On detectability of stochastic systems ⋆

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Abstract

We discuss notions of detectability for stochastic linear control systems of Itô type. A natural concept of detectability requires a non-zero output, if the state process is unstable. We show that this property can equivalently be characterized by a generalized version of the Hautus-test. The proof is based on spectral theory for positive operators.

Key words: Detectability, Observability, Stochastic systems, Positive operators, Lyapunov equation

1 Introduction

Stochastic linear control systems have attracted considerable interest in the last 40 years. In the framework of Itô equations together with the notion of mean-square stability it has been observed that problems of e.g. linear quadratic and H^∞ control can be treated quite analogously to the deterministic case, leading to generalized, but similar Riccati-type matrix equations or corresponding LMI-problems (see for instance Wonham (1970); Hinrichsen & Pritchard (1998); Yong & Zhou (1999); Petersen, Ugrinovskii, & Savkin (2000)). Still there seems to be an ongoing struggle for the appropriate concepts of and relations between stabilizability and detectability, e.g. in Drăgan, Halanay, & Stoica (1997); Tessitore (1997); Freiling & Hochhaus (2004); Zhang & Chen (2004).

Usually one defines a system to be detectable, if all its unstable modes produce a non-zero output, i.e. if vanishing of the output $y(t) = 0$ for all t implies that the state $x(t)$ converges to zero.

In the deterministic case, it follows that a system is detectable if and only if the dual system is stabilizable, which again is equivalent to the existence of an asymptotically stable linear dynamic state observer. Moreover, there is an equivalent algebraic criterion, the so called Hautus-test (or Popov-Belevich-Hautus test), which plays an important rôle in the discussion of algebraic Lyapunov and Riccati equations.

In the stochastic case, however, the property that all unstable modes produce some non-zero output is only a necessary but not a sufficient condition for stabilizability of the dual system, and the latter does not give rise to a practicable observer equation. Moreover, there is a purely algebraic way to define an analogue of the Hautus-test for stochastic systems, which turns out to be a necessary condition for stabilizability of the dual system.

Each of these properties may thus be taken as a starting point to define detectability in the stochastic case. Several authors (e.g. Da Prato & Ichikawa (1985); Tessitore (1997); Drăgan, Halanay, & Stoica (1997); Fragoso, Costa, & de Souza (1998); Freiling & Hochhaus (2004)) have chosen the second, i.e. stabilizabilty of the dual system, as a defining property. This choice, however, has some drawbacks. Firstly there is no clear interpretation with respect to dynamical properties of the underlying stochastic control system (as mentioned e.g. in Tessitore (1997)). Secondly, there is no simple equivalent algebraic criterion like the Hautus-test for deterministic systems. Thirdly, in applications to generalized algebraic Lyapunov and Riccati equations only the generalized Hautus-test is used, which is weaker than stabilizability of the dual system (e.g. Freiling & Hochhaus (2004)).

In the present note we show (Theorem 3) that the stochastic version of the Hautus criterion is equivalent to the system being detectable in the sense that all unstable modes produce some non-zero output. Thus the Hautus-test indeed

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corresponds to the natural definition of detectability. This observation is the central contribution of this note. Its proof is based on spectral theory of positive operators and has some interesting features. Moreover, an explicit example is given that detectability in this sense does not always imply stabilizability of the dual system.

A similar approach has been taken by Zhang & Chen (2004), who call a system exactly observable if all non-trivial solutions (not only the unstable ones) cause some non-zero output; they derive an equivalent Hautus-test, which again is an analogue of the Hautus-test for observability of deterministic systems. Our notion of detectability fits nicely into this framework, since it is weaker than (exact) observability, as one would expect. As an application we strengthen a result on generalized algebraic Lyapunov equations given in Zhang & Chen (2004).

2 Definition of detectability

We consider stochastic linear systems of the form

$$\begin{aligned} dx &= Ax\,dt + \sum_{j=1}^N A_j x\,dw_j\ , \\ dy &= Cx\,dt + \sum_{j=1}^N C_j x\,dw_j\ . \end{aligned} \tag{1}$$

Here $x \in \mathbb{R}^n$ is the state vector and $y \in \mathbb{R}^p$ is the measured output. The scalar Wiener processes w_j are independent, and the equations are interpreted in the Itô-sense. For a given initial state $x(0) = x_0$, we denote the corresponding solution process by $x(t, x_0)$ and the output process by $y(t, x_0)$.

