tilde{\Gamma} \geq \Gamma$. *Then* $\Lambda(X) + \Gamma \geq 0$ *implies* $\Lambda(X) + \tilde{\Gamma} \geq 0$, *and for the respective greatest solutions we have* $\tilde{X}_+ \geq X_+$.

Theorem 3.4 *Let* $\sigma(A) \subset C_-$ *and* (A, C) *observable. Set* $X_0 := 0$ *and for* $k = 0, 1, \ldots$ *define* X_{k+1} *to be the greatest solution of*

$$\Lambda(X) + \Big(\Gamma_\gamma + \Pi(X_k)\Big) \geq 0. \qquad (12)$$

If (12) is not solvable, stop and set $K = k$.
The following assertions hold:

1. $0 > X_k \geq X_{k+1}$ *for* $k < K$.
2. *The sequence* X_k *is well defined for all* $k \in N$ *and converges to a matrix* X_∞ *if and only if the inequality (9) is feasible.*
3. *If the* X_k *converge, then* X_∞ *is the greatest solution of (9), and* X_∞ *solves the corresponding rational matrix* equation *(8) (with '=' instead of '>').*

Proof: 1. Since $\Lambda(X_1) + \Gamma_\gamma \geq 0$ it follows from Theorem 3.2 that $X_1 < 0 = X_0$.
Assume $X_{k+1} \leq X_k$ and $k + 1 < K$. Then X_{k+1} and X_{k+2} respectively are the greatest solutions of

$$\begin{aligned} \Lambda(X_{k+1}) + (\Gamma_\gamma + \Pi(X_k)) &\geq 0 \\ \Lambda(X_{k+2}) + (\Gamma_\gamma + \Pi(X_{k+1})) &\geq 0. \end{aligned}$$

Since $\Gamma_\gamma + \Pi(X_{k+1}) \leq \Gamma_\gamma + \Pi(X_k)$ by our assumption, Corollary 3.3 gives $X_{k+2} \leq X_{k+1}$.
2. Now assume $\exists Y < 0 : \ \Lambda(Y) + (\Pi(Y) + \Gamma_\gamma) \geq 0$.
If $X_k \geq Y$ then also $\Pi(X_k) + \Gamma_\gamma \geq \Pi(Y) + \Gamma_\gamma$, and then by Corollary 3.3 $X_{k+1} \geq Y$ is well defined. So $X_0 \geq Y$ leads to $X_k \geq Y$ for all $k \in N$, and as (X_k) decreases monotonously, there exists the limit $X_\infty \leq X_1 < 0$.
3. If the limit X_∞ exists, it solves (9), and $X_\infty \geq Y$ for any other solution Y. What is more, every X_{k+1} solves the Riccati equation corresponding to (12), and thus in the limit X_∞ solves the rational matrix equation. □

In fact, it is not crucial to start the iteration with $X_0 = 0$. To show this we apply Theorem 2.3:

Corollary 3.5 *Let (9) be feasible and* $\tilde{X}_0 \geq X_\infty$. *For* $k > 0$ *define* $\tilde{X}_k$ *by the iteration scheme (12). Then* $\lim \tilde{X}_k = X_\infty$.

Proof: As above we see that all $\tilde{X}_k$ are well defined and $\tilde{X}_k \geq X_\infty$. If $\tilde{X}_0 < 0$ then $\tilde{X}_k \leq X_k$ for all k and thus $\lim \tilde{X}_k = X_\infty$.
It remains to show that $\tilde{X}_k < 0$ for some k. Regard the left upper block of (12):

$$\mathcal{L}_A(X) + \Pi_{A_0}(\tilde{X}_k) - C^*C \geq 0, \qquad (13)$$

Since $\sigma(A) \subset C_-$ the operator $\mathcal{L}_A$ is nonsingular and $\mathcal{L}_A^{-1}(P + C^*C) < 0$ for all $P \geq 0$. By continuity there exists an $\epsilon > 0$ such that $\mathcal{L}_A^{-1}(P + C^*C - \Pi_{A_0}(\tilde{X}_k)) < 0$ for $\tilde{X}_k \leq \epsilon I$. For arbitrary $\tilde{X}_0 \geq X_\infty$ we have from (13)

$$\tilde{X}_1 \leq -(\mathcal{L}_A^{-1}\Pi_{A_0})(\tilde{X}_0).$$

Induction yields $\tilde{X}_k \leq \Big(-(\mathcal{L}_A^{-1}\Pi_{A_0})\Big)^k(\tilde{X}_0) \leq \epsilon I$ for sufficiently large k, since by Theorem 2.3 the feasibility of (9) implies $\rho(\mathcal{L}_A^{-1}\Pi_{A_0}) < 1$. Thus $\tilde{X}_{k+1} < 0$ and the proof is complete. □

Remark 3.6 *The above algorithm produces a sequence of necessary conditions for the feasibility of (9) which by Theorem 3.2 can be written as follows:*

$$\begin{pmatrix} W(i\omega) \\ I \end{pmatrix}^* \Big(\Gamma_\gamma + \Pi(X_k)\Big) \begin{pmatrix} W(i\omega) \\ I \end{pmatrix} \overset{\forall \omega \in R}{\geq} 0 \qquad (14)$$

where $W(i\omega) := (i\omega I - A)^{-1}B$. *Inequality (14) can also be written in the form*

$$\forall \omega \in R: \ \gamma^2 I \geq G(i\omega)^* G(i\omega) - G_k(i\omega) G_k(i\omega), \qquad (15)$$

where $G(i\omega) = C(i\omega - A)^{-1}B + D$ *and* $G_k(i\omega) =$

$$\Big(I \otimes \sqrt{-X_k}\Big)\left(\begin{pmatrix} A_0^1 \\ \vdots \\ A_0^N \end{pmatrix}(i\omega - A)^{-1}B + \begin{pmatrix} B_0^1 \\ \vdots \\ B_0^N \end{pmatrix}\right).$$

References

[1] M.-D. Choi. Completely positive linear maps on complex matrices. *Linear Algebra and Its Applications*, 10:285–290, 1975.

[2] C. E. de Souza and M. D. Fragoso. On the existence of maximal solutions for generalized algebraic Riccati equations arising in stochastic control. *Systems & Control Letters*, 14:233–239, 1990.

[3] A. El Bouhtouri, D. Hinrichsen, and A. J. Pritchard. On the disturbance attenuation problem for a wide class of time invariant linear stochastic systems. *To appear in Stochastics and Stochastics Reports.*

[4] R. Khasminskij. *Stochastic Stability of Differential Equations.* Sijthoff & Noordhoff, Alphen aan den Rijn, NL, 1980.

[5] D. L. Kleinman. On the stability of linear stochastic systems. *IEEE Trans. Autom. Control*, pages 429–430, August 1969.

[6] P. Lancaster and L. Rodman. *Algebraic Riccati Equations.* Clarendon Press Oxford, 1995.

[7] H. Schneider. Positive operators and an inertia theorem. *Numerische Mathematik*, 7:11–17, 1965.

[8] W. M. Wonham. On a matrix Riccati equation of stochastic control. *SIAM J. Control*, 6(4):681–697, 1968.