A natural requirement for the system to be called detectable is that all unstable solutions x lead to some non-zero output y. Or vice-versa, if $y(t, x_0) \equiv 0$ for all t (in mean-square), then $\lim x(t, x_0) \to 0$. Here, the limit will be understood in the mean-square sense, i.e. $E\left(x(t, x_0)^T x(t, x_0)\right) \to 0$ as $t \to \infty$, where E denotes expectation. We summarize this in the following definition.

Definition 1 *System (1) is* detectable*, if for all $x_0 \in \mathbb{R}^n$*

$$E\left(y(t, x_0)^T y(t, x_0)\right) = 0 \qquad \text{for all } t \geq 0$$

implies

$$\lim_{t\to\infty} E\left(x(t, x_0)^T x(t, x_0)\right) = 0\ .$$

Remark 2 *(a) With respect to the conditions in the definition it is worthwhile to note that $E\left(y(t, x_0)^T y(t, x_0)\right) = 0$ is equivalent to $y(t, x_0) = 0$ almost surely, but convergence in mean-square is stronger than stochastic convergence (e.g. Arnold (1974)).*

(b) As was mentioned above, Definition 1 is not the only conceivable way to define detectability.

Following the definition in the references cited in the introduction, the system would be called detectable, if the dual system

$$dx = A^T x\,dt + \sum_{j=1}^N A_j^T x\,dw_j + C^T u\,dt + \sum_{j=1}^N C_j^T u\,dw_j$$

is stabilizable. This means (e.g. Willems & Willems (1976); Tessitore (1994)) that there exists a gain matrix K such that the closed-loop system

$$dx = (A + KC)^T x\,dt + \sum_{j=1}^N (A_j + KC_j)^T x\,dw_j \tag{2}$$

is mean-square stable. An observer, based on this equation would take the form of system (1) with additional input

$$\begin{aligned} d\xi &= A\xi\,dt + \sum_{j=1}^N A_j \xi\,dw_j + K(d\eta - dy) \\ d\eta &= C\xi\,dt + \sum_{j=1}^N C_j \xi\,dw_j\ . \end{aligned}$$

such that $e = \xi - x$ satisfies (2).
This observer, however, is impossible to realize, since it requires the exact reproduction of the noise terms dw_j. Moreover, it involves the stochastic differentials dy and $d\eta$, like e.g. the filtering equations derived in Gershon, Limebeer, Shaked, & Yaesh (2001).
If there is no noise term in the output equation, i.e. $y = Cx$, then the observer based on (2) *can be written more conventionally as*

$$d\xi = A\xi\,dt + \sum_{j=1}^N A_j \xi\,dw_j + K(\eta - y)\,dt\ , \quad \eta = C\xi\ ,$$

which does not involve dy and $d\eta$, but still requires dw_j.
In contrast, a realizable observer equation in this case is

$$d\xi = A\xi\,dt + K(\eta - y)\,dt\ , \quad \eta = C\xi\ .$$

Here, the equation for the error $e = \xi - x$ becomes

$$de = (A + KC)e\,dt - \sum_{j=1}^N A_j x\,dw_j\ ,$$

where, however, the stability of (2) cannot be exploited.

3 A Hautus-test for detectability

Let $H^n \subset \mathbb{R}^{n\times n}$ denote the real vector space of $n \times n$ symmetric (Hermitian) matrices. This space is endowed with the inner product $\langle X, Y\rangle = \operatorname{trace} XY$ and ordered by the regular closed convex cone H^n_+ of nonnegative matrices. For a linear operator $T : H^n \to H^n$, let $T^* : H^n \to H^n$ denote the adjoint operator. It is well known (e.g. Arnold (1974)) that for given x_0 the second moments

$$P(t) = E(x(t, x_0)x(t, x_0)^T)$$

of the solutions to (1) satisfy the deterministic linear differential equation

$$\dot{P} = (L_A + \Pi)(P) , \tag{3}$$

where $L_A, \Pi : H^n \to H^n$ are linear operators defined by

$$L_A(X) = AX + XA^T \quad \text{and} \quad \Pi(X) = \sum_{j=1}^{N} A_j X A_j^T \ . \tag{4}$$

The positive eigenvectors $X \in H^n_+$ of the operator $L_A + \Pi$ are essential in the following criterion, which is the main result of this paper.

Theorem 3 *System (1) is detectable if and only if*

$$X\left[C^T, C_1^T, \ldots, C_N^T\right] \neq 0$$

for every eigenvector $X \in H^n_+ \setminus \{0\}$ of $L_A + \Pi$ corresponding to some eigenvalue $\lambda \geq 0$.

Theorem 3 is an extension of the following Lemma 4 to stochastic systems. Lemma 4 again is a generalized version of the Hautus-test for deterministic systems.

Lemma 4 *Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$. Then the following are equivalent:*

(i) *The pair (A, C) (i.e. the deterministic system $\dot{x} = Ax$, $y = Cx$) is detectable.*
(ii) *$Cv \neq 0$ for every eigenvector v of A corresponding to some eigenvalue λ with* $\operatorname{Re}\lambda \geq 0$.
(iii) *$CX \neq 0$ for every eigenvector $X \in H^n_+ \setminus \{0\}$ of L_A corresponding to some eigenvalue $\lambda \geq 0$.*

We have a look at the proof since it anticipates some of the reasoning needed in the proof of Theorem 3.

PROOF. The equivalence of (i) and (ii) is well-known as the Hautus criterion, e.g. Hautus (1969).

To show that (iii) implies (ii) recall that

$$\sigma(L_A) = \{\lambda + \bar{\mu} \mid \lambda, \mu \in \sigma(A)\}$$

and the eigenspace of L_A corresponding to $\lambda + \bar{\mu}$ contains the matrices vw^*, where v and w are eigenvectors of A corresponding to λ and μ, respectively. If (ii) does not hold, then there exists an eigenpair (λ, v) of A with $\operatorname{Re}\lambda \geq 0$ and $Cv = 0$. With $X = vv^* \geq 0$ we therefore have $L_A(X) = 2\operatorname{Re}\lambda X$ and $CX = 0$, i.e. (iii) does not hold.

Vice versa assume that (iii) does not hold, i.e. there exist $\lambda \geq 0$ and a nonzero $X \geq 0$ with $CX = 0$ such that $L_A(X) = \lambda X$. According to e.g. Lemma 5 in Rantzer (1997) there exists a decomposition $X = \sum_{j=1}^{\operatorname{rank} X} x_j x_j^*$ with $L_A(x_j x_j^*) = \lambda x_j x_j^*$ and $Cx_j x_j^* = 0$ for all j. Hence we have at least one vector $x_1 \neq 0$ satisfying $Cx_1 = 0$ and $Ax_1 = \mu_1 x_j$ with $\operatorname{Re}\mu_1 = \lambda/2 \geq 0$. Thus (ii) is violated.

Remark 5 *It is remarkable though not really surprising that the random vibrations in the output are advantageous for detectability, because the condition in Theorem 3 is satisfied the easier, the more matrices C_j there are. Actually, it is similar for the noise terms in the state equation, because – loosely speaking – the more matrices A_j there are, the higher the rank of the eigenvectors of $L_A + \Pi$ is likely to be. Note that the situation is contrary if we think of (mean-square) stabilizability of the dual equation. In this case, the noise terms are always disadvantageous. Hence the two concepts fall the more apart, the stronger the noise terms are in an appropriate sense.*

4 Proof of Theorem 3

To prove Theorem 3 we recall some results on resolvent positive operators on H^n which have been collected e.g. in Berman, Neumann, & Stern (1989) or Damm (2004). Let H be some finite-dimensional space, ordered by a closed, solid, pointed convex cone H_+.

Definition 6 *A linear operator $T : H \to H$ is called* resolvent positive *if there exists an $\alpha_0 \in \mathbb{R}$ such that for all $\alpha \geq \alpha_0$ the resolvent $(\alpha I - T)^{-1}$ is positive, i.e.*

$$(\alpha I - T)^{-1}(H_+) \subset (H_+) \ .$$

Proposition 7 *The operator $L_A + \Pi : H^n \to H^n$ is resolvent positive.*

Proposition 8 *Let $T : H \to H$ be resolvent positive and set $\beta = \max \operatorname{Re}\sigma(T)$. Then there exists an $X \in H_+$, $X \neq 0$, such that $T(X) = \beta X$.*
Moreover, the following are equivalent:

(i) $\beta(T) < 0$,
(ii) $\exists X \in \operatorname{int} H_+ : \ T(X) < 0$,
(iii) $\forall Y \in \operatorname{int} H_+ : \ \exists X \in \operatorname{int} H_+ : \ T(X) = Y$.

Now we are ready to prove Theorem 3.

PROOF. Let us first assume that the criterion of the theorem does not hold. For notational purpose, set $C_0 = C$. Hence there exists a non-zero $X_0 \geq 0$, such that $C_j X_0 = 0$ for all $j = 0, \ldots, N$ and $(L_A + \Pi)(X_0) = \lambda X_0$ with $\lambda \geq 0$. Consider a decomposition $X_0 = \sum_{\ell=1}^n X_0^{(\ell)}$ in rank-1 matrices $X_0^{(\ell)} = x_0^{(\ell)} x_0^{(\ell)T}$ with $x_0^{(\ell)} \in \mathbb{R}^n$. If, in general, $X(t, Y_0)$ denotes the solution of the matrix differential equation

$$\dot{X} = (L_A + \Pi)(X) , \tag{5}$$

starting at Y_0, then, by construction,

$$\begin{aligned} e^{\lambda t} X_0 = X(t, X_0) &= \sum_{\ell=1}^n X(t, X_0^{(\ell)}) \\ &= \sum_{\ell=1}^n E x(t, x_0^{(\ell)}) x(t, x_0^{(\ell)})^T . \end{aligned}$$

Since $\lambda \geq 0$, at least one of the summands must not converge to zero for $t \to \infty$, i.e. for some ℓ_0 we have

$$E\|x(t, x_0^{(\ell_0)})\|^2 \overset{t\to\infty}{\not\to} 0 .$$

On the other hand, since $X(t, X_0) \geq x(t, x_0^{(\ell_0)}) x(t, x_0^{(\ell_0)})^T$ and $C_j X(t, X_0) = 0$ for all t and j, it follows that also $C_j x(t, x_0^{(\ell_0)}) x(t, x_0^{(\ell_0)})^T = 0$, i.e. $dy(t, x_0^{(\ell_0)}) = 0$. Hence $y(t, x_0^{(\ell_0)})$ is constant and can be assumed to be zero (since we can assume w.l.o.g that $y(0, x_0^{(\ell_0)}) = 0$). Therefore the system is not detectable.

Let us now assume that the criterion of the theorem holds and there exists a non-zero solution $x(t, x_0)$ with $E\|y(t, x_0)\|^2 = 0$ for all $t \geq 0$. It follows that $C_j x(t, x_0) = 0$ for all $t \geq 0$ and $j = 0, \ldots, N$. We need to show that

$$E\|x(t, x_0)\|^2 \to 0 \quad \text{as } t \to \infty . \tag{6}$$

The second-moment matrix $P(t) = E x(t, x_0) x(t, x_0)^T$ satisfies the differential equation (3). We define the set

$$H_+ = \operatorname{cl}\operatorname{conv}\{P(t) \mid t \geq 0\} \subset H_+^n ,$$

which is the closed convex hull of the positive orbit of $P(t)$. Let $H = H_+ - H_+$ be the minimal subspace of H^n containing H_+, endowed with the inner product inherited from H^n. Then H_+ is a closed solid pointed convex cone in H. By construction, both H_+ and H are invariant with respect to (5). That means $(L_A + \Pi)(H) \subset H$, and the restriction $(L_A + \Pi)\big|_H$ is resolvent positive with respect to H_+. Let β_H be the spectral bound of $(L_A + \Pi)\big|_H$. We will show that $\beta_H < 0$, which implies $P(t) \to 0$ and thus also (6).
By Proposition 8 there exists an eigenvector $X_H \in H_+$, such that

$$(L_A + \Pi)(X_H) = \beta_H X_H .$$

Since $C_j P(t) = 0$ for all $t \geq 0$ and $j = 0, \ldots, N$ we conclude $C_j X = 0$ for all $X \in H$. In particular $C_j X_H = 0$ for $j = 0, \ldots, N$. It follows now from the detectability criterion that $\beta_H < 0$, which we needed to show.

5 Some consequences

First, we verify that detectability in the sense of Definition 1 is necessary but not sufficient for the dual system to be stabilizable.

Proposition 9 *Consider system (1). If there exists a matrix K, such that the system*

$$dx = (A + KC)^T x \, dt + \sum_{j=1}^N (A_j + KC_j)^T x \, dw_j \tag{7}$$

is mean-square stable, then (1) is detectable.

PROOF. Let

$$\Pi_K : X \mapsto \sum_{j=1}^N (A_j + KC_j) X (A_j + KC_j)^T .$$

System (7) is mean-square stable, if and only if

$$\beta(L_{A+KC} + \Pi_K) < 0 . \tag{8}$$

Now assume that $(L_A + \Pi)(X) = \lambda X$ and $CX = 0$. Then

$$(L_{A+KC} + \Pi_K)(X) = (L_A + \Pi)(X) = \lambda X ,$$

contradicting (8).

For a comparison we recall the definition of exact observability. It was defined in Zhang & Chen (2004) for the case without noise terms in the output equation, but it is straightforward to restate the condition in our more general situation.

Definition 10 *System (1) is* exactly observable*, if $y(t) \neq 0$ for all $t \geq 0$ almost surely, implies $x_0 \neq 0$.*

The corresponding version of the Hautus test is the following.

Proposition 11 *System (1) is exactly observable if and only if*

$$X\Big[C^T, C_1^T, \ldots, C_N^T\Big] \neq 0$$

for every eigenvector $X \in H_+^n \setminus \{0\}$ of $L_A + \Pi$.

It generalizes the Hautus-test for observability in the same manner as Theorem 3 generalizes the Hautus test for detectability.
Obviously, exact observability implies detectability, whereas the converse is not true, as follows already from deterministic examples. In particular every stable system (with $\beta(L_A+\Pi)<0$) of the form (1) is detectable, independent of the output equation, but it is not necessarily observable. An exactly observable and hence detectable system, whose dual is not stabilizable, is given in the following example.

Example 12 *Consider the system*

$$dx = A^T x\,dt + A_0^T x\,dw\ , dy = Cx\,dt$$

with $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = C^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
Note that the pair (A,B) is controllable (resp. (A^T,C) is observable), but the dual system

$$dx = Ax\,dt + Bu\,dt + A_0x\,dw$$

is not stabilizable. To see this, we consider the operator $L_{A+BF}+\Pi : H^2 \to H^2$ with $F=[f_1,f_2]\in\mathbb{R}^{1\times 2}$. It has the Kronecker-product matrix representation

$$M_F = \begin{bmatrix} 2 & 1 & 1 & 1 \\ f_1 & 2+f_2 & 1 & 1 \\ f_1 & 1 & 2+f_2 & 1 \\ 1 & f_1 & f_1 & 2+2f_2 \end{bmatrix} .$$

To simplify the problem, we consider the restriction of the operator M_F to the three-dimensional subspace $H^2\subset\mathbb{R}^{2\times 2}$. The symmetric matrices

$$U_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ U_2 = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ U_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

constitute an orthonormal basis of this subspace. Hence, with

$$U = [\operatorname{vec} U_1, \operatorname{vec} U_2, \operatorname{vec} U_3] = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

we apply the transformation

$$\hat{M}_F = U^T M_F U = \begin{bmatrix} 2 & \sqrt{2} & 1 \\ \sqrt{2}f_1 & 3+f_2 & \sqrt{2} \\ 1 & \sqrt{2}f_1 & 2+2f_2 \end{bmatrix} .$$

A short calculation yields

$$\det(\hat{M}_F - I) = 2(f_1 - f_2 - 1)^2 \geq 0\ .$$

This shows that for all $F\in\mathbb{R}^{2\times 1}$ the matrix $\hat{M}_F$ (and thus the operator $L_{A+BF}+\Pi$) has an eigenvalue λ with $\operatorname{Re}\lambda \geq 1$. Though we can move the eigenvalues of L_{A+BF} as far as we wish into the left half plane, we cannot stabilize the operator $L_{A+BF}+\Pi$ by any $F\in\mathbb{R}^{2\times 1}$.
Nevertheless, the given system is detectable. In fact, $CX\neq 0$ for all eigenvectors of $L_A+\Pi$ (i.e. the system is even exactly observable). To show this, we consider M_F^T with $F=0$, which represents $L_{A^T}+\Pi$.
For $X\in\mathbb{R}^{2\times 2}$ we have $CX=0$, if and only if $\operatorname{vec} X = [x_1,0,x_3,0]^T$, with $x_1,x_3\in\mathbb{C}$. But the equation $M_0^T \operatorname{vec} X = \lambda \operatorname{vec} X$ obviously requires $x_3=0$ and thus also $x_1=0$. Hence, X cannot be an eigenvector of $L_{A^T}+\Pi$, if $CX=0$.

It is interesting that by the Hautus criterion, our notion of detectability can be extended to arbitrary resolvent positive operators:

Definition 13 *Let $T : H \to H$ be resolvent positive, and $Y\in H$. Then the pair (T,Y) is called* detectable*, if $\langle Y,V\rangle \neq 0$ for all eigenvectors V of T^* corresponding to an eigenvalue $\lambda\geq 0$.*

As an application, we derive another condition equivalent to the conditions (i), (ii), (iii) of Proposition 8. This result extends Theorem 6 of Zhang & Chen (2004).

Proposition 14 *Let $T : H\to H$ be resolvent positive and set $\beta(T) = \max\operatorname{Re}\sigma(T)$. Then $\beta(T)<0$ if and only if*

(iv) $\exists X,Y\in H_+ : T(X) = -Y$ *and (T,Y) is detectable*

PROOF. It is obvious that (iii) implies (iv). We prove that (iv) implies (i). So assume that (iv) holds with some $X\in H_+$ but $\beta(T)\geq 0$. Then there exists an eigenvector $V\in H_+^*$ of T^*, such that $T^*(V) = \beta(T)V$. Hence

$$0 \geq \langle -Y,V\rangle = \langle T(X),V\rangle = \langle X,T^*(V)\rangle = \beta\langle X,V\rangle \geq 0\ ,$$

i.e. $\langle Y,V\rangle = 0$ contradicting detectability.

Freiling & Hochhaus (2004) derive this result for $T = L_A+\Pi$ as in (4) and their stronger notion of detectability (i.e. stabilizability of the dual system). They use it to formulate criteria for the existence of stabilizing solutions to certain generalized Riccati equations. Without changing the proofs we may thus strengthen some of these criteria by using the weaker (and more natural) notion of detectability.

Example 15 *We consider a simple version of Lemma 5.6 from Freiling & Hochhaus (2004). It says that every nonnegative definite solution of the Riccati-type equation*

$$AX + XA^T + Q + \Pi(X) + Q - XBR^{-1}B^TX = 0$$

(with $Q \geq 0$, $R > 0$, $\Pi(X) = A_0XA_0^T$*) is* stabilizing *i.e.*

$$\sigma(L_{A-XBR^{-1}B^T} + \Pi) \subset \mathbb{C}_- \ ,$$

if $(L_A + \Pi, Q)$ *is detectable.*
For A, A_0*, and* C *as in Example 12 the stronger notion of detectability is not satisfied, such that the lemma seems to be not applicable. Knowing, however, that only our weaker notion of detectability is needed, we can still apply the result.*

More important, however, than the possibility to apply the criteria to a larger class of systems, is the fact that by Theorem 3 we obtain a better understanding of these criteria.

6 Conclusions

We briefly summarize our main observations

(i) For stochastic systems, different concepts of detectability can be thought of, which unlike in the deterministic case, are not equivalent.
(ii) Detectability defined as the property that the system always produces some non-zero output if the state process is unstable, is equivalent to a generalized Hautus criterion; the latter is an appropriate algebraic criterion to deal with generalized Lyapunov and Riccati equations.
(iii) Detectability defined as the property that the dual system is stabilizable, does not have a natural interpretation. In particular, this property does not offer a method to reconstruct the state from measurements.
(iv) It is useful to view stability, stabilizability and detectability as properties of certain resolvent positive operators.

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